Chapter 5
De Rham Cohomology of Manifolds

In this chapter, we study the topology of C^∞-manifolds. We define the de Rham cohomology of a manifold, which is the vector space of closed differential forms modulo exact forms. After sheafifying the construction, we see that the de Rham complex forms a so-called acyclic resolution of the constant sheaf $\mathbb{R}$. We prove a general result that sheaf cohomology can be computed using such resolutions, and deduce a version of de Rham's theorem that de Rham cohomology is sheaf cohomology with coefficients in $\mathbb{R}$. It follows that de Rham cohomology depends only on the underlying topology. Using a different acyclic resolution that is dual to the de Rham complex, we prove Poincaré duality. This duality makes cohomology, which is normally contravariant, into a covariant theory. We devote a section to explaining these somewhat mysterious covariant maps, called Gysin maps. We end this chapter with the remarkable Lefschetz trace formula, which in principle, calculates the number of fixed points for a map of a manifold to itself.

A systematic development of topology from the de Rham point of view is given in Bott and Tu [14].

5.1 Acyclic Resolutions

We start by reviewing some standard notions from homological algebra.

Definition 5.1.1. A complex of (sheaves of) abelian groups is a possibly infinite sequence

$$\cdots \to F^i \xrightarrow{d_i} F^{i+1} \xrightarrow{d_{i+1}} \cdots$$

of (sheaves of) groups and homomorphisms satisfying $d_{i+1}d_i = 0$.

These conditions guarantee that $\mathrm{im}(d_i) \subseteq \ker(d_{i+1})$. We denote a complex by $F^\bullet$, and we often suppress the indices on d. The ith *cohomology* of $F^\bullet$ is defined by

$$\mathscr{H}^i(F^\bullet) = \frac{\ker(d_i)}{\mathrm{im}(d_{i-1})}.$$

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(We reserve the regular font "H" for sheaf cohomology.) These groups are zero precisely when the complex is exact.

Definition 5.1.2. A sheaf $\mathscr{F}$ is called acyclic if $H^i(X,\mathscr{F}) = 0$ for all $i > 0$.

For example, flasque sheaves and soft sheaves on a metric space are acyclic.

Definition 5.1.3. An acyclic resolution of a sheaf $\mathscr{F}$ is an exact sequence

$$0 \to \mathscr{F} \to \mathscr{F}^0 \to \mathscr{F}^1 \to \cdots$$

of sheaves such that each $\mathscr{F}^i$ is acyclic.

A functor between abelian categories, such as Ab or Ab(X), need not take exact sequences to exact sequences, but it will always take complexes to complexes. In particular, given a complex of sheaves $\mathscr{F}^\bullet$, the sequence

$$\Gamma(X,\mathscr{F}^0) \to \Gamma(X,\mathscr{F}^1) \to \cdots$$

is necessarily a complex of abelian groups.

Theorem 5.1.4. *Given an acyclic resolution $\mathscr{F}^\bullet$ of $\mathscr{F}$, we have*

$$H^i(X,\mathscr{F}) \cong \mathscr{H}^i(\Gamma(X,\mathscr{F}^\bullet)).$$

Proof. Let $\mathscr{K}^{-1} = \mathscr{F}$ and $\mathscr{K}^i = \ker(\mathscr{F}^{i+1} \to \mathscr{F}^{i+2})$ for $i \geq 0$. Then there are exact sequences

$$0 \to \mathscr{K}^{i-1} \to \mathscr{F}^i \to \mathscr{K}^i \to 0$$

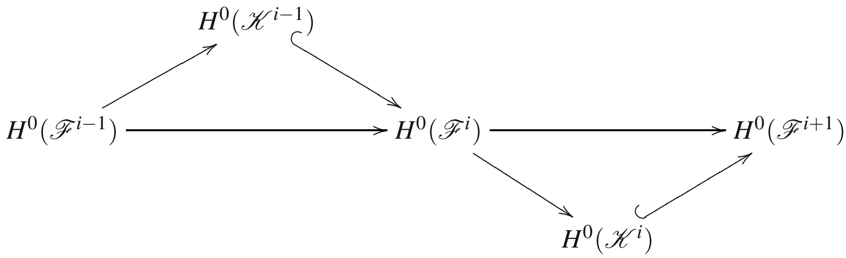
for $i \geq 0$. Since each $\mathscr{F}^i$ is acyclic, Theorem 4.2.3 implies that

$$0 \to H^0(\mathscr{K}^{i-1}) \to H^0(\mathscr{F}^i) \to H^0(\mathscr{K}^i) \to H^1(\mathscr{K}^{i-1}) \to 0 \tag{5.1.1}$$

is exact, and

$$H^j(\mathscr{K}^i) \cong H^{j+1}(\mathscr{K}^{i-1}) \tag{5.1.2}$$

for $j > 0$. We have a diagram



which is commutative, since the morphism $\mathscr{F}^{i-1} \to \mathscr{F}^i$ factors through $\mathscr{K}^{i-1}$ and so on. The oblique line in the diagram is part of (5.1.1), so it is exact. In particular, the first hooked arrow is injective. The injectivity of the second hooked arrow follows for similar reasons. Thus

$$\operatorname{im}[H^0(\mathscr{F}^i) \to H^0(\mathscr{K}^i)] = \operatorname{im}[H^0(\mathscr{F}^i) \to H^0(\mathscr{F}^{i+1})]. \tag{5.1.3}$$

Suppose that $\alpha \in H^0(\mathscr{F}^i)$ maps to 0 in $H^0(\mathscr{F}^{i+1})$. Then it maps to 0 in $H^0(\mathscr{K}^i)$. Therefore α lies in the image of $H^0(\mathscr{K}^{i-1})$. Thus

$$H^0(\mathscr{K}^{i-1}) = \ker[H^0(\mathscr{F}^i) \to H^0(\mathscr{F}^{i+1})]. \tag{5.1.4}$$

This already implies the theorem when $i = 0$. Replacing i by $i+1$ in (5.1.4) and combining it with (5.1.1) and (5.1.3) shows that

$$H^1(\mathscr{K}^{i-1}) \cong \frac{H^0(\mathscr{K}^i)}{\operatorname{im}[H^0(\mathscr{F}^i) \to H^0(\mathscr{K}^i)]} = \frac{\ker[H^0(\mathscr{F}^{i+1}) \to H^0(\mathscr{F}^{i+2})]}{\operatorname{im}[H^0(\mathscr{F}^i) \to H^0(\mathscr{F}^{i+1})]}.$$

Combining this with the isomorphisms

$$H^{i+1}(\mathscr{F}) = H^{i+1}(\mathscr{K}^{-1}) \cong H^i(\mathscr{K}^0) \cong \cdots \cong H^1(\mathscr{K}^{i-1})$$

of (5.1.2) proves the theorem for positive exponents. □

Example 5.1.5. Let $\mathscr{F}$ be a sheaf. Using the notation from Section 4.2, define $\mathbf{G}^i(\mathscr{F}) = \mathbf{G}(\mathbf{C}^i(\mathscr{F}))$. We define $d : \mathbf{G}^i(\mathscr{F}) \to \mathbf{G}^{i+1}(\mathscr{F})$ as the composition of the natural maps $\mathbf{G}^i(\mathscr{F}) \to \mathbf{C}^{i+1}(\mathscr{F}) \to \mathbf{G}^{i+1}(\mathscr{F})$. This can be seen to give an acyclic resolution of $\mathscr{F}$.

Exercises

5.1.6. Check that $\mathbf{G}^\bullet(\mathscr{F})$ gives an acyclic resolution of $\mathscr{F}$.

5.1.7. A sheaf $\mathscr{I}$ is called *injective* if given a monomorphism of sheaves $\mathscr{A} \to \mathscr{B}$, any morphism $\mathscr{A} \to \mathscr{I}$ extends to a morphism of $\mathscr{B} \to \mathscr{I}$. Show that an injective module is flasque and hence acyclic. (Hint: given an open set $U \subseteq X$, let $\mathbb{Z}_U = \ker[\mathbb{Z}_X \to \mathbb{Z}_{X-U}]$; check that $\operatorname{Hom}(\mathbb{Z}_U, \mathscr{F}) = \mathscr{F}(U)$.) Conclude that if $0 \to \mathscr{F} \to \mathscr{I}^0 \to \mathscr{I}^1 \to \cdots$ is an injective resolution, then $H^i(X, \mathscr{F}) = \mathscr{H}^i(\Gamma(X, \mathscr{I}^\bullet))$; this is usually taken as the definition of H^i.

5.1.8. A morphism of complexes is a collection of maps $\mathscr{F}^i \to \mathscr{G}^i$ commuting with the differentials d. This would induce a map on cohomology. Suppose that $\mathscr{F} \to \mathscr{F}^0 \to \cdots$ and $\mathscr{G} \to \mathscr{G}^0 \to \cdots$ are acyclic resolutions of sheaves $\mathscr{F}$ and $\mathscr{G}$, and suppose that we have a morphism $\mathscr{F} \to \mathscr{G}$ that extends to a morphism of the resolutions. Show that we can choose the isomorphisms so that the diagram

$$\begin{array}{ccc} H^i(\mathscr{F}) & \cong & \mathscr{H}^i(\Gamma(\mathscr{F}^\bullet)) \\ \downarrow & & \downarrow \\ H^i(\mathscr{G}) & \cong & \mathscr{H}^i(\Gamma(\mathscr{G}^\bullet)) \end{array}$$

commutes.

5.2 De Rham's Theorem

Let X be a C^∞ manifold and $\mathscr{E}^k = \mathscr{E}^k_X$ the sheaf of k-forms on it. Note that $\mathscr{E}^0_X = C^\infty_X$. If $U \subset X$ is a coordinate neighborhood with coordinates $x_1, \dots, x_n$, then $\mathscr{E}^k(U)$ is a free $C^\infty(U)$-module with basis

$$\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} \mid i_1 < \cdots < i_k\}.$$

Theorem 5.2.1. *There exist canonical $\mathbb{R}$-linear maps $d : \mathscr{E}^k_X \to \mathscr{E}^{k+1}_X$, called exterior derivatives, satisfying the following:*

(a) $d : \mathscr{E}^0_X \to \mathscr{E}^1_X$ is the operation introduced in Section 2.6.
(b) $d^2 = 0$.
(c) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^i \alpha \wedge d\beta$ for all $\alpha \in \mathscr{E}^i(X)$, $\beta \in \mathscr{E}^j(X)$.
(d) If $g : Y \to X$ is a C^∞ map, $g^* \circ d = d \circ g^*$.

Proof. A complete proof can be found in almost any book on manifolds (e.g., [110, 117]). We will only sketch the construction. When $U \subset X$ is a coordinate neighborhood with coordinates x_i, we can see that there is a unique operation satisfying the above rules (a) and (c) given by

$$d\left(\sum_{i_1 < \cdots < i_k} f_{i_1 \dots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) = \sum_{i_1 < \cdots < i_k} \sum_j \frac{\partial f_{i_1 \dots i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

By uniqueness, these local d's patch to define an operator on X. Taking the derivative again yields

$$\begin{aligned} &\sum_{i_1 \dots} \sum_{j,\ell} \frac{\partial^2 f_{i_1 \dots i_k}}{\partial x_j \partial x_\ell} dx_j \wedge dx_\ell \wedge dx_{i_1} \wedge \cdots dx_{i_k} \\ &= \sum_{i_1 \dots} \sum_{j<\ell} \left(\frac{\partial^2 f_{i_1 \dots i_k}}{\partial x_j \partial x_\ell} - \frac{\partial^2 f_{i_1 \dots i_k}}{\partial x_\ell \partial x_j} \right) dx_j \wedge dx_\ell \wedge dx_{i_1} \wedge \cdots dx_{i_k}, \\ &= 0 \end{aligned}$$

which proves (b). □

When $X = \mathbb{R}^3$, d can be realized as the div, grad, curl of vector calculus. The theorem tells us that $\mathscr{E}^\bullet(X)$ forms a complex, called the *de Rham complex.*

Definition 5.2.2. The de Rham cohomology groups (actually vector spaces) of X are defined by

$$H^k_{dR}(X) = \mathscr{H}^k(\mathscr{E}^\bullet(X)).$$

A differential form α is called *closed* if $d\alpha = 0$ and *exact* if $\alpha = d\beta$ for some β. Elements of de Rham cohomology are equivalence classes $[\alpha]$ represented by closed forms, where two closed forms are equivalent if they differ by an exact form.

Given a C^∞ map of manifolds $g : Y \to X$, we get a map $g^* : \mathscr{E}^*(X) \to \mathscr{E}^*(Y)$ of the de Rham complexes that induces a map g^* on cohomology. We easily have the following lemma:

Lemma 5.2.3. *$X \mapsto H^i_{dR}(X)$ is a contravariant functor from manifolds to real vector spaces.*

We compute the de Rham cohomology of Euclidean space.

Theorem 5.2.4 (Poincaré's lemma). *For all n and $k > 0$,*

$$H^k_{dR}(\mathbb{R}^n) = 0.$$

Proof. Assume, for the inductive hypothesis, that the theorem holds for $n-1$. Consider the maps $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $\iota : \mathbb{R}^{n-1} \to \mathbb{R}^n$ defined by $p(x_1, x_2, \ldots, x_n) = (x_2, \ldots, x_n)$ and $\iota(x_2, \ldots, x_n) = (0, x_2, \ldots, x_n)$. Let $R = (\iota \circ p)^*$. More explicitly, $R : \mathscr{E}^k(\mathbb{R}^n) \to \mathscr{E}^k(\mathbb{R}^n)$ is the $\mathbb{R}$-linear operator defined by

$$R(f(x_1, \ldots, x_n) dx_{i_1} \wedge \cdots \wedge dx_{i_k})$$
$$= \begin{cases} f(0, x_2, \ldots, x_n) dx_{i_1} \wedge \cdots \wedge dx_{i_k} & \text{if } 1 \notin \{i_1, i_2, \ldots\}, \\ 0 & \text{otherwise,} \end{cases}$$

where we always choose $i_1 < i_2 < \cdots$. The image of R can be identified with $p^* \mathscr{E}^k(\mathbb{R}^{n-1})$. Note that R commutes with d. Therefore if $\alpha \in \mathscr{E}^k(\mathbb{R}^n)$ is closed, $dR\alpha = Rd\alpha = 0$. By the induction assumption, $R\alpha$ is exact.

For each k, define a linear map $h : \mathscr{E}^k(\mathbb{R}^n) \to \mathscr{E}^{k-1}(\mathbb{R}^n)$ by

$$h(f(x_1, \ldots, x_n) dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \begin{cases} (\int_0^{x_1} f dx_1) dx_{i_2} \wedge \cdots \wedge dx_{i_k} & \text{if } i_1 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the fundamental theorem of calculus shows that $dh + hd = I - R$, where I is the identity. (In other words, h is homotopy from I to R.) Given $\alpha \in \mathscr{E}^k(\mathbb{R}^n)$ satisfying $d\alpha = 0$, we have $\alpha = dh\alpha + R\alpha$, which is exact. □

We have an inclusion of the sheaf of locally constant functions $\mathbb{R}_X \subset \mathscr{E}^0_X$. This is precisely the kernel of $d : \mathscr{E}^0_X \to \mathscr{E}^1_X$.

Theorem 5.2.5. *The sequence*

$$0 \to \mathbb{R}_X \to \mathscr{E}^0_X \to \mathscr{E}^1_X \to \cdots$$

is an acyclic resolution of $\mathbb{R}_X$.

Proof. Any ball is diffeomorphic to Euclidean space, and any point on a manifold has a fundamental system of such neighborhoods. Therefore, Poincaré's lemma implies that the above sequence is exact on stalks, and hence exact.

By Corollary 4.4.5, the sheaves $\mathscr{E}^k$ are soft, hence acyclic. □

Corollary 5.2.6 (De Rham's theorem). *There is an isomorphism*

$$H^k_{dR}(X) \cong H^k(X,\mathbb{R}).$$

In particular, de Rham cohomology depends only on the underlying topological space.

Recall that by our convention, $H^k(X,\mathbb{R})$ is $H^k(X,\mathbb{R}_X)$. Later on, we will work with complex-valued differential forms. Essentially the same argument shows that $H^*(X,\mathbb{C})$ can be computed using such forms.

Exercises

5.2.7. We will say that a manifold is of *finite type* if it has a finite open cover $\{U_i\}$ such that any nonempty intersection of the U_i are diffeomorphic to the ball. Compact manifolds are known to have finite type [110, pp. 595–596]. Using Mayer–Vietoris and de Rham's theorem, prove that if X is an n-dimensional manifold of finite type, then $H^k(X,\mathbb{R})$ vanishes for $k > n$, and is finite-dimensional otherwise.

5.2.8. Let X be a manifold, and let t be the coordinate along $\mathbb{R}$ in $\mathbb{R}\times X$. Consider the maps $\iota : X \to \mathbb{R}\times X$ and $p : \mathbb{R}\times X \to X$ given by $x \mapsto (0,x)$ and $(t,x) \mapsto x$. Since $(p\circ\iota) = \mathrm{id}$, conclude that $\iota^* : H^i_{dR}(\mathbb{R}\times X) \to H^i_{dR}(X)$ is surjective.

5.2.9. Continuing the notation from the previous exercise, let $R : \mathscr{E}^k(\mathbb{R}\times X) \to \mathscr{E}^k(\mathbb{R}\times X)$ be the operator $(i\circ p)^*$, and let $h : \mathscr{E}^k(\mathbb{R}\times X) \to \mathscr{E}^{k-1}(\mathbb{R}\times X)$ be the operator that is integration with respect to dt (as in the proof of the Poincaré lemma). Show that $dh + hd = I - R$. Use this to show that R induces the identity on $H^i_{dR}(\mathbb{R}\times X)$. Conclude that $\iota^* : H^i_{dR}(\mathbb{R}\times X) \to H^i_{dR}(X)$ is also injective, and therefore an isomorphism.

5.2.10. Show that $\mathbb{C}-\{0\}$ is diffeomorphic to $\mathbb{R}\times S^1$, and conclude that $H^1_{dR}(\mathbb{C}-\{0\})$ is one-dimensional. Show that $\mathrm{Re}(\frac{dz}{iz}) = \frac{-ydx+xdy}{x^2+y^2}$ generates it.

5.2.11. Let S^n denote the n-dimensional sphere. Use Mayer–Vietoris with respect to the cover $U = S^n - \{\text{north pole}\}$ and $V = S^n - \{\text{south pole}\}$ to compute $H^*(S^n,\mathbb{R})$. (Hint: show that $U\cap V \cong S^{n-1}\times\mathbb{R}$.)

5.3 Künneth's Formula

Suppose that X is a C^∞ manifold. If $\alpha \in \mathscr{E}^i(X)$ and $\beta \in \mathscr{E}^j(X)$ are closed forms, then $\alpha\wedge\beta$ is also closed, by Theorem 5.2.1. The *cup product* of the associated cohomology classes is defined by $[\alpha]\cup[\beta] = [\alpha\wedge\beta]$. This is a well-defined operation that makes de Rham cohomology into a graded ring. An extension of de Rham's theorem shows that this operation is also a topological invariant.

Theorem 5.3.1 (Multiplicative de Rham's theorem). *Under the de Rham isomorphism, the product given above coincides with the cup product in sheaf cohomology constructed in Section 4.6.*

We outline the argument, concentrating on those parts that will be needed later. First, we need a more convenient method for computing cup products. Given complexes of (sheaves of) vector spaces $(A^\bullet, d_A)$ and $(B^\bullet, d_B)$ over a field k, their tensor product is the complex

$$(A^\bullet \otimes B^\bullet)^n = \bigoplus_{i+j=n} A^i \otimes B^j$$

with differential

$$d(a \otimes b) = d_A a \otimes b + (-1)^i a \otimes d_B b, \quad a \in A^i, b \in B^j.$$

The cohomology of this is easily computed by the following result:

Theorem 5.3.2 (Algebraic Künneth formula). *If $A^\bullet$ and $B^\bullet$ are complexes of vector spaces, then*

$$H^n((A^\bullet \otimes B^\bullet)^\bullet) \cong \bigoplus_{i+j=n} H^i(A^\bullet) \otimes H^j(B^\bullet),$$

where the map (from right to left) is induced by the inclusion of $\ker d_A \otimes \ker d_B \subset \ker d$.

Proof. A proof can be found in [108, Chapter 5 §3, Lemma 1; §4] for instance. □

The next lemma is left as an exercise.

Lemma 5.3.3. *The tensor product of two soft sheaves of vector spaces is soft.*

Lemma 5.3.4. *If $\mathscr{F} \to \mathscr{F}^\bullet$ and $\mathscr{G} \to \mathscr{G}^\bullet$ are soft resolutions of sheaves of vector spaces, then $\mathscr{F} \otimes \mathscr{G} \to (\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet)^\bullet$ is again a soft resolution.*

Proof. The previous lemma shows that the sheaves $(\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet)^\bullet$ are soft. To see that it resolves $\mathscr{F} \otimes \mathscr{G}$, use Theorem 5.3.2 to obtain

$$\mathscr{H}^i((\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet)^\bullet_x) = \begin{cases} \mathscr{F}_x \otimes \mathscr{G}_x & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$ □

Choosing soft resolutions $\mathscr{F} \to \mathscr{F}^\bullet$ and $\mathscr{G} \to \mathscr{G}^\bullet$, we have a morphism of complexes

$$(\Gamma(\mathscr{F}^\bullet) \otimes \Gamma(\mathscr{G}^\bullet))^\bullet \to \Gamma((\mathscr{F}^\bullet \otimes \mathscr{G}^\bullet))^\bullet,$$

which induces a map on cohomology. The cohomology on the left decomposes into a sum of tensor products of the cohomology of $\mathscr{F}$ and $\mathscr{G}$. Thus on each summand we get a map

$$H^i(X, \mathscr{F}) \otimes H^j(X, \mathscr{G}) \to H^{i+j}(X, \mathscr{F} \otimes \mathscr{G}). \tag{5.3.1}$$

Lemma 5.3.5. *The map in* (5.3.1) *coincides with the product defined in Section 4.6.*

Proof. This hinges on the fact that the product given in (5.3.1) is well defined and satisfies the axioms of Theorem 4.6.1 by [45, pp. 255–259]. □

We can now sketch the proof of Theorem 5.3.1.

Proof. By the previous lemmas and [45, Chapter II, Theorem 6.6.1], it suffices to observe that the diagram

$$\begin{array}{ccc} \mathbb{R}\otimes\mathbb{R} & \xrightarrow{\sim} & (\mathscr{E}_X^\bullet \otimes \mathscr{E}_X^\bullet)^\bullet \\ \Big\downarrow{=} & & \Big\downarrow{\wedge} \\ \mathbb{R} & \xrightarrow{\sim} & \mathscr{E}_X^\bullet \end{array}$$

commutes. □

We can adapt these arguments to deduce a more geometric version of Künneth's formula.

Theorem 5.3.6 (Künneth formula). *Let X and Y be C^∞ manifolds. Then the product $Z = X \times Y$ is also a C^∞ manifold. Let $p : Z \to X$ and $q : Z \to Y$ denote the projections. Then the map*

$$\sum \alpha_i \otimes \beta_j \mapsto \sum p^*\alpha_i \cup q^*\beta_j$$

induces an isomorphism

$$\bigoplus_{i+j=k} H^i_{dR}(X) \otimes_{\mathbb{R}} H^j_{dR}(Y) \cong H^k_{dR}(Z).$$

Proof. The proof involves the sheaves $p^*\mathscr{E}_X^i \otimes_{\mathbb{R}} q^*\mathscr{E}_Y^j$. Their sections on basic opens are

$$p^*\mathscr{E}_X^i \otimes q^*\mathscr{E}_Y^j(U \times V) = \mathscr{E}_X^i(U) \otimes \mathscr{E}_Y^j(V).$$

These map to $\mathscr{E}_Z^{i+j}(U \times V)$ under $\kappa(\alpha \otimes \beta) = p^*\alpha \wedge q^*\beta$. Locally constant functions lie in $p^*\mathscr{E}_X^0 \otimes q^*\mathscr{E}_Y^0$. Thus we have a commutative triangle

$$\begin{array}{ccc} \mathbb{R}_Z & \overset{\iota}{\hookrightarrow} & \mathscr{E}_Z^\bullet \\ & \searrow^{\iota'} & \Big\uparrow{\kappa} \\ & & (p^*\mathscr{E}_X^\bullet \otimes q^*\mathscr{E}_Y^\bullet)^\bullet \end{array}$$

The map ι is a soft resolution, as we saw earlier. An argument similar to the proof of Lemma 5.3.4 shows that ι' is also a soft resolution. Therefore the map κ induces an isomorphism in cohomology. □

Exercises

5.3.7. Check that $[\alpha] \cup [\beta] = [\alpha \wedge \beta]$ yields a well-defined product on $H^*_{dR}(X)$.

5.3.8. Prove Lemma 5.3.3.

5.3.9. Let $e(X) = \sum(-1)^i \dim H^i(X, \mathbb{R})$ denote the Euler characteristic. Prove that $e(X \times Y) = e(X)e(Y)$.

5.3.10. Show that the cohomology ring of a torus $T = (\mathbb{R}/\mathbb{Z})^n$ is isomorphic to the exterior algebra on $\mathbb{R}^n$.

5.4 Poincaré Duality

Let X be a C^∞ manifold. Let $\mathscr{E}_c^k(X)$ denote the set of C^∞ k-forms with compact support. Clearly $d\mathscr{E}_c^k(X) \subset \mathscr{E}_c^{k+1}(X)$, so these form a complex.

Definition 5.4.1. Compactly supported de Rham cohomology is defined by $H^k_{cdR}(X) = \mathscr{H}^k(\mathscr{E}_c^\bullet(X))$.

Lemma 5.4.2. *For all n,*

$$H^k_{cdR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} \text{ if } k = n, \\ 0 \text{ otherwise.} \end{cases}$$

Proof. [14, Corollary 4.7.1]. □

This computation suggests that these groups are roughly opposite to the usual de Rham groups. There is another piece of evidence, which is that H_{cdR} behaves covariantly in certain cases. For example, given an open set $U \subset X$, a form in $\mathscr{E}_c^k(U)$ can be extended by zero to $\mathscr{E}_c^k(X)$. This induces a map $H^k_{cdR}(U) \to H^k_{cdR}(X)$.

The precise statement of duality requires the notion of orientation. An *orientation* on an n-dimensional real vector space V is a connected component of $\wedge^n V - \{0\}$ (there are two). An ordered basis $v_1, \ldots, v_n$ is positively oriented if $v_1 \wedge \cdots \wedge v_n$ lies in the given component. If V were to vary, there is no guarantee that we could choose an orientation consistently. So we make a definition:

Definition 5.4.3. An n-dimensional manifold X is called orientable if $\wedge^n T_X$ minus its zero section has two components. If this is the case, an orientation is a choice of one of these components.

Theorem 5.4.4 (Poincaré duality, version I). *Let X be a connected oriented n-dimensional manifold. Then*

$$H^k_{cdR}(X) \cong H^{n-k}(X, \mathbb{R})^*.$$

There is a standard proof of this using currents, which are to forms what distributions are to functions. However, we can get by with something much weaker. We define the space of *pseudocurrents* of degree k on an open set $U \subset X$ to be

$$\mathscr{C}^k(U) = \mathscr{E}_c^{n-k}(U)^* : = \mathrm{Hom}(\mathscr{E}_c^{n-k}(U), \mathbb{R}).$$

This is "pseudo" because we are using the ordinary (as opposed to topological) dual. We make this into a presheaf as follows. Given $V \subseteq U$, $\alpha \in \mathscr{C}_X^k(U)$, $\beta \in \mathscr{E}_c^{n-k}(V)$, define $\alpha|_V(\beta) = \alpha(\tilde{\beta})$, where $\tilde{\beta}$ is the extension of β by 0.

Lemma 5.4.5. *$\mathscr{C}_X^k$ is a sheaf.*

Proof. Let $\{U_i\}$ be an open cover of U, which we may assume is locally finite. Suppose that $\alpha_i \in \mathscr{C}_X^k(U_i)$ is a collection of sections such that $\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j}$. This means that $\alpha_i(\beta) = \alpha_j(\beta)$ if β has support in $U_i \cap U_j$. Let $\{\rho_i\}$ be a C^∞ partition of unity subordinate to $\{U_i\}$ (see §4.3). Then define $\alpha \in \mathscr{C}_X^k(U)$ by

$$\alpha(\beta) = \sum_i \alpha_i(\rho_i \beta|_{U_i}).$$

We have to show that $\alpha(\tilde{\beta}) = \alpha_j(\beta)$ for any $\beta \in \mathscr{E}_c^{n-k}(U_j)$ with $\tilde{\beta}$ its extension to U by 0. The support of $\rho_i \tilde{\beta}$ lies in $U_i \cap \mathrm{supp}(\beta) \subset U_i \cap U_j$, so only finitely many of these are nonzero. Therefore

$$\alpha(\tilde{\beta}) = \sum_i \alpha_i(\rho_i \tilde{\beta}) = \sum_i \alpha_j(\rho_i \tilde{\beta}) = \alpha_j(\beta),$$

as required. We leave it to the reader to check that α is the unique current with this property. □

Define a map $\delta : \mathscr{C}_X^k(U) \to \mathscr{C}_X^{k+1}(U)$ by $\delta(\alpha)(\beta) = (-1)^{k+1}\alpha(d\beta)$. One automatically has $\delta^2 = 0$. Thus we have a complex of sheaves.

Let X be an oriented n-dimensional manifold. Then we will recall [109] that one can define an integral $\int_X \alpha$ for any n-form $\alpha \in \mathscr{E}_c^n(X)$. Using a partition of unity, the definition can be reduced to the case that α is supported in a coordinate neighborhood U. Then we can write $\alpha = f(x_1, \ldots, x_n)dx_1 \wedge \cdots \wedge dx_n$, where the order of the coordinates is chosen so that $\partial/\partial x_1, \ldots, \partial/\partial x_n$ gives a positive orientation of T_X. Then

$$\int_X \alpha = \int_{\mathbb{R}^n} f(x_1, \ldots, x_n)dx_1 \cdots dx_n.$$

The functional $\int_X$ defines a canonical global section of $\mathscr{C}_X^0$.

Theorem 5.4.6 (Stokes's theorem). *Let X be an oriented n-dimensional manifold; then $\int_X d\beta = 0$.*

Proof. See [109]. □

Corollary 5.4.7. $\int_X \in \ker[\delta]$.

We define a map $\mathbb{R}_X \to \mathscr{C}_X^0$ by sending r to $r\int_X$. The key lemma to establish Theorem 5.4.4 is the following:

Lemma 5.4.8.
$$0 \to \mathbb{R}_X \to \mathscr{C}_X^0 \to \mathscr{C}_X^1 \to \cdots$$
is an acyclic resolution.

Proof. Lemma 5.4.2 implies that this complex is exact. Given $f \in C^\infty(U)$ and $\alpha \in \mathscr{C}^k(U)$, define
$$f\alpha(\beta) = \alpha(f\beta).$$
This makes $\mathscr{C}^k$ into a C^∞-module, and it follows that it is soft and therefore acyclic. □

We can now prove Theorem 5.4.4.

Proof. We can use the complex $\mathscr{C}_X^\bullet$ to compute the cohomology of $\mathbb{R}_X$ to obtain
$$H^i(X,\mathbb{R}) \cong \mathscr{H}^i(\mathscr{C}_X^\bullet(X)) = \mathscr{H}^i(\mathscr{E}_c^{n-\bullet}(X)^*).$$
The right-hand space is isomorphic to $H^i_{cdR}(X,\mathbb{R})^*$. This completes the proof of the theorem. □

Corollary 5.4.9. *If X is a compact oriented n-dimensional manifold, then*
$$H^k(X,\mathbb{R}) \cong H^{n-k}(X,\mathbb{R})^*.$$

The following is really a corollary of the proof.

Corollary 5.4.10. *If X is a connected oriented n-dimensional manifold, then the map $\alpha \mapsto \int_X \alpha$ induces an isomorphism*
$$\int_X : H^n_{cdR}(X,\mathbb{R}) \cong \mathbb{R}.$$

We can make the Poincaré duality isomorphism more explicit:

Theorem 5.4.11 (Poincaré duality, version II). *If $f \in H^{n-k}_{cdR}(X)^*$, then there exists a closed form $\alpha \in \mathscr{E}^k(X)$ such that $f([\beta]) = \int_X \alpha \wedge \beta$. Moreover, the class $[\alpha] \in H^k_{dR}(X)$ is unique.*

Proof. Define
$$P : \mathscr{E}_X^k \to \mathscr{C}_X^k$$
by
$$P(\alpha)(\beta) = \int_U \alpha \wedge \beta$$
for $\alpha \in \mathscr{E}^k(U)$ and $\beta \in \mathscr{E}_c^{n-k}(U)$. With the help of Stokes's theorem, we see that $\delta P(\alpha) = P(d\alpha)$. Therefore, P gives a morphism of complexes of sheaves. Note also that $P(1) = \int_X$. Thus we have a morphism of resolutions

$$\begin{array}{ccc} \mathbb{R}_X & \to & \mathscr{E}_X^\bullet \\ || & & \downarrow \\ \mathbb{R}_X & \to & \mathscr{C}_X^\bullet \end{array}$$

So the theorem follows from Exercise 5.1.8. □

Corollary 5.4.12. *The cup product (induced by $\wedge$) followed by integration gives a nondegenerate pairing*

$$H_{dR}^k(X) \times H_{cdR}^{n-k}(X) \to H_{cdR}^n(X) \cong \mathbb{R}.$$

Here is a simple example to illustrate this.

Example 5.4.13. Consider the torus $T = \mathbb{R}^n/\mathbb{Z}^n$. We will show later, in Section 8.2, that every de Rham cohomology class on T contains a unique form with constant coefficients. This will imply that there is an algebra isomorphism $H^*(T,\mathbb{R}) \cong \wedge^*\mathbb{R}^n$. Poincaré duality becomes the standard isomorphism

$$\wedge^k\mathbb{R}^n \cong \wedge^{n-k}\mathbb{R}^n.$$

Exercises

5.4.14. Prove that the Euler characteristic $\sum(-1)^i \dim H^i(X,\mathbb{R})$ is zero when X is an odd-dimensional compact orientable manifold.

5.4.15. If X is a connected oriented n-dimensional manifold, show that

$$H^n(X,\mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } X \text{ is compact,} \\ 0 & \text{otherwise.} \end{cases}$$

5.4.16.(a) Let $S^2 \subset \mathbb{R}^3$ denote the unit sphere. Show that

$$\alpha = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$$

generates $H_{dR}^2(S^2)$.

(b) The real projective plane is defined by $\mathbb{RP}^2 = S^2/i$, where $i(x,y,z) = -(x,y,z)$. This is a compact manifold. Show that $H_{dR}^2(\mathbb{RP}^2) = 0$ by identifying it with the i^*-invariant part of $H_{dR}^2(S^2)$, and conclude that it cannot be orientable.

5.4.17. Assuming the exercises of §4.3, prove that $H_c^i(X,\mathbb{R}) \cong H_{cdR}^i(X)$.

5.5 Gysin Maps

Let $f : Y \to X$ be a C^∞ map of compact oriented manifolds of dimension m and n respectively. Then we have a natural map

$$f^*: H^k_{dR}(X) \to H^k_{dR}(Y)$$

given by pulling back forms. By Poincaré duality, we can identify this with a map

$$H^{n-k}_{dR}(X)^* \to H^{m-k}_{dR}(Y)^*.$$

Dualizing and changing variables yields a map in the opposite direction,

$$f_!: H^k_{dR}(Y) \to H^{k+n-m}_{dR}(X),$$

called the Gysin homomorphism. This is characterized by

$$\int_X f_!(\alpha) \cup \beta = \int_Y \alpha \cup f^*(\beta). \tag{5.5.1}$$

Our goal is to give a more explicit description of this map. Notice that we can factor f as the inclusion of the graph $Y \to Y \times X$ given by $y \mapsto (y, f(y))$, followed by a projection $Y \times X \to X$. Therefore we only need to study what happens in these two special cases.

5.5.1 Projections

Suppose that $Y = X \times Z$ is a product of compact connected oriented manifolds. Let $p: Y \to X$ and $q: Y \to Z$ be the projections. Let $r = m - n = \dim Z$. Choose local coordinates $x_1, \ldots, x_n$ on X and $z_1, \ldots, z_r$ on Z. *Integration along the fiber* $\int_p : \mathscr{E}^k(Y) \to \mathscr{E}^{k-r}(X)$ is defined in local coordinates by

$$\sum f_{i_1,\ldots,i_{k-n}}(x_1,\ldots,x_n,z_1,\ldots,z_r)dz_1 \wedge \cdots \wedge dz_n \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{k-r}} \mapsto$$
$$\sum \left(\int f_{i_1,\ldots,i_{k-n}}(x_1,\ldots,x_n,z_1,\ldots,z_r)dz_1 \cdots dz_n \right) dx_{i_1} \wedge \cdots \wedge dx_{i_{k-r}}.$$

Note that $\int_p \alpha = 0$ if none of its terms contains $dz_1 \wedge \cdots \wedge dz_n$.

Lemma 5.5.1. *$p_!\alpha$ is represented by $\int_p \alpha$.*

Proof. Fubini's theorem in calculus gives

$$\int_Y \alpha \wedge p^*\beta = \int_X \left(\int_p \alpha \right) \wedge \beta, \tag{5.5.2}$$

so that $\int_p$ satisfies (5.5.1). □

The cohomology of Y is the tensor product of the cohomology of X and Z by the Künneth formula, Theorem 5.3.6. The Gysin map $p_!$ is simply the projection onto one of the Künneth factors,

$$H^k_{dR}(Y) \to H^{k+n-m}_{dR}(X) \otimes H^{m-n}_{dR}(Z) \cong H^{k+n-m}_{dR}(X).$$

5.5.2 Inclusions

Now suppose that $i : Y \hookrightarrow X$ is an inclusion of a closed submanifold. We need the following:

Theorem 5.5.2. *There exists an open neighborhood* Tube_Y *of* Y *in* X*, called a tubular neighbourhood. This possesses a* C^∞ *map* $\pi : \mathrm{Tube}_Y \to Y$ *that makes* Tube_Y *a locally trivial bundle over* Y*, with fibers diffeomorphic to* $\mathbb{R}^{n-m}$.

Proof. The details can be found in [110, Chapter 9, addendum]. However, we give a brief outline, since we will need to understand a bit about the construction later on. We choose a Riemannian metric on X. This amounts to a family of inner products on the tangent spaces of X; among other things, this allows one to define the length of a curve. The Riemannian distance between two points is the infimum of the lengths of curves joining the points. This is a metric in the sense of point set topology. Tube_Y is given by the set of points with Riemannian distance less than ε from Y for $0 < \varepsilon \ll 1$. In order to see the bundle structure, we give an alternative description. We can take the normal bundle N to be the fiberwise orthogonal complement to the tangent bundle T_Y in $T_X|_Y$. N inherits a Riemannian metric, and we let $\mathrm{Tube}'_Y \subset N$ be the set of points of distance less than ε from the zero section. Given a point $(y,v) \in N$, let $\gamma_{y,v}(t)$ be the geodesic emanating from y with velocity v. Then the map $(y,v) \mapsto \gamma_{y,v}(1)$ defines a diffeomorphism from Tube'_Y to Tube_Y. □

Then the map i^* can be factored as

$$H^{m-k}_{dR}(X) \to H^{m-k}_{dR}(\mathrm{Tube}_Y) \xrightarrow{\sim} H^{m-k}_{dR}(Y).$$

The second map is an isomorphism, since the fibers of π are contractible. Dualizing, we see that $i_!$ is a composition of

$$H^{k}_{dR}(Y) \xrightarrow{\sim} H^{k+n-m}_{cdR}(\mathrm{Tube}_Y) \to H^{k+n-m}_{dR}(X).$$

The first map is called the *Thom isomorphism*. The second map can be seen to be extension by zero. To get more insight into this, let $k = 0$. Then $H^0_{dR}(Y)$ has a natural generator, which is the constant function 1_Y with value 1. Under the Thom isomorphism, this maps to a class $\tau_Y \in H^{n-m}_{cdR}(\mathrm{Tube}_Y)$, called the *Thom class*. This can be represented by (any) differential form with compact support in Tube_Y, which integrates to 1 on the fibers of π. The Thom isomorphism is given explicitly by $\alpha \mapsto \pi^*\alpha \cup \tau_Y$. So to summarize:

Lemma 5.5.3. $i_!\alpha$ *is the extension of* $\pi^*\alpha \cup \tau_Y$ *to* X *by zero.*

Exercises

5.5.4. Let $j : U \to X$ be the inclusion of an open set in a connected oriented manifold. Check that the Poincaré dual of the restriction map $j^* : H^*_{dR}(X) \to H^*_{dR}(U)$ is given by extension by zero.

5.5.5. Let $\pi : \mathrm{Tube}_Y \to Y$ be a tubular neighborhood for $i : Y \hookrightarrow X$ as above. Prove that $i^*\beta = \int_\pi \tau_Y \cup \beta$ for $\beta \in \mathscr{E}^\bullet(X)$, where $\int_\pi$ is defined as above.

5.5.6. With the help of the previous exercise and (the appropriate extension of) (5.5.2), finish the proof of Lemma 5.5.3.

5.5.7. Prove the projection formula $f_!(f^*(\alpha) \cup \beta) = \alpha \cup f_!\beta$.

5.6 Fundamental Class

We can use Gysin maps to construct interesting cohomology classes. Let $i : Y \hookrightarrow X$ be a closed connected oriented m-dimensional submanifold of an n-dimensional oriented manifold.

Definition 5.6.1. The fundamental class of Y in X is $[Y] = i_! 1_Y \in H^{n-m}_{dR}(X)$.

Equivalently, $[Y]$ is the extension of τ_Y by zero. Under the duality isomorphism, $H^0_{dR}(Y) \cong H^m_{dR}(Y)^*$, 1 goes to the functional

$$\beta \mapsto \int_Y \beta,$$

and this maps to

$$\alpha \mapsto \int_Y i^*\alpha$$

in $H^m(X)^*$. Composing this with the isomorphism $H^m(X)^* \cong H^{n-m}(X)$ yields the basic relation

$$\int_Y i^*\alpha = \int_X [Y] \cup \alpha. \tag{5.6.1}$$

Let $Y, Z \subset X$ be oriented submanifolds such that $\dim Y + \dim Z = n$. Then under the duality isomorphism, $[Y] \cup [Z] \in H^n(X, \mathbb{R}) \cong \mathbb{R}$ corresponds to a number $Y \cdot Z$, called the *intersection number*. This has a geometric interpretation that we give under an extra transversality assumption that holds "most of the time." We say that Y and Z are *transverse* if $Y \cap Z$ is finite and if $T_{Y,p} \oplus T_{Z,p} = T_{X,p}$ for each p in the intersection.

Definition 5.6.2. Let Y and Z be transverse, and let $p \in Y \cap Z$. Choose ordered bases $v_1(p), \ldots, v_m(p) \in T_{Y,p}$ and $v_{m+1}(p), \ldots, v_n(p) \in T_{Z,p}$ that are positively oriented with respect to the orientations of Y and Z. The local intersection number at p is

$$i_p(Y,Z) = \begin{cases} +1 & \text{if } v_1(p), \ldots, v_m(p), v_{m+1}(p), \ldots, v_n(p) \\ & \quad \text{is a positively oriented basis of } T_{X,p}, \\ -1 & \text{otherwise.} \end{cases}$$

(This is easily seen to be independent of the choice of bases.)

Proposition 5.6.3. *If Y and Z are transverse, then $Y \cdot Z = \sum_{p \in Y \cap Z} i_p(Y,Z)$.*

Proof. Let $m = \dim Y$. Let U_p be a collection of disjoint coordinate neighborhoods for each $p \in Y \cap Z$. (Note that these U_p will be replaced by smaller neighborhoods whenever necessary.) Choose coordinates $x_1, \dots, x_n$ around each p such that Y is given by $x_{m+1} = \cdots = x_n = 0$ and Z by $x_1 = \cdots = x_m = 0$. Next construct suitable tubular neighborhoods $\pi : T \to Y$ of Y and $\pi' : T' \to Z$ of Z. Recall that these neighborhoods depend on a choice of Riemannian metric and radii $\varepsilon, \varepsilon'$. By choosing the radii small enough, we can guarantee that $T \cap T'$ lies in the union $\bigcup U_p$. Also, by modifying the metric to be Euclidean near each p, we can assume that π is locally the projection $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_m)$, and likewise for π'.

Then with the above assumptions,

$$Y \cdot Z = \int_X \tau_Y \wedge \tau_Z = \sum_p \int_{U_p} \tau_Y \wedge \tau_Z,$$

where τ_Y and τ_Z are forms representing the Thom classes of T and T'. We will assume that U_p is a ball and hence diffeomorphic to $\mathbb{R}^n$. We can view $\tau_Y|_{U_p}$ as defining a class in

$$H^0_{dR}(\mathbb{R}^m) \otimes H^{n-m}_{cdR}(\mathbb{R}^{n-m}) \cong H^{n-m}_{cdR}(\mathbb{R}^{n-m})$$

and similarly for $\tau_Z|_{U_p}$. Thus we can see that

$$\begin{aligned} \tau_Y|_{U_p} &= f(x_{m+1}, \dots, x_n) dx_{m+1} \wedge \cdots \wedge dx_n + d\eta, \\ \tau_Z|_{U_p} &= g(x_1, \dots, x_m) dx_1 \wedge \cdots \wedge dx_m + d\xi, \end{aligned}$$

with f and g compactly supported such that

$$\int_{\mathbb{R}^{n-m}} f(x_{m+1}, \dots, x_n) dx_{m+1} \cdots dx_n = \int_{\mathbb{R}^m} g(x_1, \dots, x_m) dx_1 \cdots dx_m = 1.$$

Fubini's theorem and Stokes's theorem then give

$$\int_{U_p} \tau_Y \wedge \tau_Z = i_p(Y,Z). \qquad \square$$

The proposition implies that the intersection number is an integer for transverse intersections. In fact, this is always true. There are a couple of ways to see this. One is by proving that intersections can always be made transverse without altering the intersection numbers. A simpler explanation is that the fundamental classes can actually be defined to take values in integral cohomology $H^*(X, \mathbb{Z})$. Moreover, we have a cup product pairing as indicated:

$$H^k(X, \mathbb{Z}) \times H^{n-k}(X, \mathbb{Z}) \to H^n(X, \mathbb{Z}) \cong \mathbb{Z}.$$

The classes that we have defined are images under the natural map $H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{R})$.

Example 5.6.4. Let $T = \mathbb{R}^n/\mathbb{Z}^n$, let $\{e_i\}$ be the standard basis of $\mathbb{R}^n$, and let x_i be coordinates on $\mathbb{R}^n$. If $V_I \subset \mathbb{R}^n$ is the span of $\{e_i \mid i \in I\}$, then $T_I = V_I/(\mathbb{Z}^n \cap V_I)$ is a submanifold of T. Its fundamental class is $dx_{i_1} \wedge \cdots \wedge dx_{i_d}$, where $i_1 < \cdots < i_d$ are the elements of I in increasing order. If J is the complement of I, then T_I and T_J will meet transversally at one point. Therefore $T_I \cdot T_J = \pm 1$.

Example 5.6.5. Consider complex projective space $\mathbb{P}^n_{\mathbb{C}}$. This is the basic example for us, and it will be studied further in Section 7.2. For now we just state the main results. We have

$$H^i_{dR}(\mathbb{P}^n_{\mathbb{C}}) = \begin{cases} \mathbb{R} \text{ if } 0 \le i \le 2n \text{ is even,} \\ 0 \text{ otherwise.} \end{cases}$$

Given a complex subspace $V \subset \mathbb{C}^{n+1}$, the subset $\mathbb{P}(V) \subset \mathbb{P}^n$ consisting of lines lying in V forms a submanifold, which can be identified with another projective space. If W is another subspace with $\dim W = n - \dim V + 2$ and $\dim(V \cap W) = 1$, then $\mathbb{P}(V)$ and $\mathbb{P}(W)$ will meet transversally at one point. In this case, $\mathbb{P}(V) \cdot \mathbb{P}(W)$ is necessarily $+1$ (see the exercises).

Exercises

5.6.6. Show that if $Y, Z \subset X$ are transverse complex submanifolds of a complex manifold, then $i_p(Y,Z) = 1$ for each p in the intersection. Thus $Y \cdot Z$ is the number of points of intersection.

5.6.7. Check that fundamental classes of subtori of $\mathbb{R}^n/\mathbb{Z}^n$ are described as above.

5.6.8. Let $T = \mathbb{R}^2/\mathbb{Z}^2$ and let $V, W \subset \mathbb{R}^2$ be distinct lines with rational slope. Show that the images of V and W in T are transverse. Find an interpretation for their intersection number.

5.7 Lefschetz Trace Formula

Let X be a compact n-dimensional oriented manifold with a C^∞ map $f : X \to X$. The Lefschetz formula is a formula for the number of fixed points counted appropriately. This needs to be explained. Let

$$\begin{aligned} \Gamma_f &= \{(x, f(x)) \mid x \in X\}, \\ \Delta &= \{(x,x) \mid x \in X\}, \end{aligned}$$

be the graph of f and the diagonal respectively. These are both n-dimensional submanifolds of $X \times X$ that intersect precisely at points (x,x) with $x = f(x)$. We define the “number of fixed points” as $\Gamma_f \cdot \Delta$. Since this number could be negative, we need to take this with a grain of salt. If these manifolds are transverse, we see that this can be evaluated as the sum of local intersection numbers over fixed points,

$$\sum_x i_x(\Gamma_f, \Delta),$$

by Proposition 5.6.3. In particular, $\Gamma_f \cdot \Delta$ is the true number of fixed points if each local intersection number is $+1$.

Theorem 5.7.1. *The number $\Gamma_f \cdot \Delta$ is given by*

$$L(f) = \sum_p (-1)^p \operatorname{trace}[f^* : H^p(X, \mathbb{R}) \to H^p(X, \mathbb{R})].$$

Proof. The proof will be based on the elementary observation that if F is an endomorphism of a finite-dimensional vector space with basis $\{v_i\}$ and dual basis $\{v_i^*\}$, then the matrix is given by $(v_i^*(F(v_j)))$. Therefore

$$\operatorname{trace}(F) = \sum_i v_i^*(F(v_i)).$$

For each p, choose a basis $\alpha_{p,i}$ of $H^p(X)$, and let $\alpha_{p,i}^*$ denote the dual basis transported to $H^{n-p}(X)$ under the Poincaré duality isomorphism $H^{n-p}(X) \cong H^p(X)^*$, so that

$$\int_X \alpha_{p,i} \cup \alpha_{p,j}^* = \delta_{ij}.$$

Let $\pi_i : X \times X \to X$ denote the projections. Then by Künneth's formula, $\{A_{p,i,j} = \pi_1^* \alpha_{p,i} \cup \pi_2^* \alpha_{p,j}^*\}_{p,i,j}$ and $\{A_{p,i,j}^* = (-1)^{n-p} \pi_1^* \alpha_{p,i}^* \cup \pi_2^* \alpha_{p,j}\}$ both give bases for $H^n(X \times X)$, which are dual to this in the sense that

$$\int_{X \times X} A_{p,i,j} \cup A_{p',i',j'}^* = \delta_{(p,i,j),(p',i',j')}.$$

Thus we can express

$$[\Delta] = \sum c_{p,i,j} A_{p,i,j}.$$

The coefficients can be computed by integrating against the dual basis:

$$c_{p,i,j} = \int_{X \times X} [\Delta] \cup A_{p,i,j}^* = \int_\Delta A_{p,i,j}^* = (-1)^{n-p} \int_X \alpha_{p,i} \cup \alpha_{p,j}^* = (-1)^{n-p} \delta_{ij}.$$

Therefore

$$[\Delta] = \sum_{i,p} (-1)^{n-p} \pi_1^* \alpha_{p,i} \cup \pi_2^* \alpha_{p,i}^*. \tag{5.7.1}$$

Consequently,

$$
\begin{aligned}
\Gamma_f \cdot \Delta &= \int_{\Gamma_f} [\Delta] \\
&= \sum_p (-1)^{n-p} \sum_i \int_{\Gamma_f} \pi_1^* \alpha_{p,i} \cup \pi_2^* \alpha_{p,i}^* \\
&= \sum_p (-1)^{n-p} \sum_i \int_X \alpha_{p,i} \cup f^* \alpha_{p,i}^* \\
&= \sum_p (-1)^{n-p} \operatorname{trace}[f^* : H^{n-p}(X,\mathbb{R}) \to H^{n-p}(X,\mathbb{R})] \\
&= L(f).
\end{aligned}
$$

□

Corollary 5.7.2. *If $L(f) \neq 0$, then f has a fixed point.*

Proof. If $\Gamma_f \cap \Delta = \emptyset$, then $\Gamma_f \cdot \Delta = 0$. □

Corollary 5.7.3. *$\Delta \cdot \Delta$ is the Euler characteristic $e(X)$.*

Exercises

5.7.4. We say that two C^∞ maps $f, g : X \to Y$ between manifolds are homotopic if there exists a C^∞ map $h : X \times \mathbb{R} \to Y$ such that $f(x) = h(x,0)$ and $g(x) = h(x,1)$. Using Exercise 5.2.9, show that $f^* = g^*$ if f and g are homotopic. Conclude that g has a fixed point if $L(f) \neq 0$.

5.7.5. Let $v(x)$ be C^∞ vector field on a compact manifold X. By the existence and uniqueness theorem for ordinary differential equations, there is an $\varepsilon > 0$ such that for each x there is unique curve $\gamma_x : [0,\varepsilon] \to X$ with $\gamma_x(0) = x$ and $d\gamma_x(t) = v(\gamma_x(t))$. Moreover, the map $x \mapsto \gamma_x(\delta)$ is a diffeomorphism from X to itself for every $\delta \leq \varepsilon$. Use this to show that v must have a zero if $e(X) \neq 0$.

5.7.6. Let A be a nonsingular $n \times n$ matrix. Then it acts on $\mathbb{P}^{n-1}$ by $[v] \mapsto [Av]$, and the fixed points correspond to eigenvectors. Show that A is homotopic to the identity. Use this to show that $L(A) \neq 0$, and therefore that A has an eigenvector. Deduce the fundamental theorem of algebra from this.