Chapter 19
Analogies and Conjectures

In this final chapter, we end our story by beginning another. Although we have mostly worked over $\mathbb{C}$, and occasionally over a general algebraically closed field, algebraic geometry can be done over any field. Each field has its own character: transcendental over $\mathbb{C}$, and arithmetic over fields such as $\mathbb{Q}, \mathbb{F}_p, \ldots$. It may seem that aside from a few formal similarities, the arithmetic and transcendental sides would have very little to do with each other. But in fact they are related in deep and mysterious ways. We start by briefly summarizing the results of Weil, Grothendieck, and Deligne for finite fields. Then we return to complex geometry and prove Serre's analogue of the Weil conjecture. This result inspired Grothendieck to formulate his standard conjectures. We explain some of these along with the closely related Hodge conjecture. These are among the deepest open problems in algebraic geometry.

19.1 Counting Points and Euler Characteristics

Let $\mathbb{F}_q$ be the field with $q = p^r$ elements, where p is a prime number. Consider the algebraic closure $k = \bar{\mathbb{F}}_p = \bigcup \mathbb{F}_{q^n}$. Suppose that $X \subseteq \mathbb{P}^d_k$ is a quasiprojective variety defined over $\mathbb{F}_p$, that is, assume that the coefficients of the defining equations lie in $\mathbb{F}_p$. Let $X(\mathbb{F}_{p^n})$ be the set of points of $\mathbb{P}^N_{\mathbb{F}_{p^n}}$ satisfying the equations defining X. Let $N_n(X)$ be the number of points of $X(\mathbb{F}_{p^n})$. Here are a few simple computations:

Example 19.1.1. $N_n(\mathbb{A}^m_{\mathbb{F}_p}) = p^{nm}$.

Example 19.1.2. Expressing $\mathbb{P}^m = \mathbb{A}^m \cup \mathbb{A}^{m-1} \cup \cdots$ as a disjoint union yields

$$N_n(\mathbb{P}^m_{\mathbb{F}_p}) = p^{nm} + p^{n(m-1)} + \cdots + p^n + 1.$$

Example 19.1.3. $N_n((\mathbb{P}^1_{\mathbb{F}_p})^m) = (p^n + 1)^m$.

The last two computations are based on the following obvious properties.

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(1) Additivity:
$$N_n(X) = N_n(X - Z) + N_n(Z)$$
whenever $Z \subset X$ is closed.
(2) Multiplicativity:
$$N_n(X \times Y) = N_n(X)N_n(Y).$$

Now let us return to complex geometry, so that $k = \mathbb{C}$. The Euler characteristic with respect to compactly supported cohomology is

$$\chi_c(X) = \sum (-1)^i \dim H_c^i(X, \mathbb{R}).$$

It is not difficult to compute this number in the above examples using the techniques from the earlier chapters:

$$\chi_c(\mathbb{A}^m_{\mathbb{C}}) = 1, \quad \chi_c(\mathbb{P}^m_{\mathbb{C}}) = m, \quad \chi_c((\mathbb{P}_{\mathbb{C}})^m) = 2^m.$$

This leads to the following curious observation that *if we set $p = 1$ in the above formulas, then we get χ_c*. Is there a deeper reason for this? First note that although we defined compactly supported cohomology using differential forms in Section 5.4, there is a purely topological definition that works for any locally compact Hausdorff space. We can set
$$H_c^i(X) = H^i(\bar{X}, \bar{X} - X)$$
for any (or some) compactification $\bar{X}$. Then (7.2.1) and a little diagram chasing yields the long exact sequence

$$\cdots \to H_c^i(X) \to H_c^i(X) \to H_c^i(X - Z) \to H_c^{i+1}(X) \to \cdots. \tag{19.1.1}$$

The first clue that there is a deeper relation between N_n and χ_c is the following.

Lemma 19.1.4. *The invariant χ_c is additive and multiplicative i.e.,*

$$\chi_c(X) = \chi_c(X - Z) + \chi_c(Z),$$
$$\chi_c(X \times Y) = \chi_c(X)\chi_c(Y)$$

holds.

Proof. The additivity follows immediately from (19.1.1). The multiplicativity follows from the Künneth formula

$$H_c^i(X \times Y, \mathbb{R}) = \bigoplus_{j+l=i} H_c^j(X, \mathbb{R}) \otimes H_c^l(Y, \mathbb{R}).$$ □

Suppose we start with a complex quasiprojective algebraic variety X with a fixed embedding into $\mathbb{P}^N_{\mathbb{C}}$. If the defining equations (and inequalities) have integer coefficients, then we can reduce these modulo a prime p to get a quasiprojective variety (or more accurately scheme) X_p defined over the finite field $\mathbb{F}_p$. In less prosaic terms, we have a scheme over $\operatorname{Spec} \mathbb{Z}$, and X_p is the fiber over p. (In practice, $\mathbb{Z}$ might be replaced by something bigger such as the ring of integers of a number field, but the essential ideas are the same.) Then we can count points on X_p and compare it to $\chi_c(X)$. To avoid certain pathologies, we should take p sufficiently large.

Lemma 19.1.5. *If X is expressible as a disjoint union of affine spaces, then $N_n(X_p)$ is a polynomial in p^n for sufficiently large p. Substituting $p = 1$ yields $\chi_c(X)$.*

Proof. If $X = \bigcup \mathbb{A}^{m_i}$ is a disjoint union, then we get a similar decomposition for X_p with $p \gg 0$. Therefore $N_n(X_p) = \sum p^{nm_i}$ and $\chi_c(X) = \sum 1^{nm_i}$. □

The lemma applies to the above examples of course, as well as to the larger class of toric varieties [41, p. 103], Grassmannians, and more generally flag varieties [42, 19.1.11]. Nevertheless, most varieties do not admit such decompositions (e.g., a curve of positive genus does not). So this is of limited use.

It is worth pointing out that these days, the material of this section is usually embedded into the framework of motivic integration. A succinct introduction to this is given in [81].

Exercises

19.1.6. Let $G = \mathbb{G}(2,4)$ be the Grassmannian of two-dimensional subspaces of k^4. Calculate $N_n(G)$ over $\mathbb{F}_p$ and $\chi(G)$ over $\mathbb{C}$ and compare.

19.1.7. Generalize this to $\mathbb{G}(2,n)$.

19.2 The Weil Conjectures

We may ask whether something like Lemma 19.1.5 holds for arbitrary varieties. We start by looking at an elliptic curve, which is the simplest example where the lemma does not apply.

Example 19.2.1. Let X be the elliptic curve given by the affine equation $y^2 = x^3 - 1$. This defines a smooth curve X_p over $\mathbb{F}_p$ when $p \geq 5$. So let us analyze what happens when $p = 5$. When n is odd, $5^n - 1$ is not divisible by 3. This implies that $x \mapsto x^3$ is an automorphism of $\mathbb{F}_{5^n}^*$. Therefore $y^2 + 1$ has a unique cube root. Thus $N_n(X_5) = 1 + 5^n$ if n is odd. When n is even, we can compute a few values by brute force on a machine.

p^n	N_n
5^2	$36 = 1 + 5^2 + 2 \cdot 5$
5^4	$576 = 1 + 5^4 - 2 \cdot 5^2$
5^6	$15876 = 1 + 5^6 + 2 \cdot 5^3$
5^8	$389376 = 1 + 5^8 - 2 \cdot 5^4$
5^{10}	$9771876 = 1 + 5^{10} + 2 \cdot 5^5$

So at least empirically, we have the formula

$$N_n(X_5) = \begin{cases} 1+5^n & \text{if } n \text{ odd,} \\ 1+5^n-2(-5)^{n/2} & \text{if } n \text{ even,} \end{cases}$$
$$= 1+5^n-(\sqrt{-5})^n-(-\sqrt{-5})^n.$$

Example 19.2.2. Calculating the number of points for the elliptic curve defined by $y^2 = x^3 - x$ with $p = 3$, we get

p^n	N_n
3	$4 = 1+3$
3^2	$16 = 1+3^2+2\cdot 3$
3^3	$28 = 1+3^3$
3^4	$64 = 1+3^4-2\cdot 3^2$
3^5	$244 = 1+3^5$

Then

$$N_n(X_3) = 1+3^n-(\sqrt{-3})^n-(-\sqrt{-3})^n$$

fits this data.

From these and additional examples, we observe a pattern that on an elliptic curve, $N_n = 1 + p^n - \lambda_1^n - \lambda_2^n$ for appropriate constants λ_i with order of magnitude $\sqrt{p}$. We can generate more examples by taking products of these with the previous ones. Based on this data, we may guess that in general $N_n(X_p)$ is a linear combination of powers λ_i^n, and setting $\lambda_i = 1$ yields the Euler characteristic. This turns out to be correct, but it seems to come out of nowhere. We need some guiding principle to explain these formulas. The basic insight goes back to Weil [119] (who proved a number of cases). Suppose that $X \subset \mathbb{P}^d_{\mathbb{C}}$ is a nonsingular projective variety with equations defined over $\mathbb{Z}$ as above. Let us denote by $\bar{X}_p$ the variety over the algebraic closure $\bar{\mathbb{F}}_p$ determined by reducing the equations modulo p. The Frobenius morphism $F_p : \bar{X}_p \to \bar{X}_p$ is the map that raises the coordinates to the pth power (see [60, p. 301] for a more precise description). Then $X_p(\mathbb{F}_{p^n})$ are the points of $\bar{X}_p$ fixed by F_p^n. If this were a manifold with a self-map F (satisfying appropriate transversallity conditions), then we could calculate this number using the Lefschetz trace formula. Weil conjectured that this sort of argument could be carried out in the present setting for some suitable cohomology theory. He made some additional conjectures that will be discussed a bit later. Grothendieck eventually constructed such a Weil cohomology theory—in fact several. For each prime $\ell \neq p$, he constructed functors $H^i_{et}(-,\mathbb{Q}_\ell)$ called ℓ-adic cohomology such that:

(1) $H^i_{et}(\bar{X}_p,\mathbb{Q}_\ell)$ is a vector space over the field of ℓ-adic numbers $\mathbb{Q}_\ell = (\varprojlim \mathbb{Z}/\ell^n) \otimes \mathbb{Q}$.
(2) $\dim H^i_{et}(\bar{X}_p,\mathbb{Q}_\ell)$ is the usual ith Betti number of X.
(3) F_p acts on these spaces. The action on $H^0_{et}(\bar{X}_p,\mathbb{Q}_\ell)$ is trivial, but nontrivial in general.

(4) There is a Lefschetz trace formula that implies that

$$N_n(X_p) = \sum_i (-1)^i \operatorname{trace}[F_p^{n*} : H^i_{et}(\bar{X}_p, \mathbb{Q}_\ell) \to H^i_{et}(\bar{X}_p, \mathbb{Q}_\ell)].$$

Grothendieck constructed this by generalizing sheaf cohomology. The details, which are quite involved, can be found in the books by Freitag and Kiehl [39] or Milne [85]. The last formula can be rewritten as

$$N_n(X_p) = \sum_i (-1)^i \sum \lambda_{ij}^n,$$

where λ_{ij} are the generalized eigenvalues of F_p^* on $H^i_{et}(\bar{X}_p, \mathbb{Q}_\ell)$. In the previous examples, the numbers $\pm\sqrt{-5}, \pm\sqrt{-3}$ above were precisely the eigenvalues of F_p acting on H^1. Although this is overkill, we can also use this formalism to re-prove the formulas of the last section. For example,

$$H^i_{et}(\mathbb{P}^m_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell \text{ with } F_p \text{ acting by } p^{i/2} & \text{if } i < 2m \text{ is even,} \\ 0 & \text{otherwise,} \end{cases} \tag{19.2.1}$$

gives the formula for $N_n(\mathbb{P}^m)$. Notice that the absolute values of the eigenvalues in these examples have very specific sizes. This is consistent with a deep theorem of Deligne proving the last of Weil's conjectures on the analogue of the Riemann hypothesis. (For more background and in particular what this has to do with the Riemann hypothesis, see [39], [60, Appendix C], [69], [85] and of course [25].)

Theorem 19.2.3 (Deligne). *Let X be a smooth and projective variety defined over $\mathbb{F}_p$, and $\bar{X}$ its extension to $\bar{\mathbb{F}}_p$. Then the eigenvalues of the Frobenius action on $H^i_{et}(\bar{X}, \mathbb{Q}_\ell)$ are algebraic numbers λ all of whose absolute values satisfy $|\lambda| = p^{i/2}$.*

Remark 19.2.4. This is valid over any finite field. The word "eigenvalue" above really means generalized eigenvalue, although in fact the action of F_p has been conjectured to be diagonalizable. This is still wide open.

This abstract theorem has concrete consequences. The first goes from topology to number theory, and the second goes in the opposite direction.

Corollary 19.2.5. *Let $X \subset \mathbb{P}^{N+1}$ be a smooth degree-d hypersurface defined over $\mathbb{F}_p$. Then*

$$|N_n(X) - (1 + p + \cdots + p^N)| \le b_N \cdot p^{N/2},$$

where b_N is the Nth Betti number of a smooth degree-d hypersurface in $\mathbb{P}^n_{\mathbb{C}}$.

Proof. We can assume that X comes from a hypersurface over $\mathbb{C}$ by reducing modulo a prime. By the weak Lefschetz theorem, $H^i(X, \mathbb{Q}) \cong H^i(\mathbb{P}^{N+1}, \mathbb{Q})$ for $i \in [0, 2N] - \{N\}$. So the Betti numbers of X and $\mathbb{P}^{N+1}$ are the same in this range. In fact, the action of F_p would be compatible with this isomorphism. Therefore eigenvalues would be the same in both spaces for $i \in [0, 2N] - \{N\}$. For $\mathbb{P}^{N+1}$, these are given in (19.2.1). Let λ_{jN} denote the eigenvalues on the Nth cohomology of X. Then

$$|N_1(X) - (1 + p + \cdots + p^N)| \leq |\sum_{j=1}^{b_N} \lambda_{jN}| \leq b_N \cdot p^{N/2}. \qquad \square$$

Corollary 19.2.6. *If X is determined by reducing a complex smooth projective variety mod $p \gg 0$ as above, the Betti numbers of the complex variety can be determined from $N_n(X)$.*

In most cases, the Betti numbers are easier to calculate than N_n. A nontrivial example in which the last corollary was usefully applied was given by Harder and Narasimhan [57].

When X is singular or open, then the above theorem is no longer true. Deligne [27] has shown that the eigenvalues can have varying sizes or weights independent of cohomological degree. Surprisingly, this has Hodge-theoretic meaning. If one counts the number of eigenvalues on $H^i_{et}(\bar{X}_p, \mathbb{Q}_\ell)$ of a given absolute value $p^{k/2}$, then this is the dimension of the weight-k quotient of the mixed Hodge structure on $H^i(X)$ that we touched on in §12.6. See [26] for a more precise summary of these results.

Exercises

19.2.7. Assuming Theorem 19.2.3, deduce the Hasse–Weil bound that if X is a smooth projective genus-g curve over $\mathbb{F}_p$, then $|N_n(X) - 1 - p^n| \leq 2gp^{n/2}$. (Of course this bound came first.)

19.3 A Transcendental Analogue of Weil's Conjecture

After this excursion into arithmetic, let us return to Hodge theory and prove an analogue of the Weil–Riemann hypothesis found by Serre [102]. To set up the analogy let us replace $\bar{X}$ above by a smooth complex projective variety Y, and F_p by an endomorphism $f: Y \to Y$. As for p, if we consider the effect of the Frobenius on $\mathbb{P}^N_{\bar{\mathbb{F}}_p}$, the pullback of $\mathcal{O}(1)$ under this map is $\mathcal{O}(p)$. To complete the analogy, we require the existence of a very ample line bundle $\mathcal{O}_Y(1)$ on Y, so that $f^*\mathcal{O}_Y(1) \cong \mathcal{O}_Y(1)^{\otimes q}$. We can take $c_1(\mathcal{O}_Y(1))$ to be the Kähler class ω. Then we have $f^*\omega = q\omega$.

Theorem 19.3.1 (Serre). *If $f: Y \to Y$ is a holomomorphic endomorphism of a compact Kähler manifold with Kähler class ω such that $f^*\omega = q\omega$ for some $q \in \mathbb{R}$, then q is an algebraic integer, $f^*: H^i(Y, \mathbb{Q}) \to H^i(Y, \mathbb{Q})$ is diagonalizable, and its eigenvalues are algebraic integers with absolute value $q^{i/2}$.*

Proof. The theorem holds for $H^{2n}(Y)$, since ω^n generates it. Note that q^n is the degree of f, which is necessarily a (rational) integer. Therefore q is an algebraic integer. By hypothesis, f^* preserves the Lefschetz decomposition (Theorem 14.1.1). Thus we can replace $H^i(Y)$ by primitive cohomology $P^i(Y)$. Recall from Corollary 14.1.4 that

$$\tilde{Q}(\alpha,\beta) = Q(\alpha, C\bar{\beta})$$

is a positive definite Hermitian form on $P^i(Y)$, where

$$Q(\alpha,\beta) = (-1)^{i(i-1)/2}\int \alpha\wedge\beta\wedge\omega^{n-i}.$$

Consider the endomorphism $F = q^{-i/2}f^*$ of $P^i(Y)$. We have

$$Q(F(\alpha),F(\beta)) = (-1)^{i(i-1)/2}q^{-n}\int f^*(\alpha\wedge\beta\wedge\omega^{n-i}) = Q(\alpha,\beta).$$

Moreover, since f^* is a morphism of Hodge structures, it preserves the Weil operator C. Therefore F is unitary with respect to $\tilde{Q}$, so its eigenvalues have norm 1. This gives the desired estimate on absolute values of the eigenvalues of f^*. □

Since f^* can be represented by an integer matrix, the set of its eigenvalues is a Galois-invariant set of algebraic integers. So we get a stronger conclusion that all Galois conjugates have absolute value $q^{i/2}$. This would imply that when $q = 1$ (e.g., if f is an automorphism) then these are roots of unity.

Exercises

19.3.2. Verify the above theorem for Y a complex torus, and $f : Y \to Y$ multiplication by a nonzero integer n, by direct calculation.

19.3.3. Show, by example, that if $f^*\omega$ is not a multiple of ω, then the eigenvalues of f^* on $H^i(Y)$ can have different absolute values.

19.4 Conjectures of Grothendieck and Hodge

Prior to Deligne's proof, Grothendieck [54] had suggested a strategy for carrying out a proof of the Weil–Riemann hypothesis similar to Serre's proof of the transcendental version. This required first establishing his *standard conjectures* [54, 71]. All but one of these conjectures are open over $\mathbb{C}$. The exception follows from the Hodge index theorem. For general fields, they are essentially all wide open. Grothendieck had also formulated his conjectures in order to construct his theory of *motives*, which gives a deeper explanation for some of the analogies between the worlds of arithmetic and complex geometry. So even though Deligne managed to prove the last of Weil's conjectures by another method, the problem of solving these conjectures is fundamental.

We want to spell out some of these conjectures in the complex case, and indicate their relation to a better-known Hodge conjecture [65]. Let X be an n-dimensional nonsingular complex projective variety. A codimension-p algebraic cycle is a finite formal sum $\sum n_i Z_i$, where $n_i \in \mathbb{Z}$ and $Z_i \subset X$ are codimension-p closed subvarieties. These form an abelian group $Z^p(X)$ of infinite rank. The first task is to cut it down to a more manageable size. Given a nonsingular $\iota : Z \hookrightarrow X$, we defined its fundamental class $[Z] = \iota_!(1) \in H^{2p}(X, \mathbb{Z})$. The fundamental class can be defined even when Z has singularities. This can be done in several ways (see [7]). A quick but nonelementary method is to use Hironaka's famous theorem [62] on resolution of singularities. This implies that there exists a smooth projective variety $\tilde{Z}$ with a birational map $\pi : \tilde{Z} \to Z$. Let $\tilde{i} : \tilde{Z} \to X$ denote the composition of π and the inclusion. Then set $[Z] = i_!(1) \in H^{2p}(X, \mathbb{Z})$.

Lemma 19.4.1. *This class is independent of the choice of resolution of singularities.*

Proof. Let $\tilde{Z}' \to Z$ be another resolution. Then by applying Hironaka's theorem to the fibered product, we see that there exists a third resolution $\tilde{Z}'' \to Z$ fitting into a commutative diagram

$$\begin{array}{ccc} \tilde{Z}'' & \xrightarrow{\psi} & \tilde{Z} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \tilde{Z}' & \xrightarrow{\pi'} & Z \end{array}$$

Then $i_!(1) = i_!(\psi_! 1) = (i \circ \psi)_!(1)$. Therefore $\tilde{Z}$ and $\tilde{Z}''$ give the same class. By symmetry, $\tilde{Z}'$ and $\tilde{Z}''$ also give the same class. □

We thus get a homomorphism $[] : Z^p(X) \to H^{2p}(X, \mathbb{Z})$ by sending $\sum n_i Z_i \mapsto \sum n_i [Z_i]$. The space of algebraic cohomology classes is given by

$$H^{2p}_{\text{alg}}(X, \mathbb{Z}) = \text{im}[Z^p(X) \to H^{2p}(X, \mathbb{Z})],$$

$$H^{2p}_{\text{alg}}(X, \mathbb{Q}) = \text{im}[Z^p(X) \otimes \mathbb{Q} \to H^{2p}(X, \mathbb{Q})].$$

We define the space of codimension-p Hodge cycles on X to be

$$H^{2p}_{\text{hodge}}(X) = H^{2p}(X, \mathbb{Q}) \cap H^{pp}(X)$$

and let $H^{2p}_{\text{hodge}}(X, \mathbb{Z})$ denote the preimage of $H^{pp}(X)$ in $H^{2p}(X, \mathbb{Z})$.

Lemma 19.4.2. $H^{2p}_{\text{alg}}(X, \mathbb{Z}) \subseteq H^{2p}_{\text{hodge}}(X, \mathbb{Z})$

Proof. It is enough to prove that the fundamental class $[Z]$ of a codimension-p subvariety is a Hodge class. Let $\tilde{Z} \to Z$ be a resolution of singularities, and let $i : \tilde{Z} \to X$ be the natural map. By Corollary 12.2.10, the map

$$i_! : H^0(\tilde{Z}) = \mathbb{Z} \to H^{2p}(X, \mathbb{Z})(-p)$$

is a morphism of Hodge structures. Therefore it takes 1 to a Hodge class.

In more down-to-earth terms, this amounts to the fact that for any form α of type $(r, 2n-2p-r)$ on X,

$$\int_{\tilde{Z}} i^*\alpha = 0$$

unless $r = n-p$. □

The *Hodge conjecture* asserts the converse.

Conjecture 19.4.3 (Hodge). $H^{2p}_{\mathrm{alg}}(X,\mathbb{Q})$ and $H^{2p}_{\mathrm{hodge}}(X,\mathbb{Q})$ coincide.

Note that in the original formulation, $\mathbb{Z}$ was used in place of $\mathbb{Q}$, but Atiyah and Hirzebruch have shown that this version is false [7]. It is also worth pointing out that Voisin [114] has shown that the Hodge conjecture (in various formulations) can fail for compact Kähler manifolds. On the positive side, we should mention that for $p = 1$, the conjecture is true — even over $\mathbb{Z}$ — by the Lefschetz $(1,1)$ theorem, Theorem 10.3.3. We prove that it holds for $p = \dim X - 1$.

Proposition 19.4.4. *If the Hodge conjecture holds for X in degree* $2p$ *(i.e., if* $H^{2p}_{\mathrm{alg}}(X,\mathbb{Q}) = H^{2p}_{\mathrm{hodge}}(X,\mathbb{Q})$*) with* $p < n = \dim X$*, then it holds in degree* $2n-2p$*.*

Proof. Let L be the Lefschetz operator corresponding to a projective embedding $X \subset \mathbb{P}^N$. Then for any subvariety Y, we have $L[Y] = [Y \cap H]$, where H is a hyperplane section chosen in general position. It follows that L^{n-2p} takes $H^{2p}_{\mathrm{alg}}(X)$ to $H^{2n-2p}_{\mathrm{alg}}(X)$. Moreover, the map is injective by hard Lefschetz. Thus

$$\dim H^{2p}_{\mathrm{hodge}}(X) = \dim H^{2p}_{\mathrm{alg}}(X) \le \dim H^{2n-2p}_{\mathrm{alg}}(X) \le \dim H^{2n-2p}_{\mathrm{hodge}}(X).$$

On the other hand, L^{n-2p} induces an isomorphism of Hodge structures $H^{2p}(X,\mathbb{Q})(p-n) \cong H^{2n-2p}(X,\mathbb{Q})$, and therefore an isomorphism $H^{2p}_{\mathrm{hodge}}(X) \cong H^{2n-2p}_{\mathrm{hodge}}(X)$. This forces equality of the above dimensions. □

Corollary 19.4.5. *The Hodge conjecture holds in degree* $2n-2$*. In particular, it holds for three-dimensional varieties.*

Given a cycle $Y \in Z^{n-p}(X)$, define the intersection number

$$Z \cdot Y = \int_X [Z] \cup [Y] \in \mathbb{Z}.$$

This can be defined by purely algebrogeometric methods over any field [42].

Definition 19.4.6. A cycle $Z \in Z^p(X)$ is said to be homologically equivalent to 0 if $[Z] = 0$. It is numerically equivalent to 0 if for any $Y \in Z^{n-p}(X)$ we have $Z \cdot Y = 0$. Two cycles are homologically (respectively numerically) equivalent if their difference is homologically (respectively numerically) equivalent to 0.

Numerical equivalence is a purely algebrogeometric notion, independent of any cohomology theory. (This is clearly an issue in positive characteristic where one has several equally good cohomology theories, such as the various ℓ-adic theories.) On the other hand, it is usually easier to prove things about homological equivalence. For example, $H^{2p}_{alg}(X)$, which is $Z^p(X)$ modulo homological equivalence, is finitely generated, since it is sits in the finitely generated group $H^{2p}(X,\mathbb{Z})$. Therefore $Z^p(X)$ modulo numerical equivalence is also finitely generated, because clearly homological equivalence implies numerical equivalence. The converse is one of Grothendieck's standard conjectures.

Conjecture 19.4.7 ("Conjecture D"). Numerical equivalence coincides with homological equivalence.

In order to explain the relationship to Hodge, we state another of Grothendieck's conjectures.

Conjecture 19.4.8 ("Conjecture A"). The Lefschetz operator induces an isomorphism on the spaces of algebraic cycles

$$L^i : H^{n-i}_{alg}(X,\mathbb{Q}) \to H^{n+i}_{alg}(X,\mathbb{Q})$$

There are several other conjectures, which we will not state. One of them, which is a version of the Hodge index theorem, is true over $\mathbb{C}$. The remaining conjectures are known to follow if Conjecture A is true for all X [54, 71]. These conjectures are weaker than Hodge's and are known in many more cases. For example, they are known for all abelian varieties, while Hodge is still open for this class.

Proposition 19.4.9. *If the Hodge conjecture holds for X, then Conjecture A will also hold for it. If Conjecture A holds for X, then conjecture D holds for it.*

Proof. By the hard Lefschetz theorem,

$$L^i : H^{n-i}(X,\mathbb{Q}) \to H^{n+i}(X,\mathbb{Q})$$

is an isomorphism of vector spaces. It follows that L^i gives an isomorphism $H^{n-i}(X) \cong H^{n+i}(X)(i)$ of Hodge structures. Therefore it induces an isomorphism on the spaces of Hodge cycles

$$H^{n-i}_{hodge}(X,\mathbb{Q}) \to H^{n+i}_{hodge}(X,\mathbb{Q}).$$

This is an isomorphism of the spaces of algebraic cycles, since we are assuming the Hodge conjecture.

By the Hodge index theorem, Theorem 14.1.4, we get a positive bilinear form Q on $H^i(X)$ given by

$$Q(\alpha,\beta) = \int_X \alpha \cup \beta',$$

where

$$\beta' = \sum \pm L^{n-i+j}\beta_j$$

with $\beta = \sum L^j \beta_j$ the decomposition into primitive parts in Theorem 14.1.1. Suppose that Conjecture A holds for X. Then we have a Lefschetz decomposition on the space of algebraic cycles

$$H^{2p}_{\text{alg}}(X) = \bigoplus L^j(P^{2p-2j}(X) \cap H^{2p-2j}_{\text{alg}}(X)).$$

Therefore $\beta' \in H^{2n-2p}_{\text{alg}}(X)$ whenever $\beta \in H^{2p}_{\text{alg}}(X)$. Suppose that $Z \in Z^p(X)$ is numerically equivalent to 0. Then $[Z] = 0$, since otherwise we get a contradiction $Z \cdot [Z]' = Q([Z],[Z]) > 0$. □

Further information about the Hodge conjecture, and related conjectures, can be found in Lewis [80]. See André [3] for an introduction to motives, which have been lurking behind the scenes. The historically minded reader will find a fascinating glimpse into how these ideas evolved in the letters of Grothendieck and Serre [21].

Exercises

19.4.10. Prove that the Hodge conjecture holds for products of projective spaces. (Hint: the Hodge conjecture is trivially true for a variety whose cohomology is spanned by algebraic cycles.)

19.4.11. Let X be an n-dimensional smooth projective variety. Another of Grothendieck's standard conjectures asserts that the components in $H^i(X,\mathbb{Q}) \otimes H^{2n-i}(X,\mathbb{Q})$ of the class of the diagonal $[\Delta] \in H^*(X \times X, \mathbb{Q})$ under the Künneth decomposition are algebraic. Show that this follows from the Hodge conjecture.

19.5 Problem of Computability

As we saw in this book, it is relatively straightforward to compute Hodge numbers. For things like hypersurfaces, we obtained formulas. More generally, given explicit equations for a subvariety $X \subset \mathbb{P}^n$, we may use the following strategy for computing Hodge numbers.

- View the sheaves Ω^p_X as coherent sheaves on $\mathbb{P}^n$. Specifically:

$$\Omega^p_X = \Omega^p_{\mathbb{P}^n}/(\mathscr{I}\Omega^p_{\mathbb{P}^n} + d\mathscr{I} \wedge \Omega^{p-1}_{\mathbb{P}^n}),$$

  where $\mathscr{I}$ is the ideal sheaf of X. These sheaves can be given an explicit presentation by combining this formula with the presentation

$$\mathscr{O}_{\mathbb{P}^n}(-p-2)^{\binom{n+1}{p+2}} \to \mathscr{O}_{\mathbb{P}^n}(-p-1)^{\binom{n+1}{p+1}} \to \Omega^p_{\mathbb{P}^n}$$

  coming from Corollary 17.1.3.

- Resolve these as in Theorem 15.3.9.
- Calculate cohomology using the resolution.

This can be turned into an algorithm using standard Gröbner basis techniques. (We are using the term "algorithm" somewhat loosely, but to really get one we should assume that the coefficients are given to us in some computable subfield of $\mathbb{C}$ such as $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{2},\pi)$.)

If one wanted to verify (or disprove) the Hodge conjecture in an example, one would run up against the following problem, which by contrast to the case of Hodge numbers seems extremely difficult:

Problem 19.5.1. Find an algorithm to compute the dimensions of $H^*_{\mathrm{alg}}(X,\mathbb{Q})$ and $H^*_{\mathrm{hodge}}(X,\mathbb{Q})$ given the equations (or some other explicit representation).

The following special case of the problem would already be interesting and probably very hard.

Problem 19.5.2. Find an algorithm for computing the Picard number of a surface in $\mathbb{P}^3$ or of a product of two curves.

We end with a few comments. Given a variety X, the Hodge structure on $H^k(X)$ is determined by the period matrices

$$P^p = \left(\int_{\gamma_i} \omega_j\right),$$

where γ_i is a basis of $H_k(X,\mathbb{Z})$ and ω_j a basis of $H^{k-p}(X,\Omega^p_X)$. When X is defined over $\overline{\mathbb{Q}}$, we should in principle be able to compute the entries of these matrices to any desired degree of accuracy, by combing symbolic methods with numerical ones. But this does not (appear to) help. However, Kontsevich and Zagier [74] propose that there may be an algorithm to determine whether such a number (which they call a period) is rational. More generally, we can ask for an algorithm for deciding whether any finite set of periods is linearly dependent over $\mathbb{Q}$. Such an algorithm would be instrumental in finding an algorithm for computing $\dim H^*_{\mathrm{hodge}}(X,\mathbb{Q})$.

Regarding $H^*_{\mathrm{alg}}(X)$, Tate [113] made a conjecture that can be loosely viewed as an arithmetic version of the Hodge conjecture, although there is even less evidence for it. Suppose that X is defined over $\overline{\mathbb{Q}}$ and in fact for simplicity $\mathbb{Z}$. We obtain varieties X_p defined by reducing X mod p. These are smooth for all but finitely many p. Given an algebraic cycle $Z \in Z^i(X)$ on X defined over $\mathbb{Z}$ (but this is not essential), we get induced cycles $Z_p \in Z^i(X_p)$. We can form a fundamental class in $[Z_p] \in H^{2i}_{et}(\bar{X}_p,\mathbb{Q}_\ell) \cong H^{2i}_{et}(\bar{X},\mathbb{Q}_\ell)$, and it will be an eigenvector for the Frobenius F_p with eigenvalue exactly p^i for $p \gg 0$. Tate conjectured conversely that the dimension of the intersection of the p^ith eigenspaces of F_p, as p varies, is precisely $H^{2i}_{\mathrm{alg}}(X,\mathbb{Q})$. Even without assuming the conjecture, this should give some sort of bound on $H^{2i}_{\mathrm{alg}}(X,\mathbb{Q})$. The challenge would be to make this effective.

19.6 Hodge Theory without Analysis

As we have seen, Hodge theory has a number of important consequences for the cohomology of a smooth complex projective variety X:

(1) Hodge decomposition
$$\sum_{p+q=i} h^{pq}(X) = b_i(X).$$

(2) Hodge symmetry
$$h^{pq}(X) = h^{qp}(X).$$

(3) Kodaira vanishing
$$H^i(X, \Omega_X^n \otimes L) = 0$$
for $i > 0$, $n = \dim X$, and L an ample line bundle on X.

Thanks to GAGA, we can replace (the dimensions of) the above analytic cohomology groups by their algebraic counterparts. To be precise, for $b_i(X)$ we can use the dimension of either the hypercohomology $\dim \mathbb{H}^i(X, \Omega_X^\bullet)$ or some suitable Weil cohomology such as ℓ-adic theory. A rather natural question, which occurs for example in [53], is whether these consequences can be established directly without analysis. In particular, can these be extended to arbitrary fields? First, the bad news: the answer to the second question is in general no. Counterexamples have been constructed in positive characteristic by Mumford [90], Raynaud [96], and others. In spite of this, the first question has a positive answer. Faltings [36] gave the first entirely algebraic proof of (1). This was soon followed by an easier algebraic proof of (1) and (3) by Deligne and Illusie [29], which made surprising use of characteristic-p techniques. An explanation of their proof can be found in Esnault and Viehweg's book [35].

The only thing left is see how to prove (2) without harmonic forms. In outline, first apply the decomposition (1) and hard Lefschetz (which also has an algebraic proof [27]) to get
$$h^{pq}(X) = h^{n-q,n-p}(X).$$
Now combine this with Serre duality [60, Chapter III, Corollary 7.13],
$$h^{n-q,n-p}(X) = h^{qp}(X).$$

At this point, we should remind ourselves that Hodge theory gives much more than the items (1), (2), and (3). For instance, we have seen how to associate a canonical Hodge structure to every smooth projective variety over $\mathbb{C}$. As far as the author knows, there is no purely algebraic substitute for this. Nevertheless, we can devise the following test to see how close we can get. Suppose that X is a smooth complex projective variety defined by equations $\sum a_{i_0 \dots i_n} x_0^{i_0} \cdots x_n^{i_n} = 0$ with coefficients in $\overline{\mathbb{Q}} \subset \mathbb{C}$ (or some other algebraically closed subfield). Given $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we get a new complex variety X_σ defined by $\sum \sigma(a_{i_0 \dots i_n}) x_0^{i_0} \cdots x_n^{i_n} = 0$.

Problem 19.6.1. Now suppose that Y is another smooth projective variety defined over $\overline{\mathbb{Q}}$ such that $H^i(X,\mathbb{Q}) \cong H^i(Y,\mathbb{Q})$ as Hodge structures. Show that $H^i(X_\sigma,\mathbb{Q}) \cong H^i(Y_\sigma,\mathbb{Q})$ as Hodge structures for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

If we assumed the Hodge conjecture, we would get a solution as follows. The isomorphism $H^i(X) \cong H^i(Y)$ would give a class in $H^*_{\mathrm{hodge}}(X \times Y)$, which would be an algebraic cycle α, necessarily defined over a field $\overline{\mathbb{Q}}(t_1,\ldots,t_N)$. After specializing the t_i, we can assume that α is defined over $\overline{\mathbb{Q}}$. Then $\sigma^*(\alpha)$ would induce the desired isomorphism $H^i(X_\sigma,\mathbb{Q}) \cong H^i(Y_\sigma,\mathbb{Q})$. To make this work, we really need only the following weak form of the Hodge conjecture due to Deligne [31]:

Conjecture 19.6.2 ("Hodge cycles are absolute"). If $\alpha \in \mathbb{H}^{2p}(X,\Omega^\bullet_X)$ is a rational (p,p) class, then $\sigma^*\alpha$ is a rational (p,p) class on X_σ.

This conjecture is known in many more cases than the Hodge conjecture. Although it looks rather technical, it does have some down-to-earth applications to showing that certain natural constants are algebraic numbers.