Chapter 5
Optimal Estimation and Control

Estimation and control are the two fundamental problems for dynamical systems. Many engineering design tasks can be formulated into either an estimation or control problem associated with some appropriate performance index. In order to simplify the design issues in practice, dynamical systems are usually assumed to be linear and finite-dimensional. Otherwise, various approximation methods can be applied to derive linear and finite-dimensional models with negligible modeling errors. As such, state-space representations are made possible providing computational tools for optimal design and enabling design of optimal estimators and controllers.

Estimation aims at design of state estimators that reconstruct the state vector based on measurements of the past and present input and output data. Due to the unknown and random nature of the possible disturbance at the input and the corrupting noise at the output, it is impossible to reconstruct the true state vector in real-time. Therefore, the design objective for state estimators will be minimization of the estimation error variance by assuming white noises for input disturbances and output measurement errors. The focus will be on design of optimal linear estimators.

Disturbance rejection has been the primary objective in feedback control system design in which white noises are the main concern. The emphasis will be placed on the design of state-feedback controllers to not only stabilize the feedback control system but also minimize the adverse effect due to white noise disturbances. With the variance as the performance measure, optimal control leads to linear feedback controllers that are dual to optimal linear estimators.

Without exaggeration, optimal estimation and control are the two most celebrated results in engineering system design. They have brought in not only the design algorithms but also the new design methodology that has had far reaching impacts as evidenced by the wide use of Kalman filtering and feedback control in almost every aspect of the system engineering. Nevertheless, it is the conceptual notions from linear system theory that empower the state-of-the-art design algorithms and allow applications of optimal estimation and control in engineering practice. This chapter will cover the well-known results in Kalman filtering and quadratic regulators that have enriched engineering system design. It also covers optimal output estimators and full information control that are developed more recently.

G. Gu, *Discrete-Time Linear Systems: Theory and Design with Applications*,
DOI 10.1007/978-1-4614-2281-5_5,

5.1 Minimum Variance Estimation

5.1.1 Preliminaries

As a prelude to optimal estimation for state-space systems, a simpler and more intuitive estimation problem will be investigated. Let X and Y be two random vectors with the PDFs $p_X(\mathbf{x})$ and $p_Y(\mathbf{y})$, respectively. A natural question is how knowledge of the value taken by Y can provide information about the value taken by X. In other words, how an *estimate* of $X = \hat{\mathbf{x}}$ can be made based on the observation of $Y = \mathbf{y}$? Clearly, with $Y = \mathbf{y}$ being observed the PDF of X is modified into the *conditional PDF* given by Bayes' rule:

$$p_{X|Y}(\mathbf{x}|\mathbf{y}) = \frac{p_{X,Y}(\mathbf{x},\mathbf{y})}{p_Y(\mathbf{y})} \tag{5.1}$$

assuming that $p_Y(\mathbf{y}) \neq 0$.

The quality of estimation is better measured by *maximum a posteriori* (MAP). That is, $\mathbf{x} = \hat{\mathbf{x}}$ should maximize $p_{X|Y}(\mathbf{x}|\mathbf{y})$. As a result, computation of the MAP estimate involves nonlinear optimization procedures, which is not tractable in general due to the existence of multiple peaks in $p_{X|Y}(\mathbf{x}|\mathbf{y})$ or high dimension of $\mathbf{x}$. An alternate measure is the conditional error variance $\mathrm{E}\left\{\|X-\hat{\mathbf{x}}\|^2\,|Y=\mathbf{y}\right\}$. The minimum variance estimate $X = \hat{\mathbf{x}}$ satisfies

$$\mathrm{E}\left\{\|X-\hat{\mathbf{x}}\|^2|Y=\mathbf{y}\right\} \leq \mathrm{E}\left\{\|X-\mathbf{x}\|^2|Y=\mathbf{y}\right\} \quad \forall\, \mathbf{x}. \tag{5.2}$$

The left-hand side of (5.2) is often termed the *minimum mean-squared error* (MMSE). The MMSE estimate or the minimum variance estimate has the closed-form solution which is a contrast to the MAP estimate, as shown next.

Theorem 5.1. *Let X and Y be two jointly distributed random vectors. The* MMSE *estimate $\hat{\mathbf{x}}$ of X given observation $Y = \mathbf{y}$ is uniquely specified as the conditional mean* (*by an abuse of the notation for integration*)

$$\hat{\mathbf{x}} = \mathrm{E}\{X|Y=\mathbf{y}\} = \int_{-\infty}^{\infty} \mathbf{x}\, p_{X|Y}(\mathbf{x}|\mathbf{y})\mathrm{d}\mathbf{x}. \tag{5.3}$$

Proof. Let $h(\mathbf{z}) = \mathrm{E}\left\{\|X-\mathbf{z}\|^2|Y=\mathbf{y}\right\}$ with $\mathbf{z}$ to be chosen. Then

$$\begin{aligned} h(\mathbf{z}) &= \int_{-\infty}^{\infty} \|\mathbf{x}-\mathbf{z}\|^2 p_{X|Y}(\mathbf{x}|\mathbf{y})\mathrm{d}\mathbf{x} \\ &= \int_{-\infty}^{\infty} \|\mathbf{x}\|^2 p_{X|Y}(\mathbf{x}|\mathbf{y})\mathrm{d}\mathbf{x} + \|\mathbf{z}\|^2 - 2\mathrm{Re}\left\{\mathbf{z}^*\mathrm{E}[X|Y=\mathbf{y}]\right\} \end{aligned}$$

$$
\begin{aligned}
&= \|\mathbf{z} - \mathrm{E}\{X|Y=\mathbf{y}\}\|^2 + \int_{-\infty}^{\infty} \|\mathbf{x}\|^2 p_{X|Y}(\mathbf{x}|\mathbf{y})\,\mathrm{d}\mathbf{x} - \|\mathrm{E}\{X|Y=\mathbf{y}\}\|^2 \\
&\geq \int_{-\infty}^{\infty} \|\mathbf{x}\|^2 p_{X|Y}(\mathbf{x}|\mathbf{y})\,\mathrm{d}\mathbf{x} - \|\mathrm{E}\{X|Y=\mathbf{y}\}\|^2.
\end{aligned}
$$

The minimum is achieved uniquely with $\mathbf{z} = \hat{\mathbf{x}}$ in (5.3). □

Theorem 5.1 indicates that the MMSE estimate is the same as the conditional mean. Its closed-form offers a great advantage in its computation compared with the MAP estimate. In some cases, the conditional mean can be experimentally determined which can be extremely valuable if the joint PDF of X and Y is unavailable. Clearly, the MMSE estimate is different from the MAP estimate in general unless the global maximum of $p_{X|Y}(\mathbf{x}|\mathbf{y})$ takes place at the conditional mean $\mathbf{x} = \mathrm{E}\{X|Y=\mathbf{y}\}$. The next example is instrumental.

Example 5.2. Let the two random vectors X and Y be jointly Gaussian.Then the random vector $Z = \begin{bmatrix} X^* & Y^* \end{bmatrix}^*$ is Gaussian distributed with

$$
\mathbf{m_z} = \mathrm{E}\{Z\} = \begin{bmatrix} \mathbf{m_x} \\ \mathbf{m_y} \end{bmatrix}, \quad \Sigma_{\mathbf{zz}} = \mathrm{cov}\{Z\} = \begin{bmatrix} \Sigma_{\mathbf{xx}} & \Sigma_{\mathbf{xy}} \\ \Sigma_{\mathbf{yx}} & \Sigma_{\mathbf{yy}} \end{bmatrix}.
$$

Clearly, $\Sigma_{\mathbf{xy}} = \mathrm{E}\{(\mathbf{x} - \mathbf{m_x})(\mathbf{y} - \mathbf{m_y})^*\}$ and

$$
\begin{aligned}
\mathbf{m_x} &= \mathrm{E}\{X\}, \ \ \Sigma_{\mathbf{xx}} = \mathrm{cov}\{X\} := \mathrm{E}\{(\mathbf{x} - \mathbf{m_x})(\mathbf{x} - \mathbf{m_x})^*\}, \\
\mathbf{m_y} &= \mathrm{E}\{Y\}, \ \ \Sigma_{\mathbf{yy}} = \mathrm{cov}\{Y\} := \mathrm{E}\{(\mathbf{y} - \mathbf{m_y})(\mathbf{y} - \mathbf{m_y})^*\}.
\end{aligned}
$$

Suppose that the covariance matrices $\Sigma_{\mathbf{xx}}$ and $\Sigma_{\mathbf{yy}}$ are nonsingular. Then X and Y have marginal PDFs

$$
p_X(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma_{\mathbf{xx}})}} \exp\left\{ -\frac{1}{2}(\mathbf{x} - \mathbf{m_x})^* \Sigma_{\mathbf{xx}}^{-1} (\mathbf{x} - \mathbf{m_x}) \right\}, \tag{5.4}
$$

$$
p_Y(\mathbf{y}) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma_{\mathbf{yy}})}} \exp\left\{ -\frac{1}{2}(\mathbf{y} - \mathbf{m_y})^* \Sigma_{\mathbf{yy}}^{-1} (\mathbf{y} - \mathbf{m_y}) \right\}, \tag{5.5}
$$

respectively. It is left as an exercise to show that the conditional PDF of X, given $Y = \mathbf{y}$, is

$$
\begin{aligned}
p_{X|Y}(\mathbf{x}|\mathbf{y}) &= \frac{p_{X,Y}(\mathbf{x},\mathbf{y})}{p_Y(\mathbf{y})} = \frac{p_Z(\mathbf{z})}{p_Y(\mathbf{y})} \\
&= \frac{1}{\sqrt{(2\pi)^n \det\left(\tilde{\Sigma}_{\mathbf{xx}}\right)}} \exp\left\{ -\frac{1}{2}\left(\mathbf{x} - \tilde{\mathbf{m}}_{\mathbf{x}}\right)^* \tilde{\Sigma}_{\mathbf{xx}}^{-1} \left(\mathbf{x} - \tilde{\mathbf{m}}_{\mathbf{x}}\right) \right\},
\end{aligned} \tag{5.6}
$$

where $\tilde{\mathbf{m}}_{\mathbf{x}} = \mathbf{m}_{\mathbf{x}} + \Sigma_{\mathbf{xy}}\Sigma_{\mathbf{yy}}^{-1}(\mathbf{y} - \mathbf{m}_{\mathbf{y}})$ and $\tilde{\Sigma}_{\mathbf{xx}} = \Sigma_{\mathbf{xx}} - \Sigma_{\mathbf{xy}}\Sigma_{\mathbf{yy}}^{-1}\Sigma_{\mathbf{yx}}$. Hence, the conditional PDF in (5.6) is also Gaussian. Its MAP estimate is identical to the MMSE estimate given by

$$\hat{\mathbf{x}} = \tilde{\mathbf{m}}_{\mathbf{x}} = \mathbf{m}_{\mathbf{x}} + \Sigma_{\mathbf{xy}}\Sigma_{\mathbf{yy}}^{-1}(\mathbf{y} - \mathbf{m}_{\mathbf{y}}). \tag{5.7}$$

Suppose that X and Y have the same dimension and are related by

$$Y = X + N,$$

where N is Gaussian independent of X with zero vector mean and $\Sigma_{\mathbf{nn}}$ the covariance. Then by the independence of X and N,

$$\mathbf{m}_{\mathbf{y}} = \mathbf{m}_{\mathbf{x}}, \quad \Sigma_{\mathbf{xy}} = \Sigma_{\mathbf{xx}}, \quad \Sigma_{\mathbf{yy}} = \Sigma_{\mathbf{xx}} + \Sigma_{\mathbf{nn}}.$$

The optimal estimate in (5.7) reduces to

$$\hat{\mathbf{x}} = \mathbf{m}_{\mathbf{x}} + \Sigma_{\mathbf{xx}}(\Sigma_{\mathbf{xx}} + \Sigma_{\mathbf{nn}})^{-1}(\mathbf{y} - \mathbf{m}_{\mathbf{y}}). \tag{5.8}$$

The linear form (strictly speaking it is the affine form) of the estimate in terms of the observed data $Y = \mathbf{y}$ is due to the Gaussian distribution which does not hold in general.

Example 5.2 reveals several nice properties about the Gaussian random vectors. First, one needs know only the mean and covariance matrix in order to have the complete knowledge of the PDF. Often such statistical quantities can be experimentally determined. Second, if X and Y are jointly Gaussian, then each marginal and conditional distribution is also Gaussian. Moreover, a linear combination of Gaussian random vectors is Gaussian as well. Finally, the Gaussian assumption leads to the linear form of the optimal estimate for both the MMSE and MAP criteria. Because the observed data $\mathbf{y}$ in (5.7) can be any value and is in fact random, (5.7) actually gives the expression of the optimal *estimator* (a function of the observation) for jointly Gaussian random vectors. More generally, the following result on the MMSE estimator holds.

Theorem 5.3. *Let X and Y be two jointly distributed random vectors. Then the* MMSE *estimator $\hat{X}$ of X in terms of Y is given by $\hat{X} = \mathrm{E}\{X|Y\}$.*

Proof. The difference between $\mathrm{E}\{X|Y\}$ and $\mathrm{E}\{X|Y = \mathbf{y}\}$ lies in that $\mathrm{E}\{X|Y\}$ takes the expectation over all possible values of X and Y. Hence, the MMSE estimator is more difficult to prove than the MMSE estimate. However, the following two properties of the conditional expectation are helpful:

$$\mathrm{E}_{X|Y}\{f(X,Y)|Y = \mathbf{y}\} = \mathrm{E}_{X|Y}\{f(X,\mathbf{y})|Y = \mathbf{y}\}, \tag{5.9}$$

$$\mathrm{E}_{Y}\left\{\mathrm{E}_{X|Y}[f(X,Y)|Y = \mathbf{y}]\right\} = \mathrm{E}_{X,Y}\{f(X,Y)\}, \tag{5.10}$$

where the subscripts indicate the variables with respect to which expectation is being taken. Hence, by the MMSE estimate in (5.3),

$$\mathrm{E}_{X|Y}\left\{\left\|X-\hat{X}(\mathbf{y})\right\|^2|Y=\mathbf{y}\right\} \leq \mathrm{E}_{X|Y}\left\{\left\|X-\tilde{X}(\mathbf{y})\right\|^2|Y=\mathbf{y}\right\}$$

for any other estimator $\tilde{X}(\cdot)$. On the other hand, (5.9) implies that

$$\mathrm{E}_{X|Y}\left\{\left\|X-\hat{X}(\mathbf{y})\right\|^2|Y=\mathbf{y}\right\} \leq \mathrm{E}_{X|Y}\left\{\left\|X-\tilde{X}(\mathbf{Y})\right\|^2|Y=\mathbf{y}\right\}.$$

The above inequality is preserved with expectation being taken with respect to Y. Now with the aid of (5.10), there holds

$$\mathrm{E}_{X,Y}\left\{\left\|X-\hat{X}(Y)\right\|^2\right\} \leq \mathrm{E}_{X,Y}\left\{\left\|X-\tilde{X}(Y)\right\|^2\right\}$$

which establishes the desired result. □

The next example shows that the MAP and MMSE estimators are nonlinear in general. For convenience, the estimator $\hat{X}(Y)$ is still denoted by $\hat{\mathbf{x}}$.

Example 5.4. A typical case in digital communications is when the random variables X and Y are related as $Y = X + N$. Suppose that the random variable X is binary and equiprobable with the probability

$$P_X[X=1]=0.5, \quad P_X[X=-1]=0.5.$$

The random variable N represents the additive noise which is assumed to be Gaussian distributed with zero mean and the variance σ_n^2. Suppose that X and N are independent. Then X and Y are jointly distributed. If $X = x$ (x only takes values ± 1) is transmitted, then the PDF of $Y = y$ is given by

$$p_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma_n}\exp\left\{-\frac{(y-x)^2}{2\sigma_n^2}\right\}. \tag{5.11}$$

It is easy to see that the marginal PDF for $Y = y$ is given by

$$\begin{aligned} p_Y(y) &= P_X[X=1]p_{Y|X}(y|x=1) + P_X[X=-1]p_{Y|X}(y|x=-1) \\ &= 0.5p_{Y|X}(y|x=1) + 0.5p_{Y|X}(y|x=-1). \end{aligned} \tag{5.12}$$

By an abuse of notation, the conditional probability for $X = 1$, given $Y = y$ is received, is given by

$$\begin{aligned}P_{X|Y}[X=1|Y=y] &= \frac{p_{Y|X}(y|x=1)P_X[X=1]}{p_Y(y)}\\ &= \frac{0.5p_{Y|X}(y|x=1)}{0.5p_{Y|X}(y|x=1)+0.5p_{Y|X}(y|x=-1)}\\ &= \left(1+\frac{p_{Y|X}(y|x=-1)}{p_{Y|X}(y|x=1)}\right)^{-1} = \left(1+\exp\left\{-\frac{2y}{\sigma_n^2}\right\}\right)^{-1}\end{aligned}$$

in light of (5.11) and (5.12). Similarly, the conditional probability for $X=-1$, given $Y=y$ is received, is given by

$$P_{X|Y}[X=-1|Y=y] = \left(1+\frac{p_{Y|X}(y|x=1)}{p_{Y|X}(y|x=-1)}\right)^{-1} = \left(1+\exp\left\{\frac{2y}{\sigma_n^2}\right\}\right)^{-1}.$$

It is easy to verify that $P_{X|Y}[X=1|Y=y]+P_{X|Y}[X=-1|Y=y]=1$. If $y>0$, then

$$P_{X|Y}[X=1|Y=y]>0.5, \quad P_{X|Y}[X=-1|Y=y]<0.5.$$

Because X is binary, the maximum of $P_{X|Y}[X=x|Y=y]$ for $y>0$ takes place at $X=1$. If $y<0$, then

$$P_{X|Y}[X=1|Y=y]<0.5, \quad P_{X|Y}[X=-1|Y=y]>0.5,$$

and thus, the maximum of $P_{X|Y}[X=x|Y=y]$ takes place at $X=-1$. Consequently, the MAP estimator is obtained as

$$\hat{x}_{\text{MAP}} = \begin{cases} 1, & \text{for } y>0, \\ -1, & \text{for } y<0. \end{cases} \tag{5.13}$$

This is identical to the optimal decision rule as discussed in Sect. 2.3 in the sense that the BER is minimized.

On the other hand, given received data $Y=y$ the conditional mean for X is given by (recall that X is binary valued):

$$\hat{x}_{\text{MMSE}} = \text{E}\{X|Y=y\} = \left(1+\exp\left\{-\frac{2y}{\sigma_n^2}\right\}\right)^{-1} - \left(1+\exp\left\{\frac{2y}{\sigma_n^2}\right\}\right)^{-1}.$$

Different from the MAP estimate, $\hat{x}_{\text{MMSE}}$ is not binary valued. Its values as function of the received y are plotted in the following figure where the dotted line is for the case $\sigma_n^2=1$, the dash–dotted line for $\sigma_n^2=0.1$, and the solid line for $\sigma_n^2=0.01$ (see Fig. 5.1).

As the variance σ_n^2 decreases (which corresponds to increase of the SNR), the MMSE estimate approaches the MAP estimate. It is noted that both estimators are nonlinear functions of the received data y.

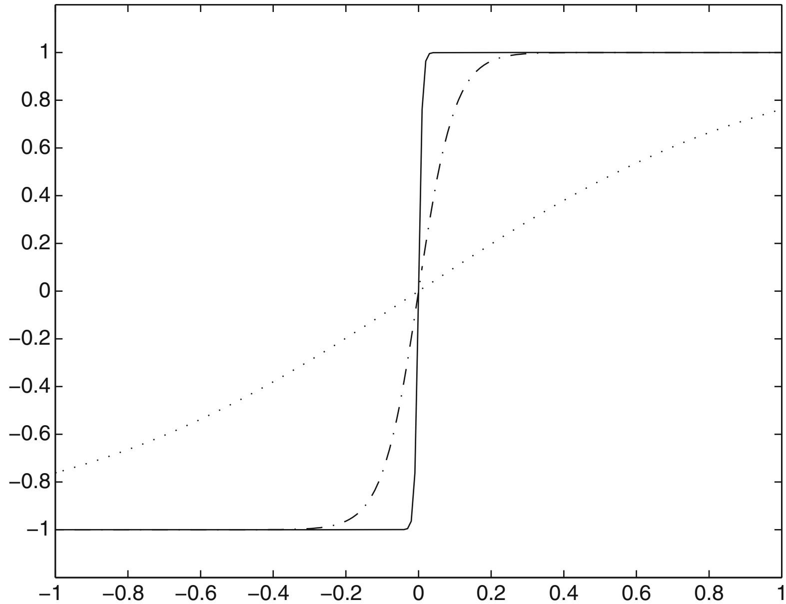

Fig. 5.1 The conditional mean estimates $\hat{x}$ as function of y

In spite of the fact that optimal estimators are nonlinear in general, the linear estimator (strictly speaking it is the affine estimator) has its appeal owing to its simplicity and mathematical tractability. Moreover, in the special case of Gaussian random vectors, both MMSE and MAP estimators are linear. Hence, there is an incentive to focus on linear estimators and search for the optimal estimator among all the linear estimators. The next theorem contains the complete result for MMSE estimators.

Theorem 5.5. *Let X and Y be two jointly distributed random vectors with*

$$\mathrm{E}\left\{\begin{bmatrix} X \\ Y \end{bmatrix}\right\} = \begin{bmatrix} \mathbf{m_x} \\ \mathbf{m_y} \end{bmatrix}, \quad \mathrm{cov}\left\{\begin{bmatrix} X \\ Y \end{bmatrix}\right\} = \begin{bmatrix} \Sigma_{\mathbf{xx}} & \Sigma_{\mathbf{xy}} \\ \Sigma_{\mathbf{yx}} & \Sigma_{\mathbf{yy}} \end{bmatrix}. \tag{5.14}$$

Then the linear MMSE *estimator for X in terms of Y is given by*

$$\hat{X} = \mathbf{m_x} + \Sigma_{\mathbf{xy}}\Sigma_{\mathbf{yy}}^{-1}(Y - \mathbf{m_y}), \tag{5.15}$$

where $\Sigma_{\mathbf{yy}}^{+}$ can be used if $\Sigma_{\mathbf{yy}}$ is singular (Problem 5.1 in Exercises). *The error covariance associated with $\hat{X}$ is unconditioned and given by*

$$\mathrm{E}\left\{\left(X - \hat{X}\right)\left(X - \hat{X}\right)^{*}\right\} = \Sigma_{\mathbf{xx}} - \Sigma_{\mathbf{xy}}\Sigma_{\mathbf{yy}}^{-1}\Sigma_{\mathbf{yx}}. \tag{5.16}$$

If X and Y are jointly Gaussian, then (5.15) is also the MMSE *estimate* $\hat{X} = \mathrm{E}\{X|Y\}$ *and is optimal among all (linear and nonlinear) estimators. Its error covariance conditioned on Y is the same as in (5.16).*

Proof. For any random vector Z its covariance satisfies

$$\mathrm{cov}\{Z\} = \mathrm{E}\{(Z - \mathbf{m_z})(Z - \mathbf{m_z})^*\} = \mathrm{E}\{ZZ^*\} - \mathbf{m_z}\mathbf{m_z}^*,$$

where $\mathbf{m_z} = \mathrm{E}\{Z\}$. As a consequence,

$$\mathrm{E}\left\{\|Z\|^2\right\} = \mathrm{Tr}\{\mathrm{cov}[Z]\} + \mathrm{Tr}\{\mathbf{m_z}\mathbf{m_z}^*\} = \mathrm{Tr}\{\mathrm{cov}[Z]\} + \|\mathbf{m_z}\|^2. \tag{5.17}$$

Now parameterize linear estimators as $\tilde{X} = FY + \mathbf{g}$ with matrix F and vector $\mathbf{g}$ free to choose. Setting the random vector

$$Z = X - \tilde{X} = X - FY - \mathbf{g}$$

yields mean $\mathbf{m_z} = \mathbf{m_x} - F\mathbf{m_y} - \mathbf{g}$ and the covariance

$$\mathrm{cov}\{Z\} = \Sigma_{\mathbf{xx}} + F\Sigma_{\mathbf{yy}}F^* - F\Sigma_{\mathbf{yx}} - \Sigma_{\mathbf{xy}}F^*.$$

Hence, the error variance $\mathrm{E}\left\{\|Z\|^2\right\} = \mathrm{E}\{\|X - FY - \mathbf{g}\|^2\}$ is given by

$$\begin{aligned}
\mathrm{E}\left\{\|Z\|^2\right\} &= \mathrm{Tr}\{\mathrm{cov}[Z]\} + \|\mathbf{m_z}\|^2 \geq \mathrm{Tr}\{\mathrm{cov}[Z]\} \\
&= \mathrm{Tr}\{\Sigma_{\mathbf{xx}} + F\Sigma_{\mathbf{yy}}F^* - F\Sigma_{\mathbf{yx}} - \Sigma_{\mathbf{xy}}F^*\} \\
&= \mathrm{Tr}\left\{\left[F - \Sigma_{\mathbf{xy}}\Sigma_{\mathbf{yy}}^{-1}\right]\Sigma_{\mathbf{yy}}\left[F - \Sigma_{\mathbf{xy}}\Sigma_{\mathbf{yy}}^{-1}\right]^* + \Sigma_{\mathbf{xx}} - \Sigma_{\mathbf{xy}}\Sigma_{\mathbf{yy}}^{-1}\Sigma_{\mathbf{yx}}\right\} \\
&\geq \mathrm{Tr}\left\{\Sigma_{\mathbf{xx}} - \Sigma_{\mathbf{xy}}\Sigma_{\mathbf{yy}}^{-1}\Sigma_{\mathbf{yx}}\right\}
\end{aligned}$$

for any F and $\mathbf{g}$ where (5.17) is used. By taking

$$F = F_{\mathrm{opt}} = \Sigma_{\mathbf{xy}}\Sigma_{\mathbf{yy}}^{-1} \quad \text{and} \quad \mathbf{g} = \mathbf{g}_{\mathrm{opt}} = \mathbf{m_x} - F_{\mathrm{opt}}\mathbf{m_y},$$

$\mathbf{m_z} = \mathbf{0}$ and $\mathrm{E}\left\{\|Z\|^2\right\} = \mathrm{Tr}\left\{\Sigma_{\mathbf{xx}} - \Sigma_{\mathbf{xy}}\Sigma_{\mathbf{yy}}^{-1}\Sigma_{\mathbf{yx}}\right\} = \mathrm{E}\left\{\|Z\|^2\right\}$ which is the unconditional error variance. Therefore, the error covariance in (5.16) holds and the linear MMSE estimator is given by

$$\hat{X} = F_{\mathrm{opt}}Y + \mathbf{g}_{\mathrm{opt}} = \mathbf{m_x} + F_{\mathrm{opt}}(Y - \mathbf{m_y})$$

which is identical to (5.15) by $F_{\mathrm{opt}} = \Sigma_{\mathbf{xy}}\Sigma_{\mathbf{yy}}^{-1}$. It is noted that the linear MMSE estimator is identical to (5.7), if $Y = \mathbf{y}$, due to the fact that the linear MMSE estimator coincides with the overall MMSE estimator (among all linear and nonlinear estimators) for the Gaussian case. □

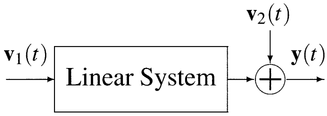

Fig. 5.2 Signal model for Kalman filtering

It should be emphasized that (5.16) is the unconditional error covariance in general. Only when X and Y are jointly Gaussian is it also the conditional error covariance. In the case when $\Sigma_{\mathbf{yy}}$ is singular, $\Sigma_{\mathbf{yy}}^{-1}$ needs to be replaced by its pseudo-inverse $\Sigma_{\mathbf{yy}}^{+}$ for which the results in Theorem 5.5 still hold. See Problem 5.1 in Exercises.

5.1.2 Kalman Filters

The design of state estimators is considerably more difficult than that of the estimator in the previous subsection due to the dynamical model for the random processes.

Consider the time-varying state-space system

$$\mathbf{x}(t+1) = A_t\mathbf{x}(t) + \mathbf{v}_1(t), \quad \mathbf{y}(t) = C_t\mathbf{x}(t) + \mathbf{v}_2(t), \tag{5.18}$$

where both $\{\mathbf{v}_1(t)\}$ and $\{\mathbf{v}_2(t)\}$ are random processes. Traditionally, $\{\mathbf{v}_1(t)\}$ is called the *process noise* and $\{\mathbf{v}_2(t)\}$ the *observation noise*. See the signal model in Fig. 5.2.

It is assumed that both $\{\mathbf{v}_1(t)\}$ and $\{\mathbf{v}_2(t)\}$ are white random processes with Gaussian distributions for each t and with zero means:

$$\mathrm{E}\{\mathbf{v}_1(t)\} = \mathbf{0}, \quad \mathrm{E}\{\mathbf{v}_2(t)\} = \mathbf{0}.$$

Since $\{\mathbf{v}_1(t)\}$ and $\{\mathbf{v}_2(t)\}$ are white, the covariance matrices are given by

$$\begin{aligned}
\mathrm{E}\{\mathbf{v}_1(t+k)\mathbf{v}_1(t)^*\} &= B_tB_t^*\delta(k),\\
\mathrm{E}\{\mathbf{v}_2(t+k)\mathbf{v}_2(t)^*\} &= D_tD_t^*\delta(k),\\
\mathrm{E}\{\mathbf{v}_2(t+k)\mathbf{v}_1(t)^*\} &= D_tB_t^*\delta(k),
\end{aligned}$$

for some matrices B_t of size $n \times m$ and D_t of size $p \times m$ where n and p are the sizes of the state vector $\mathbf{x}(t)$ and the observed output $\mathbf{y}(t)$, respectively. For this reason, the state-space model (5.18) can be equivalently written as

$$\mathbf{x}(t+1) = A_t\mathbf{x}(t) + B_t\mathbf{v}(t), \quad \mathbf{y}(t) = C_t\mathbf{x}(t) + D_t\mathbf{v}(t) \tag{5.19}$$

for some equivalent white Gaussian random process $\{\mathbf{v}(t)\}$ where

$$\mathrm{E}\{\mathbf{v}(t)\} = \mathbf{0}_m \quad \text{and} \quad \mathrm{E}\{\mathbf{v}(t+k)\mathbf{v}(t)^*\} = I_m\delta(k). \tag{5.20}$$

Basically, substitutions of $\mathbf{v}_1(t) = B_t\mathbf{v}(t)$ and $\mathbf{v}_2(t) = D_t\mathbf{v}(t)$ are employed in arriving at the state-space model (5.19).

Suppose that the initial condition $\mathbf{x}_0 = \mathbf{x}(0)$ is also random, and Gaussian distributed with mean and covariance

$$\mathrm{E}\{\mathbf{x}_0\} = \overline{\mathbf{x}}_0 \quad \text{and} \quad \mathrm{cov}\{\mathbf{x}_0\} = P_0, \tag{5.21}$$

respectively. Assume further that $\mathbf{x}_0$ is independent of $\{\mathbf{v}(t)\}$, and $\mathbf{x}_0$ and $\mathbf{v}(0)$ are jointly Gaussian. Then

$$\mathbf{x}(1) = A_0\mathbf{x}_0 + B_0\mathbf{v}(0)$$

is a linear combination of two jointly distributed Gaussian random vectors. Thus, $\mathbf{x}(1)$ is Gaussian distributed as well. By the white assumption on $\{\mathbf{v}(t)\}$ and independence of $\mathbf{x}_0$ to $\{\mathbf{v}(t)\}$, $\mathbf{x}(1)$ and $\mathbf{v}(t)$ are independent random vectors for all $t \geq 1$. As a result, $\mathbf{x}(1)$ and $\mathbf{v}(1)$ are jointly Gaussian. Hence, by the induction process, the state vectors $\{\mathbf{x}(t)\}$ are Gaussian random processes. In fact, $\mathbf{x}(t)$ and $\mathbf{v}(t)$ are jointly Gaussian for each $t \geq 0$. Optimal state estimators are concerned with the MMSE estimate of $\mathbf{x}(t+1)$ for $t \geq 0$, based on the observation data $\{\mathbf{y}(k)\}_{k=0}^{t}$. Due to the Gaussian property, such an MMSE estimator is also a MAP estimator. The solution to the optimal state estimator is the well-publicized Kalman filtering which will be studied in this subsection.

Under the Gaussian assumption, the MMSE estimator is easy to derive for $\mathbf{x}(t)$ and $\mathbf{x}(t+1)$ based on $\{\mathbf{y}(k)\}_{k=0}^{t}$ using the basic result in Theorem 5.5. Specifically in the case of $\mathbf{x}(t+1)$, denote

$$\overline{\mathbf{x}}(t+1) = \mathrm{E}\{\mathbf{x}(t+1)\}, \quad \overline{\mathscr{Y}}_t = \mathrm{E}\{\mathscr{Y}_t\},$$

where $\mathscr{Y}_t$ is the observation up to time $t \geq 0$ with an expression

$$\mathscr{Y}_t = \mathrm{vec}\left(\begin{bmatrix}\mathbf{y}(0)\ \mathbf{y}(1)\ \cdots\ \mathbf{y}(t)\end{bmatrix}\right). \tag{5.22}$$

Then $\overline{\mathbf{x}}(t+1) = A_t\overline{\mathbf{x}}(t)$. The state vector $\mathbf{x}(t+1)$ and the observed data $\mathscr{Y}_t$ are jointly Gaussian with mean $\left\{A_t\overline{\mathbf{x}}(t), \overline{\mathscr{Y}}_t\right\}$ and covariance

$$\mathrm{cov}\left\{\begin{bmatrix}\mathbf{x}(t+1)\\ \mathscr{Y}_t\end{bmatrix}\right\} = \begin{bmatrix}P_{t+1} & Z_t\\ Z_t^* & \Psi_t\end{bmatrix}. \tag{5.23}$$

It is easy to see that $\Psi_t = \mathrm{cov}\{\mathscr{Y}_t\}$, $P_{t+1} = \mathrm{cov}\{\mathbf{x}(t+1)\}$, and

$$Z_t = \mathrm{E}\left\{[\mathbf{x}(t+1) - \overline{\mathbf{x}}_{t+1}]\left[\mathscr{Y}_t - \overline{\mathscr{Y}}_t\right]^*\right\}. \tag{5.24}$$

Note that Ψ_t is nonsingular provided that $\det\left(D_k D_k^*\right) \neq 0$ for $0 \leq k \leq t$. Hence, Theorem 5.5 can be applied to compute the MMSE estimate for $\mathbf{x}(t+1)$ based on $\mathscr{Y}_t$ which is given by

$$\hat{\mathbf{x}}_{t+1|t} = A_t \overline{\mathbf{x}}_t + Z_t \Psi_t^{-1} \left(\mathscr{Y}_t - \overline{\mathscr{Y}}_t\right) \tag{5.25}$$

by $\overline{\mathbf{x}}(t+1) = A_t \overline{\mathbf{x}}(t)$. The associated error covariance according to (5.16) is

$$\Sigma_{t+1|t} = \mathrm{E}\left\{\left[\mathbf{x}(t+1) - \hat{\mathbf{x}}_{t+1|t}\right]\left[\mathbf{x}(t+1) - \hat{\mathbf{x}}_{t+1|t}\right]^*\right\} = P_{t+1} - Z_t \Psi_t^{-1} Z_t^*. \tag{5.26}$$

However, the MMSE estimator as described in (5.25) and (5.26) has no value in practice because the associated computational complexity grow with respect to the time index t. A remarkable feature of the Kalman filter is its recursive computation of the MMSE estimate $\hat{\mathbf{x}}_{k+1|k}$ and recursive update of the optimal error covariance Σ_{k+1} with complexity dependent only on the order of the state-space model in (5.19) rather than the time index.

Theorem 5.6. *Consider the state-space model in (5.19) where* $\{\mathbf{v}(t)\}$ *is the white Gaussian random process satisfying (5.20) and the initial condition* $\mathbf{x}(0) = \mathbf{x}_0$ *is also Gaussian distributed, independent of* $\{\mathbf{v}(t)\}$*, with the mean* $\overline{\mathbf{x}}_0$ *and the covariance* P_0*. Suppose*

$$B_t D_t^* = \mathbf{0}, \quad R_t = D_t D_t^* > \mathbf{0}, \;\; \forall \; t \geq 0. \tag{5.27}$$

Denote $\hat{\mathbf{x}}_{k|i}$ *as the* MMSE *estimate of* $\mathbf{x}(k)$ *based on* $\mathscr{Y}_i$*, and* $\Sigma_{k|i}$ *as the corresponding error covariance where* $k \geq i \geq 0$*. Then*

$$\hat{\mathbf{x}}_{t|t} = \hat{\mathbf{x}}_{t|t-1} + L_t\left[\mathbf{y}(t) - C_t \hat{\mathbf{x}}_{t|t-1}\right], \tag{5.28}$$

$$L_t = \Sigma_{t|t-1} C_t^* \left(R_t + C_t \Sigma_{t|t-1} C_t^*\right)^{-1}, \tag{5.29}$$

$$\Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1} C_t^* \left(R_t + C_t \Sigma_{t|t-1} C_t^*\right)^{-1} C_t \Sigma_{t|t-1}, \tag{5.30}$$

$$\hat{\mathbf{x}}_{t+1|t} = A_t \hat{\mathbf{x}}_{t|t}, \quad \Sigma_{t+1|t} = A_t \Sigma_{t|t} A_t^* + B_t B_t^*, \tag{5.31}$$

initialized by $\hat{\mathbf{x}}_{0|-1} = \overline{\mathbf{x}}_0$ *and* $\Sigma_{0|-1} = P_0$*.*

Proof. Given $\hat{\mathbf{x}}_{t|t-1}$, $\Sigma_{t|t-1}$, and observation $\mathscr{Y}_t$, it can be verified that

$$\mathrm{E}\left\{\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix} \middle| \mathscr{Y}_{t-1}\right\} = \begin{bmatrix} \hat{\mathbf{x}}_{t|t-1} \\ C_t \mathbf{x}_{t|t-1} \end{bmatrix}, \tag{5.32}$$

$$\mathrm{cov}\left\{\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix} \middle| \mathscr{Y}_{t-1}\right\} = \begin{bmatrix} \Sigma_{t|t-1} & \Sigma_{t|t-1} C_t^* \\ C_t \Sigma_{t|t-1} & R_t + C_t \Sigma_{t|t-1} C_t^* \end{bmatrix}. \tag{5.33}$$

Applying Theorem 5.5 with $X = \mathbf{x}(t)$ and $Y = \mathscr{Y}_t$ leads to the MMSE estimate $\hat{\mathbf{x}}_{t|t}$ and error covariance $\Sigma_{t|t}$ in (5.28)–(5.30) which are referred to as *measurement* *update*. Because of (5.27), the random vectors $B_t\mathbf{v}(t)$ and $D_t\mathbf{v}_t$ are uncorrelated or $\mathrm{E}\{B_t\mathbf{v}(t)[D_t\mathbf{v}_t]^*\} = B_tD_t^* = \mathbf{0}$. In fact, $B_t\mathbf{v}(t)$ and $D_t\mathbf{v}_t$ are independent of each other due to the Gauss assumption. It follows that $B_t\mathbf{v}(t)$ and $\mathbf{y}(t)$ are independent of each other. Hence

$$\mathrm{E}\{B_t\mathbf{v}(t)|\mathscr{Y}_t\} = \mathrm{E}\{B_t\mathbf{v}(t)|\mathbf{y}_t\} = \mathbf{0}.$$

The above leads to $\mathrm{E}\{\mathbf{x}(t+1)|\mathscr{Y}_t\} = A_t\hat{\mathbf{x}}_{t|t} + \mathrm{E}\{B_t\mathbf{v}(t)|\mathscr{Y}_t\} = A_t\hat{\mathbf{x}}_{t|t}$ and

$$\Sigma_{t+1|t} = A_t\Sigma_{t|t}A_t^* + B_tB_t^*$$

or (5.31) that is referred to as *time update*. The proof is now complete. □

Theorem 5.6 indicates that Kalman filtering is basically an efficient and recursive algorithm for implementing the MMSE estimator. Recall that the computational complexity for those in (5.25)–(5.26) grows with respect to the time index. It is surprising that the MMSE estimator is linear and finite-dimensional with the same order as that of the signal model in (5.19), rather than nonlinear and infinite-dimensional as one might have speculated at the beginning. Of course, such linear and finite-dimensional properties of the MMSE estimator are owing to the Gaussian assumption. If the noise process $\{\mathbf{v}(t)\}$ is not Gaussian, then the Kalman filter can only be claimed to be optimal among all linear filters of arbitrary orders in light of Theorem 5.5. In addition, its property of being an MAP estimator is lost in general.

It is observed that the Kalman filter in Theorem 5.6 actually consists of two MMSE estimators: One is the *measurement update* as described in (5.28)–(5.30), and the other is *time update* as described in (5.31). While Theorem 5.6 is the main result of Kalman filtering, Kalman filter is often referred to the recursive algorithm for computing $\hat{\mathbf{x}}_{t+1|t}$ based on $\hat{\mathbf{x}}_{t|t-1}$. The next result shows the structure of such an optimal one-step predictor. The proof is left as an exercise (Problem 5.9).

Theorem 5.7. *Denote $\Sigma_k = \Sigma_{k|k-1}$ for each integer $k \geq 1$. Under the same hypotheses of Theorem 5.6, the* MMSE *estimate $\hat{\mathbf{x}}_{t+1|t}$ for $\mathbf{x}(t+1)$ based on the observation $\mathscr{Y}_t = \{\mathbf{y}(k)\}_{k=0}^{t}$ is given recursively as*

$$\hat{\mathbf{x}}_{t+1|t} = [A_t + K_tC_t]\,\hat{\mathbf{x}}_{t|t-1} - K_t\mathbf{y}(t), \quad \hat{\mathbf{x}}_{0|-1} = \bar{\mathbf{x}}_0, \tag{5.34}$$

$$K_t = -A_t\Sigma_tC_t^*\left(R_t + C_t\Sigma_tC_t^*\right)^{-1}, \tag{5.35}$$

$$\Sigma_{t+1} = A_t\Sigma_tA_t^* + B_tB_t^* + K_tC_t\Sigma_tA_t^*, \quad \Sigma_0 = P_0. \tag{5.36}$$

It is interesting to observe that (5.34) can be written as

$$\hat{\mathbf{x}}_{t+1|t} = A_t\hat{\mathbf{x}}_{t|t-1} - K_t\left[\hat{\mathbf{y}}_{t|t-1} - \mathbf{y}(t)\right]$$

with $\hat{\mathbf{y}}_{t|t-1} = C_t\hat{\mathbf{x}}_{t|t-1}$. So, it is similar to (5.19) with only one difference in that $B_t\mathbf{v}(t)$ is replaced by $-K_t\left[\mathbf{y}(t) - C_t\hat{\mathbf{x}}_{t|t-1}\right]$. A reflection on this indicates that the vector $\left(\mathbf{y}(t) - C_t\hat{\mathbf{x}}_{t|t-1}\right)$ provides new information that is not contained in $\mathscr{Y}_{t-1}$.

For this reason, $\left\{\left(\mathbf{y}(t) - C_t\hat{\mathbf{x}}_{t|t-1}\right)\right\}$ is called *innovation sequence* which is in fact a white process (refer to Problem 5.12 in Exercises). It is also interesting to observe that the error covariance $\Sigma_{t+1|t}$ is independent of the observation $\mathscr{Y}_t$. That is, no one set of measurements helps any more than any other set to eliminate the uncertainty in $\mathbf{x}_t$. For convenience, $\Sigma_{t+1} := \Sigma_{t+1|t}$ will be used in the rest of the text. Equation (5.36) governing the error covariance is called the difference Riccati equation (DRE).

The initial covariance $\Sigma_0 = P_0$ measures the confidence of the a priori information on the initial estimate $\mathbf{x}_{0|-1} = \overline{\mathbf{x}}_0$. Small P_0 means high confidence whereas large P_0 means low confidence. In practice, the knowledge on the a priori information of $\overline{\mathbf{x}}_0$ and P_0 may not be available. In this case, $\overline{\mathbf{x}}_0 = \mathbf{0}$ and $P_0 = \rho I_n$ are often taken with $\rho > 0$ sufficiently large. However, so long as the Kalman filter is stable (to be investigated in the next subsection), the impact of $\overline{\mathbf{x}}_0$ and P_0 to the MMSE estimate $\hat{\mathbf{x}}_{t+1|t}$ will fade away as t gets large. The next result is obtained for time-invariant systems.

Proposition 5.1. *Suppose that* $(A_t, B_t, C_t, D_t) = (A, B, C, D)$ *for all* t *and the hypotheses of Theorem 5.6 hold. If* $\Sigma_0 = \mathbf{0}$*, then the solution to the* DRE *(5.36) is monotonically increasing, i.e.,* $\Sigma_{t+1} \geq \Sigma_t$ *for all* $t \geq 0$.

Proof. For $t = 0$, the DRE (5.36) gives $\Sigma_1 = BB^* \geq \Sigma_0 = \mathbf{0}$ in light of the time-invariance hypothesis. Using the induction, assume that $\Sigma_k \geq \Sigma_{k-1}$ for $k > 1$. The proof can be completed by showing $\Sigma_{k+1} \geq \Sigma_k$. Denote $\Delta_t = \Sigma_t - \Sigma_{t-1}$ for $t = k$ and $k+1$. The DRE (5.36) is equivalent to

$$\Sigma_{t+1} = A\left(I + \Sigma_t C^* R^{-1} C\right)^{-1} \Sigma_t A^* + BB^*, \quad R = DD^*.$$

See Problem 5.10 in Exercises. Taking the difference $\Delta_{k+1} = \Sigma_{k+1} - \Sigma_k$ gives

$$\begin{aligned}
\Delta_{k+1} &= A\left[\left(I + \Sigma_k C^* R^{-1} C\right)^{-1} \Sigma_k - \Sigma_{k-1}\left(I + C^* R C \Sigma_{k-1}\right)^{-1}\right] A^* \\
&= A\left(I + \Sigma_k C^* R^{-1} C\right)^{-1} \Delta_k \left(I + C^* R^{-1} C \Sigma_{k-1}\right)^{-1} A^* \\
&= A\left(I + \Sigma_{k-1} C^* R^{-1} C + \Delta_k C^* R^{-1} C\right)^{-1} \Delta_k \left(I + C^* R C \Sigma_{k-1}\right)^{-1} A^* \\
&= \overline{A}_{k-1}\left[I + \Delta_k C^* \left(R + C\Sigma_k C^*\right)^{-1} C\right]^{-1} \Delta_k \overline{A}_{k-1}^* \geq 0
\end{aligned}$$

by $\Delta_k = \Sigma_k - \Sigma_{k-1} \geq 0$ where $\overline{A}_{k-1} = A\left(I + \Sigma_{k-1} C^* R^{-1} C\right)^{-1}$. □

Before ending this subsection, the removal of the assumption (5.27) needs to be addressed. It is noted that the difference between the estimated and the true state vectors satisfies the difference equation

$$\hat{\mathbf{e}}(t+1) = (A_t + K_t C_t)\hat{\mathbf{e}}(t) + (B_t + K_t D_t)\mathbf{v}(t) \tag{5.37}$$

by taking the difference of (5.19) and (5.34) where $\hat{\mathbf{e}}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}_{t|t-1}$. This is the error equation for the associated Kalman filter under the assumption (5.27) or $B_tD_t^* = \mathbf{0}$ for all t. For the case $B_tD_t^* \neq \mathbf{0}$, it is claimed that the error equation associated with the MMSE estimate has the form

$$\hat{\mathbf{e}}(t+1) = \left(\tilde{A}_t + \tilde{K}_tC_t\right)\hat{\mathbf{e}}(t) + \left(\tilde{B}_t + \tilde{K}_tD_t\right)\mathbf{v}(t), \tag{5.38}$$

$$\tilde{A}_t = A_t - B_tD_t^*R_t^{-1}C_t, \quad \tilde{B}_t = B_t\left[I_m - D_t^*R_t^{-1}D_t\right], \tag{5.39}$$

where $\tilde{B}_tD_t^* = \mathbf{0}$. Specifically, the state-space system (5.19) can be written as

$$\mathbf{x}(t+1) = \tilde{A}_t\mathbf{x}(t) + \tilde{B}_t\mathbf{v}(t) + B_tD_t^*R_t^{-1}\mathbf{y}(t). \tag{5.40}$$

Because $\tilde{B}_tD_t^* = \mathbf{0}$, and $\mathbf{y}(t)$ is the measured output, the Kalman filter can be adapted to compute the MMSE estimate for $\mathbf{x}(t+1)$ in accordance with

$$\hat{\mathbf{x}}_{t+1|t} = \left[\tilde{A}_t + \tilde{K}_tC_t\right]\hat{\mathbf{x}}_{t|t-1} - \tilde{K}_t\mathbf{y}(t) + B_tD_t^*R_t^{-1}\mathbf{y}(t), \tag{5.41}$$

where the Kalman gain and the error covariance are given by

$$\tilde{K}_t = -\tilde{A}_t\Sigma_tC_t^*\left(R_t + C_t\Sigma_tC_t^*\right)^{-1}, \tag{5.42}$$

$$\Sigma_{t+1} = \tilde{A}_t\Sigma_t\tilde{A}_t^* + \tilde{B}_t\tilde{B}_t^* - \tilde{A}_t\Sigma_tC_t^*\left(R_t + C_t\Sigma_tC_t^*\right)^{-1}C_t\Sigma_t\tilde{A}_t^*, \tag{5.43}$$

respectively. The Kalman gain $\tilde{K}_t$ is associated with $\left(\tilde{A}_t, \tilde{B}_t\right)$, but Σ_t is the same error covariance as before. Subtracting (5.41) from (5.40) yields (5.38) as claimed earlier. On the other hand, (5.41) can be equivalently written as

$$\hat{\mathbf{x}}_{t+1|t} = (A_t + K_tC_t)\hat{\mathbf{x}}_{t|t-1} - K_t\mathbf{y}(t) \quad K_t = \tilde{K}_t - B_tD_t^*R_t^{-1}, \tag{5.44}$$

where K_t is the Kalman gain associated with (A_t, B_t). There holds

$$K_t = \tilde{K}_t - B_tD_t^*R_t^{-1} = -\left(A_t\Sigma_tC_t^* + B_tD_t^*\right)\left(R_t + C_t\Sigma_tC_t^*\right)^{-1}. \tag{5.45}$$

Therefore, the Kalman filter has the same form with a slight increase in the complexity of computing the Kalman gain and the associated DRE for the error covariance. The next result summarizes the above discussion.

Corollary 5.1. *Let $\tilde{A}_t$ and $\tilde{B}_t$ be as in (5.39). Under the same hypotheses of Theorem 5.7, except that $B_tD_t^* \neq \mathbf{0}$, the Kalman filter for $\mathbf{x}(t+1)$ based on the observation $\mathscr{Y}_t = \{\mathbf{y}(k)\}_{k=0}^t$ is given recursively by (5.44), (5.45), and (5.43) which collapse to those in Theorem 5.7 for the case $B_tD_t^* = \mathbf{0}$.*

Corollary 5.1 indicates that there is no loss of generality in focusing on the case $B_tD_t^* = \mathbf{0}$ for Kalman filtering. The case $B_tD_t^* \neq \mathbf{0}$ causes only some minor computational modifications for (linear) MMSE estimators.

5.1.3 Stability

An immediate question regarding the Kalman filter is its stability. Are Kalman filters always stable? What stability properties do Kalman filters possess? Such questions will be answered first for time-varying systems and then for time-invariant systems. The following stability result holds.

Theorem 5.8. *Let the state-space model be as in (5.19) with* $\{\mathbf{v}(t)\}$ *the white noise satisfying (5.20). For the case* $B_t D_t^* = \mathbf{0}$*, assume that* (A_t, B_t) *is uniformly stabilizable and* (C_t, A_t) *is uniformly detectable. Then the Kalman filter as described in Theorem 5.7 is asymptotically stable. For the case* $B_t D_t^* \neq \mathbf{0}$*, assume that* $\left(\tilde{A}_t, \tilde{B}_t\right)$ *is uniformly stabilizable, and* (C_t, A_t) *is uniformly detectable where* $\tilde{A}_t$ *and* $\tilde{B}_t$ *are as in (5.39). Then the Kalman filter as described in Corollary 5.1 is asymptotically stable.*

Proof. If $B_t D_t^* = \mathbf{0}$, then the stabilizability and detectability assumptions imply the existence of linear state estimation gains $\{L_t\}$ such that

$$\mathbf{x}(t+1) = (A_t + L_t C_t)\mathbf{x}(t)$$

is asymptotically stable. Hence, the difference Lyapunov equation

$$Q_{t+1} = (A_t + L_t C_t) Q_t (A_t + L_t C_t)^* + L_t R_t L_t^* + B_t B_t^*$$

has bounded nonnegative solutions $\{Q_t\}$. On the other hand, the DRE (5.36) which governs the error covariance of the Kalman filter can be written into the same form as the above difference Lyapunov equation:

$$\Sigma_{t+1} = (A_t + K_t C_t)\Sigma_t (A_t + K_t C_t)^* + B_t B_t^* + K_t R_t K_t^*.$$

See Problem 5.10 in Exercises. It is noted that $\left(A + K_t C_t, \begin{bmatrix} B_t & K_t R_t^{1/2} \end{bmatrix}\right)$ is stabilizable. In light of the discussions at the end of Chap. 3, stability of the Kalman filter is hinged to the boundedness of Σ_t as $t \to \infty$. But the Kalman filter is optimal among all linear estimators. It follows that $\mathrm{Tr}\{\Sigma_t\} \leq \mathrm{Tr}\{Q_t\}$. The Kalman filter is thus asymptotically stable. The proof for the case $B_t D_t^* \neq \mathbf{0}$ is similar, and is skipped. □

For asymptotically exponentially stable systems, the assumptions of uniform stabilizability and detectability hold. Hence, the Kalman filter preserves the stability property that is owing to the fact $\Sigma_t \leq P_t$ or the error covariance for the estimated state vector is no larger than the covariance of the state vector to be estimated, which is in turn owing to the optimality of the Kalman filter. The hypothesis on stabilizability of (A_t, B_t) in Theorem 5.8 might seem unnecessary by the argument on the existence of the stable linear estimators. However, it cannot be removed for the stability result in Theorem 5.8 to hold true. Nevertheless, this hypothesis can be weakened for stability of Kalman filters if the underlying state-space system is time invariant and the additive noise process is stationary. For this purpose, consider the following signal model

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + B\mathbf{v}(t), \quad \mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{v}(t), \tag{5.46}$$

where $\mathbf{x}(0) = \mathbf{x}_0$ with mean $\bar{\mathbf{x}}_0$ and covariance P_0. Suppose that $\mathbf{x}_0$ and $\{\mathbf{v}(t)\}$ are independently distributed and $\{\mathbf{v}(t)\}$ satisfies (5.20). Then $\{B\mathbf{v}(t)\}$ and $\{D\mathbf{v}(t)\}$ are both WSS white processes. Stability of A and $\mathrm{E}\{\mathbf{x}(t)\} = \mathbf{0}$ imply that $\{\mathbf{x}(t)\}$ can be made a stationary process, provided that $P = P_0$ satisfies the Lyapunov equation

$$P = APA^* + BB^*. \tag{5.47}$$

Assume that A is stable. Then there is a unique solution $P \geq 0$ to the above Lyapunov equation. If $P_0 \neq P$, then $\{\mathbf{x}(t)\}$ is not a WSS process in general. But, it is WSS asymptotically. Indeed, let $P_k = \mathrm{cov}\{\mathbf{x}(k)\}$ for $k \geq 0$. Then

$$P_{t+1} = AP_tA^* + BB^* = A^{t+1}P_0(A^*)^{t+1} + \sum_{k=0}^{t} A^k BB^*(A^*)^k$$

by $P_0 = \mathrm{cov}\{\mathbf{x}_0\}$. Stability of A implies that

$$P = \lim_{t\to\infty} P_{t+1} = \sum_{k=0}^{\infty} A^k BB^*(A^*)^k$$

exists and is bounded that is the unique solution to (5.47). However, if A is not a stability matrix, then $\{\mathbf{x}(t)\}$ may diverge and is thus not a WSS process in general. A somewhat surprising fact is that the error state vectors $\hat{\mathbf{e}}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}_{t|t-1}$ associated with the Kalman filter are WSS process asymptotically, provided that the Kalman filter is asymptotically stable. Stability of A is not required. Consider the following nth order LTI estimator

$$\hat{\mathbf{x}}_{t+1} = (A + KC)\hat{\mathbf{x}}_t - K\mathbf{y}(t), \quad \hat{\mathbf{x}}_0 = \bar{\mathbf{x}}(0), \tag{5.48}$$

$$K = -A\Sigma C^*\left(R + C\Sigma C^*\right)^{-1}, \tag{5.49}$$

$$\Sigma = A\left(I_n + \Sigma C^* R^{-1} C\right)^{-1}\Sigma A^* + BB^*. \tag{5.50}$$

This is the same as the Kalman filter in (5.34)–(5.36) after removing the time indexes of the matrices. The equation for the error covariance in (5.50) is called the algebraic Riccati equation (ARE).

Example 5.9. Consider the inverted pendulum system. Its state-space realization after discretization with sampling period $T_s = 0.25$ is obtained in Example 3.16 of Chap. 3. Suppose that $0.1BB^*$ is the covariance for the process noise and diag$(0.1, 0.01)$ is the covariance for the measurement noise, assuming that the noises for measurements of position and angle are uncorrelated. The Matlab command "dare" can be used to compute the solution Σ to the ARE in (5.50) and the stationary estimation gain K in (5.49). The numerical results are given by

$$\Sigma = \begin{bmatrix} 21.762 & 5.5171 & 1.6675 & 0.7219 \\ 5.5171 & 1.6269 & 0.6957 & 0.4560 \\ 1.6675 & 0.6957 & 0.4602 & 0.3910 \\ 0.7219 & 0.4560 & 0.3910 & 0.3699 \end{bmatrix}, \quad K = \begin{bmatrix} 0.4017 & -44.5166 \\ 0.4402 & -10.6004 \\ 0.4451 & -2.5400 \\ 0.4330 & -0.6398 \end{bmatrix}.$$

It can be easily verified with Matlab that the eigenvalues of $(A+KC)$ are all positive real, and strictly smaller than 1. Hence, $(A+KC)$ is a stability matrix, implying that the error for estimation of the state vector approaches zero asymptotically.

The ARE solution computed in Matlab, if it exists, is called stabilizing solution that is defined next.

Definition 5.1. The solution Σ to the ARE (5.50) is said to be stabilizing, if K as in (5.49) is stabilizing. That is, $(A+KC)$ is a stability matrix.

The stabilizing solution to ARE (5.50), if it exists, is unique (refer to Problem 5.13 in Exercises). The next result regards stability of the Kalman filter.

Theorem 5.10. *Suppose that (A,B) is stabilizable, $BD^* = \mathbf{0}$, and $R = DD^* > 0$ for the random process in (5.46) where $\{\mathbf{v}(t)\}$ is* WSS *with zero mean and identity covariance. If the* ARE *(5.50) admits a stabilizing solution, then the Kalman filter for (5.46) is asymptotically stable and its associated state estimation error vector $\hat{\mathbf{e}}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}_{t|t-1}$ is* WSS *asymptotically.*

Proof. Let Σ_t be the solution to the DRE (5.36). Since the time-invariant estimator described in (5.48)–(5.49) is a special case of the linear estimator in Problem 5.11, its error covariance $\Sigma \geq \Sigma_t \geq 0$ for all $t \geq 0$, provided that $\Sigma \geq P_0 \geq \mathbf{0}$. In this case, $\{\Sigma_t\}$ is monotonically increasing in light of Proposition 5.1 and uniformly bounded above by Σ. Hence, its limit $\overline{\Sigma}$ exists. Since the limit of the DRE (5.36) is identical to the ARE (5.50), it can be written as

$$\overline{\Sigma} = \left(A + \overline{K}C\right)\overline{\Sigma}\left(A + \overline{K}C\right)^* + \overline{K}R\overline{K}^* + BB^*,$$

where $\overline{K} = -A\overline{\Sigma}C^*\left(R + C\overline{\Sigma}C^*\right)^{-1}$. Stabilizability of (A,B) implies that $\overline{\Sigma}$ is stabilizing, and thus $\overline{\Sigma} = \Sigma$ by its uniqueness. As such the Kalman filter converges to the linear estimator as described in (5.48)–(5.49) which is stable. The fact that the Kalman filter is linear implies that its stability is independent of the initial condition $\hat{\mathbf{x}}_{0|-1} = \overline{\mathbf{x}}_0$ which in turn implies that the convergence of Σ_t to Σ is independent of the boundary condition $\Sigma_0 = P_0$. Moreover, the conditional mean and covariance associated with $\hat{\mathbf{e}}(t)$ are zero and Σ (asymptotically), respectively. So, the state error vector $\hat{\mathbf{e}}(t)$ is WSS asymptotically. The proof is completed. □

The LTI estimator in (5.48)–(5.49) is referred to as stationary Kalman filter. It is the state-space version of the Wiener filter. In lieu of the optimality properties of the Kalman filter, the stationary Kalman filter outperforms all LTI estimators of arbitrary orders. If, in addition, the noise process is Gaussian, then the stationary Kalman filter outperforms all (linear or nonlinear) time-invariant estimators. The premise is the existence of the stabilizing solution to the ARE (5.50) for which the following result provides the necessary and sufficient condition.

Theorem 5.11. *There exists a stabilizing solution to the* ARE *in (5.50), if and only if* (C,A) *is detectable and*

$$\operatorname{rank}\left\{\left[A - e^{j\omega}I_n \; B\right]\right\} = n \quad \forall \;\omega \in \mathbb{R}.$$

Different from time-varying systems with nonstationary noises, stabilizability of (A,B) is not required so long as (A,B) does not have unreachable modes on the unit circle. The proof is delayed to the next section. After the brain-storm of materials on Kalman filtering in the style of theorem/proof, it will be wise to pause for a while with readings of a few examples. It does need to be pointed out though that the results on time-invariant systems and WSS noises are established under the assumption that $BD^* = \mathbf{0}$. If the assumption does not hold, then the stationary Kalman filter is still the same as in (5.48) but the Kalman gain in (5.49) and the ARE in (5.50) need to be replaced by

$$K = -\left(A\Sigma C^* + BD^*\right)\left(R + C\Sigma C^{-1}\right)^{-1}, \tag{5.51}$$

$$\Sigma = \tilde{A}\left(I_n + \Sigma C^* R^{-1} C\right)^{-1}\Sigma\tilde{A}^* + B\left(I - D^* R^{-1} D\right)B^*, \tag{5.52}$$

respectively, where $\tilde{A} = A - BD^*R^{-1}C$. Moreover, the ARE (5.52) has a stabilizing solution, if and only if (C,A) is detectable, and

$$\operatorname{rank}\left\{\begin{bmatrix} A - e^{j\omega}I_n & B \\ C & D \end{bmatrix}\right\} = n + p \quad \forall\, \omega \in \mathbb{R}$$

with p the number of rows of C. Its proof is again delayed to the next section.

Two examples will be presented which are designed to help digest the theoretical results in this section. The first example is modified from digital communications as illustrated in the following Fig. 5.3 and discussed next.

Example 5.12. Consider estimation of the symbol $\mathbf{s}(t)$ in multiuser wireless data communications. The multipath channel is described by

$$\mathbf{r}(t) = \sum_{k=1}^{\ell} H_k(t)\mathbf{s}(t-k) \tag{5.53}$$

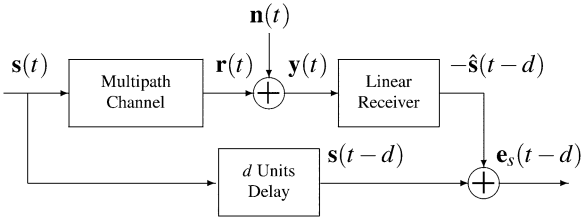

Fig. 5.3 Estimation of the symbol inputs

which is an MA model. It is assumed that the channel information or the impulse response $\{H_k(t)\}$ is known at time t and has dimension $p \times m$. The objective is to design linear receivers that estimate the symbol $\mathbf{s}(t-d)$ for some d satisfying $1 < d \le \ell$ with the minimum error variance. The design problem seems to be different from Kalman filtering but is intimately related to the estimation problem in this subsection.

First, the input symbols are assumed to be independent and have the same (equiprobable) distributions. As such, $\{\mathbf{s}(t)\}$ is white with the zero mean and covariance $\sigma_s^2 I$ where σ_s^2 is the transmission power for each symbol. For simplicity, $\sigma_s = 1$ is taken via some suitable normalization. Secondly, the channel model can be associated with a realization with the state vector

$$\mathbf{x}(t) = \text{vec}\left(\left[\,\mathbf{s}(t-1)\;\mathbf{s}(t-2)\cdots\mathbf{s}(t-\ell)\,\right]\right). \tag{5.54}$$

Denote $\mathbf{v}(t) = \left[\,\mathbf{s}(t)^*\;\mathbf{n}(t)^*\,\right]^*$. The observed signal $\mathbf{y}(t)$ at the receiver site can be described by the same state-space model in (5.19) with

$$\begin{aligned} A_t = A &= \begin{bmatrix} \mathbf{0} & \mathbf{0}_{m\times m} \\ I_{m(\ell-1)} & \mathbf{0} \end{bmatrix}, B_t = B = \begin{bmatrix} I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{m(\ell-1)\times p} \end{bmatrix} \\ C_t &= \left[\,H_1(t)\cdots H_\ell(t)\,\right], D_t = \left[\,\mathbf{0}\;\Sigma_{\mathbf{n}}^{\frac{1}{2}}\,\right], \end{aligned} \tag{5.55}$$

where $\Sigma_{\mathbf{n}} > 0$ is the covariance of $\mathbf{n}$ assumed to be white and WSS. Hence, the signal to be estimated is given by

$$\mathbf{s}(t-d) = J_d\mathbf{x}(t), \quad J_d = \left[\,\mathbf{0}\cdots\mathbf{0}\;I_m\;\mathbf{0}\cdots\mathbf{0}\,\right], \tag{5.56}$$

where $1 < d \le \ell$ and I_m is the dth block of J_d. Finally, it is noted that $\mathbf{v}(t)$ is white but not Gaussian. If $H_0(t) \equiv \mathbf{0}$, i.e., there is a pure delay in the multipath channel, then $BD_t^* = \mathbf{0}$. An application of Kalman filtering yields the optimal linear estimator for $\mathbf{s}(t-d) = J_d\mathbf{x}(t) = J_{d+1}\mathbf{x}(t+1)$ based on observations $\{\mathbf{y}(k)\}_{k=0}^t$ given by

$$\begin{aligned} \hat{\mathbf{x}}_{t+1|t} &= (A + K_tC_t)\hat{\mathbf{x}}_{t|t-1} - K_t\mathbf{y}(t), \quad K_t = -A\Sigma_tC_t^*\left(R + C_t\Sigma_tC_t^*\right)^{-1}, \\ \Sigma_{t+1} &= A\Sigma_tA^* + BB^* - A\Sigma_tC_t^*\left(R + C_t\Sigma_tC_t^*\right)^{-1}C_t\Sigma_tA^*, \end{aligned}$$

where $R = \Sigma_{\mathbf{n}}$. Any other linear estimator for $\mathbf{x}(t+1)$ has an error variance no smaller than $\text{Tr}\{\Sigma_{t+1}\}$, and the error variance for $\hat{\mathbf{s}}(t-d)$ is also the smallest among all linear receivers. Recall that $\{\mathbf{s}(t-k)\}_{k=1}^{\ell}$ are subsumed in $\mathbf{x}(t)$. It is thus concluded that $\hat{\mathbf{s}}(t-d|t) = J_d\hat{\mathbf{x}}_{t|t} = J_{d+1}\hat{\mathbf{x}}_{t+1|t}$ is an optimal linear estimate of $\mathbf{s}(t-d)$. See also Problem 5.11 in Exercises. If the channel is time invariant, then $H_k(t) = H_k$, and thus, $C_t = C \;\forall\, t$. In this case, the linear MMSE estimator converges to the stationary Kalman filter:

$$\hat{\mathbf{x}}_{t+1|t} = (A+KC)\hat{\mathbf{x}}_{t|t-1} - K\mathbf{y}(t), \;\; \hat{\mathbf{s}}(t-d|t) = J_{d+1}\hat{\mathbf{x}}_{t+1|t},$$

$$\Sigma = A\left(I_n + \Sigma C^* R^{-1} C\right)^{-1}\Sigma A^* + BB^*, \;\; K = -A\Sigma C^*\left(R + C\Sigma C^*\right)^{-1}.$$

As A is a stability matrix, the solution $\Sigma \geq 0$ exists and is stabilizing.

Example 5.12 shows that if the signals to be estimated are output of the form $\mathbf{z}(t) = J\mathbf{x}(t+1)$, then the optimal output estimator is the same as the optimal state estimator. Hence, the Kalman filter serves as the optimal linear estimator for both state and output estimation, provided that both estimators are restricted to being strictly causal. The following example regards the application of Kalman filtering to system identification.

Example 5.13. Suppose that the system is described by an ARMA model

$$y(t) = \sum_{k=1}^{n}\alpha_k y(t-k) + \sum_{k=1}^{m}\beta_k u(t-k) + \eta(t),$$

where $\{\eta(t)\}$ is white and Gaussian. Suppose that

$$\mathbf{h} = \begin{bmatrix}\alpha_1 & \cdots & \alpha_n & \beta_1 & \cdots & \beta_m\end{bmatrix}^*$$

is also Gaussian with a priori mean $\overline{\mathbf{h}}$ and covariance P. It is reasonable to assume that $\{\eta(t)\}$ and $\mathbf{h}$ are independent. The goal of system identification is to estimate the true value of $\mathbf{h}$ based on measured data $\{y(t)\}$ and the deterministic input data $\{u(t)\}$. For this purpose, consider the fictitious state-space equation

$$\mathbf{x}(t+1) = \mathbf{x}(t) = \mathbf{h}, \quad y(t) = \mathbf{q}(t)\mathbf{x}(t) + \eta(t), \tag{5.57}$$

where $\mathbf{q}(t) = \begin{bmatrix} y(t-1) & \cdots & y(t-n) & u(t-1) & \cdots & u(t-m)\end{bmatrix}$ is a row vector. This corresponds to the random process model (5.19) with $A_t = I_{n+m}, B_t = \mathbf{0}_{n+m}, \mathbf{C}_t = \mathbf{q}(t)$ and $D_t = [\text{cov}\{\eta(t)\}]^{1/2}$. An application of the Kalman filtering with $\hat{\mathbf{h}}_t = \hat{\mathbf{x}}_{t+1|t}$ yields the estimator:

$$\begin{aligned}\hat{\mathbf{h}}_t &= \hat{\mathbf{h}}_{t-1} + \Sigma_t\mathbf{q}(t)\left[R_t + \mathbf{q}(t)\Sigma_t\mathbf{q}^*(t)\right]^{-1}\left[y(t) - \mathbf{q}(t)\hat{\mathbf{h}}_{t-1}\right],\\ \Sigma_{t+1} &= \Sigma_t - \Sigma_t\mathbf{q}(t)^*\left[R_t + \mathbf{q}(t)\Sigma_t\mathbf{q}(t)^*\right]^{-1}\mathbf{q}(t)\Sigma_t, \qquad \Sigma_0 = P,\end{aligned} \tag{5.58}$$

where $\hat{\mathbf{h}}_{-1} = \overline{\mathbf{h}}$ and $R_t = \text{cov}\{\eta(t)\}$. It is noted that Σ_{t+1} is not truly the error covariance associated with $\hat{\mathbf{h}}_t$ by the fact that $C_t = \mathbf{q}(t)$ is random to which the Kalman filter in Theorem 5.7 does not apply. Hence, the estimator in (5.58) is not an MMSE estimator for $\mathbf{h}$. This also explains why the MMSE estimator in (5.58) is nonlinear in terms of the observed data $\{\mathbf{y}(k)\}_{k=t-n}^{t-1}$ in $C_t = \mathbf{q}(t)$, even though the Kalman filter is linear. On the other hand, if at time t, $\mathbf{y}(k)$ is treated as deterministic containing the realization of the noise process $\eta(k)$ for $k < t$, then the estimator in (5.58) can be interpreted as an MMSE estimator for $\mathbf{h}$. However, this interpretation is rather far-fetched. A more interesting case is when $n = 0$, i.e., the system is an MA model. In this case, the row vector $\mathbf{q}(t)$ does not contain any measured output data $\{y(k)\}_{k=0}^{t}$. Since the input $\{\mathbf{u}(t)\}$ is deterministic, the estimator in (5.58) becomes linear, and thus, the estimator in (5.58) is truly the MMSE estimator for $\mathbf{h}$ outperforming any other system identification algorithms for FIR models. Moreover, Σ_{t+1} is truly the error covariance associated with $\hat{\mathbf{h}}_t$. If the joint Gaussian assumption is dropped, then the estimator in (5.58) is the linear MMSE estimator outperforming any other linear algorithms for identification of FIR models. Nonetheless, such claims do not hold for the case $n > 0$ or the IIR models.

5.1.4 Output Estimators

The Kalman filter estimates $\mathbf{x}(t)$ or $\mathbf{x}(t+1)$ based on observations $\mathscr{Y}_t = \{\mathbf{y}(k)\}_{k=0}^{t}$ at time $t \geq 0$. A more practical problem is the output estimation or estimation of the linear combination of the state vector and the process noise at time t based on observation $\mathscr{Y}_t$. Such an estimation problem is described by the following state-space model:

$$\begin{bmatrix} \mathbf{x}(t+1) \\ \mathbf{z}(t) \\ \mathbf{y}(t) \end{bmatrix} = \begin{bmatrix} A_t & B_t \\ C_{1t} & D_{1t} \\ C_{2t} & D_{2t} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{v}(t) \end{bmatrix}, \tag{5.59}$$

where $\mathbf{v}(t)$ is the white noise process as in (5.20), the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is a random vector, and $\mathbf{z}(t)$ is the signal to be estimated. The goal is design of a linear estimator represented by state-space realization $\left(\widehat{A}_t, \widehat{B}_t, \widehat{C}_t, \widehat{D}_t\right)$ such that $\hat{\mathbf{z}}_{t|t}$, the estimate of $\mathbf{z}(t)$ based on the observation $\{\mathbf{y}(k)\}_{k=0}^{t}$, minimizes the error variance of $\mathbf{e}_z(t) = \mathbf{z}(t) - \hat{\mathbf{z}}_{t|t}$. Figure 5.4 shows the schematic diagram for output estimation that is different from the state estimation problem in Kalman filtering. In the special case of $C_{1t} = I$ and $D_{1t} = \mathbf{0}$ for all t, it aims to estimate $\mathbf{x}(t)$, based on observation $\{\mathbf{y}(k)\}_{k=0}^{t}$. On the other hand, if $C_{1t} = \mathbf{0}$ and $D_{1t} = I$ for all t, then it is an estimator for the noise process $\mathbf{v}(t)$ based on observations $\{\mathbf{y}(k)\}_{k=0}^{t}$. Therefore, the output estimation problem is more versatile and more useful in engineering practice. It turns out that among all linear estimators, the MMSE estimator can be obtained from the Kalman filter with some minor modifications.

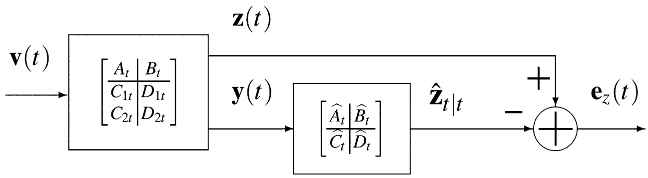

Fig. 5.4 Schematic diagram for linear output estimator

Theorem 5.14. *Let the state-space model be as in (5.59) with* $\{\mathbf{v}(t)\}$ *the white noise satisfying (5.20). Suppose that* $R_t = D_{2t}D_{2t}^*$ *is nonsingular and* $\mathbf{x}(0) = \mathbf{x}_0$ *has mean* $\overline{\mathbf{x}}_0$ *and covariance* P_0 *which is independent of* $\{\mathbf{v}(t)\}$*. Let* $\left(\tilde{A}_t, \tilde{B}_t\right)$ *be as in (5.39), i.e.,*

$$\tilde{A}_t = A_t - B_t D_{2t}^* R_t^{-1} C_{2t}, \quad \tilde{B}_t = B_t\left[I - D_{2t}^* R_t^{-1} D_{2t}\right]. \tag{5.60}$$

Then the linear MMSE estimation of $\mathbf{z}(t)$ *based on observation* $\{\mathbf{y}(k)\}_{k=0}^t$ *is given recursively by*

$$\begin{aligned} \hat{\mathbf{x}}_{t+1|t} &= [A_t + K_t C_{2t}]\,\hat{\mathbf{x}}_{t|t-1} - K_t \mathbf{y}(t), \quad \hat{\mathbf{x}}_{0|-1} = \overline{\mathbf{x}}_0, \\ \hat{\mathbf{z}}_{t|t} &= [C_{1t} + L_t C_{2t}]\,\hat{\mathbf{x}}_{t|t-1} - L_t \mathbf{y}(t), \end{aligned} \tag{5.61}$$

where the Kalman gains K_t *and* L_t *are given by*

$$\begin{bmatrix} K_t \\ L_t \end{bmatrix} = -\begin{bmatrix} A_t \Sigma_t C_{2t}^* + B_t D_{2t}^* \\ C_{1t}\Sigma_t C_{2t}^* + D_{1t} D_{2t}^* \end{bmatrix} \left(R_t + C_{2t}\Sigma_t C_{2t}^*\right)^{-1}, \tag{5.62}$$

$$\Sigma_{t+1} = \tilde{A}_t\left(I_n + \Sigma_t C_{2t}^* R_t^{-1} C_{2t}\right)^{-1} \Sigma_t \tilde{A}_t^* + \tilde{B}_t \tilde{B}_t^*, \quad \Sigma_0 = P_0. \tag{5.63}$$

Proof. The trick of the proof is to convert the output estimation to Kalman filtering. For simplicity, assume that $B_t D_{2t}^* = \mathbf{0}$ and $D_{1t} D_{2t}^* = \mathbf{0}$ for each t. Augment the state vector

$$\breve{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}(t-1) \end{bmatrix}, \quad \breve{\mathbf{x}}(0) = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{0} \end{bmatrix}.$$

Its associated a priori covariance is $\breve{P}_0 = \mathrm{diag}(P_0, \mathbf{0})$. There holds

$$\breve{\mathbf{x}}(t+1) = \breve{A}_t \breve{\mathbf{x}}(t) + \breve{B}_t \mathbf{v}(t), \quad \mathbf{y}(t) = \breve{C}_t \breve{\mathbf{x}}(t) + \breve{D}_t \mathbf{v}(t) \tag{5.64}$$

by straightforward calculation where $\breve{D}_t = D_{2t}$ and

$$\breve{A}_t = \begin{bmatrix} A_t & \mathbf{0} \\ C_{1t} & \mathbf{0} \end{bmatrix}, \quad \breve{B}_t = \begin{bmatrix} B_t \\ D_{1t} \end{bmatrix}, \quad \breve{C}_t = \begin{bmatrix} C_{2t} & \mathbf{0} \end{bmatrix}. \tag{5.65}$$

Since $\mathbf{x}(t+1)$ and $\mathbf{z}(t)$ are subsumed in $\breve{\mathbf{x}}(t+1)$, the MMSE output estimator for $\mathbf{z}(t)$ is equivalent to the MMSE estimation of $\breve{\mathbf{x}}(t+1)$, among all linear estimators, based on $\{\mathbf{y}(k)\}_{k=0}^{t}$. Recall the discussion after Example 5.12. Hence, the optimal solution is the Kalman filter in Theorem 5.7 for the random process in (5.64) due to $\breve{B}_t\breve{D}_t^* = \mathbf{0}$ by the hypothesis $B_tD_{2t}^* = \mathbf{0}$ and $D_{1t}D_{2t}^* = \mathbf{0}$ leading to the DRE and Kalman gain:

$$\breve{\Sigma}_{t+1} = \breve{A}_t\left(I+\breve{\Sigma}_t\breve{C}_t^*R_t^{-1}\breve{C}_t\right)^{-1}\breve{\Sigma}_t\breve{A}_t^* + \breve{B}_t\breve{B}_t^*, \tag{5.66}$$

$$\breve{K}_t = -\breve{A}_t\breve{\Sigma}_t\breve{C}_t^*\left(R_t+\breve{C}_t\Sigma_t\breve{C}_t^*\right)^{-1}. \tag{5.67}$$

Partition $\breve{\Sigma}_k$ into a 2×2 block matrix with Σ_k the $(1,1)$ block which is the error covariance for $\hat{\mathbf{x}}_{k|k-1}$ at $k=t$ and $k=t+1$. Then the $(1,1)$ block of the DRE (5.66) and the Kalman gain in (5.67) are obtained as

$$\begin{aligned}\Sigma_{t+1} &= A_t\left(I+\Sigma_tC_{2t}^*R_t^{-1}C_{2t}\right)^{-1}\Sigma_tA_t^* + B_tB_t^*,\\ \breve{K}_t &= \begin{bmatrix}K_t\\ L_t\end{bmatrix} = -\begin{bmatrix}A_t\Sigma_tC_{2t}^*,\\ C_{1t}\Sigma_tC_{2t}^*\end{bmatrix}(R_t+C_{2t}\Sigma_tC_{2t}^*)^{-1}\end{aligned}$$

which are the same as in (5.63) and (5.62), respectively, for the case $B_tD_{2t}^* = \mathbf{0}$. It follows that the Kalman filter for $\breve{\mathbf{x}}(t+1)$ in the system (5.64) is given by

$$\begin{aligned}\begin{bmatrix}\hat{\mathbf{x}}_{t+1|t}\\ \hat{\mathbf{z}}_{t|t}\end{bmatrix} &= \left(\breve{A}_t+\breve{K}_t\breve{C}_t\right)\begin{bmatrix}\hat{\mathbf{x}}_{t|t-1}\\ \hat{\mathbf{z}}_{t-1|t-1}\end{bmatrix} - \breve{K}_t\mathbf{y}(t),\\ &= \begin{bmatrix}A_t+K_tC_{2t}\\ C_{1t}+L_tC_{2t}\end{bmatrix}\hat{\mathbf{x}}_{t|t-1} - \begin{bmatrix}K_t\\ L_t\end{bmatrix}\mathbf{y}(t),\end{aligned}$$

by substitution of the expressions in (5.65). The above are the same as the linear MMSE output estimator in (5.61) for the case $B_tD_{2t}^* = \mathbf{0}$. If $B_tD_{2t}^* \neq \mathbf{0}$ and $D_{1t}D_{2t}^* \neq \mathbf{0}$, the same procedure can be carried out using the Kalman filtering results in Corollary 5.1 that will lead to the linear MMSE estimator in (5.61) with the Kalman gains in (5.62). The details are omitted here and left as an exercise (Problem 5.15). □

Theorem 5.14 indicates that the optimal output estimate $\hat{\mathbf{z}}_{t|t}$ is a linear function of the optimal state estimate $\hat{\mathbf{x}}_{t|t-1}$ in light of (5.61) and the associated error covariance is irrelevant to C_{1t} and D_{1t}. In this sense, optimal output estimation is equivalent to optimal state estimation. The realization of the linear MMSE output estimator in (5.61) is given by

$$\left(\widehat{A}_t,\widehat{B}_t,\widehat{C}_t,\widehat{D}_t\right) = (A_t+K_tC_{2t}, -K_t, C_{1t}+L_tC_{2t}, -L_t)$$

which has the same order as the original state-space model in (5.59). Its input is $\mathbf{y}(t)$, and output is $\hat{\mathbf{z}}_{t|t}$ as shown in Fig. 5.4. In light of the Kalman filtering, the linear estimator in (5.61) is optimal among all linear estimators with arbitrary orders. If, in addition, the noise process $\{\mathbf{v}(t)\}$ and the initial condition $\mathbf{x}_0$ are independent, and jointly Gaussian, then the linear estimator in (5.61) is optimal among all possible output estimators, including those nonlinear ones. The next example is related to Example 5.12, and illustrates the utility of output estimators.

Example 5.15. A commonly seen state-space model in applications is

$$\mathbf{x}(t) = A_t\mathbf{x}(t-1) + B_t\mathbf{v}(t), \quad \mathbf{y}(t) = C_t\mathbf{x}(t) + D_t\mathbf{v}(t) \tag{5.68}$$

that appears differently from the ones discussed in this chapter thus far. It will be shown that the results on output estimation are applicable to derive the optimal state estimator for the model in (5.68).

For simplicity, assume that $B_{t+1}D_t^* = \mathbf{0}$ and $R_t = D_tD_t^* > \mathbf{0}$. Under the same hypotheses on white and Gaussian $\mathbf{v}(t)$, and on the initial state $\mathbf{x}(0)$ that has mean $\overline{\mathbf{x}}_0$ and covariance P_0, Theorem 5.14 can be used to derive the equations for measurement update:

$$\hat{\mathbf{x}}_{t|t} = [I + L_tC_t]\,\hat{\mathbf{x}}_{t|t-1} - L_t\mathbf{y}(t), \tag{5.69}$$

$$\Sigma_{t|t} = \Sigma_t - \Sigma_tC_t^*\left(R_t + C_t\Sigma_tC_t^*\right)^{-1}C_t\Sigma_t, \tag{5.70}$$

initialized by $\hat{\mathbf{x}}_{0|-1} = \overline{\mathbf{x}}_0$ and covariance $\Sigma_0 = P_0$ where

$$L_t = -\Sigma_tC_t^*\left(R_t + C_t\Sigma_tC_t^*\right)^{-1}. \tag{5.71}$$

Recall $\Sigma_t = \Sigma_{t|t-1}$. Moreover, the time update equations can be obtained as

$$\hat{\mathbf{x}}_{t+1|t} = A_{t+1}\hat{\mathbf{x}}_{t|t}, \quad \Sigma_{t+1} = A_{t+1}\Sigma_{t|t}A_{t+1}^* + B_{t+1}B_{t+1}^*. \tag{5.72}$$

Specifically, the state-space model in (5.68) can be rewritten as

$$\mathbf{x}(t+1) = A_{t+1}\mathbf{x}(t) + B_{t+1}\mathbf{v}(t+1), \quad \mathbf{y}(t) = C_t\mathbf{x}(t) + D_t\mathbf{v}(t). \tag{5.73}$$

Because $B_{t+1}\mathbf{v}(t+1)$ and $D_t\mathbf{v}(t)$ are uncorrelated, replacing A_t by A_{t+1} and B_t by B_{t+1} in Theorem 5.14 leads to the optimal state estimator or one-step predictor $\hat{\mathbf{x}}_{t+1|t} = \mathrm{E}\{\mathbf{x}(t+1)|\mathscr{Y}_t\}$:

$$\hat{\mathbf{x}}_{t+1|t} = [A_{t+1} + K_tC_t]\,\hat{\mathbf{x}}_{t|t-1} - K_t\mathbf{y}(t) \tag{5.74}$$

initialized by $\hat{\mathbf{x}}_{0|-1} = \overline{\mathbf{x}}_0$ where $K_t = A_{t+1}L_t$ is the Kalman gain. In addition with $\Sigma_0 = P_0$, the associated error covariance Σ_{t+1} for $t \geq 0$ can be computed according to the DRE

$$\Sigma_{t+1} = A_{t+1}\Sigma_t \left(I + C_t^* R_t^{-1} C_t \Sigma_t\right)^{-1} A_{t+1}^* + B_{t+1}B_{t+1}^*. \tag{5.75}$$

For $\hat{\mathbf{x}}_{t|t} = \mathrm{E}\{\mathbf{x}(t)|\mathscr{Y}_t\}$, setting $\mathbf{z}(t) = \mathbf{x}(t)$ leads to the optimal estimator in (5.69) by $C_{1t} = I$, $D_{1t} = \mathbf{0}$, and $C_{2t} = C_t$. The fact of $K_t = A_{t+1}L_t$ and optimal estimate in (5.69) yield time update equations in (5.72). A comparison of Σ_{t+1} in (5.72) with the one in (5.75) shows

$$\Sigma_{t|t} = \Sigma_t \left(I + C_t^* R_t^{-1} C_t \Sigma_t\right)^{-1} \tag{5.76}$$

that is the same error covariance in (5.70).

The linear MMSE output estimator as in Theorem 5.14 has the same stability properties as those of the Kalman filter by the fact that they have the identical covariance matrices for the state vectors. Hence, all the results in the previous subsection apply to the linear MMSE output estimators, which will not be repeated here except for the following. Suppose that the state-space realization in (5.59) is independent of time t. Let $\tilde{A} = A - BD_2^* R^{-1} C_2$, $R = D_2 D_2^*$, and $\tilde{B} = B\left(I - D_2^* R^{-1} D_2\right)$. If the ARE

$$\Sigma = \tilde{A}\left(I_n + \Sigma C_2^* R^{-1} C_2\right)^{-1} \Sigma \tilde{A}^* + \tilde{B}\tilde{B}^* \tag{5.77}$$

has a stabilizing solution $\Sigma \geq 0$, then the output estimator in (5.61) is asymptotically stable, in light of Theorem 5.10. In this case, the output estimator in (5.61) converges asymptotically to the following time-invariant system

$$\begin{aligned} \hat{\mathbf{x}}_{t+1|t} &= [A + KC_2]\,\hat{\mathbf{x}}_{t|t-1} - K\mathbf{y}(t), \quad \hat{\mathbf{x}}_{0|-1} = \overline{\mathbf{x}}_0, \\ \hat{\mathbf{z}}_{t|t} &= [C_1 + LC_2]\,\hat{\mathbf{x}}_{t|t-1} - L\mathbf{y}(t), \end{aligned} \tag{5.78}$$

where K and L have the same expressions as in (5.62) with the time index t removed. One may employ the time-invariant estimator (5.78) directly for output estimation which can be computed off-line in order to reduce the computational complexity in its implementation. Clearly, the estimator in (5.78) admits the transfer matrix, denoted by $\mathbf{F}(z)$ and given by

$$\mathbf{F}(z) = -\left[L + (C_1 + LC_2)(zI_n - A - KC_2)^{-1} K\right]. \tag{5.79}$$

This section is concluded with an example on Wiener filtering (see Fig. 5.5).

Example 5.16. (Wiener filtering) Consider the signal model as in Fig. 4.5 where $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$ are independent white noises of zero mean and identity covariance, and $\mathbf{G}_1(z)$ and $\mathbf{G}_2(z)$ are causal and stable rational transfer matrices. Wiener filtering aims to design a LTI filter $\mathbf{W}(z)$ which estimates $\mathbf{z}(t-m)$, the output of $\mathbf{G}_1(z)$, based on observations $\mathbf{y}(k)$ for all $k \leq t$ and some integer m. It is termed as smoothing, if $m > 0$ (estimation of the past output), filtering, if $m = 0$ (estimation of the present output), and prediction, if $m < 0$ (estimation of the future output). It is claimed

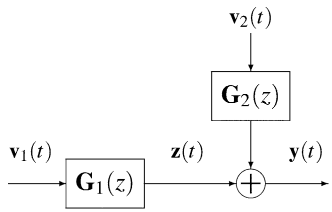

Fig. 5.5 Signal model for Wiener filtering

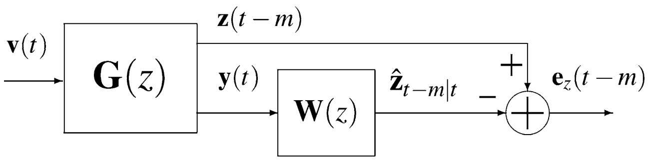

Fig. 5.6 Wiener filter as an output estimator

that these three estimation problems can all be cast into the output estimation as illustrated in the figure below, provided that $z^{-m}\mathbf{G}_1(z)$ is causal.

Indeed, if $z^{-m}\mathbf{G}_1(z)$ is causal, then

$$\mathbf{G}(z) = \begin{bmatrix} z^{-m}\mathbf{G}_1(z) & \mathbf{0} \\ \mathbf{G}_1(z) & \mathbf{G}_2(z) \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C_1 & D_1 \\ C_2 & D_2 \end{array}\right]$$

for some realization matrices. Let $\mathbf{v}(t) = \begin{bmatrix} \mathbf{v}_1(t) & \mathbf{v}_2(t) \end{bmatrix}'$. Then Wiener filtering can be converted into output estimation as shown in Fig. 5.6. Consequently, the results on output estimation can be applied to design the stationary optimal estimator represented by $\mathbf{W}(z)$ which is the required Wiener filter. If $z^{-m}\mathbf{G}_1(z)$ is not causal, decompose

$$z^{-m}\mathbf{G}_1(z) = \mathbf{G}_A(z) + \mathbf{G}_C(z)$$

with $\mathbf{G}_C(z)$ causal and $\mathbf{G}_A(z)$ strictly anticausal. A state-space realization of $\mathbf{G}(z)$ can again be obtained with $z^{-m}\mathbf{G}_1(z)$ replaced by $\mathbf{G}_C(z)$. It can be shown (Problem 5.19 in Exercises) that the Wiener filter does not depend on $\mathbf{G}_A(z)$.

5.2 Minimum Variance Control

A common control problem is *disturbance rejection*. Engineering systems are designed to operate in various environments where unknown and random disturbances are unavoidable and detrimental to the system performances. Disturbance rejection aims to design effective control laws that suppress the devastating effects of the disturbances and ensure that the system operates as desired. Often it results in feedback control laws. This section investigates the case when disturbances are white noise processes and the variance is the performance measure for control system design.

The system under consideration is described by the state-space model

$$\mathbf{x}(t+1) = A_t\mathbf{x}(t) + B_{1t}\mathbf{v}(t) + B_{2t}\mathbf{u}(t), \quad \mathbf{z}(t) = C_t\mathbf{x}(t) + D_t\mathbf{u}(t), \tag{5.80}$$

where $\mathbf{v}(t)$ is the white noise disturbance with the same statistics as in (5.20), $\mathbf{u}(t)$ is the control input signal, and $\mathbf{z}(t)$ is the output to be controlled. The initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is assumed to be random and has mean $\bar{\mathbf{x}}_0$ and covariance P_0. Due to the random nature and the dynamic impact of the initial condition, $\mathbf{x}_0$ is accounted as part of the disturbance. The objective is to design a control law $\mathscr{U}_T = \{\mathbf{u}(t)\}_{t=0}^{T-1}$ that minimizes

$$V_{[0,T)} = \sum_{t=0}^{T-1} V_t, \quad V_t = \mathrm{E}\left\{\|\mathbf{z}(t)\|^2 \,|\, \mathscr{U}_T\right\} = \mathrm{Tr}\left(\mathrm{E}\left\{\mathbf{z}(t)\mathbf{z}(t)^* \,|\, \mathscr{U}_T\right\}\right) \tag{5.81}$$

with V_t the variance of the controlled output.

The aforementioned control problem is very different from the estimation problem in the previous section, but the two are closely related. In fact, there exists a duality relation between the linear minimum variance control and the linear minimum variance estimation. As a fortiori, optimal disturbance rejection can be obtained from the Kalman filtering. However, such a derivation may blur out the distinctions between control and estimation and is thus not adopted in this text. Instead, linear minimum variance control will be derived independently. The duality will be interpreted at a later stage to deepen the understanding of the resultant optimal feedback control.

5.2.1 Linear Quadratic Regulators

Before tackling the problem of disturbance rejection, design of linear quadratic regulators (LQRs) will be studied. The LQR is a deterministic control problem. Yet, its solution coincides with that for disturbance rejection. Let

$$\mathbf{x}(t+1) = A_t\mathbf{x}(t) + B_t\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \neq 0. \tag{5.82}$$

It is desirable to regulate the state vector $\mathbf{x}(T)$ to the origin $\mathbf{0}$ in finite time $T > 0$ through some suitable control action $\{\mathbf{u}(t)\}_{t=0}^{T-1}$. However, the exact regulation to $\mathbf{x}(T) = \mathbf{0}$ may not be feasible in some finite time T. Even if it is feasible, the cost of control input can be prohibitively high. Hence, it is appropriate to consider the quadratic performance index

$$J_T(t_0) = \mathbf{x}(T)^* Q_T \mathbf{x}(T) + \sum_{t=t_0}^{T-1} \mathbf{x}(t)^* Q_t \mathbf{x}(t) + \mathbf{u}(t)^* R_t \mathbf{u}(t) \tag{5.83}$$

for $t_0 = 0$ which provides the mechanism of trade-offs between the regulation of $\mathbf{x}(t)$ and the energy constraint on $\mathbf{u}(t)$. The weighting matrix $Q_t = Q_t^* \geq 0$ represents the penalty on the state vector and thus the quality of regulation, and $R_t = R_t^* > 0$ shows the penalty on the control input and thus the measure of the energy at time t. For convenience, $J_T = J_T(0)$ is used.

The hypothesis on the weighting matrices in J_T implies that

$$Q_t = C_t^* C_t, \quad R_t = D_t^* D_t, \quad D_t^* C_t = \mathbf{0} \tag{5.84}$$

for some C_t of dimension $p \times n$ and D_t of dimension $p \times m$ with n the order of the state-space model in (5.82) and m the dimension of the control input. Let $\mathbf{z}(t) = C_t \mathbf{x}(t) + D_t \mathbf{u}(t)$. Then together with the state-space equation (5.82),

$$\begin{bmatrix} \mathbf{x}(t+1) \\ \mathbf{z}(t) \end{bmatrix} = \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \tag{5.85}$$

which represents the system model. The decomposed LQR cost at time t is

$$\|\mathbf{z}(t)\|^2 = \mathbf{x}(t)^* Q_t \mathbf{x}(t) + \mathbf{u}(t)^* R_t \mathbf{u}(t).$$

For any square matrix $X_{t+1} = X_{t+1}^* \geq \mathbf{0}$ with size $n \times n$, let

$$W_{t+1} = \mathbf{x}(t+1)^* X_{t+1} \mathbf{x}(t+1) + \|\mathbf{z}(t)\|^2$$

be the candidate Lyapunov function for $0 \leq t < T$. Then

$$\begin{aligned}
W_{t+1} &= \mathbf{x}(t+1)^* X_{t+1} \mathbf{x}(t+1) + \mathbf{z}(t)^* \mathbf{z}(t) \\
&= \begin{bmatrix} \mathbf{x}(t+1)^* & \mathbf{z}(t)^* \end{bmatrix} \begin{bmatrix} X_{t+1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \mathbf{x}(t+1) \\ \mathbf{z}(t) \end{bmatrix} \\
&= \left(\begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \right)^* \begin{bmatrix} X_{t+1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \left(\begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} \right) \\
&= \begin{bmatrix} \mathbf{x}(t)^* & \mathbf{u}(t)^* \end{bmatrix} \begin{bmatrix} A_t^* X_{t+1} A_t + Q_t & A_t^* X_{t+1} B_t \\ B_t^* X_{t+1} A_t & R_t + B_t^* X_{t+1} B_t \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix},
\end{aligned}$$

where (5.85) and the relations $R_t = D_t^* D_t$, $Q_t = C_t^* C_t$, and $D_t^* C_t = \mathbf{0}$ are used. Let $\Psi_t = A_t^* X_{t+1} A_t + Q_t, \Omega_t = B_t^* X_{t+1} A_t$, and $\Theta_t = R_t + B_t^* X_{t+1} B_t$. Because $\Theta_t > 0$ by $R_t > 0$, the Schur decomposition

$$\begin{bmatrix} \Psi_t & \Omega_t^* \\ \Omega_t & \Theta_t \end{bmatrix} = \begin{bmatrix} I & \Omega_t^*\Theta_t^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} X_t & 0 \\ 0 & \Theta_t \end{bmatrix} \begin{bmatrix} I & 0 \\ \Theta_t^{-1}\Omega_t & I \end{bmatrix}$$

holds (refer to Problem 5.8 in Exercises) with $X_t = \left(\Psi_t - \Omega_t^*\Theta_t^{-1}\Omega_t\right)$ the Schur complement. The Lyapunov function candidate W_{t+1} can now be written into

$$W_{t+1} = \left[\, \mathbf{x}(t)^* \;\; (\mathbf{u}(t) - F_t\mathbf{x}(t))^* \,\right] \begin{bmatrix} X_t & 0 \\ 0 & \Theta_t \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) - F_t\mathbf{x}(t) \end{bmatrix},$$

where $F_t = -\Theta_t^{-1}\Omega_t = -\left(R_t + B_t^*X_{t+1}B_t\right)^{-1} B_t^*X_{t+1}A_t$ and X_t satisfies

$$\begin{aligned} X_t &= A_t^*X_{t+1}A_t + Q_t - A_t^*X_{t+1}B_t\left(R_t + B_t^*X_{t+1}B_t\right)^{-1} B_t^*X_{t+1}A_t \\ &= A_t^*X_{t+1}\left(I_n + B_tR_t^{-1}B_t^*X_{t+1}\right)^{-1} A_t + Q_t \end{aligned} \tag{5.86}$$

which is a DRE dual to the estimation DRE as in Theorem 5.7. Because $\Theta_t > 0$, the minimum value of W_{t+1} is achieved by setting

$$\mathbf{u}(t) = \mathbf{u}_{\text{opt}}(t) := F_t\mathbf{x}(t), \quad F_t = -\left(R_t + B_t^*X_{t+1}B_t\right)^{-1} B_t^*X_{t+1}A_t \tag{5.87}$$

for which $W_{t+1} = \mathbf{x}(t)^*X_t\mathbf{x}(t)$ is the minimum possible.

Let $X_T = Q_T$ be the boundary condition and $\{X_t\}_{t=0}^{T-1}$ be the solution to the DRE in (5.86). Then an induction process can be applied to W_{t+1} with $t = T-1, T-2, \ldots, 0$ in the performance index (5.83) yielding

$$\begin{aligned} J_T &= \mathbf{x}(T)^*X_T\mathbf{x}(T) + \sum_{k=0}^{T-1} \|\mathbf{z}(k)\|^2 = W_T + \sum_{k=0}^{T-2} \|\mathbf{z}(k)\|^2 \\ &\geq \mathbf{x}(T-1)^*X_{T-1}\mathbf{x}(T-1) + \sum_{k=0}^{T-2} \|\mathbf{z}(k)\|^2 \\ &= W_{T-1} + \sum_{k=0}^{T-3} \|\mathbf{z}(k)\|^2 \geq \cdots \geq \mathbf{x}(0)^*X_0\mathbf{x}(0), \end{aligned}$$

where the fact $W_{t+1} \geq \mathbf{x}(t)^*X_t\mathbf{x}(t)$ is used to arrive at the lower bound of J_T. It is noted that the lower bound $\mathbf{x}_0^*X_0\mathbf{x}_0$ for J_T is achievable by employing the optimal control law $\left\{\mathbf{u}_{\text{opt}}(t)\right\}_{t=0}^{T-1}$ as in (5.87) which constitutes the optimal solution to the LQR control problem. The above derivations are summarized into the following result.

Theorem 5.17. *Suppose that $Q_t \geq 0$ and $R_t > 0$ for $0 \leq t < T$. Let $\{X_t\}_{t=0}^{T-1}$ be the solution to the* DRE *in (5.86) with the boundary condition $X_T = Q_T \geq 0$. Then the optimal control law minimizing the performance index J_T in (5.83) and subject* *to the dynamic equation (5.82) is given by* $\{\mathbf{u}_{\text{opt}}(t)\}_{t=0}^{T-1}$ *in (5.87). The associated minimum performance index of* J_T *is* $\mathbf{x}_0^* X_0 \mathbf{x}_0$ *with* $\mathbf{x}_0 \neq \mathbf{0}$ *the initial condition of the state vector* $\mathbf{x}(t)$*.*

It is surprising that the optimal control law for the LQR problem is linear and static. After all, one might have expected nonlinear and dynamic control laws as the possible optimal solution. On the other hand, the feedback structure of the control law is more or less expected. The optimal feedback gains $\{F_t\}_{t=0}^{T-1}$ as in (5.87) are functions of the solution to the DRE in (5.86) which has the form of backward recursion. Such a backward recursion is deeply rooted in the *principle of optimality* which states that an optimal control policy over the time interval $[k,\ T)$ for $0 < k < T$ constitutes the optimal control policy over the time interval $[0,\ T)$ regardless of the states and control inputs before the time k. Indeed, denote $\mathscr{U}_{[t_0,t_f)} = \{\mathbf{u}(t)\}_{t=t_0}^{t_f-1}$ with $t_f > t_0$. There holds

$$\min_{\mathscr{U}_{[0,T)}} J_T(0) = \min_{\mathscr{U}_{[0,T)}} \{J_k(0) + J_T(k)\} = \min_{\mathscr{U}_{[0,k)}} \left\{ J_k(0) + \min_{\mathscr{U}_{[k,T)}} J_T(k) \right\}$$

for $0 < k < T$ in light of the causality of the state-space equation. That is, minimization of $J_T(0)$ can be carried out in two stages with stage 1 for minimization of $J_T(k)$ over all possible $\mathscr{U}_{[k,T)}$ and stage 2 for minimization of $J_k(0) + \min\left\{J_T(k) :\ \mathscr{U}_{[k,T)}\right\}$ over all possible $\mathscr{U}_{[0,k)}$. The repeated use of this two-stage method for $k = T-1$, $T-2, \ldots$ is what is called *dynamic programming* and is employed in the derivation of the optimal control law for the LQR problem. It is worth emphasizing that, by again causality,

$$\min_{\mathscr{U}_{[0,T)}} J_T(0) \neq \min_{\mathscr{U}_{[k,T)}} \left\{ J_T(k) + \min_{\mathscr{U}_{[0,k)}} J_k(0) \right\} = \min_{\mathscr{U}_{[k,T)}} J_T(k) + \min_{\mathscr{U}_{[0,k)}} J_k(0).$$

One should also realize that the principle of optimality applies to more broad optimal control problems beyond the LQR for linear state-space systems.

Example 5.18. As an application of the LQR control, consider the tracking problem for the state-space system (5.82) with output $\mathbf{y}(t) = C_t\mathbf{x}(t)$. Given a desired output trajectory $\tilde{\mathbf{y}}(\cdot)$, how does one design a state-feedback control law which minimizes the tracking error and consumes the minimum energy? A reasonable measure is the quadratic performance index

$$J_T = \sum_{t=0}^{T-1} [\mathbf{y}(t) - \tilde{\mathbf{y}}(t)]^* Q_1(t) [\mathbf{y}(t) - \tilde{\mathbf{y}}(t)] + \mathbf{x}(t)^* Q_2(t)\mathbf{x}(t) + \mathbf{u}(t)^* R_t \mathbf{u}(t), \quad (5.88)$$

where $R_t > 0$, $Q_1(t) > 0$, and $Q_2(t) = I - C_t^* \left(C_t C_t^*\right)^{-1} C_t$ for all t assuming that rows of C_t are linearly independent. Let the dimension of $\mathbf{y}(t)$ be $p < n$ with n the dimension of the state vector $\mathbf{x}(t)$. A small tracking error imposes only the p

constraints on the state vector or in the subspace spanned by p columns of C_t^*. The weighting matrix $Q_2(t)$ regulates the state vectors with the remaining $(n-p)$ constraints or in the null space of C_t. The weighting factors can be changed through adjusting $Q_1(t)$ and R_t. The terminal penalty $Q_1(T)$ is omitted for convenience. The desired output trajectory $\tilde{\mathbf{y}}(\cdot)$ is assumed to be the output of a LTI model

$$\mathbf{w}(t+1) = A_w \mathbf{w}(t), \quad \tilde{\mathbf{y}}(t) = H\mathbf{w}(t).$$

Such a model includes step functions, sinusoidal signals, and their linear combinations. Together with the system model (5.82), there holds

$$\tilde{\mathbf{x}}(t+1) = \tilde{A}_t \mathbf{x}(t) + \tilde{B}_t \mathbf{u}(t), \quad \delta\mathbf{y}(t) = \tilde{C}_t \tilde{\mathbf{x}}(t),$$

where $\tilde{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{x}(t)^* & \mathbf{w}(t)^* \end{bmatrix}^*$, $\delta\mathbf{y}(t) = \mathbf{y}(t) - \tilde{\mathbf{y}}(t)$, and thus,

$$\tilde{A}_t = \begin{bmatrix} A_t & \mathbf{0} \\ \mathbf{0} & A_w \end{bmatrix}, \quad \tilde{B}_t = \begin{bmatrix} B_t \\ \mathbf{0} \end{bmatrix}, \quad \delta C_t = \begin{bmatrix} C_t & -H \end{bmatrix}.$$

It follows that the performance index J_T in (5.88) can be written into the same form as in (5.83) with $\mathbf{x}(t)$ replaced by $\tilde{\mathbf{x}}(t)$ leading to

$$Q_t = \begin{bmatrix} Q_2(t) + C_t^* Q_1(t) C_t & -C_t^* Q_1(t) H \\ -H^* Q_1(t) C_t & H^* Q_1(t) H \end{bmatrix}.$$

Hence, the LQR control law in Theorem 5.17 can be readily applied.

In reference to the LQR control, a similar solution approach can be adopted to the minimum variance control. Recall the expression of V_t in (5.81) and the augmented performance index

$$V_{[t_0,T)} = V_{t_0} + V_{t_1} + \cdots + V_{T-1}, \quad 0 \le t_0 < T. \tag{5.89}$$

Then the control law $\{\mathbf{u}(t)\}_{t=0}^{T-1}$ minimizing $V_{[0,T)}$ is reminiscent of LQR control law as shown next.

Theorem 5.19. *Consider the state-space system (5.82) where $\{\mathbf{v}(t)\}$ is white satisfying (5.20) and $\mathbf{x}(0) = \mathbf{x}_0$ is a random vector independent of $\{\mathbf{v}(t)\}$ with mean $\overline{\mathbf{x}}_0$ and covariance P_0. Suppose that $R_t = D_t^* D_t > 0$ and $D_t^* C_t = \mathbf{0}$ for all t. Let $Q_t = C_t^* C_t$, and $\{X_t\}_{t=0}^{T-1}$ be the solution to the* DRE *in (5.86) with the boundary condition $X_T = \mathbf{0}$ and $B_t = B_{2t}$. Then the optimal control law minimizing $V_{[0,T)}$ in (5.89) is the same as $\mathbf{u}_{\rm opt}(t)$ in (5.87). Denote $\mathscr{U}_T = \{\mathbf{u}(t)\}_{t=0}^{T-1}$. Let $\Sigma_t = \mathrm{E}\{\mathbf{x}(t)\mathbf{x}(t)^* | \mathscr{U}_T\}$. Then $\Sigma_0 = P_0$ and*

$$\Sigma_{t+1} = (A_t + B_{2t}F_t)\Sigma_t(A_t + B_{2t}F_t)^* + B_{1t}B_{1t}^*. \tag{5.90}$$

The minimum variance for the controlled output over $[0,\ T)$ *is given by*

$$\min_{\mathscr{U}_T} V_{[0,T)} = \sum_{t=0}^{T-1} \mathrm{Tr}\{(C_t + D_t F_t)\Sigma_t (C_t + D_t F_t)^*\}. \tag{5.91}$$

Proof. It is noted that the state-space equation in (5.82) can be written as

$$\begin{bmatrix} \mathbf{x}(t+1) \\ \mathbf{z}(t) \end{bmatrix} = \begin{bmatrix} A_t & B_{2t} \\ C_t & D_t \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} + \begin{bmatrix} B_{1t} \\ \mathbf{0} \end{bmatrix} \mathbf{v}(t) \tag{5.92}$$

which is similar to (5.85). Let $W_{k+1} = \mathrm{E}\{\mathbf{x}(k+1)^* X_{k+1} \mathbf{x}(k+1) | \mathscr{U}_T\} + V_k$. Since the solution to the DRE (5.86) with the boundary condition $X_T = \mathbf{0}$ satisfies $X_t \geq \mathbf{0}$ (refer to Problem 5.21 in Exercises), W_{k+1} is nonnegative for $0 \leq k < T$. Similar to the derivation for the LQR control, there holds

$$\begin{aligned} W_{k+1} &= \mathrm{E}\{\mathbf{x}(k+1)^* X_{k+1} \mathbf{x}(k+1) + \mathbf{z}(k)^* \mathbf{z}(k) | \mathscr{U}_T\} \\ &= \mathrm{E}\left\{ \begin{bmatrix} \mathbf{x}(k+1)^* & \mathbf{z}(k)^* \end{bmatrix} \begin{bmatrix} X_{k+1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{z}(k) \end{bmatrix} \middle| \mathscr{U}_T \right\} \\ &= \mathrm{Tr}\{B_{1k}^* X_{k+1} B_{1k}\} + \mathrm{E}\left\{ \begin{bmatrix} \mathbf{x}(k) \\ \delta\mathbf{u}(k) \end{bmatrix}^* \begin{bmatrix} X_k & 0 \\ 0 & \Theta_k \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \delta\mathbf{u}(k) \end{bmatrix} \middle| \mathscr{U}_T \right\} \end{aligned}$$

by the independence of $\mathbf{u}(k)$ and $\mathbf{x}(k)$ to $\mathbf{v}(k)$ where $\Theta_k = R_k + B_{2k}^* X_k B_{2k}$, $\delta\mathbf{u}(k) = \mathbf{u}(k) - \mathbf{u}_{\mathrm{opt}}(k)$, and $\mathbf{u}_{\mathrm{opt}}(k)$ is defined as in (5.87) with $B_k = B_{2k}$. It follows that $\mathbf{u}(k) = \mathbf{u}_{\mathrm{opt}}(k)$ minimizes W_{k+1} for $k = T-1, T-2, \ldots, 0$ and thus $V_{[0,T)}$ in (5.89). Indeed, with $X_T = \mathbf{0}$,

$$\begin{aligned} V_{[0,T)} &= \sum_{k=0}^{T-1} V_k = W_T + \sum_{k=0}^{T-2} \mathrm{E}\{\mathbf{z}(k)^* \mathbf{z}(k) | \mathscr{U}_T\} \\ &\geq \mathrm{Tr}\left\{B_{1(T-1)}^* X_T B_{1(T-1)}\right\} + W_{T-1} + \sum_{k=0}^{T-3} \mathrm{E}\{\mathbf{z}(k)^* \mathbf{z}(k) | \mathscr{U}_T\} \\ &\geq \cdots \geq \sum_{k=0}^{T-1} \mathrm{Tr}\{B_{1k}^* X_{k+1} B_{1k}\} + \mathrm{E}\{\mathbf{x}(0)^* X_0 \mathbf{x}(0) | \mathscr{U}_T\}. \end{aligned}$$

The lower bound for $V_{[0,T)}$ is achieved by $\mathscr{U}_T = \{\mathbf{u}_{\mathrm{opt}}(t)\}_{t=0}^{T-1}$ which also minimizes V_t for all $t \in [0,\ T)$ in light of the principle of optimality. The expression in (5.91) can be easily verified by direct computation. □

It is observed that B_{1t} has no influence on the optimal feedback gain $\{F_t\}$ although it changes the performance index V_t. This feature is important and induces the duality between the minimum variance control and the Kalman filtering for which F_t is dual to the Kalman gain K_t, and the backward control DRE (5.86) is dual to the forward filtering DRE (5.36). For this reason, many properties of the Kalman filter also hold for the minimum variance control. In particular, the condition $D_t^* C_t = \mathbf{0}$ can be removed as shown next.

Corollary 5.2. *Under the same hypotheses as in Theorem 5.19 except that* $D_t^* C_t \neq \mathbf{0}$*, the optimal control law minimizing* V_t *in (5.89) is given by* $\mathbf{u}(t) = \mathbf{u}_{\text{opt}}(t) = F_t \mathbf{x}(t)$ *with*

$$F_t = -\left(R_t + B_{2t}^* X_{t+1} B_{2t}\right)^{-1} \left(B_{2t}^* X_{t+1} A_t + D_t^* C_t\right), \tag{5.93}$$

$$X_t = \tilde{A}_t^* X_{t+1} \left(I_n + B_{2t} R_t^{-1} B_{2t}^* X_{t+1}\right)^{-1} \tilde{A}_t + \tilde{C}_t^* \tilde{C}_t, \tag{5.94}$$

where $\tilde{A}_t = \left(A_t - B_{2t} R_t^{-1} D_t^* C_t\right)$, $\tilde{C}_t = \left(I - D_t R_t^{-1} D_t^*\right) C_t$*, and* $X_T = \mathbf{0}$.

Again the control gain F_t in (5.93) is dual to the filtering gain K_t in (5.45), and the control DRE in (5.94) is dual to the filtering DRE in (5.43).

Proof. Introduce the variable substitution $\mathbf{u}(t) = -R_t^{-1} D_t^* C_t \mathbf{x}(t) + \tilde{\mathbf{u}}(t)$ with $\tilde{\mathbf{u}}(t)$ to be designed. Then (5.92) is changed into

$$\begin{bmatrix} \mathbf{x}(t+1) \\ \mathbf{z}(t) \end{bmatrix} = \begin{bmatrix} \tilde{A}_t & B_{2t} \\ \tilde{C}_t & D_t \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \tilde{\mathbf{u}}(t) \end{bmatrix} + \begin{bmatrix} B_{1t} \\ \mathbf{0} \end{bmatrix} \mathbf{v}(t).$$

The result in Theorem 5.19 can then be applied to obtain the DRE in (5.94) and the optimal control gain as $\tilde{F}_t = -\left(R_t + B_{2t}^* X_t B_{2t}\right)^{-1} B_{2t}^* X_t \tilde{A}_t$. Hence, $\mathbf{u}(t) = -R_t^{-1} D_t^* C_t \mathbf{x}(t) + \tilde{F}_t \mathbf{x}(t) = F_t \mathbf{x}(t)$ with

$$\begin{aligned} F_t &= \tilde{F}_t - R_t^{-1} D_t^* C_t = -\Theta_t^{-1} \left[B_{2t}^* X_t \tilde{A}_t + (R_t + B_{2t}^* X_t B_{2t}) R_t^{-1} D_t^* C_t\right] \\ &= -\Theta_t^{-1} \left[B_{2t}^* X_t \left(A_t - B_{2t} R_t^{-1} D_t^* C_t\right) + D_t^* C_t + B_{2t}^* X_t B_{2t} R_t^{-1} D_t^* C_t\right] \\ &= -\left(R_t + B_{2t}^* X_t B_{2t}\right)^{-1} \left(B_{2t}^* X_t A_t + D_t^* C_t\right) \end{aligned}$$

which is identical to (5.93) where $\Theta_t = (R_t + B_{2t}^* X_t B_{2t})$ is used. □

Corollary 5.2 shows that it has no loss of generality to study the minimum variance control for the case $D_t^* C_t = \mathbf{0}$ from which the results can be easily carried to the case $D_t^* C_t \neq \mathbf{0}$. Moreover, there is no loss of generality to study the LQR problem in place of the minimum variance control. Both result in the same linear feedback control law. Hence, the rest of the section will focus on the LQR problem under the condition $D_t^* C_t = \mathbf{0}$ for simplicity.

5.2.2 Stability

State-feedback control was briefly discussed in Chap. 3 in connection with the notion of stabilizability. The LQR control is an effective way to design state-feedback control laws for the system model in (5.82) and is aimed at minimizing the quadratic performance index (5.83). A more general LQR control problem can be found in Problem 5.24 in Exercises that includes the cross term for the performance index. It is natural to study stability of the closed-loop system for the LQR feedback control which is governed by

$$\mathbf{x}(t+1) = [A_t + B_t F_t]\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{5.95}$$

with F_t given in Theorem 5.17. An important question to be answered is under what conditions the closed-loop system (5.95) is asymptotically or exponentially stable as $T \to \infty$ in the performance index J_T. It needs to be pointed out that difficulties exist in computing the LQR control law in the limiting case because of the time-varying realization for the system in (5.82) and time-varying weighting matrices in the performance index. Nevertheless, theoretical analysis can be made to obtain similar stability results to those for the Kalman filter. The next result is dual to Theorem 5.8 in the previous section but with strengthened stability property.

Theorem 5.20. *For the state-space model in (5.82) and the performance index in (5.83), assume that* (A_t, B_t) *is uniformly stabilizable and* (C_t, A_t) *is uniformly detectable with* $Q_t = C_t^* C_t$. *Then the closed-loop system (5.95) for the LQR control as described in Theorem 5.17 is exponentially stable as* $T \to \infty$.

The proof of this theorem is again left as an exercise (Problem 5.26). It is noted that exponential stability can be concluded for the LQR control different from that for the Kalman filter which is only asymptotically stable. Its reason lies in the Lyapunov stability criteria as discussed in Chap. 3. Recall that the filtering DRE in (5.36) can be written as the forward Lyapunov difference equation (Problem 5.10 in Exercises) for which the result of Lemma 3.4 can only ensure the asymptotic stability. On the other hand, the control DRE in (5.86) can be written as the backward Lyapunov difference equation (Problem 5.23 in Exercises) for which the result of Theorem 3.37 can in fact ensure the exponential stability.

While a stronger stability result holds for the LQR control than that for the Kalman filter, optimal state-feedback gain is difficult to compute for the limiting case $T \to \infty$. The exception is the stationary LQR control when the realization and the weighting matrices in the performance index J_T are all time invariant, and the time horizon $T \to \infty$. Consider the ARE

$$\begin{aligned} X &= A^*XA - A^*XB\left(R + B^*XB\right)^{-1} B^*XA + C^*C \\ &= A^*\left(I_n + XBR^{-1}B^*\right)^{-1} XA + C^*C, \end{aligned} \tag{5.96}$$

where $R = D^*D$ and $D^*C = \mathbf{0}$. The above is the same as the DRE in (5.86) except that all the time indices are removed. It is often called the control ARE, versus the filtering ARE (5.50) for the stationary Kalman filtering.

Example 5.21. Consider the flight control system introduced in Problem 1.11 in Exercises of Chap. 1. Its state-space realization after discretization with sampling period $T_s = 0.025$ is obtained in Example 3.17 of Chap. 3. Suppose that the LQR control is employed with $Q = C'C$ and $R = I$. The Matlab command “dare” can be used to compute the solution X to the ARE in (5.96) that is given by

$$X = \begin{bmatrix} 112.3476 & 18.2824 & 164.7632 & 25.6960 & -102.8198 \\ 18.2824 & 43.9026 & 38.9169 & 8.1070 & -24.7280 \\ 164.7632 & 38.9169 & 382.4594 & 81.6639 & -213.8244 \\ 25.6960 & 8.1070 & 81.6639 & 23.1625 & -42.9162 \\ -102.8198 & -24.7280 & -213.8244 & -42.9162 & 129.6328 \end{bmatrix}.$$

The corresponding stationary state-feedback gain is obtained as

$$F = -\left(R + B^*XB\right)^{-1} B^*XA = \begin{bmatrix} 0.1230 & 0.0096 & -0.0520 & -0.0853 & -0.0399 \\ -0.4477 & -1.0709 & -0.9530 & -0.1986 & 0.6062 \\ 0.8648 & 0.2892 & 2.9645 & 0.8771 & -1.5290 \end{bmatrix}.$$

It can be easily verified with Matlab that $(A + BF)$ is a stability matrix by examining its eigenvalues, implying that the state vector under this LQR control approaches zero asymptotically.

A similar notion to that for Kalman filtering is defined next.

Definition 5.2. The solution X to the *ARE* (5.96) is said to be stabilizing, if the state-feedback gain $F = -\left(R + B^*XB\right)^{-1} B^*XA$ is stabilizing.

With the feedback gain $F = -\left(R + B^*XB^*\right)^{-1} B^*XA$, the ARE in (5.96) can be written into the Lyapunov equation (refer to Problem 5.23 in Exercises)

$$X = (A + BF)^* X (A + BF) + C^*C + F^*RF. \tag{5.97}$$

The following is the stability result for the stationary LQR control.

Theorem 5.22. *Let $Q = C^*C$ and (C, A) be detectable. If the* ARE *(5.96) admits a stabilizing solution X, then the solution $\{X_t(T)\}$ to the* DRE

$$X_t(T) = A^* X_{t+1}(T) \left[I_n + BR^{-1}B^* X_{t+1}(T)\right]^{-1} A + Q, \quad X_T(T) = \mathbf{0} \tag{5.98}$$

converges to X as $T \to \infty$. In this case, the closed-loop system

$$\mathbf{x}(t+1) = (A + BF)\mathbf{x}(t), \quad F = -\left(R + B^*XB\right)^{-1} B^*XA \tag{5.99}$$

for the stationary LQR control is stable.

Proof. The solution to (5.98) satisfies (refer to Problem 5.31 in Exercises):

$$X_{t+1}(T) \le X_t(T) = X_{t+1}(T+1),$$

where $X_{t+1}(T+1)$ is the solution to the same DRE in (5.98) with T replaced by $T+1$. Hence, $\{X_t(T)\}$ is monotonically increasing with respect to T. Let $J_T(t)$, $t \geq 0$, be the performance index associated with DRE (5.98). Then

$$0 \leq \lim_{T\to\infty} J_T(t) = \lim_{T\to\infty} \mathbf{x}(t)^* X_t(T)\mathbf{x}(t) \leq \mathbf{x}(t)^* X \mathbf{x}(t)$$

implying $0 \leq X_t(T) = X_t(T)^* \leq X \; \forall t < T$ by the fact that the stabilizing solution X is maximal (Problem 5.27 in Exercises). Thus, it has a unique limit $\overline{X} = \overline{X}^*$ satisfying $0 \leq \overline{X} \leq X$. Since the ARE (5.96) is the limit of the DRE (5.98), $\overline{X}$ is a solution to (5.96) which is the same as the Lyapunov equation (5.97) with X replaced by $\overline{X}$ and F replaced by $\overline{F} = -\left(R + B^*\overline{X}B^*\right)^{-1} B^*\overline{X}A$. The detectability of (C,A) and $\overline{X} \geq 0$ imply that $\left(A + B\overline{F}\right)$ is a stability matrix in light of the Lyapunov stability result or $\overline{X}$ is a stabilizing solution to the ARE in (5.96). By the uniqueness of the stabilizing solution to the ARE (Problem 5.13 in Exercises), $\overline{X} = X$. It follows that the closed-loop system in (5.99) is stable. □

Theorem 5.22 offers a numerical algorithm for computing the unique stabilizing solution X to the ARE (5.96) through computing iteratively the solution to (5.98). That is, one may set $X^{(0)} = X_T(T) = \mathbf{0}$ then compute

$$X^{(i+1)} = A^* \left[I_n + BR^{-1}B^*X^{(i)}\right]^{-1} X^{(i)}A + Q$$

for $i = 1, 2, \ldots$ until $\left\|X^{(N+1)} - X^{(N)}\right\| \leq \varepsilon$ with $\varepsilon > 0$ some prespecified error tolerance and then take $X = X^{(N+1)}$. The next result answers under what condition there exists a stabilizing solution to the ARE (5.96). Since the ARE (5.50) for the stationary Kalman filter is dual to the ARE (5.96), it also provides the proof for Theorem 5.11.

Theorem 5.23. *Let $Q = C^*C$ and $R > 0$. There exists a stabilizing solution to the* ARE *in (5.96), if and only if (A, B) is stabilizable and*

$$\operatorname{rank}\left\{\begin{bmatrix} A - e^{j\omega}I_n \\ C \end{bmatrix}\right\} = n \quad \forall\; \omega \in \mathbb{R}. \tag{5.100}$$

Proof. It is obvious that stabilizability of (A, B) is a necessary condition for the ARE (5.96) to have a stabilizing solution. To confirm that (5.100) is also a necessary condition, assume on the contrary that (5.100) does not hold but the ARE (5.96) admits a stabilizing solution X. The Lyapunov form of the ARE in (5.97) implies that $(A + BF)$ is a stability matrix with F as in (5.99), and thus, $X = X^* \geq 0$. Since (5.100) does not hold,

$$A\mathbf{q} = e^{j\theta}\mathbf{q}, \quad C\mathbf{q} = 0 \tag{5.101}$$

for some θ real and $\mathbf{q} \neq \mathbf{0}$. That is, (C, A) has at least one unobservable mode on the unit circle. Multiplying both sides of the ARE in (5.96) by $\mathbf{q}^*$ from left and $\mathbf{q}$ from right, and using the relation in (5.101) yield

$$\mathbf{q}^* XB \left(R + B^* XB\right)^{-1} B^* X\mathbf{q} = 0 \quad \Longrightarrow \quad B^* X\mathbf{q} = \mathbf{0}.$$

By the expression of F, the above leads to

$$(A+BF)\mathbf{q} = \left[A - BB^* \left(R + B^* XB\right)^{-1} B^* XA\right] \mathbf{q} = \mathrm{e}^{j\theta}\mathbf{q}.$$

So $\mathrm{e}^{j\theta}$ remains an eigenvalue of $A+BF$ contradicting the stabilizing assumption on X. This concludes the necessity of (5.100).

For the sufficiency part of the proof, assume that (A,B) is stabilizable and (5.100) holds. Then some F_0 exists such that $(A+BF_0)$ is a stability matrix. It is claimed that the following recursion

$$X_i = (A+BF_i)^* X_i (A+BF_i) + F_i^* RF_i + Q, \tag{5.102}$$

$$F_{i+1} = -\left(R + B^* X_i B\right)^{-1} B^* X_i A, \quad i = 0, 1, \ldots, \tag{5.103}$$

converges to the stabilizing solution X of the ARE (5.96). The proof of the claim proceeds in three steps. At the first step, it will be shown that stability of $(A+BF_i)$ implies stability of $(A+BF_{i+1})$ for $i \geq 0$. For this purpose, rewrite (5.102) as (refer to Problem 5.27 in Exercises)

$$\begin{aligned} X_i &= A^* \left(I_n + X_i BR^{-1} B^*\right)^{-1} X_i A + Q + \Delta_F(i)^* \left[R + B^* X_i B\right] \Delta_F(i) \\ &= (A+BF_{i+1})^* X_i (A+BF_{i+1}) + F_{i+1}^* RF_{i+1} + Q \\ &\quad + \Delta_F(i)^* \left[R + B^* X_i B\right] \Delta_F(i) \end{aligned} \tag{5.104}$$

with $\Delta_F(i) = F_{i+1} - F_i$ where (5.97) is used with X replaced by X_i and F by F_{i+1} to obtain the second equality. Now suppose that

$$(A+BF_{i+1})\,\mathbf{v} = \lambda\mathbf{v}, \quad |\lambda| \geq 1. \tag{5.105}$$

Multiplying both sides of (5.104) by $\mathbf{v}^*$ from left and $\mathbf{v}$ from right yields

$$\left(1 - |\lambda|^2\right) \mathbf{v}^* X_i \mathbf{v} = \mathbf{v}^* \left[F_{i+1}^* RF_{i+1} + Q + \Delta_F(i)^* \left(R + B^* X_i B\right) \Delta_F(i)\right] \mathbf{v},$$

where (5.105) is used. Because the left-hand side ≤ 0 by $|\lambda| \geq 1$ and positivity of X_i due to stability of $(A+BF_i)$ and the right-hand side ≥ 0 by positivity of R, Q, and X_i, it is concluded that $|\lambda| = 1$ and

$$C\mathbf{v} = \mathbf{0}, \quad F_{i+1}\mathbf{v} = \mathbf{0}, \quad \Delta_F(i)\mathbf{v} = \mathbf{0} \quad \Longrightarrow \quad F_i\mathbf{v} = \mathbf{0}.$$

The above together with (5.105) imply $A\mathbf{v} = \lambda\mathbf{v}$ which in turn implies

$$(A+BF_i)\mathbf{v} = A\mathbf{v} = \lambda\mathbf{v}.$$

Because $(A+BF_i)$ is a stability matrix, $\mathbf{v} = \mathbf{0}$ concluding that λ is not an eigenvalue of $(A+BF_{i+1})$. As λ with $|\lambda| \geq 1$ is arbitrary, $(A+BF_{i+1})$ is also a stability matrix. The fact that $(A+BF_0)$ is a stability matrix implies that F_{i+1} in (5.103) is stabilizing for each $i \geq 0$. As a second step, it is noted that (5.104) and the definition of X_{i+1} imply

$$\Delta_X(i) = (A+BF_{i+1})^*\Delta_X(i)(A+BF_{i+1}) + \Delta_F(i)^*(R+B^*X_iB)\Delta_F(i)$$

with $\Delta_X(i) = X_i - X_{i+1}$. Stability of $(A+BF_{i+1})$ implies that $\Delta_X(i) \geq \mathbf{0}$ or $\{X_i\}$ is a decreasing matrix sequence. Since $X_i \geq \mathbf{0}$ by stability of $(A+BF_i)$, the recursion in (5.102) and (5.103) converges with limits $X \geq \mathbf{0}$ satisfying the ARE (5.96) and F as given in (5.99). Finally, as $(A+BF_i)$ is stable for all $i \geq 0$, the n eigenvalues of $(A+BF_i)$ converge to the n eigenvalues of $(A+BF)$ on the closed unit disk. The condition (5.100) prohibits any eigenvalues of $(A+BF)$ from being on the unit circle because if it does, then

$$(A+BF)\mathbf{v} = \mathrm{e}^{j\theta}\mathbf{v}, \quad \mathbf{v} \neq \mathbf{0}$$

for some θ real. Multiplying both sides of (5.97) by $\mathbf{v}^*$ from left and $\mathbf{v}$ from right leads to $F\mathbf{v} = \mathbf{0}$ and $C\mathbf{v} = \mathbf{0}$, and thus $A\mathbf{v} = \mathrm{e}^{j\theta}\mathbf{v}$ with the same argument as before. This contradicts the condition (5.100). The proof is now complete. □

The proof of Theorem 5.23 shows that the condition (5.100) is indispensable. Stabilizability of (A,B) alone does not ensure that the LQR problem is well posed. If the condition (5.100) is violated, then any unobservable mode of (C,A) on the unit circle does not contribute to the LQR performance index. Thus, in this case, even if the ARE (5.96) admits a solution $X = X^* \geq 0$, the optimal performance index (for the stationary LQR control)

$$J_{\text{opt}} = \sum_{t=0}^{\infty} \mathbf{x}(t)^*Q\mathbf{x}(t) + \mathbf{u}(t)^*R\mathbf{u}(t) = \sum_{t=0}^{\infty} \mathbf{x}(t)^*\left(Q + F^*RF\right)\mathbf{x}(t) = \mathbf{x}_0^*X\mathbf{x}_0$$

and stability of $(A+BF)$ cannot be achieved simultaneously. The reason lies in the facts that stabilization of any unobservable mode of (C,A) on the unit circle will increase the energy cost of the control input by $R > 0$ and that such unstable modes of (C,A) do not contribute to the LQR performance index anyway. This is illustrated in the following example.

Example 5.24. Consider the stationary LQR control with

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and $R = 1$. Clearly, (A,B) is stabilizable, but the condition (5.100) does not hold. It can be verified that with $F_0 = \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix}$, $(A+BF_0)$ is a stability matrix. The recursive algorithm as in (5.102) and (5.103) gives

$$X_i = \begin{bmatrix} \frac{1}{2^{i+2}-1} & 0 \\ 0 & 1 \end{bmatrix}, \quad A + BF_i = \begin{bmatrix} \frac{1}{2^{i+1}} - 1 & 0 \\ 0 & 0 \end{bmatrix}$$

for $0 \leq i < \infty$. Hence, $(A + BF_i)$ is stable for any finite i and $X_i \geq \mathbf{0}$ is monotonically decreasing. However, the limits

$$\lim_{i\to\infty} X_i = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \lim_{i\to\infty} F_i = \lim_{k\to\infty} \begin{bmatrix} \frac{1}{2^{i+1}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

and thus $(A + BF_i) \to A$ as $i \to \infty$ which is unstable.

Example 5.24 leads to the deduction that if (C,A) has unobservable modes strictly outside the unit circle but the stabilizability of (A,B) and (5.100) hold, then the ARE has more than one nonnegative definite solutions. One is the stabilizing solution X. There is at least one more, denoted by X_u, which is not stabilizing. That is, the unobservable modes of (C,A) strictly outside the unit circle are not stabilized by $F_u = -(R + B^*X_uB)^{-1} B^*X_uA$. Since the unstable modes strictly outside the unit circle do not contribute to the performance index by the hypothesis, $X \geq X_u \geq 0$. In this case, the maximal solution of the ARE is always the stabilizing solution. It is now clear why the detectability of (C,A) is required in Theorem 5.22, without which $X_T(t)$ may converge to X_u as $T \to \infty$ with $t \geq 0$ finite.

The next result states the solution to the general stationary LQR control.

Corollary 5.3. *For* $\mathbf{x}(t+1) = A\mathbf{x}(t) + B\mathbf{u}(t)$ *with* $\mathbf{x}(0) = \mathbf{x}_0 \neq \mathbf{0}$*, let*

$$J_\infty = \sum_{t=0}^{\infty} \|C\mathbf{x}(t) + D\mathbf{u}(t)\|^2, \quad R = D^*D > 0, \quad D^*C \neq \mathbf{0}.$$

Let $\tilde{A} = \left(A - BR^{-1}D^*C\right)$ *and* $\tilde{C} = \left(I - DR^{-1}D^*\right)C$*. Suppose that the* ARE

$$X = \tilde{A}^* \left(I + XBR^{-1}B^*\right)^{-1} X\tilde{A} + \tilde{C}^*\tilde{C} \tag{5.106}$$

has a stabilizing solution X*. Then the optimal control law is given by*

$$\mathbf{u}(t) = F\mathbf{x}(t), \quad F = -(R + B^*XB)^{-1}(B^*XA + D^*C) \tag{5.107}$$

which is stabilizing and minimizes J_∞*. Moreover, the* ARE *(5.106) admits a stabilizing solution, if and only if* (A,B) *is stabilizable and*

$$\operatorname{rank}\left\{ \begin{bmatrix} A - e^{j\omega}I_n & B \\ C & D \end{bmatrix} \right\} = n + m \quad \forall\ \omega \in \mathbb{R}, \tag{5.108}$$

where m *is the dimension of the input and* n *the dimension of the state vector.*

Proof. Since the general LQR control is the same as that for Theorems 5.22 and 5.23 with A replaced by $\tilde{A}$ and C replaced by $\tilde{C}$, the proof of the first part of the corollary is simple and skipped. For the second part of the corollary, it is noted that stabilizability of $\left(\tilde{A}, B\right)$ is the same as stabilizability of (A, B), and thus, the proof can be completed by showing that the condition

$$\operatorname{rank}\left\{\begin{bmatrix} \tilde{A} - \mathrm{e}^{j\omega} I_n \\ \tilde{C} \end{bmatrix}\right\} = n \tag{5.109}$$

is equivalent to (5.108). It is straightforward to compute

$$\begin{bmatrix} I_n & -BR^{-1}D^* \\ \mathbf{0} & I - DR^{-1}D^* \\ \mathbf{0} & R^{-1}D^* \end{bmatrix} \begin{bmatrix} A - \mathrm{e}^{j\omega} I_n & B \\ C & D \end{bmatrix} = \begin{bmatrix} \tilde{A} - \mathrm{e}^{j\omega} I_n & \mathbf{0} \\ \tilde{C} & \mathbf{0} \\ R^{-1}D^*C & I_m \end{bmatrix}.$$

The first matrix on the left is an elementary matrix that does not alter the rank of the second matrix on the left. It follows that

$$\operatorname{rank}\left\{\begin{bmatrix} A - \mathrm{e}^{j\omega} I_n & B \\ C & D \end{bmatrix}\right\} = m + \operatorname{rank}\left\{\begin{bmatrix} \tilde{A} - \mathrm{e}^{j\omega} I_n \\ \tilde{C} \end{bmatrix}\right\},$$

and hence, the condition (5.108) is equivalent to the one in (5.109). □

5.2.3 Full Information Control

In minimum variance control, the controlled output $\mathbf{z}(t)$ in (5.80) does not involve the disturbance input. This is the main reason why the optimal feedback control law is a function of only $\mathbf{x}(t)$. Suppose that the state-space model and the controlled output are specified respectively by

$$\begin{aligned} \mathbf{x}(t+1) &= A_t \mathbf{x}(t) + B_{1t}\mathbf{v}(t) + B_{2t}\mathbf{u}(t), \\ \mathbf{z}(t) &= C_t \mathbf{x}(t) + D_{1t}\mathbf{v}(t) + D_{2t}\mathbf{u}(t). \end{aligned} \tag{5.110}$$

It can be expected that the optimal feedback control law will be a function of not only $\mathbf{x}(t)$ but also of $\mathbf{v}(t)$ which is the white noise process satisfying (5.20). Such a control law is termed *full information control*. One needs to keep in mind that often, in the practice of feedback control, both $\mathbf{x}(t)$ and $\mathbf{v}(t)$ are not measurable directly for which output estimators in the previous section can be employed to provide information on $\mathbf{x}(t)$ and $\mathbf{v}(t)$. The next result provides the optimal solution to full information control.

Theorem 5.25. *Consider the state-space system (5.110) where* $\{\mathbf{v}(t)\}$ *is the white noise process satisfying (5.20). Suppose that* $R_t = D_{2t}^* D_{2t} > 0$. *Let*

$$\tilde{A}_t = A_t - B_{2t} R_t^{-1} D_{2t}^* C_t, \quad \tilde{C}_t = \left[I - D_{2t} R_t^{-1} D_{2t}^*\right] C_t.$$

Let X_t *be the solution to the* DRE *(5.94). Then the optimal control law that minimizes* $\mathrm{E}\{\|\mathbf{z}(t)\| \,|\, \mathscr{U}_T\}$ *with* $\mathscr{U}_T = \{\mathbf{u}(t)\}_{t=0}^{T-1}$ *is* $\mathbf{u}(t) = F_{1t}\mathbf{x}(t) + F_{2t}\mathbf{v}(t)$ *with*

$$\begin{aligned} F_{1t} &= -\left(R_t + B_{2t}^* X_{t+1} B_{2t}\right)^{-1}\left(B_{2t}^* X_{t+1} A_t + D_{2t}^* C_t\right), \\ F_{2t} &= -\left(R_t + B_{2t}^* X_{t+1} B_{2t}\right)^{-1}\left(B_{2t}^* X_{t+1} B_{1t} + D_{2t}^* D_{1t}\right). \end{aligned} \tag{5.111}$$

Theorem 5.25 for full information control is dual to Theorem 5.14 for output estimation. Its proof is similar to that of Theorem 5.14 and is thus left as an exercise (Problem 5.33).

It is noted that the closed-loop system for (5.110) under the full information control law (5.111) is given by

$$\begin{aligned} \mathbf{x}(t+1) &= (A_t + B_{2t}F_{1t})\mathbf{x}(t) + (B_{1t} + B_{2t}F_{2t})\mathbf{v}(t), \\ \mathbf{z}(t) &= (C_t + D_{2t}F_{1t})\mathbf{x}(t) + (D_{1t} + D_{2t}F_{2t})\mathbf{v}(t). \end{aligned} \tag{5.112}$$

The above is dual to (5.143) in Exercises for output estimation. The optimality of the full information control shows that the static feedback gains (F_{1t}, F_{2t}) in (5.111) outperform any other controllers such as dynamic or nonlinear ones in minimization of $\mathbf{E}\left\{\|\mathbf{z}(t)\|^2 \,|\mathscr{U}_T\right\}$ under the white noise disturbance $\{\mathbf{v}(t)\}$ for all $t \in [0, T)$. This observation is important as shown in the next example.

Example 5.26. In wireless data communications, the processing burden at the receiver site is sometimes shifted to the transmitter site which often has more computational power for the downlink channels (from the station to the cellular users). A precoder is designed at the transmitter site to compensate the distorted channel so that the receivers can pick up the data directly without further digital processing. The block diagram below shows the use of such precoders in data communications where the state-space model with realization (A_t, B_t, C_t, D_t) represents the (downlink) wireless channel which is asymptotically stable. For simplicity, the additive noise at the receiver site is taken to be zero, and $\det(D_t^* D_t) \neq 0$ is assumed for each t.

Our objective is to design the linear precoder that minimizes the error variance of $\mathbf{e}_s(t)$ under the assumption that the transmitted signal $\mathbf{s}(t)$ is white with zero mean and identity covariance. It is claimed that any linear, causal, and stable precoder has the form

$$\mathbf{x}_p(t+1) = (A_t + B_t F_t)\,\mathbf{x}_p(t) + B_t \mathbf{w}(t), \quad \mathbf{u}(t) = F_t \mathbf{x}_p(t) + \mathbf{w}(t) \tag{5.113}$$

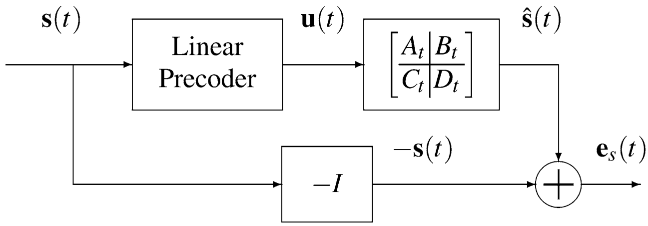

Fig. 5.7 Precoder in data detection

for some asymptotically stabilizing F_t and $\mathbf{w}(t) = Q(t,k) \star \mathbf{s}(t)$ with $\{Q(t,k)\}$ the impulse response of some causal and stable LTV system at time t. Indeed, given a linear, causal, and stable precoder with impulse response $\{G(t,k)\}$, consider the inverse of the system in (5.113):

$$\tilde{\mathbf{x}}_p(t+1) = A_t\tilde{\mathbf{x}}_p(t) + B_t\mathbf{u}(t), \quad \mathbf{w}(t) = -F_t\tilde{\mathbf{x}}_p(t) + \mathbf{u}(t).$$

Denote the impulse response of the above system by $L(t,k)$. Then $Q(t,k) = L(t,k) \star G(t,k)$ is causal and stable. Thus, $G(t,k)$ can be implemented by (5.113) with $\mathbf{w}(t) = Q(t,k) \star \mathbf{s}(t)$. The channel is now described by

$$\begin{aligned}\mathbf{x}(t+1) &= A_t\mathbf{x}(t) + B_t\mathbf{u}(t) = A_t\mathbf{x}(t) + B_tF_t\mathbf{x}_p(t) + B_tQ_t\mathbf{s}(t),\\ \hat{\mathbf{s}}(t) &= C_t\mathbf{x}(t) + D_t\mathbf{u}(t) = C_t\mathbf{x}(t) + D_tF_t\mathbf{x}_p(t) + D_tQ_t\mathbf{s}(t),\end{aligned}$$

by (5.113) and $\mathbf{w}(t) = Q_t\mathbf{s}(t)$ where $Q(t,k) = Q_t$ is taken as a static gain for each t temporarily. The overall system in Fig. 5.7 has a realization

$$\left[\begin{array}{cc|c} A_t & B_tF_t & B_tQ_t \\ \mathbf{0} & A_t + B_tF_t & B_tQ_t \\ \hline C_t & D_tF_t & D_tQ_t - I \end{array}\right] \implies \left[\begin{array}{c|c} A_t + B_tF_t & B_tQ_t \\ \hline C_t + D_tF_t & D_tQ_t - I \end{array}\right]$$

after using the similarity transform

$$T = \begin{bmatrix} I & -I \\ \mathbf{0} & I \end{bmatrix}$$

to eliminate the unreachable modes. Therefore, the overall system in Fig. 5.7 is described by

$$\begin{aligned}\hat{\mathbf{x}}(t+1) &= (A_t + B_tF_t)\,\hat{\mathbf{x}}(t) + B_tQ_t\mathbf{s}(t),\\ \mathbf{e}_s(t) &= (C_t + D_tF_t)\,\hat{\mathbf{x}}(t) + (D_tQ_t - I)\,\mathbf{s}(t),\end{aligned}$$

which has the same form as in (5.112) by taking $B_{1t} = \mathbf{0}$, $B_{2t} = B_t$, $D_{1t} = -I$, and $D_{2t} = D_t$. Hence, the results in Theorem 5.25 for optimal full information control can be applied to compute the optimal precoder gains F_t and Q_t. It is noted that the use of dynamic gains $Q(t,k)$ do not improve its performance any further.

The closed-loop system for full information control as in Theorem 5.25 admits the same stability properties as those for LQR control and minimum variance control in light of the fact that they share the same DRE (5.94). Hence, all the stability results in the previous subsection apply to the case of full information control which will not be repeated here. In the case of stationary full information control, the realization matrices in both the state-space model and the controlled signal are all time invariant and the time horizon $T \to \infty$ for the performance index. It can be expected that the DRE (5.94) converges to the ARE

$$X = \tilde{A}^* \left(I + XB_2R^{-1}B_2^*\right)^{-1} X\tilde{A} + \tilde{C}^*\tilde{C} \tag{5.114}$$

with $R = D_2^*D_2$ which is identical to (5.106) except that B is replaced by B_2 and D by D_2. In this case, the transfer matrix from $\mathbf{v}(t)$ to $\mathbf{z}(t)$ is given by

$$\mathbf{T}(z) = (D_1 + D_2F_2) + (C + D_2F_1)(zI - A - B_2F_1)^{-1}(B_1 + B_2F_2),$$

where F_1 and F_2 are the same as in (5.111) with all the time indices removed. It is interesting to note that with the white noise disturbance $\{\mathbf{v}(t)\}$ WSS satisfying (5.20), there holds $\mathbf{E}\left\{\|\mathbf{z}(t)\|^2\right\} = \|\mathbf{T}\|_2^2$ where

$$\|\mathbf{T}\|_2 = \sqrt{\mathrm{Tr}\left\{(D_1 + D_2F_2)^*(D_1 + D_2F_2) + (B_1 + B_2F_2)^*X(B_1 + B_2F_2)\right\}}$$

with X the stabilizing solution to (5.114). By the optimality of the solution to full information control, $\|\mathbf{T}\|_2$ is minimized by static feedback controllers F_1 and F_2. In fact, dynamic feedback controllers do not outperform static feedback controllers for stationary full information control.

5.3 LTI Systems and Stationary Processes

This section intends to explore further optimal estimation and control for LTI state-space models and stationary white noises. As shown in the previous two sections, both Kalman filters and LQR controllers tend to stationary ones as the time horizon approaches to infinity. Hence, the results for optimal estimation and control can have frequency domain interpretations which will help deepen the understanding of the results in the previous two sections. Several results will be presented which have applications to various problems in design of communication and control systems in later chapters.

5.3.1 Spectral Factorizations

A PSD transfer matrix $\Psi(z)$ has the form

$$\Psi(z) = \sum_{k=-\infty}^{\infty} \Gamma_k z^{-k}, \quad \Gamma_k^* = \Gamma_{-k}, \tag{5.115}$$

and $\Psi(e^{j\omega}) \geq \mathbf{0}$ for all real ω. There exist spectral factorizations

$$\Psi(z) = \mathbf{H}_L(z)\mathbf{H}_L(z)^{\sim} = \mathbf{H}_R(z)^{\sim}\mathbf{H}_R(z), \tag{5.116}$$

where $\mathbf{H}_L(z)$ and $\mathbf{H}_R(z)$ are both causal, stable, and minimum phase. The transfer matrices $\mathbf{H}_L(z)$ and $\mathbf{H}_R(z)$ are called left and right spectral factors of $\Psi(z)$, respectively. This section shows how Kalman filtering and LQR control can be used to compute spectral factorizations.

Recall the random process in form of state-space model

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + B\mathbf{v}(t), \quad \mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{v}(t), \tag{5.117}$$

where $\mathbf{v}(t)$ is the white noise satisfying (5.20) and $\mathbf{y}(t)$ is the observed output. Assume that $\mathbf{x}(0) = \mathbf{x}_0$ is independent of $\{\mathbf{v}(t)\}$, has mean $\bar{\mathbf{x}}_0 = 0$, and covariance P satisfying the Lyapunov equation (5.47). Then the white noise hypothesis on $\mathbf{v}(t)$ implies that the PSD of the observed output is

$$\Psi_{\mathbf{y}}(\omega) = \mathbf{G}\left(e^{j\omega}\right)\mathbf{G}\left(e^{j\omega}\right)^*, \quad \mathbf{G}(z) = D + C(zI - A)^{-1}B. \tag{5.118}$$

The zero mean initial condition for $\mathbf{x}_0$ yields the ACS of $\mathbf{y}(t)$ given by

$$R_{\mathbf{y}}(k) = \mathrm{E}\{\mathbf{y}(t)\mathbf{y}(t-k)^*\} = \begin{cases} CA^{k-1}(APC^* + BD^*), & k > 0, \\ R + CPC^*, & k = 0, \\ (CPA^* + D^*B)(A^*)^{k-1}C^*, & k < 0, \end{cases}$$

by Problem 5.7 in Exercises. Hence, $\Psi_{\mathbf{y}}(\omega)$ is the Fourier transform of $\{R_{\mathbf{y}}(k)\}$ which exists, if A is a stability matrix. Let $\tilde{A} = A - BD^*R^{-1}C$ and $R = DD^* > 0$. Then the associated filtering ARE is (5.52) which is copied below:

$$\Sigma = \tilde{A}\left(I_n + \Sigma C^* R^{-1} C\right)^{-1} \Sigma \tilde{A}^* + B\left(I - D^* R^{-1} D\right) B^*.$$

Lemma 5.1. *Consider the state-space system in (5.117) with $\mathbf{v}(t)$ of dimension m and $\mathbf{y}(t)$ of dimension p. Assume that $m \geq p$ and $R = DD^* > 0$. Let K be the Kalman gain as in (5.51) with Σ satisfying (5.52). Then the* PSD *$\Psi_{\mathbf{y}}(\omega)$ as in (5.118) has the expression*

$$\Psi_{\mathbf{y}}(\omega) = \left[I - C(\mathrm{e}^{j\omega}I - A)^{-1}K\right](R + C\Sigma C^*)\left[I - C(\mathrm{e}^{j\omega}I - A)^{-1}K\right]^*. \tag{5.119}$$

Proof. Direct state-space computations yield

$$\begin{aligned}
\mathbf{W}(z) &= \left[I + C(zI - A - KC)^{-1}K\right]\mathbf{G}(z) \\
&= \left[\begin{array}{c|c} A+KC & K \\ \hline C & I \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \qquad \left\{T = \left[\begin{array}{c|c} I & 0 \\ \hline I & I \end{array}\right]\right\} \\
&= \left[\begin{array}{cc|c} A & \mathbf{0} & B \\ KC & A+KC & KD \\ \hline C & C & D \end{array}\right] = \left[\begin{array}{c|c} A+KC & B+KD \\ \hline C & D \end{array}\right],
\end{aligned} \tag{5.120}$$

where the similarity transform T is used to eliminate the unobservable subsystem. It is claimed that

$$\mathbf{F}(z) = \left[I + C(zI - A - KC)^{-1}K\right] = \left[I - C(zI - A)^{-1}K\right]^{-1} \tag{5.121}$$

is a "whitening" filter in the sense that

$$\Phi_{\mathbf{W}}(z) = W(z)W(z)^{\sim} = R + C\Sigma C^*. \tag{5.122}$$

Indeed, denote $A_K = A + KC$, $B_K = B + KD$, and $\Pi = B_K B_K^*$. Then the ARE (5.52) can be written as

$$\begin{aligned}
\Pi &= B_K B_K^* = (B + KD)(B + KD)^* = \Sigma - A_K \Sigma A_K^* \\
&= (zI - A_K)\Sigma\left(z^{-1}I - A_K^*\right) + (zI - A_K)\Sigma A_K^* + A_K\Sigma\left(z^{-1}I - A_K^*\right).
\end{aligned}$$

Multiplying both sides of the above equation by $C(zI - A_K)^{-1}$ from left and $\left(z^{-1}I - A_K^*\right)^{-1}C^*$ from right gives

$$\begin{aligned}
\Phi_{\Pi}(z) &= C(zI - A_K)^{-1}B_K B_K^*\left(z^{-1}I - A_K^*\right)C^* \\
&= C\Sigma C^* + C(zI - A_K)^{-1}A_K\Sigma C^* + C\Sigma A_K^*\left(z^{-1}I - A_K^*\right)^{-1}C^*.
\end{aligned}$$

It follows from the state-space realization of $\mathbf{W}(z)$ that

$$\begin{aligned}
\Phi_{\mathbf{W}}(z) &= R + C(zI - A_K)^{-1}B_K D^* + DB_K^*\left(z^{-1}I - A_K^*\right)^{-1}C^* + \Phi_{\Pi}(z) \\
&= R + C\Sigma C^* + C(zI - A_K)^{-1}(B_K D^* + A_K\Sigma C^*) \\
&\quad + (B_K D^* + A_K\Sigma C^*)^*\left(z^{-1}I - A_K^*\right)^{-1}C^*.
\end{aligned}$$

By the expression of the Kalman gain,

$$\begin{aligned} B_K D^* + A_K \Sigma C^* &= (B+KD)D^* + (A+KC)\Sigma C^* \\ &= BD^* + A\Sigma C^* + K(DD^* + C\Sigma C^*) = \mathbf{0}. \end{aligned}$$

Therefore, substituting the expression of $\Phi_\Pi(z)$ into $\Phi_{\mathbf{W}}(z)$ yields (5.122) concluding the fact that $\mathbf{F}(z)$ is a "whitening" filter. In light of (5.120) or $\mathbf{G}(z) = \left[I + C(zI - A - KC)^{-1}K\right]^{-1}\mathbf{W}(z) = \left[I - C(zI-A)^{-1}K\right]\mathbf{W}(z)$,

$$\Psi_{\mathbf{y}}(\omega) = \left[I - C\left(e^{j\omega}I - A\right)^{-1}K\right]\Phi_{\mathbf{W}}\left(e^{j\omega}\right)\left[I - C\left(e^{j\omega}I - A\right)^{-1}K\right]^*$$

which is the same as (5.119). The proof is thus completed. □

In the case when A is a stability matrix and the stabilizing solution to the ARE (5.52) exits, then $I - C(zI-A)^{-1}K$ is not only causal and stable but also admits a causal and stable inverse. Let $R + C\Sigma C^* = \Omega\Omega^*$ be the Cholesky factorization and $\mathbf{G}_o(z) = \left[I - C(zI-A)^{-1}K\right]\Omega$. Then

$$\Phi(z) = \mathbf{G}(z)\mathbf{G}(z)^\sim = \mathbf{G}_o(z)\mathbf{G}_o(z)^\sim, \tag{5.123}$$

and thus, $\mathbf{G}_o(z)$ is the left *spectral factor* of $\Phi(z)$. Kalman filtering provides an algorithm to compute spectral factorization of $\Phi(z) = \mathbf{G}(z)\mathbf{G}(z)^\sim$. Conversely, (left) spectral factorization can be used to compute the Kalman filtering gain K by the expression of (5.119). It is noted that $\mathbf{F}(z)$ in (5.121) satisfies the following state-space equation

$$\hat{\mathbf{x}}_{t+1|t} = (A+KC)\hat{\mathbf{x}}_{t|t-1} - K\mathbf{y}(t), \quad \delta\mathbf{y}(t) = \mathbf{y}(t) - C\hat{\mathbf{x}}_{t|t-1}, \tag{5.124}$$

where $\hat{\mathbf{x}}_{k|k-1}$ is the stationary MMSE estimate of $\mathbf{x}(k)$ based on the observation up to time $(k-1)$. Hence, the output of $\mathbf{F}(z)$ is the innovation sequence.

Example 5.27. In the traditional Wiener filtering (refer to Example 5.16), a whitening filter is designed first to obtain the innovation sequence, and an estimator is then designed for smoothing, filtering, or prediction. The whitening filter can clearly be obtained using the spectral factorization for

$$\Phi(z) = \mathbf{G}_1(z)\mathbf{G}_1(z)^\sim + \mathbf{G}_2(z)\mathbf{G}_2(z)^\sim = \mathbf{G}_o(z)\mathbf{G}_o(z)^\sim$$

by the fact that $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$ are independent of each other and have zero means. Thus, $\Psi_{\mathbf{y}}(\omega) = \mathbf{G}_1\left(e^{j\omega}\right)\mathbf{G}_1\left(e^{j\omega}\right)^* + \mathbf{G}_2\left(e^{j\omega}\right)\mathbf{G}_2\left(e^{j\omega}\right)^*$. To proceed, a realization (A,B,C,D) for $\left[\mathbf{G}_1(z)\ \mathbf{G}_2(z)\right]$ needs to be obtained before applying Lemma 5.1 for computing the whitening filter $\mathbf{F}_o(z) = \mathbf{G}_o^{-1}(z)$. It is noted that Wiener filtering can be approached by Kalman filtering, if $\mathbf{G}_2(z) = D$ and $\mathbf{G}_1(z) = C(zI-A)^{-1}B$ in Example 5.16. In this case,

$$\Phi(z) = R + C(zI-A)^{-1}BB^*\left(z^{-1}I - A^*\right)^{-1}C^*$$

which is identical to $\Psi_{\mathbf{y}}(\omega)$ at $z = e^{j\omega}$ as in (5.118) provided that $BD^* = \mathbf{0}$. Hence, Kalman filtering can be employed to compute the whitening filter for Wiener filtering. If in addition $\mathbf{v}_1(t) = \mathbf{v}_2(t) = \mathbf{v}(t)$, then Wiener filtering for $m = 1$ in Fig. 5.5 coincides with Kalman filtering in Fig. 5.2. Recall that optimal output estimation is the same as the optimal state estimation for one-step prediction or strictly causal filtering.

The next result is dual to Lemma 5.1,and thus, the proof is omitted.

Lemma 5.2. *Let $\tilde{\mathbf{G}}(z) = D + C(zI - A)^{-1}B$. Assume that $R = D^*D > 0$, A is a stability matrix and the* ARE *(5.106) has a unique stabilizing solution $X \geq 0$ so that F in (5.107) is stabilizing. There holds factorization*

$$\tilde{\mathbf{G}}(z)^{\sim}\tilde{\mathbf{G}}(z) = \left[I - F(zI - A)^{-1}B\right]^{\sim}(R + B^*XB)\left[I - F(zI - A)^{-1}B\right]. \tag{5.125}$$

Let $R + B^*XB = \tilde{\Omega}^*\tilde{\Omega}$, and $\tilde{\mathbf{G}}_{\mathrm{o}}(z) = \tilde{\Omega}\left[I - F(zI - A)^{-1}B\right]$. Then

$$\Phi(z) = \tilde{\mathbf{G}}(z)^{\sim}\tilde{\mathbf{G}}(z) = \tilde{\mathbf{G}}_{\mathrm{o}}(z)^{\sim}\tilde{\mathbf{G}}_{\mathrm{o}}(z). \tag{5.126}$$

Hence, $\tilde{\mathbf{G}}_{\mathrm{o}}(z)$ is a right spectral factor of $\Phi(z)$. Spectral factors $\mathbf{G}_{\mathrm{o}}(z)$ in (5.123) and $\tilde{\mathbf{G}}_{\mathrm{o}}(z)$ in (5.126) are also called *outers* because both are stable and their inverses are analytic outside the unit circle. Moreover,

$$\mathbf{G}_{\mathrm{i}}(z) = \mathbf{G}_{\mathrm{o}}^{-1}(z)\mathbf{G}(z), \quad \tilde{\mathbf{G}}_{\mathrm{i}}(z) = \tilde{\mathbf{G}}(z)\tilde{\mathbf{G}}_{\mathrm{o}}^{-1}(z) \tag{5.127}$$

satisfy $\mathbf{G}_{\mathrm{i}}(z)\mathbf{G}_{\mathrm{i}}(z)^{\sim} = I$ and $\tilde{\mathbf{G}}_{\mathrm{i}}(z)^{\sim}\tilde{\mathbf{G}}_{\mathrm{i}}(z) = I$. Hence, all transmission zeros of $\mathbf{G}_{\mathrm{i}}(z)$ and $\tilde{\mathbf{G}}_{\mathrm{i}}(z)$ are unstable, or their inverses are analytic inside the unit circle. For this reason, $\tilde{\mathbf{G}}_{\mathrm{i}}(z)$ is called *inner* and $\mathbf{G}_{\mathrm{i}}(z)$ called *co-inner*. In light of (5.127) and Lemmas 5.1 and 5.2,

$$\mathbf{G}(z) = \mathbf{G}_{\mathrm{o}}\mathbf{G}_{\mathrm{i}}(z), \quad \tilde{\mathbf{G}}(z) = \tilde{\mathbf{G}}_{\mathrm{i}}(z)\tilde{\mathbf{G}}_{\mathrm{o}}(z)$$

which are termed inner-outer factorizations. The next result is thus true.

Theorem 5.28. *Let A be a stability matrix and D have size $p \times m$.* (i) *If $p \leq m, R = DD^* > 0$, and the* ARE *(5.52) admits a unique solution $\Sigma \geq 0$, then $\mathbf{G}(z) = D + C(zI - A)^{-1}B$ admits inner-outer factorization $\mathbf{G}(z) = \mathbf{G}_{\mathrm{o}}\mathbf{G}_{\mathrm{i}}(z)$ with*

$$\mathbf{G}_{\mathrm{o}} = \left[\begin{array}{c|c} A & K\Omega \\ \hline -C & \Omega \end{array}\right], \quad \mathbf{G}_{\mathrm{i}}(z) = \left[\begin{array}{c|c} A + KC & B + KD \\ \hline \Omega^{-1}C & \Omega^{-1}D \end{array}\right], \tag{5.128}$$

where K is the Kalman gain as defined in (5.51) and $\Omega = (R + C\Sigma C^*)^{1/2}$. (ii) *If $p \geq m, R = D^*D > 0$, and the* ARE *(5.106) admits a unique solution $X \geq 0$, then $\tilde{\mathbf{G}}(z) = D + C(zI - A)^{-1}B$ admits inner-outer factorization $\tilde{\mathbf{G}}(z) = \tilde{\mathbf{G}}_{\mathrm{i}}(z)\tilde{\mathbf{G}}_{\mathrm{o}}(z)$ with*

$$\tilde{\mathbf{G}}_{\mathrm{o}} = \left[\begin{array}{c|c} A & -B \\ \hline \tilde{\Omega}F & \tilde{\Omega} \end{array}\right], \quad \tilde{\mathbf{G}}_{\mathrm{i}}(z) = \left[\begin{array}{c|c} A+BF & B\tilde{\Omega}^{-1} \\ \hline C+DF & D\tilde{\Omega}^{-1} \end{array}\right], \tag{5.129}$$

where F is defined as in (5.107) and $\tilde{\Omega} = (R+B^*XB)^{1/2}$.

Remark 5.1. The hypothesis $R > \mathbf{0}$ in Theorem 5.28 can be weakened to

$$\text{(i) rank}\left\{\begin{bmatrix} C & D \end{bmatrix}\right\} = p, \quad \text{(ii) rank}\left\{\begin{bmatrix} B \\ D \end{bmatrix}\right\} = m, \tag{5.130}$$

respectively, even if D may not have the full rank. Indeed, for (i) there holds

$$\mathbf{G}(z)\mathbf{G}(z)^{\sim} = \left[I - C(zI-A)^{-1}K\right](R + C\Sigma C^*)\left[I - C(zI-A)^{-1}K\right]^{\sim} \tag{5.131}$$

in light of (5.119) in Lemma 5.1. Hence, (i) of (5.130) implies that $\mathbf{G}(z)$ has normal rank equal to p that in turn implies that $(R + C\Sigma C^*)$ is nonsingular. Similarly, for (ii) there holds

$$\mathbf{G}(z)^{\sim}\mathbf{G}(z) = \left[I - F(zI-A)^{-1}B\right]^{\sim}(R + B^*XB)\left[I - F(zI-A)^{-1}B\right] \tag{5.132}$$

that is dual to (5.131). Hence, (ii) of (5.130) implies that $(R + B^*XB)$ is nonsingular. Consequently, the formulas in Theorem 5.28 for computing inner-outer factorizations are valid under the weak conditions in (5.130).

5.3.2 Normalized Coprime Factorizations

Coprime factorizations have been studied in Sect. 3.2.3. For a given plant model

$$\mathbf{P}(z) = D + C(zI-A)^{-1}B \tag{5.133}$$

coprime factorizations search for $\{\mathbf{M}(z), \mathbf{N}(z)\}$ and $\{\tilde{\mathbf{M}}(z), \tilde{\mathbf{N}}(z)\}$ which are stable transfer matrices such that

$$\mathbf{P}(z) = \mathbf{M}(z)^{-1}\mathbf{N}(z) = \tilde{\mathbf{N}}(z)\tilde{\mathbf{M}}(z)^{-1}$$

and the augmented transfer matrices

$$\tilde{\mathbf{G}}(z) = \begin{bmatrix} \tilde{\mathbf{M}}(z) \\ \tilde{\mathbf{N}}(z) \end{bmatrix}, \quad \mathbf{G}(z) = \begin{bmatrix} \mathbf{M}(z) & \mathbf{N}(z) \end{bmatrix} \tag{5.134}$$

void zeros on and outside the unit circle. In other words, $\tilde{\mathbf{G}}(z)$ and $\mathbf{G}(z)$ are outers. Normalized coprime factorizations search for coprime factors such that $\tilde{\mathbf{G}}(z)$ and

$\mathbf{G}(z)$ are not only outers but also inner and co-inner respectively:

$$\tilde{\mathbf{G}}(z)^{\sim}\tilde{\mathbf{G}}(z) = \tilde{\mathbf{M}}(z)^{\sim}\tilde{\mathbf{M}}(z) + \tilde{\mathbf{N}}(z)^{\sim}\tilde{\mathbf{N}}(z) = I,$$
$$\mathbf{G}(z)\mathbf{G}(z)^{\sim} = \mathbf{M}(z)\mathbf{M}(z)^{\sim} + \mathbf{N}(z)\mathbf{N}(z)^{\sim} = I.$$

Such $\mathbf{G}(z)$ and $\tilde{\mathbf{G}}(z)$ are termed *power complementary* in the signal processing literature. The following result shows that normalized coprime factorizations can be solved via Kalman filtering and LQR control.

Theorem 5.29. *Denote $R_{\rm o} = I + DD^*/\tilde{R}_{\rm o} = I + D^*D$ for $\mathbf{P}(z)$ in (5.133).*

(i) *Assume that (C,A) is detectable and (A,B) has no unreachable modes on the unit circle. Let $A_{\rm o} = A - BD^*R_{\rm o}^{-1}C$. Then the following* ARE

$$\Sigma = A_{\rm o}\Sigma\left(I + C^*R_{\rm o}^{-1}C\Sigma\right)^{-1}A_{\rm o}^* + B\tilde{R}_{\rm o}^{-1}B^* \tag{5.135}$$

admits a unique stabilizing solution $\Sigma = \Sigma^* \geq \mathbf{0}$. A state-space realization of the normalized (right) coprime factors is given by

$$\mathbf{G}(z) = \begin{bmatrix} \mathbf{M}(z) & \mathbf{N}(z) \end{bmatrix} = \left[\begin{array}{c|cc} A + KC & K & B + KD \\ \hline \Omega_{\rm o}^{-1}C & \Omega_{\rm o}^{-1} & \Omega_{\rm o}^{-1}D \end{array}\right], \tag{5.136}$$

where $K = -(A\Sigma C^* + BD^*)(R_{\rm o} + C\Sigma C^*)^{-1}$ and $\Omega_{\rm o} = (R_{\rm o} + C\Sigma C^*)^{1/2}$.

(ii) *Assume that (A,B) is stabilizable and (C,A) has no unobservable modes on the unit circle. Let $\tilde{A}_{\rm o} = A - B\tilde{R}_{\rm o}^{-1}D^*C$. Then the following* ARE

$$X = \tilde{A}_{\rm o}^*X\left(I + B\tilde{R}_{\rm o}^{-1}B^*X\right)^{-1}\tilde{A}_{\rm o} + C^*R_{\rm o}^{-1}C \tag{5.137}$$

admits a unique stabilizing solution $X = X^* \geq \mathbf{0}$. A state-space realization of the normalized (left) coprime factors is given by

$$\tilde{\mathbf{G}}(z) = \begin{bmatrix} \tilde{\mathbf{M}}(z) \\ \tilde{\mathbf{N}}(z) \end{bmatrix} = \left[\begin{array}{c|c} A + BF & B\tilde{\Omega}_{\rm o}^{-1} \\ \hline F & \tilde{\Omega}_{\rm o}^{-1} \\ C + DF & D\tilde{\Omega}_{\rm o}^{-1} \end{array}\right], \tag{5.138}$$

where $F = -\left(\tilde{R}_{\rm o} + B^*XB\right)^{-1}(B^*XA + D^*C)$ and $\tilde{\Omega}_{\rm o} = \left(\tilde{R}_{\rm o} + B^*XB\right)^{1/2}$.

Proof. For (i), the pair $\{\mathbf{M}(z), \mathbf{N}(z)\}$ in (5.136) is a pair of left coprime factors for K is stabilizing. To show that $\{\mathbf{M}(z), \mathbf{N}(z)\}$ is normalized, denote $B_{\mathbf{v}} = \begin{bmatrix} \mathbf{0} & B \end{bmatrix}$ and $D_{\mathbf{v}} = \begin{bmatrix} I & D \end{bmatrix}$. Let

$$\mathbf{T}(z) = D_{\mathbf{v}} + C(zI - A)^{-1}B_{\mathbf{v}} \tag{5.139}$$

and associate $\mathbf{T}(z)$ with the following Kalman filtering problem:

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + B_{\mathbf{v}}\mathbf{v}(t), \quad \mathbf{y}(t) = C\mathbf{x}(t) + D_{\mathbf{v}}\mathbf{v}(t),$$

where $\mathbf{v}(t)$ is an independent white noise process with zero mean and identity covariance. Applying the results of the stationary Kalman filter yields the ARE (5.135) and the required Kalman gain K which is stabilizing by the hypothesis. In light of the proof of Lemma 5.1, the filter

$$\mathbf{W}(z) = [I + C(zI - A - KC)^{-1}K]\mathbf{T}(z) = \left[\begin{array}{c|c} A+KC & B_{\mathbf{v}} + KD_{\mathbf{v}} \\ \hline C & D_{\mathbf{v}} \end{array}\right]$$
$$= \left[\begin{array}{c|cc} A+KC & K & B+KD \\ \hline C & I & D \end{array}\right]$$

has the white PSD. That is, $\mathbf{W}(z)\mathbf{W}(z)^{\sim} = \Omega_{\mathrm{o}}\Omega_{\mathrm{o}}^* = R_{\mathrm{o}} + C\Sigma C^*$, and hence, $\Omega_{\mathrm{o}}^{-1}\mathbf{W}(z)$ is co-inner and has the same realization as in (5.136). It follows that $\{\mathbf{M}(z), \mathbf{N}(z)\}$ is a pair of the normalized left coprime factors. Since (ii) is dual to (i), the proof for (ii) is similar and omitted. □

For the given left and right normalized coprime factors in (5.136) and (5.138), respectively, the following result gives their respective reachability and observability gramians.

Theorem 5.30. *Consider Theorem 5.29. The reachability gramian P and observability gramian Q of $\mathbf{G}(z)$ as in (5.136) are given respectively by:*

$$P = \Sigma, \quad Q = (I + X\Sigma)^{-1}X \tag{5.140}$$

while the reachability gramian $\tilde{P}$ and observability gramian $\tilde{Q}$ of $\tilde{\mathbf{G}}(z)$ as in (5.138) are given respectively by

$$\tilde{P} = (I + \Sigma X)^{-1}\Sigma, \quad \tilde{Q} = X. \tag{5.141}$$

Proof. By definition the controllability gramian of $\mathbf{G}(z)$ in (5.136) satisfies

$$P = (A+KC)^* P (A+KC) + (B_{\mathbf{v}} + KD_{\mathbf{v}})(B_{\mathbf{v}} + KD_{\mathbf{v}})^*$$

with $B_{\mathbf{v}} = \begin{bmatrix} \mathbf{0} & B \end{bmatrix}$ and $D_{\mathbf{v}} = \begin{bmatrix} I & D \end{bmatrix}$. The above is the same as the ARE (5.135) if $P = \Sigma$. Hence, Σ is indeed the controllability gramian of $\mathbf{G}(z)$. Now assume temporarily that $\det(A_{\mathrm{o}}) \neq 0$ and $\det(A+KC) \neq 0$. Since $D^* R_{\mathrm{o}}^{-1} = \tilde{R}_{\mathrm{o}}^{-1} D^*$, $A_{\mathrm{o}} = \tilde{A}_{\mathrm{o}}$. The ARE in (5.137) can then be written as

$$\begin{bmatrix} -X & I \end{bmatrix} S \begin{bmatrix} I \\ X \end{bmatrix} = \mathbf{0}, \quad S = \begin{bmatrix} A_{\mathrm{o}} + \Gamma (A_{\mathrm{o}}^*)^{-1}\Pi & -\Gamma (A_{\mathrm{o}}^*)^{-1} \\ -(A_{\mathrm{o}}^*)^{-1}\Pi & (A_{\mathrm{o}}^*)^{-1} \end{bmatrix},$$

where $\Pi = C^* R_o^{-1} C$ and $\Gamma = B\tilde{R}_o^{-1}B^*$ (refer to Appendix A). Denote

$$T = \begin{bmatrix} I & \Sigma \\ \mathbf{0} & I \end{bmatrix} \implies T^{-1} = \begin{bmatrix} I & -\Sigma \\ \mathbf{0} & I \end{bmatrix}.$$

Let $\tilde{S} = TST^{-1}$. The ARE in (5.137) can be written as

$$\begin{bmatrix} -Z & I \end{bmatrix} \tilde{S} \begin{bmatrix} I \\ Z \end{bmatrix} = \mathbf{0}, \quad Z = (I + X\Sigma)^{-1}X. \tag{5.142}$$

Direction computation yields

$$\tilde{S} = \begin{bmatrix} \tilde{S}_{11} & \tilde{S}_{12} \\ \tilde{S}_{21} & \tilde{S}_{22} \end{bmatrix} = \begin{bmatrix} A_o + (\Gamma - \Sigma)(A_o^*)^{-1}\Pi & \mathbf{0} \\ -(A_o^*)^{-1}\Pi & (A_o^*)^{-1}(\Pi\Sigma + I) \end{bmatrix}$$

due to $\tilde{S}_{12} = -\left[A_o\Sigma(I+\Pi\Sigma)^{-1}A_o^* + \Gamma - \Sigma\right](A_o^*)^{-1}(\Pi\Sigma + I) = \mathbf{0}$ by the ARE in (5.135). On the other hand, the results on Kalman filtering with the dynamic model in (5.139) show that

$$A + KC = A_o(I + \Pi\Sigma)^{-1} \implies \tilde{S}_{22} = [(A+KC)^*]^{-1}.$$

Since the ARE in (5.135) can be written as $\Gamma - \Sigma = -A_o\Sigma(I+\Pi\Sigma)^{-1}A_o^*$,

$$\tilde{S}_{11} = A_o - A_o\Sigma(I+\Pi\Sigma)^{-1}\Pi = A_o - A_o\Sigma C^*(R_o + C\Sigma C^*)^{-1}C = A + KC.$$

Finally, by the expression of $\tilde{S}_{22}$,

$$\begin{aligned} \tilde{S}_{21} &= -(A_o^*)^{-1}\Pi = -[(A+KC)^*]^{-1}(I+\Pi\Sigma)^{-1}\Pi \\ &= -[(A+KC)^*]^{-1}C^*(R_o + C\Sigma C^*)C = -[(A+KC)^*]^{-1}C_\Omega^* C_\Omega, \end{aligned}$$

where $C_\Omega = \Omega_o^{-1}C$. Substituting the above into (5.142) yields

$$\begin{aligned} \mathbf{0} &= \begin{bmatrix} -Z & I \end{bmatrix} \begin{bmatrix} A + KC & \mathbf{0} \\ -[(A+KC)^*]^{-1}C_\Omega^* C_\Omega & [(A+KC)^*]^{-1} \end{bmatrix} \begin{bmatrix} I \\ Z \end{bmatrix} \\ &= -Z(A+KC) + [(A+KC)^*]^{-1}Z - [(A+KC)^*]^{-1}C_\Omega^* C_\Omega. \end{aligned}$$

Multiplying the above by $(A+KC)^*$ from left leads to

$$Z = (A+KC)^* Z (A+KC) + C_\Omega^* C_\Omega$$

which verifies that $Q = Z = (I + X\Sigma)^{-1}X$ is the observability gramian of $\mathbf{G}(z)$. If A and $(A+KC)$ are singular, then A and $(A+KC)$ can be perturbed to A_ε and $A_{\varepsilon K}$, respectively, by adding εI such that both are nonsingular. Similar proof can thus be adopted to obtain the observability gramian Z_ε. The limit $\varepsilon \to 0$ can be taken to conclude the proof for the case when A and $(A+KC)$ are singular. As (5.141) is dual to (5.140), its proof is skipped. □

Notes and References

There are many papers and books on optimal control for continuous-time systems. See [5, 39, 57, 58, 74, 122, 126] for a sample of references. For linear discrete-time systems, readers are referred to [1, 7, 11, 16, 25, 68, 69] for a glimpse of work on optimal control. For optimal estimation or filtering, most of work has been focused on discrete-time systems, except the Kalman–Bucy filter [60]. Many books are available with [8, 54] as the representative.

Exercises

5.1. Let X be a random vector of dimension $n > 1$ that is Gaussian distributed with mean zero and covariance $\Sigma_{\mathbf{xx}}$. Suppose that $\Sigma_{\mathbf{xx}}$ has rank $m < n$. Show that its PDF has the form

$$p_X(X=\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^m \prod_{i=1}^m \sigma_i^2}} \exp\left\{-\frac{1}{2}\mathbf{x}^* \Sigma_{\mathbf{xx}}^+ \mathbf{x}\right\},$$

where $\Sigma_{\mathbf{xx}}^+$ is the pseudoinverse of $\Sigma_{\mathbf{xx}}$ and $\{\sigma_i^2\}_{i=1}^m$ are the m nonzero singular values of $\Sigma_{\mathbf{xx}}$. (*Hint:* Consider first $\Sigma_{\mathbf{xx}} = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_m^2, 0, \ldots, 0)$ and then extend it to the general case.)

5.2. Suppose that the system is described by state-space model

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + B\mathbf{v}(t), \quad \mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{v}(t),$$

where $\{\mathbf{v}(t)\}$ is a WSS white noise with mean zero and covariance Q_v. Let $\mathbf{H}(z) = D + C(zI-A)^{-1}B$ be the transfer matrix. Show that

$$\begin{aligned}\|\mathbf{y}\|_{\mathscr{P}} = \left\|\mathbf{H}Q_v^{1/2}\right\|_2 &:= \sqrt{\mathrm{Tr}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{H}\left(\mathrm{e}^{j\omega}\right) Q_v \mathbf{H}\left(\mathrm{e}^{j\omega}\right)^* \,\mathrm{d}\omega\right\}} \\ &= \sqrt{\mathrm{Tr}\{CPC^* + DQ_vD^*\}},\end{aligned}$$

where $P = APA^* + BQ_dB^*$ is the covariance of the state vector $\mathbf{x}(t)$. Recall that $\|\cdot\|_{\mathscr{P}}$ is the power norm as defined by (2.49) in Chap. 2.

5.3. Consider the nth order state-space system

$$\mathbf{x}(t+1) = A\mathbf{x}(t), \quad \mathbf{y}(t) = C\mathbf{x}(t) + \mathbf{v}(t)$$

with $\mathbf{x}(0) = \mathbf{x}_0 \neq \mathbf{0}$ and $\mathbf{v}(t)$ the measurement noise. Assume that (C, A) is observable. Let $\mathscr{O}_\ell$ be the observability matrix of size $\ell \geq n$ and

$$\mathscr{Y}_\ell = \text{vec}\left\{ \left[\mathbf{y}(0)\ \mathbf{y}(1)\ \cdots\ \mathbf{y}(\ell-1) \right] \right\}.$$

Show that the estimate $\hat{\mathbf{x}}_0$ which minimizes the estimation error $\|\mathscr{Y}_\ell - \mathscr{O}_\ell \hat{\mathbf{x}}_0\|$ is given by $\hat{\mathbf{x}}_0 = \left(\mathscr{O}_\ell^* \mathscr{O}_\ell\right)^{-1} \mathscr{O}_\ell^* \mathscr{Y}_\ell$.

5.4. Prove the expression for the conditional PDF in (5.6). What modifications are needed for the PDFs of X and Y and for the conditional PDF in (5.6), if the dimensions of X and Y are different from each other?

5.5. Let X and Y be two jointly distributed random variables. Let $\hat{x}$ be the optimal estimate, given observation $Y = y$, such that

$$\text{E}\{|X - \hat{x}|\ |Y = y\} \leq \text{E}\{|X - z|\ |Y = y\} \quad \forall\, z.$$

That is, $\hat{x}$ minimizes the absolute error of the estimation. Show that $\hat{x}$ is the median of the conditional density $p_{X|Y}(x|y)$; i.e.,

$$P_{X|Y}[X \leq \hat{x}|y] = P_{X|Y}[X \geq \hat{x}|y] = 0.5.$$

5.6. Let X and Y be jointly distributed. If $\text{E}\{XY^*\} = \mathbf{0}$, then X and Y are termed orthogonal. Show that the linear MMSE estimate $\hat{X}$ in (5.15) as in Theorem 5.5 satisfies the orthogonality condition

$$\text{E}\left\{ \left(X - \hat{X}\right) Y^* \right\} = \mathbf{0}.$$

Give a geometric interpretation for the above orthogonality condition.

5.7. Suppose that $B_t D_t^* \neq \mathbf{0}$ for the random process in (5.19). Show that for $t \geq k$,

$$\begin{aligned} Q_{t,k} &= \text{E}\left\{ [\mathbf{y}(t) - \bar{\mathbf{y}}_t]\, [\mathbf{y}(k) - \bar{\mathbf{y}}_k]^* \right\} \\ &= C_t \Phi_{t,k} P_k C_k^* + C_t \Phi_{t,k+1} B_k D_k^* + D_t D_t^* \delta(t-k), \\ \Gamma_{t,k} &= \text{E}\left\{ [\mathbf{x}(t) - \bar{\mathbf{x}}_t]\, [\mathbf{y}(k) - \bar{\mathbf{y}}_k]^* \right\} = \Phi_{t,k} P_k C_k^* + \Phi_{t,k+1} B_k D_k^*. \end{aligned}$$

5.8. Suppose that Ψ and Θ are both square and hermitian, which may not necessarily have the same dimensions. Assume $\Psi > 0$. Show that

$$Z^{-1} = \begin{bmatrix} \Psi & \Omega^* \\ \Omega & \Theta \end{bmatrix}^{-1} = \begin{bmatrix} \Psi^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \Psi^{-1}\Omega^* \\ -I \end{bmatrix} \nabla^{-1} \begin{bmatrix} \Omega\Psi^{-1} & -I \end{bmatrix}$$

whenever Z is also square and hermitian positive definite, where

$$\nabla = \Theta - \Omega\Psi^{-1}\Omega^*$$

is called *Schur complement*. (*Hint:* Use factorization

$$Z = \begin{bmatrix} \Psi & \Omega^* \\ \Omega & \Theta \end{bmatrix} = \begin{bmatrix} I & \mathbf{0} \\ \Omega\Psi^{-1} & I \end{bmatrix} \begin{bmatrix} \Psi & \mathbf{0} \\ \mathbf{0} & \nabla \end{bmatrix} \begin{bmatrix} I & \Psi^{-1}\Omega^* \\ \mathbf{0} & I \end{bmatrix}$$

to compute the inverse of Z). What if $\Theta > 0$ but Ψ is singular?

5.9. Prove Theorem 5.7.

5.10. (i) Use the matrix inversion formula (refer to Appendix A)

$$\left(F + HJ^{-1}G\right)^{-1} = F^{-1} - F^{-1}H\left(J + GF^{-1}H\right)^{-1}GF^{-1}$$

to show that with the Kalman gain in (5.35),

$$A_t + K_tC_t = A_t\left(I_n + \Sigma_tC_t^*R_t^{-1}C_t\right)^{-1}.$$

(ii) Show that the DRE (5.36) can be equivalently written as

$$\begin{aligned}\Sigma_{t+1} &= A_t\left(I_n + \Sigma_tC_t^*R_t^{-1}C_t\right)^{-1}\Sigma_tA_t^* + B_tB_t^* \\ &= (A_t + K_tC_t)\,\Sigma_t\,(A_t + K_tC_t)^* + B_tB_t^* + K_tR_tK_t^*.\end{aligned}$$

(iii) Show that for $B_tD_t^* \neq \mathbf{0}$, the DRE in (5.43) can be written as

$$\begin{aligned}\Sigma_{t+1} &= \tilde{A}_t\left(I_n + \Sigma_tC_t^*R_t^{-1}C_t\right)^{-1}\Sigma_t\tilde{A}_t^* + B_t\left(I_m - D_t^*R_t^{-1}D_t\right)B_t^* \\ &= (A_t + K_tC_t)\,\Sigma_t\,(A_t + K_tC_t)^* + (B_t + K_tD_t)\,(B_t + K_tD_t)^*,\end{aligned}$$

where $\tilde{A}_t = A_t - B_tD_t^*R_t^{-1}C_t$ and $K_t = -(A_t\Sigma_tC_t^* + B_tD_t^*)(R_t + C_t\Sigma_tC_t^*)^{-1}$.

5.11. Consider linear estimator

$$\tilde{\mathbf{x}}_{t+1} = (A + L_tC_t)\tilde{\mathbf{x}}_t - L_t\mathbf{y}(t), \quad \tilde{\mathbf{x}}_0 = \bar{\mathbf{x}}_0$$

for the process in (5.19). Let $Q_t = \mathrm{E}\{[\mathbf{x}(t) - \tilde{\mathbf{x}}_t][\mathbf{x}(t) - \tilde{\mathbf{x}}_t]^*\}$ be its error covariance. Show that $Q_t \geq \Sigma_t$ for all $t \geq 0$ with Σ_t the error covariance for the Kalman filter.

5.12. (Kalman filter as a whitening filter) For the random process described in (5.19), consider the linear estimator of the form

$$\hat{\mathbf{x}}(t+1) = (A_t + L_t C_t)\hat{\mathbf{x}}(t) - L_t \mathbf{y}(t), \quad \delta\mathbf{y}(t) = \mathbf{y}(t) - \hat{\mathbf{y}}(t)$$

with $\hat{\mathbf{y}}(t) = C_t\hat{\mathbf{x}}(t)$ and $\hat{\mathbf{x}}(0) = \overline{\mathbf{x}}_0$. Note that $\{\delta\mathbf{y}(t)\}$ is the innovation sequence. Show that the output process $\{\delta\mathbf{y}(t)\}$ is white (i.e.,

$$\mathrm{E}\{\delta\mathbf{y}(t)\delta\mathbf{y}(t-k)^*\} = \mathrm{E}\{\delta\mathbf{y}(t)\delta\mathbf{y}(t)^*\}\delta(k)$$

for all t and k), if and only if

$$(B_t + L_t D_t)D_t^* + (A_t + L_t C_t)X_t C_t^* = \mathbf{0},$$

where $X_t = \mathrm{E}\{[\mathbf{x}(t) - \hat{\mathbf{x}}(t)][\mathbf{x}(t) - \hat{\mathbf{x}}(t)]^*\}$ is the error covariance. Show also in this case that L_t is necessarily the Kalman gain K_t as in Corollary 5.1 and $\hat{\mathbf{x}}(t+1) = \hat{\mathbf{x}}_{t+1|t}$ is the linear MMSE estimate of $\mathbf{x}(t+1)$ based on $\mathscr{Y}_t$.

5.13. Show that the stabilizing solution to the ARE (5.50), if it exists, is unique. (*Hint:* Assume Σ_1 and Σ_2 are both stabilizing solutions to (5.50). Show that:

$$\Delta_\Sigma = (A + K_1 C)\Delta_\Sigma (A + K_2 C)^*, \quad \Delta_\Sigma = \Sigma_1 - \Sigma_2,$$

where $K_i = -A\Sigma_i C^*(R + C\Sigma_i C^*)^{-1}$ for $i = 1, 2$.)

5.14. For Example 5.12, find the optimal linear receiver in the case $H_0(t) \neq \mathbf{0}$ and discuss its performance in comparison with that of the optimal linear receivers designed in Example 5.15 assuming that H_k are the same for $1 \leq k \leq \ell$.

5.15. Prove Theorem 5.14 for the case $B_t D_{2t}^* \neq \mathbf{0}$.

5.16. Suppose that the random process has the state-space form:

$$\mathbf{x}(t+1) = A_t\mathbf{x}(t) + B_t\mathbf{v}(t), \quad \mathbf{y}(t) = C_t\mathbf{x}(t) + D_t\mathbf{v}(t),$$

where $\mathbf{v}(t)$ is a white process satisfying (5.20) and $\mathbf{x}(0) = \mathbf{x}_0$ is random and independent of $\{\mathbf{v}(t)\}$ with mean $\overline{\mathbf{x}}_0$ and covariance P_0. (i) Find the linear MMSE estimator for $\mathbf{x}(t)$ based on observation $\{\mathbf{y}(k)\}_{k=0}^{t}$. (ii) Find the linear MMSE estimator for $\mathbf{v}(t)$ based on observation $\{\mathbf{y}(k)\}_{k=0}^{t}$. (*Hint:* Use Theorem 5.14).

5.17. Use Simulink toolbox to program and simulate data detection for a SISO channel with gains $h_k = 1/\sqrt{5}$ for $0 \leq k \leq \ell = 4$. The symbol detector is the linear receiver (based on Kalman filter) followed by a quantizer $Q_n(\cdot) = \mathrm{sign}(\cdot)$. The observation noise $\{v(t)\}$ can be generated by normal distributed uncorrelated or white random variables with variance 0.1. The data block of the same size can generated in a similar way followed by $Q_n(\cdot) = \mathrm{sign}(\cdot)$ to produce ± 1 sequence. It is emphasized that the data and noise sequences are uncorrelated. In the context of Example 5.15, do the following:

(i) Design an MMSE estimator to estimate $s(t-m)$ with $m = 2\ell$ followed by a quantizer based on observation of the channel output up to time t.
(ii) Simulate and access the average performance of the detector by counting the number of detection errors in each block of 10^4 data assuming that the receiver knows the first ℓ transmitted data.

5.18. For the output estimator in Theorem 5.14, show that the output error variance $\mathrm{E}\left\{\|\mathbf{e}_z(t)\|^2\right\}$ is given by

$$\mathrm{Tr}\left\{(D_{1t}+L_tD_{2t})(D_{1t}+L_tD_{2t})^* + (C_{1t}+L_tC_{2t})\Sigma_t(C_{1t}+L_tC_{2t})^*\right\}$$

(*Hint:* Let $\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}_{k|k-1}$ for $k = t, t+1$. Show first that

$$\begin{aligned} \mathbf{e}(t+1) &= (A_t + K_tC_{2t})\mathbf{e}(t) + (B_t + K_tD_{2t})\mathbf{v}(t) \\ \mathbf{e}_z(t) &= (C_{1t} + L_tC_{2t})\mathbf{e}(t) + (D_{1t} + L_tD_{2t})\mathbf{v}(t) \end{aligned} \tag{5.143}$$

and then compute the variance of $\mathbf{e}_z(t)$.)

5.19. Consider the equivalent Wiener filtering as in Fig. 5.6 where $\mathbf{G}_1(z)$ and $\mathbf{G}_2(z)$ are both stable and causal. Suppose that $z^{-m}\mathbf{G}_1(z)$ is noncausal in the case $m < 0$. Decompose

$$z^{-m}\mathbf{G}_1(z) = \mathbf{G}_C(z) + \mathbf{G}_A(z),$$

where $\mathbf{G}_C(z)$ is causal and $\mathbf{G}_A(z)$ is anticausal. Show that the optimal estimation for the output of $\mathbf{G}_A(z)$ with white noise input is zero and thus conclude that the optimal estimate $\hat{\mathbf{z}}_{t|t-m}$ is independent of $\mathbf{G}_A(z)$. Provide a design procedure for the optimal output estimation. (*Hint:* Use the result from the solution to Problem 2.19.)

5.20. Consider the nth order state-space system

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + B\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \neq \mathbf{0}.$$

Assume that (A, B) is controllable with $\mathscr{C}_\ell$ the controllability matrix of size $\ell > n$. Let

$$\tilde{\mathscr{U}}_\ell = \mathrm{vec}\left\{\left[\,\mathbf{u}(\ell-1)\ \mathbf{u}(\ell-2)\ \cdots\ \mathbf{u}(0)\,\right]\right\}.$$

Show that the control input $\tilde{\mathscr{U}}_\ell = -\mathscr{C}_\ell^*(\mathscr{C}_\ell\mathscr{C}_\ell^*)^{-1}A^\ell\mathbf{x}_0$ has the minimum energy $\|\tilde{\mathscr{U}}_\ell\|^2$ among all possible control inputs which brings $\mathbf{x}(\ell)$ to the origin.

5.21. Use direct computation to show that the solution X_t to the DRE (5.86) with boundary condition $X_T = \mathbf{0}$ satisfies

$$X_0 \geq X_1 \geq \cdots X_{T-1} \geq X_T = \mathbf{0}.$$

5.22. Show that the closed-loop system under the LQR control (5.87) is $\mathbf{x}(t+1) = \left(I + B_t R_t^{-1} B_t^* X_{t+1}\right)^{-1} A_t \mathbf{x}(t)$. Show also that the DRE (5.86) can be written as

$$X_t = A_t^* X_{t+1} \left(I + B_t R_t^{-1} B_t^* X_{t+1}\right)^{-1} A_t + Q_t, \quad X_T = Q_T.$$

5.23. Show that the control DRE in (5.86) can be written into:

$$X_t = (A_t + B_t F_t)^* X_{t+1} (A_t + B_t F_t) + C_t^* C_t + F_t^* R_t F_t$$

with F_t in (5.87) being the optimal state-feedback gain for the LQR control. If the control law $\mathbf{u}(t) = G_t \mathbf{x}(t)$ is used with $G_t \neq F_t$, show that the solution to

$$Y_t = (A_t + B_t G_t)^* Y_{t+1} (A_t + B_t G_t) + C_t^* C_t + G_t^* R_t G_t$$

satisfies $Y_t \geq X_t$ for $0 \leq t \leq T$ where $X_T = Y_T = \mathbf{0}$ is assumed.

5.24. For the system model in (5.82), let the controlled output be $\mathbf{z}(t) = C_t \mathbf{x}(t) + D_t \mathbf{u}(t)$ and the performance index be

$$J_T = \mathbf{x}(T)^* Q_T \mathbf{x}(T) + \sum_{t=0}^{T-1} \|\mathbf{z}(t)\|^2. \tag{5.144}$$

Denote $\tilde{A}_t = A_t - B_{2t} R_t^{-1} D_t^* C_t$ and $\tilde{C}_t = \left(I - D_t \left(D_t^* D_t\right)^{-1} D_t^*\right) C_t$. Show that the control law which minimizes J_T is given by $\mathbf{u}(t) = F_t \mathbf{x}(t)$ where

$$\begin{aligned} F_t &= -\left(R_t + B_t^* X_{t+1} B_t\right)^{-1} \left(B_{2t}^* X_{t+1} A_t + D_t^* C_t\right), \quad R_t = D_t^* D_t > 0 \\ X_t &= \tilde{A}_t^* X_{t+1} \left(I + B_t R_t^{-1} B_t^* X_{t+1}\right)^{-1} \tilde{A}_t + \tilde{C}_t^* \tilde{C}_t, \qquad X_T = Q_T. \end{aligned}$$

5.25. Let the state-space model be as in (5.82). Show that if the open-loop system $\mathbf{x}(t+1) = A_t \mathbf{x}(t)$ is asymptotically (exponentially) stable, then the closed-loop system (5.95) for the LQR control as described in Theorem 5.17 is asymptotically (exponentially) stable as $T \to \infty$.

5.26. Prove Theorem 5.20.

5.27. For any stabilizing state-feedback gain F, show that

$$\begin{aligned} X &= (A + BF)^* X (A + BF) + F^* R F + Q \\ &= A^* \left(I_n + X B R^{-1} B^*\right)^{-1} X A + Q + \Delta_F^* (R + B^* X B) \Delta_F, \end{aligned}$$

where $\Delta_F = F + (R + B^* X B)^{-1} B^* X A$. Establish that (a) the LQR control law minimizes $\mathrm{Tr}\{X\}$ and (b) the stabilizing solution to the ARE (5.96) is maximal among all possible nonnegative solutions to the ARE (5.96).

5.28. Consider $\mathbf{x}(t+1) = A\mathbf{x}(t) + B\mathbf{u}(t)$ and assume that $(A+BF)$ is a stability matrix for some state-feedback gain F. (i) Let $\mathbf{u}(t) = F\mathbf{x}(t)$ be the state-feedback control law. Show that

$$J(F) = \sum_{t=0}^{\infty} \|\mathbf{u}(t)\|^2 + \|C\mathbf{x}(t)\|^2 = \mathbf{x}_0' X \mathbf{x}_0,$$

where $X = (A+BF)^* X(A+BF) + F^*F + C^*C$ and $\mathbf{x}(0) = \mathbf{x}_0$ is the initial condition. (ii) Let $X_{\rm m} \geq \mathbf{0}$ be the stabilizing solution to

$$X_{\mathbf{m}} = A^* X_{\mathbf{m}} (I + BB^* X_{\mathbf{m}})^{-1} A + C^*C.$$

Show that $X_{\rm m} \leq X$.

5.29. Suppose that the ARE (5.96) admits a stabilizing solution. Show that X is positive definite, if and only if all stable modes of (C,A) are observable where $Q = C^*C$.

5.30. (i) Let $\tilde{A} = A - BR^{-1}D^*C$ and $\tilde{C} = \left[I - DR^{-1}D^*\right]C$ with $R = D^*D > \mathbf{0}$. Show that the ARE $X = \tilde{A}^* X \left[I + BR^{-1}B^*X\right]^{-1} \tilde{A} + \tilde{C}^*\tilde{C}$ can be equivalently written as the following ARE:

$$X = A^*XA - (A^*XB + C^*D)(R + B^*XB)^{-1}(B^*XA + D^*C) + C^*C.$$

(ii) What are the equivalent AREs in the case of optimal estimation?

5.31. Consider the DRE

$$X_t(T) = A^* X_{t+1}(T) \left[I_n + BR^{-1}B^* X_{t+1}(T)\right]^{-1} A + Q, \quad X_T(T) = \mathbf{0}.$$

Show that $\{X_t(T)\}$ satisfy $X_t(T) \geq X_{t+1}(T)$ for $0 \leq t < T$. (*Hint:* Use the same idea as in the proof of Proposition 5.1).

5.32. Suppose that (A,B) is stabilizable and the condition (5.100) holds. Construct a numerical example for which the ARE (5.96) has a solution $X_u \geq 0$ that is not stabilizing.

5.33. Prove Theorem 5.25. (*Hint:* Consider the augmented state vector $\check{\mathbf{x}}(t) = \left[\, \mathbf{x}(t)^* \ \mathbf{v}(t)^* \,\right]^*$ and then convert the full information control to the state-feedback control problem as in Theorem 5.19.)

5.34. Let (A,B,C,D) be a minimal realization of $\mathbf{G}(z)$ with $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, and $C \in \mathbb{C}^{p\times n}$. Assume that D has full rank. (i) If $p \leq m$, show that $z = z_0$ is a transmission zero of $\mathbf{G}(z)$, if and only if z_0 is an unreachable mode of $\left(\tilde{A}, \tilde{B}\right)$ with

$$\tilde{A} = A - BD^*(DD^*)^{-1}C, \quad \tilde{B} = B\left(I - D^*(DD^*)^{-1}D\right).$$

(ii) If $p \geq m$, show that $z = z_0$ is a transmission zero of $\mathbf{G}(z)$, if and only if z_0 is an unobservable mode of $\left(\tilde{C}, \tilde{A}\right)$ with

$$\tilde{A} = A - B\left(D^*D\right)^{-1} D^*C, \quad \tilde{C} = \left(I - D\left(D^*D\right)^{-1} D^*\right) C.$$

5.35. Let $\mathbf{G}(z)$ in (5.118) be stable. (i) Show that

$$\Phi(z) = \mathbf{G}(z)\mathbf{G}(z)^{\sim} = DD^* + CPC^* + C(zI - A)^{-1}L + L^*\left(z^{-1}I - A^*\right)^{-1} C^*,$$

where $L = APC^* + BD^*$ and $P = APA^* + BB^*$. (ii) Show that

$$\Phi(z) = \mathbf{G}(z)^{\sim}\mathbf{G}(z) = D^*D + B^*QB + H(zI - A)^{-1}B + B^*\left(z^{-1}I - A^*\right)^{-1} H^*,$$

where $H = B^*QA + D^*C$ and $Q = A^*QA + C^*C$.

5.36. Let $\mathbf{G}(z) = D + C(zI - A)^{-1}B$ with A a stability matrix, $R = DD^*$ nonsingular and $BD^* = \mathbf{0}$. (i) Show that

$$\mathbf{G}\left(e^{j\omega}\right)\mathbf{G}\left(e^{j\omega}\right)^* = R + C\left(e^{j\omega}I - A\right)^{-1} BB^*\left(e^{-j\omega}I - A^*\right)^{-1} C^* \geq R.$$

(ii) Use Lemma 5.1 and (i) to show that for each real ω,

$$\left[I - C(e^{j\omega}I - A)^{-1}K\right]\left(R + C\Sigma C^*\right)\left[I - C(e^{j\omega}I - A)^{-1}K\right]^* \geq R.$$