# Chapter 12
# Hodge Structures and Homological Methods

Our next goal is to make the Hodge decomposition functorial with respect to holomorphic maps. This is not immediate, since the pullback of a harmonic form along a holomorphic map is almost never harmonic. The trick is to state things in a way that depends only on the complex structure: a cohomology class is of type $(p,q)$ if it can be represented by a form with $p$ $dz_i$'s and by a form with $q$ $d\bar{z}_j$'s. Of course, just making a definition is not enough. There is something to be proved. The main ingredients are the previous Hodge decomposition for harmonic forms together with some homological algebra, which we develop here.

Although projective manifolds are Kähler, there are examples of algebraic manifolds that are not. One benefit of this homological approach is that it will allow us to extend the decomposition to these manifolds where harmonic theory alone would be insufficient.

The articles and books by Deligne [24], Griffiths and Schmid [50], Peters and Steenbrink [95], and Voisin [115, 116] cover this material in more detail.

## 12.1 Pure Hodge Structures

It is useful to isolate the purely linear algebraic features of the Hodge decomposition. We define a *pure real Hodge structure* of weight $m$ to be a finite-dimensional complex vector space with a real structure $H_{\mathbb{R}}$, and a bigrading

$$H = \bigoplus_{p+q=m} H^{pq}$$

satisfying $\bar{H}^{pq} = H^{qp}$. We generally use the same symbol for Hodge structure and the underlying vector space. A (pure weight $m$) Hodge structure is a real Hodge structure $H$ together with a choice of a finitely generated abelian group $H_{\mathbb{Z}}$ and an isomorphism $H_{\mathbb{Z}} \otimes \mathbb{R} \cong H_{\mathbb{R}}$. Even though the abelian group $H_{\mathbb{Z}}$ may have torsion, it

D. Arapura, *Algebraic Geometry over the Complex Numbers*, Universitext,
DOI 10.1007/978-1-4614-1809-2_12, © Springer Science+Business Media, LLC 2012

is helpful to think of it as a "lattice" in $H_{\mathbb{R}}$. Rational Hodge structures are defined in a similar way.

Before continuing with the abstract development of Hodge structures, we need to ask the obvious question. Why is it useful to consider these things? More specifically, why is it useful for algebraic geometry? To answer, we observe that algebraic varieties tend to come in families. For example, we may simply allow the coefficients of the defining equations to vary. Thus varieties tend to come with natural "continuous" parameters. The cohomological invariants considered up to now are discrete. Hodge structures, however, also have continuous parameters that sometimes match those coming from geometry. The simplest example is very instructive. Start with an elliptic curve $X_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ with $\tau$ in the upper half plane $H$. We can identify $\iota : H^1(X_\tau,\mathbb{C}) \cong \mathbb{C}^2$ by mapping a closed differential form $\alpha$ to $(\int_0^1 \alpha, \int_0^\tau \alpha)$, where the paths of integration are lines. Then

$$H^{10} = \mathbb{C}\iota(dz) = \mathbb{C}(1,\tau).$$

So in this case, we recover the basic parameter $\tau$ and therefore the curve itself from its Hodge structure.

Given a pure Hodge structure, define the Hodge filtration by

$$F^p H_{\mathbb{C}} = \bigoplus_{p' \geq p} H^{pq}.$$

In many situations, the Hodge filtration is the more natural object to work with. This determines the bigrading thanks to the following lemma:

**Lemma 12.1.1.** *If $H$ is a pure Hodge structure of weight $m$, then*

$$H_{\mathbb{C}} = F^p \oplus \bar{F}^{m-p+1}$$

*for all $p$. Conversely, if $F^\bullet$ is a descending filtration satisfying $F^a = H_{\mathbb{C}}$ and $F^b = 0$ for some $a,b \in \mathbb{Z}$ and satisfying the above identity, then*

$$H^{pq} = F^p \cap \bar{F}^q$$

*defines a pure Hodge structure of weight $m$.*

The most natural examples of Hodge structures come from compact Kähler manifolds: if $H_{\mathbb{Z}} = H^i(X,\mathbb{Z})$ with the Hodge decomposition on $H^i(X,\mathbb{C}) \cong H_Z \otimes \mathbb{C}$, then we get a Hodge structure of weight $i$. It is easy to manufacture other examples. For every integer $i$, there is a rank-one Hodge structure $\mathbb{Z}(i)$ of weight $-2i$. Here the underlying space is $\mathbb{C}$, with $H^{(-i,-i)} = \mathbb{C}$ and lattice $H_{\mathbb{Z}} = (2\pi\sqrt{-1})^i\mathbb{Z}$ (these factors should be ignored on first reading). The collection of Hodge structures forms a category HS, where a morphism is a linear map $f$ preserving the lattices and the bigradings. In particular, morphisms between Hodge structures with different weights must vanish. This category has the following operations: direct sums of Hodge structures of the same weight (we will eventually relax this), and (unrestricted) tensor products and duals. Explicitly, given Hodge structures $H$ and $G$ of weights $n$ and $m$, their tensor product $H \otimes_{\mathbb{Z}} G$ is equipped with a weight-$(n+m)$ Hodge structure with bigrading

$$(H \otimes G)^{pq} = \bigoplus_{\substack{p'+p''=p \\ q'+q''=q}} H^{p'q'} \otimes G^{p''q''}.$$

If $m = n$, their direct sum $H \oplus G$ is equipped with the weight-$m$ Hodge structure

$$(H \oplus G)^{pq} = \bigoplus_{p+q=m} H^{pq} \oplus G^{pq}.$$

The dual $H^* = \mathrm{Hom}(H, \mathbb{Z})$ is equipped with a weight-$(-n)$ Hodge structure with bigrading

$$(H^*)^{pq} = (H^{-p,-q})^*.$$

The operation $H \mapsto H(i) = H \otimes \mathbb{Z}(i)$ is called the Tate twist. It has the effect of leaving $H$ unchanged and shifting the bigrading by $(-i,-i)$.

## Exercises

**12.1.2.** Show that there are no free rank-one pure Hodge structures of odd weight, and up to isomorphism a unique rank-one Hodge structure for every even weight.

**12.1.3.** Show that $\mathrm{Hom}_{\mathrm{HS}}(\mathbb{Z}(0), H^* \otimes G) \cong \mathrm{Hom}_{\mathrm{HS}}(H, G)$.

**12.1.4.** Prove Lemma 12.1.1.

**12.1.5.** Given a $g$-dimensional complex torus $T$, use Exercise 10.3.9, that $T \cong \mathrm{Alb}(T)$, to conclude that $T$ can be recovered from the Hodge structure $H = H^1(T)$.

## 12.2 Canonical Hodge Decomposition

The Hodge decomposition involved harmonic forms, so it is tied up with the Kähler metric. It is possible to reformulate it so as to make it independent of the choice of metric. Let us see how this works for a compact Riemann surface $X$. We have an exact sequence

$$0 \to \mathbb{C}_X \to \mathscr{O}_X \to \Omega^1_X \to 0,$$

and we saw in Lemma 6.2.8 that the induced map

$$H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C})$$

is injective. If we define

$$\begin{aligned} F^0H^1(X,\mathbb{C}) &= H^1(X,\mathbb{C}), \\ F^1H^1(X,\mathbb{C}) &= \mathrm{im}[H^0(X,\Omega_X^1) \to H^1(X,\mathbb{C})], \\ F^2H^1(X,\mathbb{C}) &= 0, \end{aligned}$$

then this together with the isomorphism $H^1(X,\mathbb{C}) = H^1(X,\mathbb{Z}) \otimes \mathbb{C}$ determines a pure Hodge structure of weight 1. To see this, choose a metric, which is automatically Kähler because $\dim X = 1$. Then $H^1(X,\mathbb{C})$ is isomorphic to a direct sum of the space of harmonic $(1,0)$-forms, which maps to $F^1$, and the space of harmonic $(0,1)$-forms, which maps to $\bar{F}^1$.

Before proceeding with the higher-dimensional version, we need some facts from homological algebra. Let

$$C^\bullet = \to \cdots C^a \xrightarrow{d} C^{a+1} \to \cdots$$

be a complex of vectors spaces (or modules or ...). It is convenient to allow the indices to vary over $\mathbb{Z}$, but we will require that it be *bounded below*, which means that $C^a = 0$ for all $a \ll 0$. Let us suppose that each $C^i$ is equipped with a filtration $F^pC^i \supseteq F^{p+1}C^i \supseteq \cdots$, that is preserved by $d$, i.e., $dF^pC^i \subseteq F^pC^{i+1}$. This implies that each $F^pC^\bullet$ is a subcomplex. We suppose further that $F^\bullet$ *biregular*, which means that for each $i$ there exist $a$ and $b$ with $F^aC^i = C^i$ and $F^bC^i = 0$. We get a map on cohomology

$$\phi^p : \mathscr{H}^\bullet(F^pC^\bullet) \to \mathscr{H}^\bullet(C^\bullet),$$

and we let $F^p\mathscr{H}^\bullet(C^\bullet)$ be the image. Define $Gr^p\mathscr{H}^i(C^\bullet) = F^p\mathscr{H}^i(C^\bullet)/F^{p+1}\mathscr{H}^i(C^\bullet)$. When $C^\bullet$ is a complex of vector spaces, there are noncanonical isomorphisms

$$\mathscr{H}^i(C^\bullet) = \bigoplus_p Gr^p\mathscr{H}^i(C^\bullet).$$

The filtration is said to be *strictly compatible with differentials* of $C^\bullet$, or simply just *strict*, if all the $\phi^p$'s are injective. Let $Gr_F^pC^\bullet = Gr^pC^\bullet = F^pC^\bullet/F^{p+1}C^\bullet$. Then we have a short exact sequence of complexes

$$0 \to F^{p+1}C \to F^pC \to Gr^pC \to 0,$$

from which we get a connecting map $\delta : \mathscr{H}^i(Gr^pC^\bullet) \to \mathscr{H}^{i+1}(F^{p+1}C^\bullet)$. This can be described explicitly as follows. Given $\bar{x} \in \mathscr{H}^i(Gr^pC^\bullet)$, it can be lifted to an element $x \in F^pC^i$ such that $dx \in F^{p+1}C^{i+1}$. Then $\delta(\bar{x})$ is represented by $dx$.

**Proposition 12.2.1.** *The following are equivalent:*

*(1) F is strict.*
*(2)* $F^pC^{i+1} \cap dC^i = dF^pC^i$ *for all i and p.*
*(3) The connecting maps* $\delta : \mathscr{H}^i(Gr^pC^\bullet) \to \mathscr{H}^{i+1}(F^{p+1}C^\bullet)$ *vanish for all i and p.*

*Proof.* This proof is due to Su-Jeong Kang.

(1)$\Rightarrow$ (2). Suppose that $z \in F^pC^{i+1} \cap dC^i$. Then $z = dx \in F^pC^{i+1}$ for some $x \in C^i$. Thus we have $z \in \ker[d : F^pC^{i+1} \to F^pC^{i+2}]$. Let $\bar{z} \in \mathscr{H}^{i+1}(F^pC^\bullet)$ denote the cohomology class of $z$. Note that $\phi^p(\bar{z}) = 0$, since $z = dx$. Hence from the assumption that $F$ is strict, $\bar{z} = 0$ in $\mathscr{H}^{i+1}(F^pC^\bullet)$, or equivalently $z = dy$ for some $y \in F^pC^i$. This shows that $F^pC^{i+1} \cap dC^i \subseteq dF^pC^i$. The reverse inclusion is clear.

(2) $\Rightarrow$ (3). Let $\bar{x} \in \mathscr{H}^i(Gr^pC^\bullet)$. This lifts to an element $x \in F^pC^i$ with $dx \in F^{p+1}C^{i+1}$ as above. Then from the assumption (2), we have

$$dx \in F^{p+1}C^{i+1} \cap dC^i = dF^{p+1}C^i.$$

Since $\delta(\bar{x})$ is represented by $dx$, we have $\delta(\bar{x}) = 0$ in $\mathscr{H}^{i+1}(F^{p+1}C^\bullet)$.

(3) $\Rightarrow$ (1). For each $i$, $\phi^p$ can be expressed as finite a composition

$$\mathscr{H}^i(F^pC^\bullet) \to \mathscr{H}^i(F^{p-1}C^\bullet) \to \mathscr{H}^i(F^{p-2}C^\bullet) \to \cdots .$$

These maps are all injective by assumption, since their kernels are the images of the connecting maps. $\square$

**Corollary 12.2.2.** *$Gr^p\mathscr{H}^i(C^\bullet)$ is a subquotient of $\mathscr{H}^i(Gr^pC^\bullet)$, which means that there is a diagram*

$$\mathscr{H}^i(Gr^pC^\bullet) \supseteq I^{i,p} \to Gr^p\mathscr{H}^i(C^\bullet)$$

*where the last map is onto. Isomorphisms $Gr^p\mathscr{H}^i(C^\bullet) \cong I^{i,p} \cong \mathscr{H}^i(Gr^pC^\bullet)$ hold for all $i, p$ if and only if $F$ is strict.*

*Proof.* Let $I^{i,p} = \mathrm{im}[\mathscr{H}^i(F^pC^\bullet) \to \mathscr{H}^i(Gr^pC^\bullet)]$. Then the surjection $\mathscr{H}^i(F^pC^\bullet) \to Gr^p\mathscr{H}^i(C^\bullet)$ factors through $I$. The remaining statement follows from (3) and a diagram chase. $\square$

**Corollary 12.2.3.** *Suppose that $C^\bullet$ is a complex of vector spaces over a field such that* $\dim \mathscr{H}^i(Gr^pC^\bullet) < \infty$ *for all $i, p$. Then*

$$\dim \mathscr{H}^i(C^\bullet) \le \sum_p \dim \mathscr{H}^i(Gr^pC^\bullet),$$

*and equality holds for all $i$ if and only if $F$ is strict, in which case we also have*

$$\dim F^p\mathscr{H}^i(C^\bullet) = \sum_{p' \ge p} \dim \mathscr{H}^i(Gr^{p'}C^\bullet).$$

*Proof.* We have

$$\dim \mathscr{H}^i(C^\bullet) = \sum_p \dim Gr^p\mathscr{H}^i(C^\bullet) \le \sum_p \dim I^{ip} \le \sum_p \dim \mathscr{H}^i(Gr^pC^\bullet),$$

and equality is equivalent to strictness of $F$ by the previous corollary. The last statement is left as an exercise. $\square$

These results are usually formulated in terms of spectral sequences, which we have chosen to avoid. In this language, the last corollary says that $F$ is strict if and only if the associated spectral sequence degenerates at $E_1$. This is partially explained in the exercises.

Let $X$ be a complex manifold. Then the de Rham complex $\mathscr{E}^\bullet(X)$ has a filtration called the Hodge filtration:

$$F^p\mathscr{E}^\bullet(X) = \sum_{p'\geq p} \mathscr{E}^{p'q}(X).$$

Its conjugate equals

$$\bar{F}^q\mathscr{E}^\bullet(X) = \sum_{q'\geq q} \mathscr{E}^{pq'}(X).$$

**Theorem 12.2.4.** *If $X$ is compact Kähler, the Hodge filtration is strict. The associated filtration $F^\bullet H^i(X,\mathbb{C})$, on cohomology, gives a Hodge structure*

$$H^i(X,\mathbb{Z})\otimes\mathbb{C}\cong H^i(X,\mathbb{C}) = \bigoplus_{p+q=i} H^{pq}(X)$$

*of weight i, where*

$$H^{pq}(X) = F^p\mathscr{H}^i(X,\mathbb{C})\cap\bar{F}^qH^i(X,\mathbb{C})\cong H^q(X,\Omega_X^p).$$

*Proof.* Dolbeault's theorem (Corollary 9.2.3) implies that $\mathscr{H}^q(Gr^p\mathscr{E}^\bullet(X)) = \mathscr{H}^q(\mathscr{E}_X^{(p,\bullet)}(X))$ is isomorphic to $H^q(X,\Omega_X^p)$. Therefore $F$ is strict by Corollary 12.2.3 and the Hodge decomposition. By conjugation, we see that $\bar{F}$ is also strict. Furthermore, these facts together with Corollary 12.2.3 give

$$\dim F^pH^i(X,\mathbb{C}) = h^{p,i-p}(X) + h^{p+1,i-p-1}(X) + \cdots \tag{12.2.1}$$

and

$$\dim \bar{F}^{i-p+1}H^i(X,\mathbb{C}) = h^{p-1,i-p+1}(X) + h^{p-2,i-p+2}(X) + \cdots. \tag{12.2.2}$$

A cohomology class lies in $F^pH^i(X,\mathbb{C})$ (respectively $\bar{F}^{i-p+1}H^i(X,\mathbb{C})$) if and only if it can be represented by a form in $F^p\mathscr{E}^\bullet(X)$ (respectively $\bar{F}^{i-p+1}\mathscr{E}^\bullet(X)$). Thus $H^i(X,\mathbb{C})$ is the sum of these subspaces. Using (12.2.1) and (12.2.2), we see that it is a direct sum. Therefore the filtrations determine a pure Hodge structure of weight $i$ on $H^i(X,\mathbb{C})$. □

Even though harmonic theory is needed to verify that this Hodge structure, it should be clear that it involves only the holomorphic structure and not the metric. Thus we have obtained a canonical Hodge decomposition. The word canonical is really synonymous with functorial:

**Corollary 12.2.5.** *If $f : X \to Y$ is a holomorphic map of compact Kähler manifolds, then the pullback map $f^* : H^i(Y,\mathbb{Z}) \to H^i(X,\mathbb{Z})$ is compatible with the Hodge structures.*

**Corollary 12.2.6.** *If $X$ is compact Kähler, the maps*

$$H^q(X,\Omega_X^p) \to H^q(X,\Omega_X^{p+1})$$

*induced by differentiation vanish. In particular, global holomorphic differential forms on $X$ are closed.*

*Proof.* This follows from strictness, as will be explained in the exercises. □

This corollary, and hence the theorem, can fail for compact complex non-Kähler manifolds. An explicit example is described in the exercises.

**Theorem 12.2.7.** *If $X$ is a compact Kähler manifold, the cup product*

$$H^i(X)\otimes H^j(X) \to H^{i+j}(X)$$

*is a morphism of Hodge structures.*

The proof comes down to the observation that

$$F^p\mathscr{E}^\bullet \wedge F^q\mathscr{E}^\bullet \subseteq F^{p+q}\mathscr{E}^\bullet.$$

For the corollaries, we work with rational Hodge structures. We have compatibility with Poincaré duality:

**Corollary 12.2.8.** *If $\dim X = n$, then Poincaré duality gives an isomorphism of Hodge structures*

$$H^i(X) \cong [H^{2n-i}(X)^*](-n).$$

We have compatibility with the Künneth formula:

**Corollary 12.2.9.** *If $X$ and $Y$ are compact Kähler manifolds, then*

$$\bigoplus_{i+j=k} H^i(X)\otimes H^j(Y) \cong H^k(X\times Y)$$

*is an isomorphism of Hodge structures.*

We have compatibility with the Gysin map:

**Corollary 12.2.10.** *If $f : X \to Y$ is a holomorphic map of compact Kähler manifolds of dimension $n$ and $m$ respectively, the Gysin map is a morphism*

$$H^i(X) \to H^{i+2(m-n)}(Y)(n-m).$$

## Exercises

**12.2.11.** Finish the proof of Corollary 12.2.3.

**12.2.12.** Let $C^\bullet$ be a bounded-below complex with biregular filtration $F^\bullet$. Define $E_1^{pq} = H^{p+q}(Gr^p C^\bullet)$ and $d_1 : E_1^{pq} \to E_1^{p+1,q}$ as the connecting map associated to

$$0 \to Gr^{p+1}C^\bullet \to F^pC^\bullet/F^{p+2}C^\bullet \to Gr^pC^\bullet \to 0.$$

Show that $d_1 = 0$ if $F^\bullet$ is strict. (The converse is not quite true, as we will see shortly.)

**12.2.13.** When $C^\bullet = \mathscr{E}^\bullet(X)$ with its Hodge filtration, show that $E_1^{pq} \cong H^q(X, \Omega_X^p)$ and that $d_1$ is induced by $\alpha \mapsto \partial\alpha$ on $\bar{\partial}$-cohomology. Conclude that these maps vanish when $X$ is compact Kähler.

**12.2.14.** Continuing the notation from Exercise 12.2.12. Suppose that $d_1 = 0$ for all indices. Construct a map $d_2 : E_1^{pq} \to E_1^{p+2,q-1}$ that fits into the commutative diagram

$$\begin{array}{ccccc} & & H^{p+q}(Gr^p) & \stackrel{d_2}{\searrow} & \\ & & \uparrow & & \\ H^{p+q}(F^p/F^{p+3}) & \longrightarrow & H^{p+q}(F^p/F^{p+2}) & \longrightarrow & H^{p+q+1}(Gr^{p+2}) \end{array}$$

Show that $d_2 = 0$ if $F^\bullet$ is strict. Optional messy part: If $d_1 = d_2 = 0$, define $d_3 : E_1^{pq} \to E_1^{p+3,q-2}$ etc. in the same way, and check that strictness is equivalent to the vanishing of whole lot.

**12.2.15.** Given a commutative ring $R$, let $U_3(R)$ be the space of upper triangular $3 \times 3$ matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with entries in $R$. The Iwasawa manifold is the quotient $U_3(\mathbb{C})/U_3(\mathbb{Z} + \mathbb{Z}\sqrt{-1})$. Verify that this is a compact complex manifold with a nonclosed holomorphic form $dz - xdy$.

## 12.3 Hodge Decomposition for Moishezon Manifolds

A compact complex manifold need not have any nonconstant meromorphic functions at all. At the other extreme, a compact manifold $X$ is called *Moishezon* if its field of meromorphic functions is as large as possible, that is, if it has transcendence degree equal to $\dim X$ (this is the maximum possible by a theorem of Siegel [104]). This is a very natural class of manifolds, which includes smooth proper algebraic varieties. Moishezon manifolds need not be Kähler; explicit examples due to Hironaka can be found in [60, Appendix B]. Nevertheless, Theorem 12.2.4 holds for these manifolds. A somewhat more general result is true. Let us say that a holomorphic map between complex manifolds is bimeromorphic if it is a biholomorphism between dense open sets.

**Theorem 12.3.1.** *Suppose that $X$ is a compact complex manifold for which there exist a compact Kähler manifold and a surjective holomorphic bimeromorphic map $f : \tilde{X} \to X$. Then $X$ possesses a canonical Hodge decomposition on cohomology described exactly as in Theorem 12.2.4.*

**Corollary 12.3.2.** *The Hodge decomposition holds for Moishezon manifolds.*

*Proof.* Moishezon [88] proved that that there exists a bimeromorphic map $\tilde{X} \to X$, in fact a blowup, with $\tilde{X}$ smooth projective. □

We have already proved a special case of Serre duality for Kähler manifolds. In fact, the result holds for a general compact complex manifold $X$. There is a pairing

$$\langle , \rangle : H^q(X, \Omega_X^p) \otimes H^{n-q}(X, \Omega_X^{n-p}) \to \mathbb{C} \tag{12.3.1}$$

induced by

$$(\alpha, \beta) \to \int_X \alpha \wedge \beta.$$

**Theorem 12.3.3.** *Suppose $X$ is a compact complex manifold. Then*
*(a) (Cartan)* $\dim H^q(X, \Omega_X^p) < \infty$.
*(b) (Serre) The pairing* (12.3.1) *is perfect.*

*Proof.* Both results can be deduced from the Hodge decomposition theorem for the $\bar{\partial}$-operator, which works regardless of the Kähler condition. See [49]. □

We outline the proof of Theorem 12.3.1. Further details can be found in [23, 28].

*Proof.* Let $n = \dim X$. There is a map

$$f^* : H^q(X, \Omega_X^p) \to H^q(\tilde{X}, \Omega_{\tilde{X}}^p)$$

that is induced by the map $\alpha \mapsto f^*\alpha$ of $(p,q)$-forms. We claim that the map $f^*$ is injective. To see this, define a map

$$f_* : H^q(\tilde{X}, \Omega_{\tilde{X}}^p) \to: H^q(X, \Omega_X^p),$$

analogous to the Gysin map, as the adjoint $\langle f_*\alpha, \beta \rangle = \langle \alpha, f^*\beta \rangle$. We leave it as an exercise to check that $f_* f^*(\alpha) = \alpha$. This proves injectivity of $f^*$ as claimed. By similar reasoning, $f^* H^i(X, \mathbb{C}) \to H^i(\tilde{X}, \mathbb{C})$ is also injective.

We claim that $F$ is strict. As we saw in previous exercises (12.2.12,12.2.14), this is equivalent to the vanishing of the differentials $d_1, d_2 \ldots$. We check only the first case, but the same reasoning works in general. Consider the commutative diagram

$$\begin{array}{ccc} H^q(\Omega_X^p) & \xrightarrow{d_1} & H^q(\Omega_X^{p+1}) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H^q(\Omega_{\tilde{X}}^p) & \xrightarrow{d_1} & H^q(\Omega_{\tilde{X}}^{p+1}) \end{array}$$

Since the bottom $d_1$ vanishes, the same goes for the top.

The filtration $\bar{F}$ can also be shown to be strict. We can now argue as in the proof of Theorem 12.2.4 that the filtrations give a Hodge structure on $H^i(X)$. □

## Exercises

**12.3.4.** Check that $\langle f^*\alpha, f^*\beta\rangle = \langle\alpha,\beta\rangle$, and deduce the identity $f_*f^*\alpha = \alpha$ used above.

## 12.4 Hypercohomology*

At this point, it is convenient to give a generalization of the constructions from Chapter 4. Recall that a complex of sheaves is a possibly infinite sequence of sheaves

$$\cdots \to \mathscr{F}^i \xrightarrow{d^i} \mathscr{F}^{i+1} \xrightarrow{d^{i+1}} \cdots$$

satisfying $d^{i+1}d^i = 0$. We say that the complex is *bounded* (below) if finitely many of these sheaves are nonzero (or if $\mathscr{F}^i = 0$ for $i \ll 0$). Given any sheaf $\mathscr{F}$ and natural number $n$, we get a bounded complex $\mathscr{F}[n]$ consisting of $\mathscr{F}$ in the $-n$th position, and zeros elsewhere. The collection of bounded (respectively bounded below) complexes of sheaves on a space $X$ form a category $C^b(X)$ (respectively $C^+(X)$), where a morphism of complexes $f : \mathscr{E}^\bullet \to \mathscr{F}^\bullet$ is defined to be a collection of sheaf maps $\mathscr{E}^i \to \mathscr{F}^i$ that commute with the differentials. This category is abelian. We define additive functors $\mathscr{H}^i : C^+(X) \to \mathrm{Ab}(X)$

$$\mathscr{H}^i(\mathscr{F}^\bullet) = \ker(d^i)/\operatorname{im}(d^{i-1}).$$

A morphism $f : \mathscr{E}^\bullet \to \mathscr{F}^\bullet$ in $C^+(X)$ is a called a *quasi-isomorphism* if it induces isomorphisms $\mathscr{H}^i(\mathscr{E}^\bullet) \cong \mathscr{H}^i(\mathscr{F}^\bullet)$ on all the sheaves.

**Theorem 12.4.1.** *Let $X$ be a topological space. Then there are additive functors $\mathbb{H}^i : C^+(X) \to \mathrm{Ab}$, with $i \in \mathbb{N}$, such that*

*(1) For any sheaf $\mathscr{F}$, $\mathbb{H}^i(X, \mathscr{F}[n]) = H^{i+n}(X, \mathscr{F})$.*
*(2) If $0 \to \mathscr{E}^\bullet \to \mathscr{F}^\bullet \to \mathscr{G}^\bullet \to 0$ is exact, then there is an exact sequence*

$$0 \to \mathbb{H}^0(X, \mathscr{E}^\bullet) \to \mathbb{H}^0(X, \mathscr{F}^\bullet) \to \mathbb{H}^0(X, \mathscr{G}^\bullet) \to \mathbb{H}^1(X, \mathscr{E}^\bullet) \to \cdots.$$

*(3) If $\mathscr{E}^\bullet \to \mathscr{F}^\bullet$ is a quasi-isomorphism, then the induced map $\mathbb{H}^i(X, \mathscr{E}^\bullet) \to \mathbb{H}^i(X, \mathscr{F}^\bullet)$ is an isomorphism.*

$\mathbb{H}^i(X, \mathscr{E}^\bullet)$ is called the $i$th hypercohomology group of $\mathscr{E}^\bullet$.

*Proof.* We outline the proof. Further details can be found in [44], [68], or [118]. We start by redoing the construction of cohomology for a single sheaf $\mathscr{F}$. The functor $\mathbf{G}$ defined in Section 4.1, gives a flasque sheaf $\mathbf{G}(\mathscr{F})$ with monomorphism $\mathscr{F} \to \mathbf{G}(\mathscr{F})$. The sheaf $\mathbf{C}^1(\mathscr{F})$ is the cokernel of this map. Applying $\mathbf{G}$ again yields a sequence

$$\mathscr{F} \to \mathbf{G}(\mathscr{F}) \to \mathbf{G}(\mathbf{C}^1(\mathscr{F})).$$

By continuing as above, we get a resolution by flasque sheaves

$$\mathscr{F} \to \mathbf{G}^0(\mathscr{F}) \to \mathbf{G}^1(\mathscr{F}) \to \cdots.$$

Theorem 5.1.4 shows that $H^i(X,\mathscr{F})$ is the cohomology of the complex $\Gamma(X,\mathbf{G}^\bullet(\mathscr{F}))$, and this gives a clue how to generalize the construction. The complex $\mathbf{G}^\bullet$ is functorial. So given a complex

$$\cdots \to \mathscr{F}^i \xrightarrow{d} \mathscr{F}^{i+1} \to \cdots,$$

we get a commutative diagram

$$\begin{array}{ccccc}
\cdots & \mathscr{F}^i & \xrightarrow{d} & \mathscr{F}^{i+1} & \cdots \\
 & \downarrow & & \downarrow & \\
 & \mathbf{G}^0(\mathscr{F}^i) & \xrightarrow{d} & \mathbf{G}^0(\mathscr{F}^{i+1}) & \\
 & \downarrow \partial & & \downarrow \partial & \\
 & \cdots & & \cdots &
\end{array}$$

We define the total complex

$$\mathscr{T}^i(\mathscr{F}^\bullet) = \bigoplus_{p+q=i} \mathbf{G}^p(\mathscr{F}^q)$$

with a differential $\delta = d + (-1)^q\partial$. We can now define

$$\mathbb{H}^i(X,\mathscr{F}^\bullet) = H^i(\Gamma(X,\mathscr{T}^\bullet(\mathscr{G}))).$$

When applied to $\mathscr{F}[n]$, this yields $H^i(\Gamma(X,G^\bullet(\mathscr{F}))[n])$, which as we have seen is $H^i(X,\mathscr{F})$, and this proves (1).

(2) can be deduced from the exact sequence

$$0 \to \mathscr{T}^\bullet(\mathscr{F}^\bullet) \to \mathscr{T}^\bullet(\mathscr{G}^\bullet) \to \mathscr{T}^\bullet(\mathscr{F}^\bullet) \to 0$$

given in the exercises.

We now turn to the last statement, and prove it for bounded complexes. For any complex $\mathscr{E}^\bullet$ of sheaves (or elements of an abelian category), we can introduce the truncation operator given by the subcomplex

$$\tau_{\le p}\mathscr{E}^i = \begin{cases} \mathscr{E}^p & \text{if } i < p, \\ \ker(\mathscr{E}^p \to \mathscr{E}^{p+1}) & \text{if } i = p, \\ 0 & \text{otherwise.} \end{cases}$$

Truncation yields an increasing filtration $\tau_{\le p}$ or a decreasing filtration $\tau_{\le -p}$. The key property is given in the following lemma:

**Lemma 12.4.2.** *There is an exact sequence of complexes*

$$0 \to \tau_{\le q-1}\mathscr{E}^\bullet \to \tau_{\le q}\mathscr{E}^\bullet \to \mathscr{H}^q(\mathscr{E}^\bullet)[-q] \to 0$$

*for each q.*

A quasi-isomorphism $\mathscr{E}^\bullet \to \mathscr{F}^\bullet$ induces a quasi-isomorphism $\tau_{\le q}\mathscr{E}^\bullet \to \tau_{\le q}\mathscr{F}^\bullet$ for each $q$. Thus the lemma can be applied to get a diagram with exact rows:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathbb{H}^i(\tau_{\le q-1}\mathscr{E}^\bullet) & \longrightarrow & \mathbb{H}^i(\tau_{\le q}\mathscr{E}^\bullet) & \longrightarrow & H^{i-q}(\mathscr{H}^q(\mathscr{E}^\bullet)) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow \cong & \\ \cdots \longrightarrow & \mathbb{H}^i(\tau_{\le q-1}\mathscr{F}^\bullet) & \longrightarrow & \mathbb{H}^i(\tau_{\le q}\mathscr{F}^\bullet) & \longrightarrow & H^{i-q}(\mathscr{H}^q(\mathscr{E}^\bullet)) & \longrightarrow \cdots \end{array}$$

Thus (3) follows by induction on $q$. □

The precise relationship between the various (hyper) cohomology groups is usually expressed by the spectral sequence

$$E_1^{pq} = H^q(X, \mathscr{E}^p) \Rightarrow \mathbb{H}^{p+q}(\mathscr{E}^\bullet).$$

There are a number of standard consequences that we can prove directly. The first is a refinement of Theorem 5.1.4.

**Corollary 12.4.3.** *If $\mathscr{E}^\bullet$ is a bounded complex of acyclic sheaves, then $\mathbb{H}^i(X, \mathscr{E}^\bullet) = H^i(\Gamma(X, \mathscr{E}^\bullet))$.*

*Proof.* There is a map of complexes $\Gamma(X, \mathscr{E}^\bullet) \to \Gamma(X, \mathscr{T}^\bullet(\mathscr{F}^\bullet))$ inducing a map $H^i(\Gamma(X, \mathscr{E}^\bullet)) \to \mathbb{H}^i(X, \mathscr{E}^\bullet)$. We have to check that this is an isomorphism. We do this by induction on the length, or number of nonzero terms, of $\mathscr{E}^\bullet$. With the help of the "stupid" filtration,

$$\sigma^p\mathscr{E}^\bullet = \mathscr{E}^{\ge p} = \cdots \to 0 \to \mathscr{E}^p \to \mathscr{E}^{p+1} \to \cdots$$

is gotten by dropping the first $p-1$ terms of the complex. We have an exact sequence

$$0 \to \mathscr{E}^{\ge p+1} \to \mathscr{E}^{\ge p} \to \mathscr{E}^p[-p] \to 0 \tag{12.4.1}$$

leading to a commutative diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^i(\Gamma(X,\mathscr{E}^{\geq p+1})) & \longrightarrow & H^i(\Gamma(X,\mathscr{E}^{\geq p})) & \longrightarrow & H^i(\mathscr{E}^p[-p]) & \longrightarrow \cdots \\
& \downarrow \cong & & \downarrow f & & \downarrow \cong & \\
\longrightarrow & \mathbb{H}^i(X,\mathscr{E}^{\geq p+1}) & \longrightarrow & \mathbb{H}^i(X,\mathscr{E}^{\geq p}) & \longrightarrow & \mathbb{H}^i(X,\mathscr{E}^p[-p]) & \longrightarrow
\end{array}$$

with exact rows. The arrows marked by $\cong$ are isomorphisms by induction. Therefore $f$ is an isomorphism by the 5-lemma. □

**Corollary 12.4.4.** *Suppose that $\mathscr{E}^\bullet$ is a bounded complex of sheaves of vector spaces. Then*

$$\dim \mathbb{H}^i(\mathscr{E}^\bullet) \leq \sum_{p+q=i} \dim H^q(X,\mathscr{E}^p).$$

*Proof.* The corollary follows by induction on the length (number of nonzero entries) of $\mathscr{E}^\bullet$ using the long exact sequences on hypercohomology coming from (12.4.1). □

**Corollary 12.4.5.** *Suppose that $\mathscr{E}^\bullet$ is a bounded complex with $H^q(X,\mathscr{E}^p) = 0$ for all $p+q=i$. Then $\mathbb{H}^i(\mathscr{E}^\bullet) = 0$.*

We can extract one more corollary, using Lemma 11.3.3.

**Corollary 12.4.6.** *If $\sum \dim H^q(X,\mathscr{E}^p) \leq \infty$, then*

$$\sum (-1)^i \dim \mathbb{H}^i(\mathscr{E}^\bullet) = \sum (-1)^{p+q} \dim H^q(X,\mathscr{E}^p).$$

In order to facilitate the computation of hypercohomology, we need a criterion for when two complexes are quasi-isomorphic. We will say that a filtration

$$\mathscr{E}^\bullet \supseteq F^p\mathscr{E}^\bullet \supseteq F^{p+1}\mathscr{E}^\bullet \supseteq \cdots$$

is finite (of length $\leq n$) if $\mathscr{E}^\bullet = F^a\mathscr{E}^\bullet$ and $F^{a+n}\mathscr{E}^\bullet = 0$ for some $a$.

**Lemma 12.4.7.** *Let $f : \mathscr{E}^\bullet \to \mathscr{F}^\bullet$ be a morphism of bounded complexes. Suppose that $F^p\mathscr{E}^\bullet$ and $G^p\mathscr{F}^\bullet$ are finite filtrations by subcomplexes such that $f(F^p\mathscr{E}^\bullet) \subseteq G^p\mathscr{F}^\bullet$. If the induced maps*

$$Gr_F^p(\mathscr{E}^\bullet) \to Gr_G^p(\mathscr{F}^\bullet)$$

*are quasi-isomorphisms for all $p$, then $f$ is a quasi-isomorphism.*

## Exercises

**12.4.8.** If $\mathscr{F}^\bullet$ is a bounded complex with zero differentials, show that $H^i(X,\mathscr{F}^\bullet) = \oplus_j H^{i-j}(X,\mathscr{F}^j)$.

**12.4.9.** Prove Lemma 12.4.7 by induction on the length.

## 12.5 Holomorphic de Rham Complex*

Let $X$ be a $C^\infty$ manifold. We can resolve $\mathbb{C}_X$ by the complex of $C^\infty$ forms $\mathscr{E}_X^\bullet$. In other words, $\mathbb{C}_X$ and $\mathscr{E}_X^\bullet$ are quasi-isomorphic. Since $\mathscr{E}_X^\bullet$ is acyclic, it follows that

$$H^i(X,\mathbb{C}_X) = \mathbb{H}^i(X,\mathbb{C}_X[0]) \cong \mathbb{H}^i(X,\mathscr{E}_X^\bullet) \cong H^i(\Gamma(X,\mathscr{E}_X^\bullet)).$$

We have just re-proved de Rham's theorem.

Now suppose that $X$ is a (not necessarily compact) complex manifold. Then we define a subcomplex

$$F^p\mathscr{E}_X^\bullet = \sum_{p'\geq p} \mathscr{E}_X^{p'q}.$$

The image of the map

$$\mathbb{H}^i(X,F^p\mathscr{E}_X^\bullet) \to \mathbb{H}^i(X,\mathscr{E}_X^\bullet)$$

is the filtration introduced just before Theorem 12.2.4. We want to reinterpret this purely in terms of holomorphic forms. We define the holomorphic de Rham complex by

$$\mathscr{O}_X \to \Omega_X^1 \to \Omega_X^2 \to \cdots.$$

We have a natural map $\Omega_X^\bullet \to \mathscr{E}_X^\bullet$ that takes $\sigma^p$ to $F^p$, where $\sigma^p\Omega_X^\bullet = \Omega_X^{\geq p}$. Dolbeault's Theorem 9.2.3 implies that $F^p/F^{p+1}$ is quasi-isomorphic to $\sigma^p/\sigma^{p+1} = \Omega_X^p[-p]$. Therefore, Lemma 12.4.7 implies that $\Omega_X^\bullet \to \mathscr{E}_X^\bullet$, and more generally $\sigma^p\Omega_X^\bullet \to F^p\mathscr{E}_X^\bullet$, are quasi-isomorphisms.

**Lemma 12.5.1.** *$H^i(X,\mathbb{C}) \cong \mathbb{H}^i(X,\Omega_X^\bullet)$ and $F^pH^i(X,\mathbb{C})$ is the image of $\mathbb{H}^i(X,\Omega_X^{\geq p})$.*

When $X$ is compact Kähler, Theorem 12.2.4 implies that the map

$$\mathbb{H}^i(X,\Omega_X^{\geq p}) \to \mathbb{H}^i(X,\Omega_X^\bullet)$$

is injective.

From Corollaries 12.4.4, 12.4.5, 12.4.6 we obtain the following result.

**Corollary 12.5.2.** *If $X$ is compact, the ith Betti number satisfies*

$$b_i(X) \leq \sum_{p+q=i} \dim H^q(X,\Omega_X^p),$$

*and the Euler characteristic satisfies*

$$e(X) = \sum(-1)^i b_i(X) = \sum(-1)^{p+q}\dim H^q(X,\Omega_X^p).$$

**Corollary 12.5.3.** *If $H^q(X,\Omega_X^p) = 0$ for all $p+q=i$, then $H^i(X,\mathbb{C}) = 0$.*

The next corollary uses the notion of Stein manifold that will be discussed later, in Section 16.1. For the time being, we note that Stein manifolds include smooth affine varieties. The above results give nontrivial topological information for this class of manifolds.

**Corollary 12.5.4.** *Let $X$ be a Stein manifold, or in particular a smooth affine variety. Then $H^i(X,\mathbb{C}) = 0$ for $i > \dim X$.*

*Proof.* This follows from Theorem 16.3.3. □

## Exercises

**12.5.5.** Suppose that $H^i(X,\mathscr{F}) = 0$ for $i > N$ and any locally free sheaf $\mathscr{F}$. Show that $b_i(X) = 0$ for $i > N + \dim X$.

**12.5.6.** Show that the inequality in Corollary 12.5.2 is strict for the Iwasawa manifold defined in Exercise 12.2.15).

## 12.6 The Deligne–Hodge Decomposition*

We fix the following: a smooth hypersurface (also called a smooth divisor) $X \subset Y$ in a projective smooth variety. Let $U = Y - X$. Our goal is to understand the cohomology and Hodge theory of $U$. This can be calculated using $C^\infty$ differential forms $\mathscr{E}_U^\bullet$, but it will more useful to compute this with forms having controlled singularities. We define $\Omega_Y^p(*X)$ to be the sheaf of meromorphic $p$-forms that are holomorphic on $U$. This is not coherent, but it is a union of coherent subsheaves $\Omega_Y^p(mX)$ of mermorphic $p$-forms with at worst poles of order $m$ along $X$. We also define $\Omega_Y^p(\log X) \subset \Omega_Y^p(1X)$ as the subsheaf of meromorphic forms $\alpha$ such that both $\alpha$ and $d\alpha$ have simple poles along $X$. If we choose local coordinates $z_1,\dots,z_n$ so that $X$ is defined by $z_1 = 0$, then the sections of $\Omega_Y^p(\log X)$ are locally spanned as an $\mathscr{O}_X$ module by

$$\{dz_{i_1} \wedge \cdots \wedge dz_{i_p} \mid i_j > 1\} \cup \left\{ \frac{dz_1 \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_p}}{z_1} \right\};$$

$\Omega_X^\bullet(\log D) \subset \Omega_Y^\bullet(*X)$ is a subcomplex.

**Proposition 12.6.1.** *There are isomorphisms*

$$H^i(U,\mathbb{C}) \cong \mathbb{H}^i(Y,\Omega_Y^\bullet(\log X)) \cong \mathbb{H}^i(Y,\Omega_Y^\bullet(*X)).$$

*Proof.* Details can be found in [49, pp. 449–454]. The key point is to show that the inclusions

$$\Omega_Y^\bullet(\log X) \subset \Omega_Y^\bullet(*X) \subset j_*\mathscr{E}_U^\bullet \tag{12.6.1}$$

are quasi-isomorphisms, where $j : U \to Y$ is the inclusion. This can be reduced to a calculation in which $Y$ is replaced by a disk with coordinate $z$, and with $X$ corresponding to the origin. Then (12.6.1) becomes

$$
\begin{array}{ccccc}
\mathscr{O}_D & \longrightarrow & \sum_{n\in\mathbb{Z}} \mathscr{O}_D z^n & \longrightarrow & C^\infty_{D-0} \\
\downarrow d & & \downarrow d & & \downarrow d \\
\mathscr{O}_D \frac{dz}{z} & \longrightarrow & \sum_{n\in\mathbb{Z}} \mathscr{O}_D z^n dz & \longrightarrow & \mathscr{E}^1_{D-0} \\
 & & & & \downarrow d \\
 & & & & \mathscr{E}^2_{D-0}
\end{array}
$$

The cohomology in each column is $\mathbb{C}$ in degrees $0,1$, and horizontal maps induce isomorphisms between these. □

The spaces in the proposition on the right carry natural filtrations. The Hodge filtration

$$F^p H^{i+1}(U) = \mathrm{im}[\mathbb{H}^{i+1}(0 \to \Omega_Y^p(\log X) \to \Omega_Y^{p+1}(\log X) \to \cdots) \to \mathbb{H}^{i+1}(\Omega_Y^\bullet(\log X))]$$

and the pole filtration induced by

$$\mathrm{Pole}^p H^{i+1}(U) = \mathrm{im}[\mathbb{H}^{i+1}(\cdots \to 0 \to \Omega_Y^p(X) \to \Omega_Y^{p+1}(2X) \to \cdots) \to \mathbb{H}^{i+1}(\Omega_Y^\bullet(*X))].$$

It follows more or less immediately that $F^p \subseteq \mathrm{Pole}^p$. Equality need not hold in general, but it does in an important case studied later, in Section 17.5.

In order to relate this to the cohomology of $X$, we use residues. We have a map

$$\mathrm{Res} : \Omega_Y^p(\log X) \to \Omega_X^{p-1}, \tag{12.6.2}$$

called the *Poincaré residue* map, given by

$$\mathrm{Res}\left(\alpha \wedge \frac{dz_1}{z_1}\right) = \alpha|_X.$$

Res commutes with $d$. Therefore it gives a map of complexes

$$\Omega_Y^\bullet(\log X) \to \Omega_X^\bullet[-1],$$

where $[-1]$ indicates a shift of indices by $-1$. This induces a map

$$H^i(U,\mathbb{C}) = \mathbb{H}^i(\Omega_Y^\bullet(\log X)) \to H^{i-1}(X,\mathbb{C}).$$

After normalizing this by a factor of $\frac{1}{2\pi\sqrt{-1}}$, it takes integer cohomology to integer cohomology (modulo torsion). This can be described topologically as a composition

$$H^i(U) \to H^i(\mathrm{Tube}) \xrightarrow{\sim} H^{i-1}(X),$$

where Tube is a tubular neighborhood and the second map is the inverse of the Thom isomorphism, §5.5. The residue map is an epimorphism of sheaves, and the kernel is precisely the sheaf of holomorphic forms. So we have an exact sequence

$$0 \to \Omega_Y^\bullet \to \Omega_Y^\bullet(\log X) \to \Omega_X^\bullet[-1] \to 0, \tag{12.6.3}$$

which leads to a long exact sequence

$$\cdots \to H^q(\Omega_Y^p) \to H^q(\Omega_Y^p(\log X)) \to H^q(\Omega_X^{p-1}) \to H^{q+1}(\Omega_Y^p) \to \cdots \tag{12.6.4}$$

$$\cdots \to H^i(Y,\mathbb{C}) \to H^i(U,\mathbb{C}) \to H^{i-1}(X,\mathbb{C}) \xrightarrow{\gamma} H^{i+1}(Y,\mathbb{C}) \to \cdots . \tag{12.6.5}$$

The second is called the Gysin sequence. Indeed, $\gamma$ is the Gysin map.

**Theorem 12.6.2 (Deligne).** *The Hodge filtration on $H^i(U,\mathbb{C})$ is strict, i.e., the maps*

$$\mathbb{H}^{i+1}(0 \to \Omega_Y^p(\log X) \to \Omega_Y^{p+1}(\log X) \to \cdots) \to \mathbb{H}^{i+1}(\Omega_Y^\bullet(\log X))$$

*are injective. In particular, there is a (noncanonical) decomposition*

$$H^i(U,\mathbb{C}) \cong \bigoplus_{p+q=i} H^q(Y,\Omega_Y^p(\log X)).$$

*Proof.* By Corollary 12.2.3, it is enough to prove that

$$\dim H^i(U) = \sum_{p+q=i} \dim H^q(Y,\Omega_Y^p(\log X)).$$

From (12.6.4) and (12.6.5), we get

$$\begin{aligned}\dim H^q(\Omega_Y(\log X)) = \dim\ker[H^q(\Omega_X^{p-1}) \to H^{q+1}(\Omega_Y^p)]\\ + \dim\operatorname{im}[H^{q-1}(\Omega_X^{p-1}) \to H^q(\Omega_Y^p)]\end{aligned}$$

and

$$\dim H^i(U) = \dim\ker[H^{i-1}(X) \to H^{i+1}(Y)] + \dim\operatorname{im}[H^{i-2}(X) \to H^i(Y)].$$

Combining the last equation with Corollaries 12.2.5 and 12.2.10 shows that

$$\begin{aligned}\dim H^i(U) &= \sum \dim\ker[H^q(\Omega_X^{p-1}) \to H^{q+1}(\Omega_Y^p)]\\ &\quad + \sum \dim\operatorname{im}[H^{q-1}(\Omega_X^{p-1}) \to H^q(\Omega_Y^p)]\\ &= \sum \dim H^q(Y,\Omega_Y^p(\log X)).\end{aligned}$$

□

**Corollary 12.6.3 (Weak Kodaira vanishing).** *If $X$ is a smooth divisor such that $U = X - Y$ is affine, then $H^i(Y,\Omega_Y^n \otimes \mathscr{O}_Y(X)) = 0$ for $i > 0$.*

*Proof.* By Theorem 12.6.2 and Corollary 12.5.4,

$$\dim H^i(Y,\Omega_Y^n(\log X)) \le \dim H^{i+n}(U,\mathbb{C}) = 0$$

when $i > 0$. A direct calculation shows that

$$\Omega^n_Y(\log X) = \Omega^n_Y(X) \cong \Omega^n_Y \otimes \mathscr{O}_Y(X). \qquad \square$$

Here is a more useful form.

**Corollary 12.6.4 (Kodaira vanishing).** *If L is an ample line bundle, then*

$$H^i(Y, \Omega^n_Y \otimes L) = 0$$

*for* $i > 0$.

*Proof.* Here is the outline. By assumption, $L^{\otimes m} = \mathscr{O}_Y(1)$ for some $m > 0$. By Bertini's theorem, we can choose a hyperplane $H \subset \mathbb{P}^n$ such that $X = Y \cap H$ is smooth. Note that $\mathscr{O}(X) \cong L^{\otimes m}$ and $Y - X$ is affine. So when $m = 1$, we can apply the previous corollary. In general, one can construct a nonsingular cover $\pi : Y' \to Y$ branched over $X$, which locally is given by $y^m = f$, where $f = 0$ is the local equation for $X$. A precise construction can be found in [77, vol. I, Proposition 4.1.6], along with the proof of the following properties:

1. The set-theoretic preimage $X' = \pi^{-1}X$ is again smooth,
2. $L' = \pi^* L$ has a smooth section vanishing along $X'$ without multiplicity, or to be more precise, $\mathscr{O}(X') \cong L'$.
3. The cohomology $H^i(Y, \Omega^n_Y \otimes L)$ injects into $H^i(Y', \Omega^n_{Y'} \otimes L')$.

The last property follows from [77, Lemma 4.1.14] plus Serre duality [60, Chapter III, Corollary 7.7]. Thus the result follows from the previous corollary applied to $(Y', X')$. $\square$

*Remark 12.6.5.* Kodaira proved a slightly different statement, where ampleness was replaced by positivity in a differential-geometric sense (cf. [49, Chapter 1§2]). This form was used in the proof of the Kodaira embedding theorem, Theorem 10.1.11. The embedding theorem then implies that positivity and ampleness are, a posteriori, equivalent conditions for line bundles.

Deligne [24] proved (a refinement of) Theorem 12.6.2 en route to constructing a *mixed Hodge structure* on cohomology. This is roughly something given by gluing pure Hodge structures of different weights together. More formally, a mixed Hodge structure is given by a lattice $H$ with two filtrations $W$ and $F$ defined over $\mathbb{Q}$ and $\mathbb{C}$ respectively so that $F$ induces a pure rational Hodge structure of weight $k$ on $W_k/W_{k-1}$ for each $k$. In the case of $H^i(U)$, where $U$ is the complement of a smooth divisor in a smooth projective variety $X$, we have

$$W_k H^i(U, \mathbb{Q}) = \begin{cases} 0 & \text{if } k < i, \\ \operatorname{im} H^i(Y, \mathbb{Q}) & \text{if } k = i, \\ H^i(U, \mathbb{Q}) & \text{otherwise.} \end{cases}$$

A fairly detailed introduction to mixed Hodge theory can be found in the book by Peters and Steenbrink [95]. Since we can barely scratch the surface, we will be content to give a simple example to indicate the power of these ideas.

*Example 12.6.6.* Given an elliptic curve $E$, we have seen that $H^1(E)$ with its Hodge structure determines $E$. Now remove the origin 0 and a nonzero point $p$ and consider the mixed Hodge structure on $H^1(E - \{0, p\})$. This determines the complement $E - \{0, p\}$ by the work of Carslon [16].

## Exercises

**12.6.7.** Work out (12.6.4) and (12.6.5) explicitly when $Y = \mathbb{P}^2$ and $X$ is a smooth curve of degree $d$.

**12.6.8.** Show that forms in $H^0(Y, \Omega_Y^p(\log X))$ are closed. Is this necessarily true for forms in $H^0(Y, \Omega_Y^p(kX))$?

**12.6.9.** Verify the isomorphism $\Omega_Y^n(\log X) \cong \Omega_Y^n \otimes \mathscr{O}_Y(X)$ used in the proof of Corollary 12.6.3.