Chapter 9
Toward Hodge Theory for Complex Manifolds

From now on, we are going to work almost exclusively with complex-valued functions and forms. So we revise our notation accordingly. Given a C^∞ manifold X, let C^∞_X (respectively $\mathscr{E}^k_X$) now denote the space of complex-valued C^∞ functions (respectively k-forms). We write $C^\infty_{X,\mathbb{R}}$ (or $\mathscr{E}^k_{X,\mathbb{R}}$) for the space of real-valued functions or forms. Let us say that a complex-valued form is exact, closed, or harmonic if its real and imaginary parts both have this property. Then de Rham's theorem and Hodge's theorem carry over almost word for word: $H^k(X,\mathbb{C})$ is isomorphic to the space of complex closed k-forms modulo exact forms, and if X is compact and oriented with a Riemannian metric, then it is also isomorphic to the space of complex harmonic k-forms. This can be checked easily by working with the real and imaginary parts separately.

To go deeper, we should ask how de Rham and Hodge theory interact with the holomorphic structure when X is a complex manifold. This is really a central question in complex algebraic geometry. In this chapter, which is really a warmup for the next, we take the first few steps toward answering this. Here we concentrate on some special cases such as Riemann surfaces and tori, which can be handled without explicitly talking about Kähler metrics. In these cases, we will see that the answer is as nice as one can hope for. We will see, for instance, that the genus of a Riemann surface, which a priori is a topological invariant, can be interpreted as the number of linearly independent holomorphic 1-forms.

9.1 Riemann Surfaces Revisited

Fix a compact Riemann surface X with genus g, which we can define to be one-half of the first Betti number. In this section, we tie up a loose end from Chapter 6, by proving Proposition 6.2.9, that

$$g = \dim H^0(X,\Omega^1_X) = \dim H^1(X,\mathscr{O}_X).$$

D. Arapura, *Algebraic Geometry over the Complex Numbers*, Universitext,
DOI 10.1007/978-1-4614-1809-2_9,

This will be an easy application of the Hodge theorem. In order to use it, we need to choose a Riemannian metric that is a C^∞ family of inner products on the tangent spaces. We will also impose a compatibility condition that multiplication by $\mathrm{i} = \sqrt{-1}$ preserves the angles determined by these inner products. To say this more precisely, view X as a two-dimensional real C^∞ manifold. Choosing an analytic local coordinate $z = x + \mathrm{i}y$ in a neighborhood of U, the vectors $v_1 = \frac{\partial}{\partial x}$ and $v_2 = \frac{\partial}{\partial y}$ give a basis (or frame) of the real tangent sheaf $\mathscr{T}_X$ of X restricted to U. The automorphism $J_p : \mathscr{T}_X|_U \to \mathscr{T}_X|_U$ represented by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in the basis v_1, v_2 is independent of this basis, and hence globally well defined. A Riemannian metric $(,)$ is said to be compatible with the complex structure, or *Hermitian*, if the transformations J_p are orthogonal. In terms of the basis v_1, v_2 this forces the matrix of the bilinear form $(,)$ to be a positive multiple of I by some function h. In coordinates, the metric would be represented by a tensor $h(x,y)(dx \otimes dx + dy \otimes dy)$. The volume form is represented by $h dx \wedge dy$. It follows that $*dx = dy$ and $*dy = -dx$. In other words, $*$ is the transpose of J, which is independent of h. Once we have $*$, we can define all the operators from the last chapter.

Standard partition of unity arguments show that Hermitian metrics always exist. For our purposes, one metric is as good as any other, so we simply choose one.

Lemma 9.1.1. *A* 1*-form is harmonic if and only if its* $(1,0)$ *and* $(0,1)$ *parts are respectively holomorphic and antiholomorphic.*

Proof. Given a local coordinate $z = x + \mathrm{i}y$,

$$*dz = *(dx + \mathrm{i}dy) = dy - \mathrm{i}dx = -\mathrm{i}dz, \tag{9.1.1}$$

and similarly

$$*d\bar{z} = \mathrm{i}d\bar{z}. \tag{9.1.2}$$

If α is a $(1,0)$-form, then $d\alpha = \bar{\partial}\alpha$. Thus α is holomorphic if and only if it is closed if and only if $d\alpha = d * \alpha = 0$. The last condition is equivalent to harmonicity by Lemma 8.2.4. By a similar argument a $(0,1)$-form is antiholomorphic if and only if it is harmonic. This proves one direction.

By (9.1.1) and (9.1.2) the $(1,0)$ and $(0,1)$ parts of a 1-form are linear combinations of α and $*\alpha$. Thus if α is harmonic, then so are its parts. $\square$

Corollary 9.1.2. $\dim H^0(X, \Omega^1_X)$ *equals the genus of* X.

Proof. By the Hodge theorem, the first Betti number $2g$ is the dimension of the space of harmonic 1-forms, which decomposes into a direct sum of the spaces of holomorphic and antiholomorphic 1-forms. Both these spaces have the same dimension, since conjugation gives a real isomorphism between them. Therefore

$$2g = 2\dim H^0(X,\Omega_X^1). \qquad \square$$

Lemma 9.1.3. *The images of Δ and $\partial\bar{\partial}$ on $\mathscr{E}^2(X)$ coincide.*

Proof. On the space of 2-forms, we have $\Delta = -d * d*$. Computing in local coordinates yields

$$\partial\bar{\partial} f = -\frac{\mathrm{i}}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy$$

and

$$d * df = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy,$$

which implies the lemma. $\square$

Proposition 9.1.4. *The map $H^1(X,\mathscr{O}_X) \to H^1(X,\Omega_X^1)$ induced by d vanishes.*

Proof. We use the descriptions of these spaces as $\bar{\partial}$-cohomology groups provided by Corollary 6.2.5. Given $\alpha \in \mathscr{E}^{01}(X)$, let $\beta = d\alpha$. We have to show that β lies in the image of $\bar{\partial}$. Theorem 8.2.5 shows that we can write $\beta = H(\beta) + \Delta G(\beta)$. Since β is exact, we can conclude that $H(\beta) = 0$ by Corollary 8.2.6. Therefore β lies in the image of $\partial\bar{\partial} = -\bar{\partial}\partial$. $\square$

Corollary 9.1.5. *The map $H^1(X,\mathbb{C}) \to H^1(X,\mathscr{O}_X)$ is surjective, and* $\dim H^1(X,\mathscr{O}_X)$ *coincides with the genus of X.*

Proof. The surjectivity is immediate from the exact sequence

$$H^1(X,\mathbb{C}) \to H^1(X,\mathscr{O}_X) \to H^1(X,\Omega_X^1).$$

The second part follows from the equation

$$\dim H^1(X,\mathscr{O}_X) = \dim H^1(X,\mathbb{C}) - \dim H^0(X,\Omega_X). \qquad \square$$

Exercises

9.1.6. Show that $H^1(X,\mathbb{C}) \to H^1(X,\mathscr{O}_X)$ can be identified with the projection of harmonic 1-forms to antiholomorphic $(0,1)$-forms.

9.1.7. Calculate the spaces of harmonic and holomorphic one-forms explicitly for an elliptic curve.

9.1.8. Show that the pairing $(\alpha,\beta) \mapsto \int_X \alpha \wedge \overline{\beta}$ is positive definite.

9.2 Dolbeault's Theorem

We now extend the results from Riemann surfaces to higher dimensions. Given an n-dimensional complex manifold X, let $\mathscr{O}_X$ denote the sheaf of holomorphic functions. We can regard X as a $2n$-dimensional (real) C^∞ manifold as explained in Section 2.2. As explained in the introduction, $\mathscr{E}_X^k$ will now denote the sheaf of C^∞ complex-valued k-forms. We have

$$\mathscr{E}_X^k(U) = \mathbb{C} \otimes_{\mathbb{R}} \mathscr{E}_{X,\mathbb{R}}^k(U).$$

By a *real structure* on a complex vector space V, we mean a real vector space $V_{\mathbb{R}}$ and an isomorphism $\mathbb{C} \otimes V_{\mathbb{R}} \cong V$. This gives rise to a $\mathbb{C}$-antilinear involution $v \mapsto \bar{v}$ given by $\overline{a \otimes v} = \bar{a} \otimes v$. Conversely, such an involution gives rise to the real structure $V_{\mathbb{R}} = \{v \mid \bar{v} = v\}$. In particular, $\mathscr{E}_X^k(U)$ has a natural real structure.

The sheaf of holomorphic p-forms Ω_X^p is a subsheaf of $\mathscr{E}_X^p$ stable under multiplication by $\mathscr{O}_X$. This sheaf is locally free as an $\mathscr{O}_X$-module. If $z_1, \dots, z_n$ are holomorphic coordinates defined on an open set $U \subset X$, then

$$\{dz_{i_1} \wedge \cdots \wedge dz_{i_p} \mid i_1 < \cdots < i_p\}$$

gives a basis for $\Omega_X^p(U)$. To simplify our formulas, we let $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, where $I = \{i_1, \dots, i_p\}$.

Definition 9.2.1. Let $\mathscr{E}_X^{(p,0)}$ denote the C^∞ submodule of $\mathscr{E}_X^p$ generated by Ω_X^p. Let $\mathscr{E}_X^{(0,p)} = \overline{\mathscr{E}_X^{(p,0)}}$ and $\mathscr{E}_X^{(p,q)} = \mathscr{E}_X^{(p,0)} \wedge \mathscr{E}_X^{(0,q)}$.

In local coordinates, $\{dz_I \wedge d\bar{z}_J \mid \#I = p, \#J = q\}$ gives a basis of $\mathscr{E}_X^{(p,q)}(U)$. All of the operations of Section 6.2 can be extended to the higher-dimensional case. The operators

$$\partial : \mathscr{E}_X^{(p,q)} \to \mathscr{E}_X^{(p+1,q)}$$

and

$$\bar{\partial} : \mathscr{E}_X^{(p,q)} \to \mathscr{E}_X^{(p,q+1)}$$

are given locally by

$$\partial\left(\sum_{I,J} f_{I,J} dz_I \wedge d\bar{z}_J\right) = \sum_{I,J} \sum_{i=1}^{n} \frac{\partial f_{I,J}}{\partial z_i} dz_i \wedge dz_I \wedge d\bar{z}_J,$$

$$\bar{\partial}\left(\sum_{I,J} f_{I,J} dz_I \wedge d\bar{z}_J\right) = \sum_{I,J} \sum_{j=1}^{n} \frac{\partial f_{I,J}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J.$$

The identities

$$d = \partial + \bar{\partial}, \quad \partial^2 = \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0, \tag{9.2.1}$$

hold.

Theorem 6.2.2 has the following extension.

Theorem 9.2.2. *Let $D \subset \mathbb{C}^n$ be an open polydisk (i.e., a product of disks). Given $\alpha \in \mathscr{E}^{(p,q)}(\bar{D})$ with $\bar{\partial}\alpha = 0$, there exists $\beta \in \mathscr{E}^{(p,q-1)}(D)$ such that $\alpha = \bar{\partial}\beta$.*

Proof. See [49, pp. 25–26]. □

Corollary 9.2.3 (Dolbeault's theorem I). *For any complex manifold X,*

(a)

$$0 \to \Omega_X^p \to \mathscr{E}_X^{(p,0)} \xrightarrow{\bar{\partial}} \mathscr{E}_X^{(p,1)} \xrightarrow{\bar{\partial}} \cdots$$

is a soft resolution.

(b)

$$H^q(X, \Omega_X^p) \cong \frac{\ker[\bar{\partial} : \mathscr{E}^{(p,q)}(X) \to \mathscr{E}^{(p,q+1)}]}{\operatorname{im}[\bar{\partial} : \mathscr{E}^{(p,q-1)}(X) \to \mathscr{E}^{(p,q)}]}.$$

Proof. Since exactness can be checked on the stalks, there is no loss in assuming $X = D$ for (a). The only thing not stated above is that Ω_X^p is the kernel of the $\bar{\partial}$ operator on $\mathscr{E}_X^{(p,0)}$. This is a simple calculation. Given a $(p,0)$-form $\sum_I f_I dz_I$,

$$\bar{\partial}\left(\sum_I f_I dz_I\right) = \sum_I \sum_{j=1}^n \frac{\partial f_I}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I = 0$$

if and only it is holomorphic. Thus the sheaves $\mathscr{E}_X^{(p,\bullet)}$ give a resolution, which is soft since these are modules over C_X^∞.

(b) is now a consequence of Theorem 5.1.4. □

In the sequel, we will refer to elements of $\ker \bar{\partial}$ (or $\operatorname{im} \bar{\partial}$) as $\bar{\partial}$-exact (or $\bar{\partial}$-closed).

Exercises

9.2.4. Check the identities (9.2.1).

9.2.5. Give an explicit description of the map $H^i(X, \mathbb{C}) \to H^i(X, \mathscr{O}_X)$ induced by inclusion $\mathbb{C}_X \to \mathscr{O}_X$ as a projection from de Rham cohomology to $\bar{\partial}$-cohomology.

9.3 Complex Tori

A *complex torus* is a quotient $X = V/L$ of a finite-dimensional complex vector space by a lattice (i.e., a discrete subgroup of maximal rank). Thus it is both a complex manifold and a torus. After choosing a basis, we may identify V with $\mathbb{C}^n$. Let $z_1, \ldots, z_n$ be the standard complex coordinates on $\mathbb{C}^n$, and let $x_i = \operatorname{Re}(z_i)$, $y_i = \operatorname{Im}(z_i)$.

We give X the flat metric induced by the Euclidean metric on V. Recall that harmonic forms with respect to this are the forms with constant coefficients (Example 8.2.7).

Lemma 9.3.1. *A holomorphic form on X has constant coefficients and is therefore harmonic.*

Proof. The coefficients of a holomorphic form $\sum f_I dz_I$ are holomorphic functions on X. These are constant because X is compact. □

This can be refined.

Proposition 9.3.2. *$H^q(X,\Omega_X^p)$ is isomorphic to the space of (p,q)-forms with constant coefficients.*

Corollary 9.3.3. *Set*

$$H^{(p,q)} = \bigoplus_{\#I=p,\#J=q} \mathbb{C}dz_I \wedge d\bar{z}_J.$$

Then $H^q(X,\Omega_X^p) \cong H^{(p,q)} \cong \wedge^p\mathbb{C}^n \otimes \wedge^q\mathbb{C}^n$.

The isomorphism in the corollary is highly noncanonical. A more natural identification is

$$H^q(X,\Omega_X^p) \cong \wedge^p V^* \otimes \wedge^q \bar{V}^*,$$

where V^* is the usual dual, and $\bar{V}^*$ is the set of antilinear maps from V to $\mathbb{C}$.

The proof of Proposition 9.3.2 hinges on a certain identity between Laplacians that we now define. The space of forms carries inner products as in Section 8.2, where X is equipped with the flat metric. Let ∂^* and $\bar{\partial}^*$ denote the adjoints to ∂ and $\bar{\partial}$ respectively. These will be calculated explicitly below. We can define the ∂- and $\bar{\partial}$-Laplacians by

$$\Delta_\partial = \partial^*\partial + \partial\partial^*,$$
$$\Delta_{\bar{\partial}} = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*.$$

Lemma 9.3.4. $\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_\partial$.

We give two proofs. The first is by direct calculation.

Proof. Let

$$\alpha = \sum_{I,J} \alpha_{IJ} dz_I \wedge d\bar{z}_J.$$

Then

$$\begin{aligned}
\Delta_{\bar{\partial}}(\alpha) &= -2\sum_{I,J,i} \frac{\partial^2 \alpha_{IJ}}{\partial z_i \partial \bar{z}_i} dz_I \wedge d\bar{z}_J \\
&= -\frac{1}{2}\sum_{I,J,i}\left(\frac{\partial^2 \alpha_{IJ}}{\partial x_i^2} + \frac{\partial^2 \alpha_{IJ}}{\partial y_i^2}\right) dz_I \wedge d\bar{z}_J \\
&= \frac{1}{2}\Delta(\alpha).
\end{aligned}$$

A similar calculation holds for $\Delta_\partial(\alpha)$. □

This implies Proposition 9.3.2:

Proof. By Dolbeault's theorem,

$$H^q(X,\Omega_X^p) \cong \frac{\ker[\bar{\partial} : \mathscr{E}^{(p,q)}(X) \to \mathscr{E}^{(p,q+1)}]}{\operatorname{im}[\bar{\partial} : \mathscr{E}^{(p,q-1)}(X) \to \mathscr{E}^{(p,q)}]}.$$

Let α be a $\bar{\partial}$-closed (p,q)-form. Decompose

$$\alpha = \beta + \Delta\gamma = \beta + 2\Delta_{\bar{\partial}}\gamma = \beta + \bar{\partial}\gamma_1 + \bar{\partial}^*\gamma_2$$

with β harmonic, which is possible by Theorem 8.2.5. We have

$$\|\bar{\partial}^*\gamma_2\|^2 = \langle \gamma_2, \bar{\partial}\bar{\partial}^*\gamma_2\rangle = \langle \gamma_2, \bar{\partial}\alpha\rangle = 0.$$

It is left as an exercise to check that β is of type (p,q), and that it is unique. Therefore the $\bar{\partial}$-class of α has a unique representative by a constant (p,q)-form. □

We will sketch a second proof of Lemma 9.3.4. Although it is much more complicated than the first, it has the advantage of generalizing nicely to Kähler manifolds. We introduce a number of auxiliary operators. Let i_k and $\bar{i}_k$ denote contraction with the vector fields $2\frac{\partial}{\partial z_k}$ and $2\frac{\partial}{\partial \bar{z}_k}$. Thus for example, $i_k(dz_k \wedge \alpha) = 2\alpha$. If we choose our Euclidean metric so that monomials in dx_i, dy_j are orthonormal, then the contractions i_k and $\bar{i}_k$ can be checked to be adjoints to $dz_k\wedge$ and $d\bar{z}_k\wedge$. Let

$$\omega = \frac{\sqrt{-1}}{2}\sum dz_k \wedge d\bar{z}_k = \sum dx_k \wedge dy_k, \quad L\alpha = \omega \wedge \alpha,$$

and

$$\Lambda = -\frac{\sqrt{-1}}{2}\sum \bar{i}_k i_k.$$

The operators L and Λ are adjoint. Using integration by parts (see [49, p. 113]), we get explicit formulas

$$\partial^*\alpha = -\sum \frac{\partial}{\partial \bar{z}_k} i_k\alpha, \quad \bar{\partial}^*\alpha = -\sum \frac{\partial}{\partial z_k}\bar{i}_k\alpha,$$

where the derivatives above are taken coefficient-wise.

Let $[A,B] = AB - BA$ denote the commutator. Then we have the following first-order Kähler identities:

Proposition 9.3.5.

(a) $[\Lambda, \bar{\partial}] = -\sqrt{-1}\partial^*$.
(b) $[\Lambda, \partial] = \sqrt{-1}\bar{\partial}^*$.

Proof. We check the second identity on the space of $(1,1)$-forms. The general case is more involved notationally but not essentially harder; see [49, p. 114]. There are two cases. First suppose that $\alpha = f dz_j \wedge d\bar{z}_k$, with $j \neq k$. Then

$$\begin{aligned}
[\Lambda,\partial]\alpha &= \Lambda\partial\alpha \\
&= \Lambda\left(\sum_m \frac{\partial f}{\partial z_m} dz_m \wedge dz_j \wedge d\bar{z}_k\right) \\
&= 2\sqrt{-1}\frac{\partial f}{\partial z_k} dz_j \\
&= \sqrt{-1}\bar{\partial}^*\alpha.
\end{aligned}$$

Next suppose that $\alpha = f dz_k \wedge d\bar{z}_k$. Then

$$\begin{aligned}
[\Lambda,\partial]\alpha &= \Lambda\left(\sum_{m\neq k} \frac{\partial f}{\partial z_m} dz_m \wedge dz_k \wedge d\bar{z}_k\right) - \partial\left(-2\sqrt{-1}f\right) \\
&= -2\sqrt{-1}\left(\sum_{m\neq k} \frac{\partial f}{\partial z_m} dz_m - \sum_m \frac{\partial f}{\partial z_m} dz_m\right) \\
&= 2\sqrt{-1}\frac{\partial f}{\partial z_k} dz_k \\
&= \sqrt{-1}\bar{\partial}^*\alpha.
\end{aligned}$$

□

We now give a second proof of Lemma 9.3.4. Upon substituting the first-order identities into the definitions of the various Laplacians, some remarkable cancellations take place:

Proof. We first establish $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$,

$$\sqrt{-1}(\partial\bar{\partial}^* + \bar{\partial}^*\partial) = \partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\partial = \partial\Lambda\partial - \partial\Lambda\partial = 0.$$

Similarly, we have $\partial^*\bar{\partial} + \bar{\partial}\partial^* = 0$.
Next expand Δ,

$$\begin{aligned}
\Delta &= (\partial+\bar{\partial})(\partial^*+\bar{\partial}^*) + (\partial^*+\bar{\partial}^*)(\partial+\bar{\partial}) \\
&= (\partial\partial^* + \partial^*\partial) + (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) + (\partial\bar{\partial}^* + \bar{\partial}^*\partial) + (\partial^*\bar{\partial} + \bar{\partial}\partial^*) \\
&= (\partial\partial^* + \partial^*\partial) + (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) + (\partial\bar{\partial}^* + \bar{\partial}^*\partial) + (\partial^*\bar{\partial} + \bar{\partial}\partial^*) \\
&= \Delta_\partial + \Delta_{\bar{\partial}}.
\end{aligned}$$

Finally, we check $\Delta_\partial = \Delta_{\bar{\partial}}$,

$$\begin{aligned}
-\sqrt{-1}\Delta_\partial &= \partial(\Lambda\bar{\partial} - \bar{\partial}\Lambda) + (\Lambda\bar{\partial} - \bar{\partial}\Lambda)\partial \\
&= \partial\Lambda\bar{\partial} - \partial\bar{\partial}\Lambda + \Lambda\bar{\partial}\partial - \bar{\partial}\Lambda\partial \\
&= (\partial\Lambda - \Lambda\partial)\bar{\partial} + \bar{\partial}(\partial\Lambda - \Lambda\partial) \\
&= -\sqrt{-1}\Delta_{\bar{\partial}}.
\end{aligned}$$

□

Exercises

9.3.6. Check the first order-Kähler identities (Proposition 9.3.5) on the space of all 2-forms.

9.3.7. Show that $\beta + \bar{\partial}\gamma = 0$ forces $\beta = 0$ if β is harmonic.

9.3.8. Suppose that $\alpha = \beta + \bar{\partial}\gamma$ and that α is of type (p,q) and β harmonic. By decomposing $\beta = \sum \beta^{(p',q')}$ and $\gamma = \sum \gamma^{(p',q')}$ into (p',q') type and using the previous exercise, prove that β is of type (p,q) and unique.