Chapter 10
Graph Enumeration

10.1 Introduction

Graph enumeration is a study in graph theory that deals with counting non-isomorphic graphs with a particular property. Harary and Palmer [60] provide an excellent introduction to the topic of graph enumeration. On counting labelled cubic graphs, there has been a series of results, most notably Read [88], Read [89], Wormald [104], and Wormald [105], which collectively present various approaches for counting labelled cubic graphs, and labelled cubic graphs with a given connectivity. In comparison to labelled cubic graphs, the numeration of unlabelled cubic graphs is a significantly more challenging problem [105]. Robinson [90] presents a method to count unlabelled cubic graphs.

In this chapter, we present the starting point of a possible approach to resolving Conjecture 9.1 on the prevalence of cubic bridge graphs. The idea is, if we can find a method, possibly recursive, to count all cubic bridge graphs of a given order, and use the same method to count the number of cubic bridge graphs of the same order, then we can evaluate the ratio of cubic bridge graphs to cubic graphs. Then, we can compare this ratio to the ratio determined in Robinson and Wormald [91], which also states the striking result that almost all cubic graphs are Hamiltonian.

Here, we derive a new recursive formula for the cardinality of unlabelled cubic graphs with bridges, weighted by the number of orbits of bridges in each graph. In the process of deriving the formula, we introduce a new graph property: *subdivision-equivalent edges*. Two edges in a graph are *subdivision-equivalent* if the two graphs G and H obtained by subdividing each edge respectively are isomorphic. In a sense, subdivision-equivalent edges are the opposite of *pseudo-similar edges*. Two edges in a graph are *pseudo-similar* if the two graphs G and H obtained by removing each edge respectively are isomorphic, but there is no automorphism of the original graph that maps

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one edge to another. Pseudo-similar edges have received a lot of attention due to their relationship to the well-known Reconstruction Conjecture in graph theory [75], which states that graphs are uniquely determined by their set of subgraphs.

10.2 Subdivision-equivalent Edges

Wormald [105] presents formulae to enumerate labelled cubic graphs with a given connectivity. In particular, the author determines recurrence relations to count connected cubic graphs, 2-connected cubic graphs and 3-connected cubic graphs. Recall that all cubic bridge graphs are 1-connected, but not 2- or 3-connected. Consequently, subtracting the number of labelled 2-connected cubic graphs from the total number of labelled cubic graphs of the same order gives the number of labelled cubic bridge graphs. This, together with the fact that the expected number of non-trivial automorphisms of a labelled cubic graph of order N approaches zero as N tends to infinity [106], gives us an asymptotic number of unlabelled bridge graphs. However, subtracting one recurrence relation from another recurrence relation, in this case, results in a formula that is not easy to interpret or manipulate. For the sake of completeness, we include here the two aforementioned recurrence relations.

Proposition 10.1. [105] *For $N = 2n$, the following two statements hold.*
(i) The number of labelled cubic graphs of order N is

$$\frac{(2n)!}{3n2^n}\hat{r}_n,$$

where $\hat{s}_0 = \hat{s}_1 = 0, \hat{s}_2 = 1$, and

$$\begin{aligned}\hat{s}_n &= 3n\hat{s}_{n-1} + 4\hat{s}_{n-2} + 2\hat{s}_{n-3} \\ &\quad + \sum_{i=2}^{n-3} \hat{s}_i \left(\hat{s}_{n-1-i} - 2\hat{s}_{n-2-i} - 2\hat{s}_{n-3-i}\right), \quad \textit{for } n \geq 3, \\ \hat{r}_n &= \hat{s}_n - 2\hat{s}_{n-1} - 2\hat{s}_{n-2}, \quad \textit{for } n \geq 2.\end{aligned}$$

(ii) The number of labelled 2-connected cubic graphs of order N is

$$\frac{(2n)!}{3n2^n}\widetilde{r}_n,$$

where $\widetilde{s}_1 = 0, \widetilde{s}_2 = 1$, and

$$\widetilde{s}_n = 3n\widetilde{s}_{n-1} + 2\widetilde{s}_{n-2} + (3n-1)\sum_{i=2}^{n-3} \widetilde{s}_1\widetilde{s}_{n-1-i}, \quad \textit{for } n \geq 3,$$

$$\widetilde{r}_n = \widetilde{s}_n - 2\widetilde{s}_{n-1}, \qquad \textit{for } n \geq 2.$$

Automorphisms, Isomorphisms, Orbits and Similarities Consider two graphs G and H. Let $V(G)$ and $V(H)$ be the sets of vertices, and $E(G)$ and $E(H)$ be the sets of edges in G and H, respectively. Recall that the graphs G and H are *isomorphic* if there exists a bijection $f : V(G) \mapsto V(H)$ such that for every edge $(u, v) \in E(G)$, the edge $(f(u), f(v)) \in E(H)$, and they are said to be *non-isomorphic* otherwise. An *automorphism* of a graph G is an *isomorphism* of G with itself [59]. Two edges e_1 and e_2 in G are *similar* if there is an automorphism that maps e_1 to e_2. Similar edges are said to be in the same *orbit*. An orbit of edges in G is a set of edges in G such that every edge in the set is similar to any other edge in the set.

Example 10.1 *The following two graphs G and H are isomorphic:*

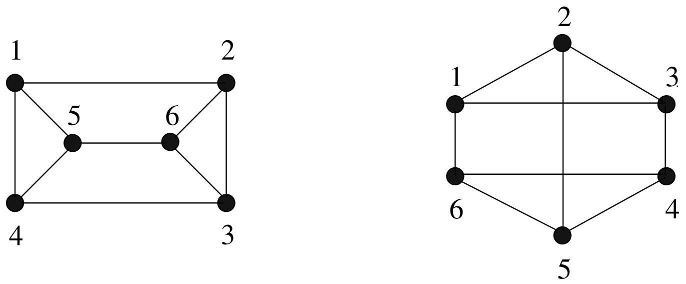

Fig. 10.1: G and H are isomorphic

This is because there exists at least one bijection f: $V(G) \mapsto V(H)$, with $f(1) = 6, f(2) = 1, f(3) = 3, f(4) = 4, f(5) = 4,$ and $f(6) = 2$, such that for every $(u, v) \in E(G)$, the edge $(f(u), f(v)) \in E(H)$. An automorphism of G is $g : V(G) \mapsto V(G)$, with $g(1) = 3, g(2) = 4, g(3) = 1, g(4) = 2, g(5) = 6,$ and $g(6) = 5$. Under this particular automorphism, the edge $(1, 2)$ is mapped into $(3, 4)$, so they are similar. These two edges are also in the same orbit. Note that there are other edges in this orbit, and there are other orbits of edges in this graph.

Pseudo-Similarity and Removal-Similarity. Let $G - e_2$ be the original graph G with the edge e_2 removed. Then $V(G) = V(G - e_2)$ and $E(G) = E(G - e_2) \backslash \{e_2\}$. If e_1 and e_2 are similar, then $G - e_1$ and $G - e_2$ are isomorphic graphs. Two edges e_1 and e_2 are *pseudosimilar* if $G - e_1$ and $G - e_2$ are isomorphic graphs but e_1 and e_2 are not similar in G. *Edge-similarity* and *edge-pseudosimilarity* are collectively known as *edge-removal-similarity*. The parallel concepts of *vertex-similarity*, *vertex-pseudo-similarity* and *vertex-removal-similarity* are defined analogously. As we are only concerned about edge-similarity, edge-pseudosimilarity, and edge-removal-similarity, for convenience, we will drop the prefix *edge-* whenever confusion cannot arise.

We present here a recursive formula, (10.1), to determine the number of cubic bridge graphs of a given order N, weighted by the number of orbits of bridges in these graphs. This means that, when counting cubic bridge graphs of any given order, we count the number of cubic bridge graphs that have one orbit of bridges once, two orbits of bridges twice, and so on.

Remark 10.1. For example, consider three cubic bridge graphs G_1, G_2 and G_3, where G_1 has two bridges, G_2 has three bridges, and G_3 has one bridge. Furthermore, assume that two bridges in G_1 are not in the same orbit (hence G_1 has two orbits of bridges), that all three bridges in G_2 are in the same orbit (hence G_2 has one orbit of bridges). In our method, we count G_1 twice, G_2 once and G_3 once. Hence, the total number of cubic bridge graphs of a given order N, weighted by the number of orbits of bridges in these graphs, is an overestimate of the total number of N-vertex cubic bridge graphs. An open problem is to estimate the difference between these two counts, which we conjecture to be approaching zero as N tends to infinity.

In the process of deriving the aforementioned formula (10.1), we introduce the useful notion of *subdivision-equivalent* edges. Consider an edge $e = (s,t) \in E(G)$. The edge e is *subdivided* if we add a new vertex w to $V(G)$ and replace the edge e with two edges (s,w) and (w,t).

Definition 10.2. *Given a graph G, let G_1 and G_2 be two resulting graphs obtained by subdividing edges e_1 and e_2 of G, respectively. Then, the two edges e_1 and e_2 are* subdivision-equivalent *if and only if G_1 and G_2 are isomorphic.*

Let $\mathsf{Eqe}(G)$ denote the number of sets of subdivision-equivalent edges in G, and let $\mathsf{Eqe}(\mathcal{R})$ denote $\sum_i \mathsf{Eqe}(R_i)$, if $\mathcal{R}$ is a set of graphs R_i.

Example 10.3 *Consider the labelled envelope graph G.*

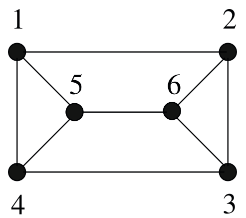

Fig. 10.2: The labelled envelope graph G

Subdividing $(1,2)$ and $(3,4)$ gives us the two graphs G_1 and G_2 in Figs 10.3 and 10.4, respectively. In G, the edges $(1,2)$ and $(3,4)$ are subdivision-equivalent, as the resulting graphs G_1 and G_2 obtained after subdividing these edges respectively are isomorphic. There are other pairs of subdivision-equivalent edges in this graph.

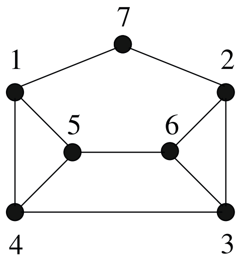

Fig. 10.3: G_1

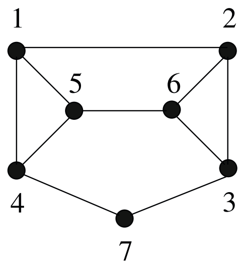

Fig. 10.4: G_2

One of our, still open, problems concerns the question of whether, for regular graphs, two edges are subdivision-equivalent if and only if they are removal-similar.

However, in a general graph, subdivision-equivalent edges are not necessarily removal-similar and vice versa. The following counterexample was provided by Brendan McKay, through private communication. We consider two graphs: in the first one, a pair of edges are removal-similar but not subdivision-equivalent, and in the second, a pair of edges are subdivision-equivalent but not removal-similar.

Example 10.4 *Edges a and c in G are removal-similar but not subdivision-equivalent:*

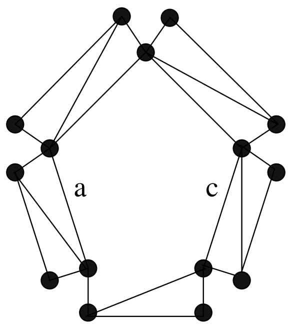

Fig. 10.5: Graph G

The resulting graphs after removing a and c, respectively, are isomorphic (see Figure 10.6), but the resulting graphs after subdividing a and c, respectively, are non-isomorphic (see Figure 10.7).

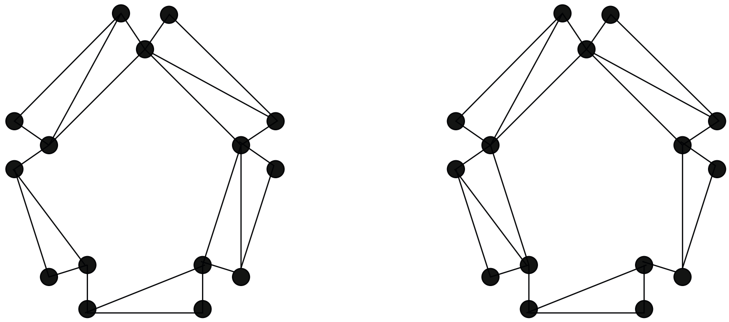

Fig. 10.6: The resulting graphs are isomorphic

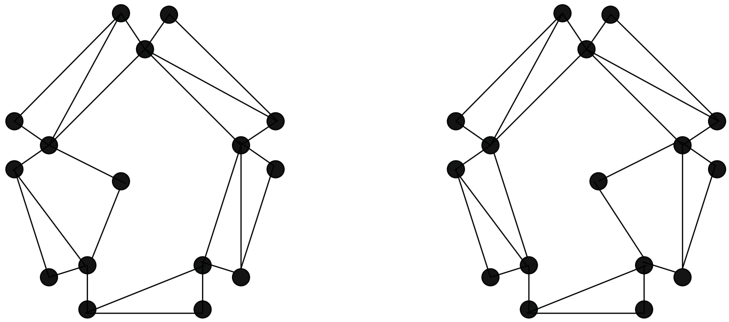

Fig. 10.7: The resulting graphs are non-isomorphic

In the graph Π below, the edges a and c are subdivision-equivalent but not removal-similar:

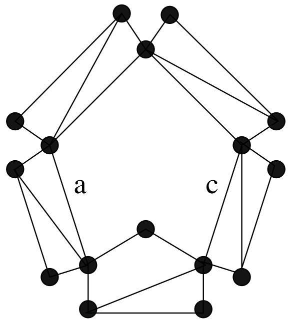

Fig. 10.8: Graph Π

The resulting graphs and after subdividing a and c, respectively, are isomorphic:

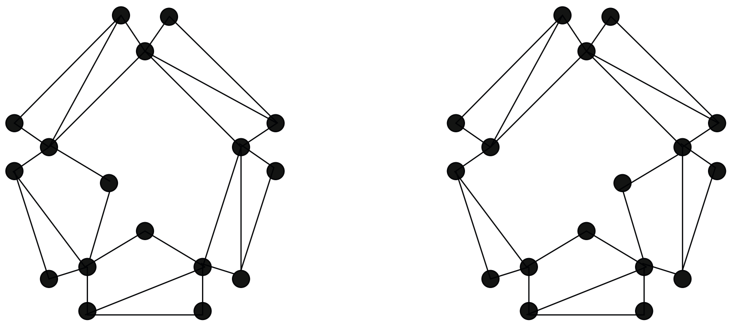

Fig. 10.9: The resulting graphs are isomorphic

The resulting graphs after removing a and c, respectively, are non-isomorphic:

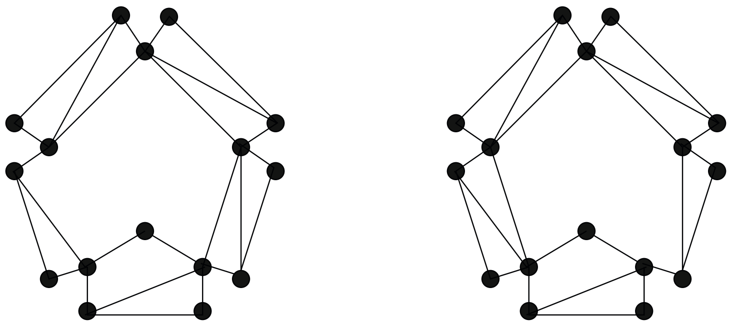

Fig. 10.10: The resulting graphs are non-isomorphic

10.3 Enumerating Cubic Bridge Graphs

We denote by C_N and B_N respectively the sets of cubic graphs and cubic bridge graphs of order N. Our recursive formula for the total number of cubic bridge graphs of order N, weighted by the number of orbits of bridges in each graph, is dependent on $\mathsf{Eqe}(C_{M_i})$ and the number of cubic bridge graphs of various orders $M < N$.

Consider a cubic bridge graph G of order N with a bridge (u, v), displayed in Figure 10.11. Let us denote two vertices that are connected to u (beside v) by u_1 and u_2, and two vertices that are connected to v (beside u) by v_1 and v_2.

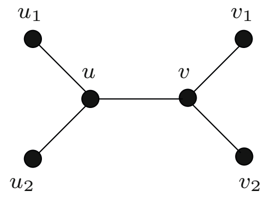

Fig. 10.11: A bridge (u, v)

When a vertex t is removed from a graph, it is assumed that all edges with t as one of its ends are also removed from the graph. Let G_1 and G_2 be the two resulting, disjoint, components if we remove two vertices u and v, and assume that $u_1 \in V(G_1)$ while $v_1 \in V(G_2)$. As (u, v) is a bridge in G, we have $V(G_1 \cup G_2) = V(G) \backslash \{u, v\}$, $u_2 \in V(G_1)$, $v_2 \in V(G_2)$ and $|V(G_1)| + |(V(G_2)| + 2 = N$.

Example 10.5 *Recall the only cubic bridge graph of order 10:*

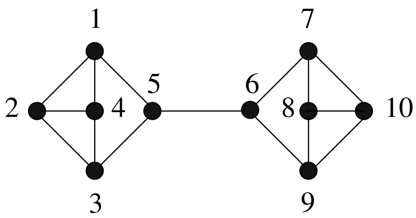

Fig. 10.12: The only cubic bridge graph of order 10

In this graph, the edge $(5, 6)$ *is a bridge. Removing this bridge disconnects the original graph, resulting in two separate components* G_1 *and* G_2*, in Figures 10.13 and 10.13, respectively.*

In any cubic graph of order N, the number of edges is $3N/2$. Therefore, N has to be even, and $|V(G_1)|$ and $|V(G_2)|$ have to be both odd or both even.

Lemma 10.1. $|V(G_1)|$ *and* $|V(G_2)|$ *are both even.*

Proof. Suppose $|V(G_1)|$ and $|V(G_2)|$ are both odd. Let $|V(G_1)| = 2k+1, k \geq 2$. In $V(G_1)$, there are $2k-1$ vertices with degrees of three and two vertices,

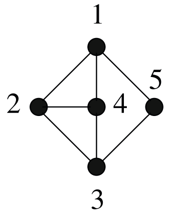

Fig. 10.13: G_1

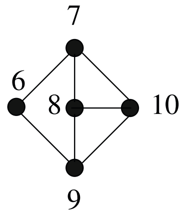

Fig. 10.14: G_2

u_1 and u_2, with degrees of two. The number of edges in G_1 is $(3(2k-1)+4)/2$, which implies that $2k-1$ is an even number, resulting in a contradiction. □

For $k, q \geq 2$, let $|V(G_1)| = 2k$ and $|V(G_1)| = 2q$, where $2k + 2q + 2 = N$. Note that it is not possible to construct G_1 (or G_2) if $k = 1$ (or $q = 1$). Consequently, smallest possible cubic bridge graphs are of order 10. Recall that, in this case, a bridge graph is one of two cubic non-Hamiltonian graphs of order 10, the other one being the famous Petersen graph.

Theorem 10.1. *The number of cubic bridge graphs of order N, weighted by the number of orbits of bridges in each graph, is*

$$\sum_i f(k_i, q_i) + |B_{N-4}| + |B_{N-2}|, \tag{10.1}$$

where (k_i, q_i) are all possible unordered pairs of integers $k_i, q_i \geq 2, 2k_i + 2q_i + 2 = N$, and

$$f(k_i, q_i) = \begin{cases} \mathsf{Eqe}(C_{2k_i})\mathsf{Eqe}(C_{2q_i}) \text{ if } k_i \neq q_i, \\ \frac{1}{2}\mathsf{Eqe}(C_{2k_i})\,[\mathsf{Eqe}(C_{2k_i}) + 1] \text{ if } k_i = q_i. \end{cases} \tag{10.2}$$

Proof. Consider a cubic bridge graph G of order N with a bridge (u, v) as described in Figure 10.11. Again, let G_1 and G_2 be the two resulting, disjoint, components if we remove two vertices u and v and without loss of generality, assume that $u_1 \in V(G_1)$ while $v_1 \in V(G_2)$.

There are now three cases to consider:

1. both edges (u_1, u_2) and (v_1, v_2) are not in $E(G)$,
2. only one of them is $E(G)$, and
3. both edges are in $E(G)$.

For $i = 1, 2, 3$, let B_N^i denote the set of cubic bridge graphs that have an orbit of bridges in each Case i. We count the number of cubic bridge graphs of order N that have an orbit of bridges of Cases 1, 2, and 3, respectively. Naturally, a cubic bridge graph might have two or more orbits of bridges, each of which might be in a different or the same case. In an abuse of terminology, we will simply refer a cubic bridge graph that has an orbit of bridges of Case i as *a cubic bridge graph of Case i*. Adding these three numbers together gives us the total number of cubic bridge graphs of order N where each graph with k orbits of bridges is counted k times.

For example, consider a cubic bridge graph H_1 with three bridges, none of which is in the same orbit of another. Therefore, H_1 has three orbits of bridges. Assume that each of these bridges is in a different case, then H_1 is a cubic bridge graph of Case 1, of Case 2 and also of Case 3. In our counting, H_1 is counted three times, once in each case. Consider a different cubic bridge graph H_2 with three bridges e_1, e_2 and e_3, where e_1 and e_2 are in the same orbit, and e_3 is in a different orbit, but both orbits of bridges are in Case 1. Then H_2 is counted twice in Case 1.

Case 1. Here, $(u_1, u_2), (v_1, v_2) \notin E(G)$.

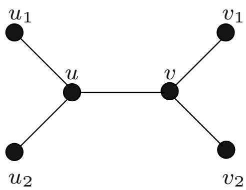

Fig. 10.15: Neighbour around the bridge (u, v)—Case 1

In G_1 and G_2, inserting two new edges (u_1, u_2) and (v_1, v_2) results in two cubic graphs $\bar{G}_1$ and $\bar{G}_2$ of orders $2k$ and $2q$, respectively. Therefore, if we choose any two cubic graphs of orders $2k$ and $2q$, subdivide an edge in each graph (consequently introducing two new vertices), and connect these two vertices (thereby creating a bridge), we obtain a cubic bridge graph of order $2k+2q+2 = N$. For each G, the number of non-isomorphic graphs obtained by subdividing an edge is precisely the number $\mathsf{Eqe}(G)$ of sets of equivalent edges in G.

Therefore,

$$|B_N^1| = \sum_{i=1}^{t} f(k_i, q_i), \tag{10.3}$$

where t is the number of possible unordered pairs of integers (k_i, q_i) such that $2k_i + 2q_i + 2 = N$; $k_i, q_i \geq 2$, and

$$f(k_i, q_i) = \begin{cases} \mathsf{Eqe}(C_{2k_i})\mathsf{Eqe}(C_{2q_i}), & \text{if } k_i \neq q_i, \\ \frac{1}{2}\mathsf{Eqe}(C_{2k_i})\left[\mathsf{Eqe}(C_{2k_i}) + 1\right], & \text{if } k_i = q_i. \end{cases} \tag{10.4}$$

Note that the difference in the formulae of the case for $k_i \neq q_i$ and the case for $k_i = q_i$ is due to the fact that, for the former, we are counting pairs of objects from two different sets, whereas for the latter, we are counting pairs of objects from the same set.

Case 2. Here, $(u_1, u_2) \in E(G), (v_1, v_2) \notin E(G)$. Let $u_3, u_4 \notin \{u, u_2\}$ be the third vertices to which u_1 and u_2 are adjacent, respectively. Now there are two subcases: $u_3 = u_4$ (Figure 10.16) and $u_3 \neq u_4$ (Figure 10.17).

Case 2a. Here, $u_3 = u_4$. Let $u_5 \notin \{u_1, u_2\}$ be the third vertex to which u_3 is adjacent.

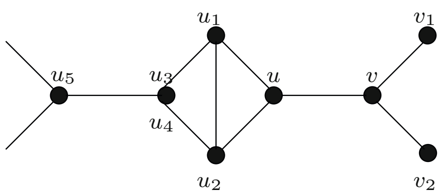

Fig. 10.16: Neighbour around the bridge (u, v) in G—Case 2a

Since $V(G_1) \cap V(G_2) = \emptyset$, (u_3, u_5) is also a bridge. Contracting G by removing the set of "diamond" vertices $\{u, u_1, u_2, u_3\}$ and connecting v to u_5 results in a bridge graph of order $N - 4$ which belongs to either Case 1 or Case 2, since $(v_1, v_2) \notin E(G)$. Therefore, if we choose any cubic bridge graph of order $N - 4$ that belongs to either Case 1 or Case 2, insert a diamond set of four vertices and their associated edges like $\{u, u_1, u_2, u_3\}$ depicted in Figure 10.16 into the bridge, we obtain a cubic bridge graph of order N.

Therefore,

$$|B_N^{2a}| = |B_{N-4}^1| + |B_{N-4}^2|. \tag{10.5}$$

Since the smallest possible cubic bridge graphs are of order 10, the smallest possible cubic bridge graphs in Case 2a are of order 14.

Case 2b. Here, $u_3 \neq u_4$ (see Figure 10.17). Contracting G by removing the set of vertices $\{u_1, u_2\}$ and connecting u to u_3 and u_4 results in a bridge graph of order $N-2$ which belongs to either Case 1 or Case 2, since $(v_1, v_2) \notin E(G)$. Therefore, if we choose any cubic bridge graph of order $N - 2$ that belongs to either Case 1 or Case 2, insert two vertices and their associated edges like $(u_1, u_2), (u_1, u)$ and (u_2, u) depicted in Figure 10.17 into one side of the bridge (the side that has a connecting edge between two vertices), we obtain a cubic bridge graph of order N.

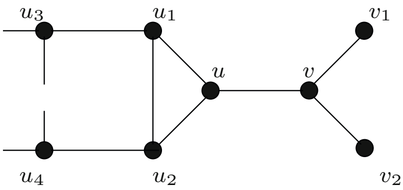

Fig. 10.17: Neighbour around the bridge (u, v) in G—Case 2b

Therefore,

$$|B_N^{2b}| = |B_{N-2}^1| + |B_{N-2}^2|. \tag{10.6}$$

Since the smallest possible cubic bridge graphs are of order 10, the smallest possible cubic bridge graphs in Case 2b are of order 12.

Case 3. Here, $(u_1, u_2), (v_1, v_2) \in E(G)$. Let $u_3, u_4 \notin \{u, u_2\}$ be the "third vertices" to which u_1 and u_2 are adjacent, respectively. Now there are two subcases: $u_3 = u_4$ and $u_3 \neq u_4$.

Case 3a. Here, $u_3 = u_4$. This case can be analysed in a manner similar to Case 2a. Let $u_5 \notin \{u_1, u_2\}$ be the "third vertex" that u_3 is adjacent to.

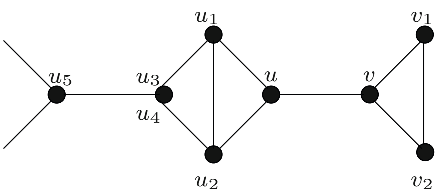

Fig. 10.18: Neighbour around the bridge (u, v) in G—Case 3a

Since $V(G_1) \cap V(G_2) = \emptyset$, (u_3, u_5) is also a bridge. Contracting G by removing the diamond set of vertices $\{u, u_1, u_2, u_3\}$ and connecting v to u_5 results in a bridge graph of order $N - 4$ which belongs to either Case 2 or Case 3, since $(v_1, v_2) \in E(G)$. Therefore, if we choose any cubic bridge graph of order $N - 4$ that belongs to either Case 2 or Case 3, insert a diamond set of four vertices and their associated edges like $\{u, u_1, u_2, u_3\}$ depicted in Figure 10.18 into the bridge, we obtain a cubic bridge graph of order N.

Note that, the set of cubic bridge graphs of order N obtained by applying such a construction to any cubic bridge graph in B^2_{N-4} is equivalent to the set of cubic bridge graphs of order N obtained by applying the construction in Case 2a to any cubic bridge graph in B^2_{N-4}.

Therefore,

$$|B^{2a}_N| + |B^{3a}_N| = |B^1_{N-4}| + |B^2_{N-4}| + |B^3_{N-4}| = |B_{N-4}|. \tag{10.7}$$

Since the smallest possible cubic bridge graphs are of order 10, the smallest possible cubic bridge graph in Case 3a are of order 14.

Case 3b. Here, $u_3 \neq u_4$ (see Figure 10.19).

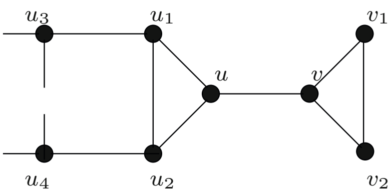

Fig. 10.19: Neighbour around the bridge (u, v) in G—Case 3b

Contracting G by removing the set of vertices $\{u_1, u_2\}$ and connecting u to u_3 and u_4 results in a bridge graph of order $N - 2$ which belongs to either Case 2 or Case 3, since $(v_1, v_2) \notin E(G)$. Therefore, if we choose any cubic bridge graph of order $N - 2$ that belongs to either Case 2 or Case 3, insert a set of two vertices and their associated edges like $\{u_1, u_2\}$ depicted in Figure 10.19 into one side of the bridge (the side that has a connecting edge between two vertices), we obtain a cubic bridge graph of order N.

Note that, the set of cubic bridge graphs of order N obtained by applying such a construction to any cubic bridge graph in B^2_{N-2} is equivalent to the set of cubic bridge graphs of order N obtained from applying the construction in Case 2b to any cubic bridge graph in B^2_{N-2}.

Therefore,

$$|B^{2b}_N| + |B^{3b}_N| = |B^1_{N-2}| + |B^2_{N-2}| + |B^3_{N-2}| = |B_{N-2}|. \tag{10.8}$$

Since the smallest possible cubic bridge graphs are of order 10, the smallest possible cubic bridge graphs in this case are of order 12.

From (10.3),(10.7), and (10.8), we obtain

$$\begin{aligned}|B_N^1| + |B_N^{2a}| + |B_N^{3a}| + |B_N^{2b}| + |B_N^{3b}| &= |B_N^1| + |B_N^2| + |B_N^3| \\ &= \sum_i f(k_i, q_i) + |B_{N-4}| + |B_{N-2}|.\end{aligned}$$

□

In this chapter we outlined one attempt at counting cubic bridge graphs of a given order, and in the process introduced a new graph property, subdivision-equivalent edges. Arguably, the recursive relation described in Theorem 10.1 has potential to become an important component of a practical counting algorithm. However, to date, we do not have any tools for counting, or even estimating, the number of non-bridge non-Hamiltonian graphs of a given order. Thus, it is not yet clear whether Theorem 10.1 can be helpful in resolving Conjecture 9.1 concerning the prevalence of bridge graphs in the population of cubic non-Hamiltonian graphs. Perhaps, a fruitful approach to tackling this conjecture might be to explore a possibility of using results in this chapter together with newly proposed “genetic theory” of cubic graphs (see Baniasadi *et al.* [8]).

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