Chapter 11
A Little Algebraic Surface Theory

Let us return to geometry armed with what we have learned so far. We have already looked at Riemann surfaces (which will be referred to as complex curves from now on) in some detail. So we consider the next step up. A nonsingular complex surface is a two-dimensional complex manifold. By an algebraic surface, we will mean a two-dimensional nonsingular projective variety. So in particular, they are Kähler manifolds. In this chapter, we will present a somewhat breezy account of surface theory, concentrating on topics that illustrate the general theorems from the previous chapters.

Much more systematic introductions to algebraic surface theory can be found in the books by Barth, Peters, and Van de Ven [9] and Beauville [10].

11.1 Examples

The basic discrete invariant of a curve is its genus. For algebraic surfaces, there are several numbers that play a similar role. We can use the Hodge numbers. By the symmetry properties considered earlier, there are only three that matter: h^{10}, h^{20}, h^{11}. The first two are traditionally called (and denoted by) the *irregularity* ($q = h^{10}$) and *geometric genus* ($p_g = h^{20}$). The basic topological invariants are the Betti numbers, which by Poincaré duality and the Hodge decomposition theorem can be expressed as

$$b_1 = b_3 = 2q, \quad b_2 = 2p_g + h^{11}.$$

It is also convenient to consider the Euler characteristic,

$$e = b_0 - b_1 + b_2 - b_3 + b_4 = 2 - 4q + 2p_g + h^{11}.$$

Let us now start our tour.

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Example 11.1.1. The most basic example is the projective plane $X = \mathbb{P}^2 = \mathbb{P}^2_{\mathbb{C}}$. We computed the Betti numbers $b_1 = 0, b_2 = 1$ in Section 7.2. Therefore $q = p_g = 0$ and $h^{11} = 1$.

Example 11.1.2. If $X = C_1 \times C_2$ is a product of two nonsingular curves of genus g_1 and g_2, then by Künneth's formula (Theorem 5.3.6 and Corollary 10.2.6),

$$q = h^{10}(X) = h^{10}(C_1)h^{00}(C_2) + h^{00}(C_1)h^{10}(C_2) = g_1 + g_2.$$

Similarly, $p_g = g_1 g_2$ and $h^{11} = 2g_1 g_2 + 2$.

Example 11.1.3. As a special case of the previous example, when we have a product of a curve C with $\mathbb{P}^1$, the invariants are $q = g$, $p_g = 0$, and $h^{11} = 2$. More generally, we can consider ruled surfaces over C, which are $\mathbb{P}^1$-bundles that are locally isomorphic to $U_i \times \mathbb{P}^1$, for a Zariski open cover $\{U_i\}$ of C. (See Section 14.5 for a bit more explanation of what this means.) We will see shortly that the invariants are the same as above, although they are not generally products. When $C = \mathbb{P}^1$, there are, up to isomorphism, countably many ruled surfaces. Here is a simple description. Let $C_n \subset \mathbb{P}^n$ be the closure of $\{(t, t^2, \ldots, t^n) \mid t \in \mathbb{C}\}$. Choose a point $p_0 \in \mathbb{P}^n - C_n$. Let F_n be the set of pairs $(q, p) \in \mathbb{P}^n \times C_n$ such that q lies on the line connecting p_0 to q.

Example 11.1.4. Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d. Then $q = 0$. We will list the first few values of the remaining invariants:

d	p_g	h^{11}
2	0	2
3	0	7
4	1	20
5	4	45
6	10	86

These can be calculated using formulas given later (17.3.4).

A method of generating new examples from old is by blowing up. We start by describing the blowup of $\mathbb{C}^2$ at 0:

$$\mathrm{Bl}_0\mathbb{C}^2 = \{(x, \ell) \in \mathbb{C}^2 \times \mathbb{P}^1 \,|\, x \in \ell\}.$$

The projection $p_1 : \mathrm{Bl}_0\mathbb{C}^2 \to \mathbb{C}^2$ is one-to-one away from $0 \in \mathbb{C}^2$. This can be generalized to yield the blowup $\mathrm{Bl}_p X \to X$ of a surface X at the point p. Let $B \subset X$ be a coordinate ball centered at p. After identifying B with a ball in $\mathbb{C}^2$ centered at 0, we can let $\mathrm{Bl}_0 B$ be the preimage of B in $\mathrm{Bl}_0\mathbb{C}^2$. The boundary of $\mathrm{Bl}_0 B$ can be identified with the boundary of B. Thus we can glue $X - B \cup \mathrm{Bl}_0 B$ to form $\mathrm{Bl}_p X$. When X is algebraic, $\mathrm{Bl}_p X$ is again algebraic by Exercise 2.4.23.

Let us compute $H^*(\mathrm{Bl}_p X, \mathbb{Z})$. Set $Y = \mathrm{Bl}_p X$ and compare Mayer–Vietoris sequences:

$$
\begin{array}{ccccc}
H^i(X) & \longrightarrow & H^i(X-B') \oplus H^i(B) & \longrightarrow & H^i(X-B' \cap B) \\
\downarrow & & \downarrow & & \downarrow = \\
H^i(Y) & \longrightarrow & H^i(Y-\mathrm{Bl}_p B') \oplus H^i(\mathrm{Bl}_p B) & \longrightarrow & H^i(Y-\mathrm{Bl}_p B' \cap \mathrm{Bl}_p B)
\end{array}
$$

where $B' \subset B$ is a smaller ball. We thus have the following result:

Lemma 11.1.5. $H^1(\mathrm{Bl}_p X) \cong H^1(X)$ *and* $H^2(\mathrm{Bl}_p X) = H^2(X) \oplus \mathbb{Z}$.

Corollary 11.1.6. *q and p_g are invariant under blowing up.* $h^{11}(\mathrm{Bl}_p X) = h^{11}(X) + 1$.

Proof. The lemma implies that $b_1 = 2q$ is invariant and $b_2(Y) = b_1(X) + 1$. Since $b_2 = 2p_g + h^{11}$, the only possibilities are $h^{11}(Y) = h^{11}(X) + 1$, $p_g(Y) = p_g(X)$, and $p_g(Y) < p_g(X)$. The last inequality means that there is a nonzero holomorphic 2-form on X that vanishes on $X - p$, but this is impossible. □

A *birational map* $\kappa : X \dashrightarrow Y$ is simply an isomorphism in the category of varieties and rational maps. In more explicit terms, it is given by an isomorphism of Zariski open sets $X \supset U \cong V \subset Y$. Blowups and their inverses ("blowdowns") are examples of birational maps. Two varieties are birationally equivalent if a birational map exists between them. For example, any two ruled surfaces over $\mathbb{P}^1$ are birationally equivalent to each other and to $\mathbb{P}^2$, because they all contain $\mathbb{A}^2$.

For surfaces, the structure of birational maps is explained by the following theorem:

Theorem 11.1.7 (Castelnuovo). *Any birational map between algebraic surfaces is given by a finite sequence of blowups and blowdowns.*

Proof. See [60, 9]. □

Corollary 11.1.8. *The numbers q and p_g depend only on the birational equivalence class of the surface.*

This implies that $p_g = g, q = 0$ for ruled surfaces over a genus-g curve, as claimed above.

Blowing up of singular points figures in the proof of the next important theorem.

Theorem 11.1.9 (Zariski). *Given a singular algebraic surface Y, there exist a nonsingular surface X and a morphism $\pi : X \to Y$, called a resolution of singularities, that is an isomorphism away from the singular points.*

Proof. See [9]. □

Corollary 11.1.10 (Zariski). *If $f : X \dashrightarrow V$ is a rational function from an algebraic surface to a variety V, then there is a finite sequence of blowups $Y \to X$ such that f extends to a holomorphic map $f' : Y \to V$.*

Proof. We can construct Y by resolving singularities of the closure of the graph of f. □

Analogues of Zariski's and Castelnuovo's theorems in higher dimensions have been established by Hironaka and Włodarczyk respectively. These are much harder.

An *elliptic surface* is a surface X that admits a surjective morphism $f : X \to C$ to a smooth projective curve such that all nonsingular fibers are elliptic curves.

Example 11.1.11. A simple example of an elliptic surface is given as follows: choose two distinct nonsingular cubics $E_0, E_1 \subset \mathbb{P}^2$ defined by $f_0(x,y,z)$ and $f_1(x,y,z)$. These generate a pencil of cubics $E_t = V(tf_1 + (1-t)f_0)$ with $t \in \mathbb{P}^1$. Define

$$X = \{(p,t) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid p \in E_t\}.$$

Projection to $\mathbb{P}^1$ makes this an elliptic surface. X can be identified with the blowup of $\mathbb{P}^2$ at the nine points $E_0 \cap E_1$. So $q = p_g = 0$ and $h^{11} = 10$.

Example 11.1.12. Consider the family of elliptic curves in Legendre form

$$\mathscr{E} = \{([x,y,z],t) \in \mathbb{P}^2 \times \mathbb{C} - \{0,1\} \mid y^2z - x(x-z)(x-tz) = 0\} \to \mathbb{C}.$$

The above equation is meaningful if $t = 0, 1$, and it defines a rational curve with a single node. By introducing $s = t^{-1}$, we get an equation

$$sy^2z - x(x-z)(sx-z) = 0$$

that defines a union of lines when $s = 0$. In this way, we can extend $\mathscr{E}$ to a surface $\mathscr{E}' \to \mathbb{P}^1$. Unfortunately, $\mathscr{E}'$ is singular, and it is necessary to resolve singularities to get a nonsingular surface $\bar{\mathscr{E}}$ containing $\mathscr{E}$ (we can take the minimal desingularization, which for our purposes means that $b_2(\bar{\mathscr{E}})$ is chosen as small as possible).

Exercises

11.1.13. Finish the proof of Lemma 11.1.5.

11.1.14. Given a ruled surface X over a curve C, check that $e(X) = e(\mathbb{P}^1)e(C)$, and use this to verify that $h^{11}(X) = 2$.

11.1.15. Show that there is a nonsingular quartic $X \subset \mathbb{P}^3$ containing a line ℓ, which we can assume to be $x_2 = x_3 = 0$. Show that the map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ defined by $[x_0, \ldots, x_3] \mapsto [x_0, x_1]$ determines a morphism $X \to \mathbb{P}^1$ that makes it an ellptic surface.

11.1.16. Given a smooth projective curve C, the symmetric product is given by $S^2C = C \times C/\sigma$, where σ is the involution interchanging factors. This has the structure of a smooth algebraic surface such that $H^i(S^2C, \mathbb{Q}) = H^i(C \times C, \mathbb{Q})^\sigma$. Compute the Betti and Hodge numbers for S^2C.

11.2 The Neron–Severi Group

Let X be an algebraic surface. The image of the first Chern class map

$$c_1 : \mathrm{Pic}(X) \to H^2(X,\mathbb{Z})$$

is the *Neron–Severi* group $\mathrm{NS}(X)$. The rank of this group is called the *Picard number* $\rho(X)$. By Lefschetz's Theorem 10.3.1, $\mathrm{NS}(X) = H^2(X,\mathbb{Z}) \cap H^{11}(X)$. Therefore $\rho \leq h^{11}$ with equality if $p_g = 0$.

A divisor on X is a finite integer linear combination $\sum n_i D_i$ of possibly singular irreducible curves $D_i \subset X$. We can define a line bundle $\mathscr{O}_X(D)$ as we did for Riemann surfaces in Section 6.3. If f_i are local equations of $D_i \cap U$ in some open set U, then

$$\mathscr{O}_X(D)(U) = \mathscr{O}_X(U)\frac{1}{f_1^{n_1} f_2^{n_2} \dots}$$

is a fractional ideal. In particular, when $n_i = 1$, $\mathscr{O}_X(-D)$ is the ideal sheaf of D, and an ideal sheaf of a subscheme supported on D when $n_i \geq 0$.

Lemma 11.2.1. *If D_i are smooth curves, then $c_1(\mathscr{O}_X(\sum n_i D_i)) = \sum n_i [D_i]$.*

Proof. This is an immediate consequence of Theorem 7.5.8. □

When D is singular, we simply define its fundamental class to be $c_1(\mathscr{O}_X(D))$.

The cup product pairing

$$H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to H^4(X,\mathbb{Z}) \cong \mathbb{Z}$$

restricts to a pairing on $\mathrm{NS}(X)$ denoted by "$\cdot$". Note that the last isomorphism follows from a stronger form of Poincaré duality than what we proved earlier [61].

Lemma 11.2.2. *Given a pair of transverse smooth curves D and E,*

$$D \cdot E = \int_X c_1(\mathscr{O}(D)) \cup c_1(\mathscr{O}(E)) = \int_D c_1(\mathscr{O}_X(E))|_D = \#(D \cap E).$$

Proof. By Lemma 11.2.1 and Proposition 5.6.3, $D \cdot E$ is a sum of local intersection numbers $i_p(D,E)$. The numbers $i_p(D,E)$ are always $+1$ in this case by Exercise 5.6.6. □

If the intersection of the curves D and E is finite but not transverse, it is still possible to give a geometric meaning to the above product. Choose local coordinates centered at p, and local equations f and g for D and E respectively.

Definition 11.2.3. The local intersection number is given by $i_p(D,E) = \dim \mathscr{O}_p/(f,g)$. (This depends only on the ideals (f) and (g), so it is well defined.)

Proposition 11.2.4. *If D,E are curves such that $D \cap E$ is finite, then*

$$D \cdot E = \sum_{p \in X} i_p(D,E).$$

Proof. We assume for simplicity that D and E are smooth, although this argument can be made to work in general. As in the proof of Proposition 5.6.3, the number on the left is given by

$$\int_X \tau_D \wedge \tau_E$$

for appropriate representatives τ_D, τ_E for the Thom classes. In particular, we assume that the supports are small enough that it breaks up into a sum of integrals over disjoint coordinate neighborhoods of $p \in D \cap E$.

We need a convenient expression for the Thom classes. We first note that if $\rho : \mathbb{R}^+ \to [0,1]$ is a cutoff function, 1 in a neighborhood of 0 and 0 in a neighborhood of ∞, then $-\frac{1}{2\pi} d\rho(r) \wedge d\theta$ gives a local expression for the Thom class of $0 \in \mathbb{R}^2$. Thus after choosing local equations of D, E at $p \in D \cap E$ as above, we can assume that (locally) $\tau_D = \tau_f$ and $\tau_E = \tau_g$, where

$$\tau_f = -\frac{1}{2\pi\sqrt{-1}} d\rho(|f|) \wedge \frac{df}{f}.$$

Let $h(z_1, z_2) = (f(z_1, z_2), g(z_1, z_2))$. It maps a small ball $0 \in U \subset \mathbb{C}^2$ to another small ball $0 \in U'$. The *degree* of h is the number of points in the fiber $h^{-1}(y)$ for almost all y. This coincides with $i_p(E, D)$ by Lemma 1.3.3. Computing the integral by a change of variables gives

$$\int_U \tau_f \wedge \tau_g = \int_{U'} h^*(\tau_f \wedge \tau_g) = (\deg h) \int \tau_{z_1} \wedge \tau_{z_2} = \deg h = i_p(E, D). \qquad \square$$

Example 11.2.5. Recall that $H^2(\mathbb{P}^2, \mathbb{Z}) = \mathbb{Z}$, and the generator of $H^2(\mathbb{P}^2, \mathbb{Z})$ is the class of the line $[L]$. Since $[L] = [L']$ for any other line, we have $L^2 = L \cdot L' = 1$, where $L^2 = L \cdot L$.

Example 11.2.6. $H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z}) = \mathbb{Z}^2$ with generators given by fundamental classes of the horizontal and vertical lines $H = \mathbb{P}^1 \times \{0\}$ and $V = \{0\} \times \mathbb{P}^1$. We see that $H^2 = V^2 = 0$ and $H \cdot V = 1$.

Given a curve $D \subset \mathbb{P}^2$ defined by a polynomial f, we let $\deg D = \deg f$.

Corollary 11.2.7 (Bézout). *If D, E are curves on $\mathbb{P}^2$ with a finite intersection, then*

$$\sum_{p \in X} i_p(D, E) = \deg(D) \deg(E).$$

Proof. We have $[D] = c_1(\mathscr{O}(\deg D)) = (\deg D)[L]$, and likewise $[E] = (\deg E)[L]$. Therefore $D \cdot E = \deg D \deg E (L^2) = \deg D \deg E$. $\square$

Corollary 11.2.8. *If D, E are distinct irreducible curves, then $D \cdot E \geq 0$, and equality holds only if they are disjoint.*

This nonnegativity can fail when $D = E$. For example, by Corollary 5.7.3, the diagonal in a product of curves $\Delta \subset C \times C$ has negative self-intersection as soon as the genus of C is greater than 1. From Lemma 11.2.1, we obtain the following:

Lemma 11.2.9. *If D is a smooth curve, then*

$$D^2 = \int_D c_1(\mathscr{O}_X(D)) = \deg(\mathscr{O}_D(D)).$$

Given a surjective morphism $f : X \to Y$ of algebraic surfaces, the pullback $f^* : H^2(Y,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ preserves the Neron–Severi group and the intersection pairing. This can be interpreted directly in the language of divisors. Given an irreducible divisor D on Y, we can make the set-theoretic preimage $f^{-1}D$ into a divisor by pulling back the ideal, i.e.,

$$\mathscr{O}_X(-f^{-1}D) = \mathrm{im}[f^*\mathscr{O}_Y(-D) \to \mathscr{O}_X].$$

We extend this operation to all divisors by linearity. The operation satisfies $f^*\mathscr{O}_Y(D) = \mathscr{O}_X(f^{-1}D)$. Since c_1 is functorial, we get the following:

Lemma 11.2.10. *f^{-1} is compatible with f^* on* $\mathrm{NS}(X)$.

Exercises

11.2.11. Let $X = C \times C$ be the product of a curve with itself. Consider the divisors $H = C \times \{p\}$, $V = \{p\} \times C$ and the diagonal Δ. Compute their intersection numbers, and show that these are linearly independent in $\mathrm{NS}(X) \otimes \mathbb{Q}$. Thus the Picard number is at least 3. Show that this is at least 4 if C admits a nontrivial automorphism with the appropriate conditions.

11.2.12. Let $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be an elliptic curve, and let $X = E \times E$. Show that the Picard number is 3 for "most" τ, but that it is 4 for $\tau = \sqrt{-1}$.

11.2.13. The ruled surface F_1 can be described as the blowup $\pi : F_1 \to \mathbb{P}^2$ of $\mathbb{P}^2$ at some point p. Let L_1 be a line in $\mathbb{P}^2$ containing p, and L_2 another line not containing p. Show that $\pi^*L_1 = \pi^*L_2$ and $\pi^*L_1 = E + F$, where $E = \pi^{-1}(p)$ and F is the closure in Y of $L_1 - \{p\}$ (F is called the strict transform of L_1). Use all of this to show that $E^2 = -1$. Conclude that F_1 and $\mathbb{P}^1 \times \mathbb{P}^1$ are not isomorphic.

11.2.14. A divisor is called (very) ample if $\mathscr{O}_X(H)$ is. If H is ample, then prove that $H^2 > 0$ and that $H \cdot C > 0$ for any curve $C \subset X$. This pair of conditions characterizes ampelness (Nakai–Moishezon). Show that the first condition alone is not sufficient.

11.3 Adjunction and Riemann–Roch

In this section, we introduce two of the most basic tools of surface theory. The first result, called the adjunction formula, computes the genus of a curve on a surface.

To set things up, recall that the canonical divisor K of a smooth projective curve is a divisor such that $\mathscr{O}_C(K) \cong \Omega^1_C$. Since this determines K uniquely up to linear equivalence, we can talk about *the* canonical divisor class K_C. A canonical divisor K_X (class) on a surface X is divisor such that $\mathscr{O}_X(K_X) \cong \Omega^2_X$. The linear equivalence class is again well defined. For the present, we need only its image in $\mathrm{NS}(X)$, and this can be defined to be $c_1(\Omega^2_X)$.

Theorem 11.3.1. *If C is a smooth curve of genus g on an algebraic surface X, then*

$$2g-2=(K_X+C)\cdot C.$$

Proof. Let $\Omega^2_X(\log C)$ be the $\mathscr{O}_X$-module generated by rational 2-forms of the form $\frac{dz_1\wedge dz_2}{f}$, where f is a local equation for C. This is the same as the tensor product $\Omega^2_X \otimes \mathscr{O}_X(C)$. Such expressions can be rewritten as $\alpha\wedge\frac{df}{f}$, with α holomorphic. We define the residue map $\Omega^2_X(\log C)\to\Omega^1_c$ by sending $\alpha\wedge\frac{df}{f}$ to $\alpha|_C$. The kernel consists of the holomorphic differentials Ω^2_X, leading to a sequence

$$0\to\Omega^2_X\to\Omega^2_X(\log C)\to\Omega^1_C\to 0, \tag{11.3.1}$$

which is seen to be exact. The holomorphic forms Ω^2_X are spanned locally by $f\frac{dz_1\wedge dz_2}{f}$. Thus we can identify the inclusion in (11.3.1) with the tensor product of the map $\mathscr{O}_X(-C)\to\mathscr{O}_X$ with $\Omega^2_X\otimes\mathscr{O}_X(C)$. The cokernel of the map just described is $\mathscr{O}_C\otimes\Omega^2_X(C)$, that is, the restriction of $\Omega^2_X\otimes\mathscr{O}_X(C)$ to C. So in summary, Ω^1_C is isomorphic to the restriction of $\Omega^2_X\otimes\mathscr{O}_X(C)$ to C. Therefore

$$\int_C c_1(\Omega^1_C)=\int_C c_1(\Omega^2_X\otimes\mathscr{O}_X(C))=\int_C (c_1(\Omega^2_X)+c_1(\mathscr{O}_X(C))=(K_X+C)\cdot C.$$

The left side is the degree of the K_C, but this is $2g-2$ by Proposition 6.3.7. □

We can use this to recover the formula for the genus of a degree-d curve $C\subset\mathbb{P}^2$. Since $\mathrm{NS}(\mathbb{P}^2)\subseteq H^2(\mathbb{P}^2)=\mathbb{Z}$, we can identify $K_{\mathbb{P}^2}$ with an integer k. Therefore

$$g=\frac{1}{2}(k+d)d+1.$$

When $d=1$, we know that $g=0$, so $k=-3$. Thus

$$g=(d-1)(d-2)/2.$$

A fundamental, and rather difficult, problem in algebraic geometry is to estimate $\dim H^0(X,\mathscr{O}_X(D))$. As a first step, one can calculate the Euler characteristic

$$\chi(\mathscr{O}_X(D))=\sum(-1)\dim H^i(X,\mathscr{O}_X(D))$$

by the Riemann–Roch formula given below. The higher cohomologies can then be controlled in some cases by other techniques. The advantage of the Euler characteristic is the additivity property:

Lemma 11.3.2. *If* $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ *is an exact sequence of sheaves with* $\sum \dim H^i(X,\mathscr{F}_j) < \infty$, *then* $\chi(\mathscr{F}_2) = \chi(\mathscr{F}_1) + \chi(\mathscr{F}_3)$.

This is a consequence of following elementary lemma:

Lemma 11.3.3. *If*

$$\cdots \to A^i \to B^i \to C^i \to A^{i+1} \to \cdots$$

is a finite sequence of finite-dimensional vector spaces,

$$\sum(-1)^i \dim B^i = \sum(-1)^i \dim A^i + \sum(-1)^i \dim C^i.$$

Theorem 11.3.4 (Riemann–Roch). *If D is a divisor on a surface X, then*

$$\chi(\mathscr{O}_X(D)) = \frac{1}{2} D\cdot(D-K_X) + \chi(\mathscr{O}_X).$$

Proof. We prove this under the assumption that $D = \sum_i n_i D_i$ is a sum of smooth curves. (In fact, by a simple trick, it is always possible to reduce to this case. The basic idea can be found, for example, in the proof of [60, Chapter V, Theorem 1.1].) By induction, it suffices to prove Riemann–Roch for $D = D' \pm C$, where the formula holds for D' and C is smooth. The idea is to do induction on $\sum |n_i|$. We will use the Riemann–Roch theorem for C as given in Exercise 6.3.16. We treat the case of $D = D' + C$, leaving the remaining case for the exercises. Tensoring the sequence

$$0 \to \mathscr{O}_X(-C) \to \mathscr{O}_X \to \mathscr{O}_C \to 0$$

by $\mathscr{O}(D)$ yields

$$0 \to \mathscr{O}_X(D') \to \mathscr{O}_X(D) \to \mathscr{O}_C(D) \to 0.$$

Therefore, using this together with the adjunction formula and Riemann–Roch on C, we obtain

$$\begin{aligned}
\chi(\mathscr{O}_X(D)) &= \chi(\mathscr{O}_X(D')) + \chi(\mathscr{O}_C(D)) \\
&= \frac{1}{2} D'(D'-K) + \chi(\mathscr{O}_X) + \deg(D|_C) + 1 - g(C) \\
&= \frac{1}{2} D'(D'-K) + C\cdot D - \frac{1}{2} C(C+K) + \chi(\mathscr{O}_X) \\
&= \frac{1}{2} D(D-K) + \chi(\mathscr{O}_X).
\end{aligned}$$

□

Exercises

11.3.5. Find a formula for the genus of a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ in terms of its bidegree.

11.3.6. Given a morphism $f: C \to D$, calculate the self-intersection of the graph Γ_f^2.

11.3.7. Do the case $D = D' - C$ in the proof of Riemann–Roch.

11.3.8. Find a formula for $D \cdot C$ in terms of $\chi(\mathscr{O}(D)), \chi(\mathscr{O}(C)), \chi(\mathscr{O}(C+D))$ and $\chi(\mathscr{O}_X)$.

11.3.9. Use the formula of the previous exercise to give another proof of Proposition 11.2.4.

11.4 The Hodge Index Theorem

The next result is Hodge-theoretic, so we work with a compact Kähler surface X. Let ω denote the Kähler form.

Lemma 11.4.1. *If α is a harmonic 2-form, then $\omega \wedge \alpha$ is again harmonic.*

Proof. By the Kähler identities, it is enough to prove that $\bar{\partial}(\omega \wedge \alpha) = 0$, which is trivially true, and $\bar{\partial}^*(\omega \wedge \alpha) = 0$. By Proposition 9.3.5 and some calculation,

$$\bar{\partial}^*(\omega \wedge \alpha) = C_1(\Lambda\partial - \partial\Lambda)(\omega \wedge \alpha) = C_1\partial\Lambda(\omega \wedge \alpha) = C_2\partial\alpha = 0$$

for appropriate constants C_1, C_2. □

Then the form ω is a closed real $(1,1)$-form. Therefore the Kähler class $[\omega]$ is an element of $H^{11}(X) \cap H^2(X,\mathbb{R})$.

Theorem 11.4.2 (Hodge index theorem). *Let X be a compact Kähler surface. Then the restriction of the cup product to $(H^{11}(X) \cap H^2(X,\mathbb{R})) \cap (\mathbb{R}[\omega])^{\perp}$ is negative definite.*

Proof. Around each point we have a neighborhood U such that $\mathscr{E}^{(1,0)}(U)$ is a free module. By Gram–Schmid, we can find an orthonormal basis $\{\phi_1, \phi_2\}$ for it. In this basis, over U, we have

$$\omega = \frac{\sqrt{-1}}{2}(\phi_1 \wedge \bar{\phi}_1 + \phi_2 \wedge \bar{\phi}_2)$$

and the volume form

$$d\mathrm{vol} = \frac{\omega^2}{2} = -\frac{1}{4}\phi_1 \wedge \bar{\phi}_1 \wedge \phi_2 \wedge \bar{\phi}_2,$$

using Exercise 10.1.18. It follows from this and the previous lemma that $d$vol is the unique harmonic 4-form up to scalar multiples. Choose an element $\alpha \in (H^{11}(X) \cap H^2(X,\mathbb{R}))$ and represent it by a harmonic real $(1,1)$-form. Then over U,

$$\alpha = \sqrt{-1}\sum a_{ij}\phi_i \wedge \bar{\phi}_j$$

with

$$a_{ji} = \bar{a}_{ij}. \tag{11.4.1}$$

By the previous lemma,

$$\alpha \wedge \omega = 2(a_{11} + a_{22})d\text{vol}$$

is also a harmonic 4-form, and therefore a scalar multiple of $d$vol. If α is chosen in $(\mathbb{R}[\omega])^{\perp}$, then $\int \alpha \wedge \omega = 0$, so that

$$a_{11} + a_{22} = 0. \tag{11.4.2}$$

Combining (11.4.1) and (11.4.2) yields

$$\alpha \wedge \alpha = -8(|a_{11}|^2 + |a_{12}|^2)d\text{vol},$$

so globally, $\alpha \wedge \alpha$ is a negative multiple of $d$vol. Therefore

$$\int_X \alpha \wedge \alpha < 0. \qquad \square$$

Corollary 11.4.3. *If H is an ample divisor on an algebraic surface, the intersection pairing is negative definite on* $(\mathrm{NS}(X) \otimes \mathbb{R}) \cap (\mathbb{R}[H])^{\perp}$.

Proof. By Corollary 10.1.10, $[H]$ is a Kähler class. $\square$

Corollary 11.4.4. *If H, D are divisors on an algebraic surface such that $H^2 > 0$ and $D \cdot H = 0$, then $D^2 < 0$ unless $[D] = 0$.*

Proof. This is an exercise in linear algebra using the fact that the intersection form on $(\mathrm{NS}(X) \otimes \mathbb{R})$ has signature $(+1, -1, \ldots, -1)$. $\square$

Exercises

11.4.5. Prove that the restriction of the cup product to $(H^{20}(X) + H^{02}(X)) \cap H^2(X, \mathbb{R})$ is positive definite.

11.4.6. Conclude that (the matrix representing) the cup product pairing has $2p_g + 1$ positive eigenvalues. Therefore p_g is a topological invariant.

11.4.7. Let $f : X \to Y$ be a morphism from a smooth algebraic surface to a possibly singular projective surface. Consider the set $\{D_i\}$ of irreducible curves that map to points under f. Prove a theorem of Mumford that the matrix $(D_i \cdot D_j)$ is negative definite.

11.5 Fibered Surfaces*

Let us say that a surface X is *fibered* if it admits a nonconstant holomorphic map $f : X \to C$ to a nonsingular curve. For example, ruled surfaces and elliptic surfaces are fibered. Not all surfaces are fibered. However, any surface can be fibered after blowing up: Choose a nontrivial rational function $X \dashrightarrow \mathbb{P}^1$, then blow up X to get a morphism. The fiber over $p \in C$ is the closed set $f^{-1}(p)$. Let z be a local coordinate at p. Then $f^{-1}(p)$ is defined by the equation $f^*z = 0$. We can regard $f^{-1}(p)$ as a divisor $\sum n_i D_i$, where D_i are its irreducible components and n_i is the order of vanishing of f^*z along D_i. Call a divisor on X *vertical* if its irreducible components are contained in the fiberes.

Lemma 11.5.1. *Suppose that $f : X \to C$ is a fibered surface.*

(a) If $F = f^{-1}(p)$ and D is a vertical divisor, then $F \cdot D = 0$. In particular, $F^2 = 0$.
(b) If D is vertical, then $D^2 \leq 0$.

Proof. Since the fundamental class of $f^{-1}(p)$ is independent of p, we can assume that F and D are disjoint. This proves (a).

Suppose that $D^2 > 0$, when combined with (a), we would get a contradiction to the Hodge index theorem. □

Corollary 11.5.2. *A necessary condition for a surface to be fibered is that there be an effective divisor with $F^2 = 0$. In particular, $\mathbb{P}^2$ is not fibered.*

We can give a complete characterization of surfaces fibered over curves of genus greater than one.

Theorem 11.5.3 (Castelnuovo–de Franchis). *Suppose that X is an algebraic surface. A necessary and sufficient condition for X to admit a nonconstant holomorphic map to a smooth curve of genus $g \geq 2$ is that there exist two linear independent forms $\omega_i \in H^0(X, \Omega_X^1)$ such that $\omega_1 \wedge \omega_2 = 0$.*

Proof. The necessity is easy. If $f : X \to C$ is a holomorphic map onto a curve of genus at least 2, then it possesses at least two linearly independent holomorphic 1-forms ω_i'. Set $\omega_i = f^*\omega_i'$. By writing this in local coordinates, we see that these are nonzero. But $\omega_1 \wedge \omega_2 = f^*(\omega_1' \wedge \omega_2') = 0$.

We will sketch the converse. A complete proof can be found in [9, pp. 123–125]. Choosing local coordinates, we can write

$$\omega_i = f_i(z_1, z_2)dz_1 + g_i(z_1, z_2)dz_2.$$

The condition $\omega_1 \wedge \omega_2 = 0$ forces

$$(f_1 g_2 - f_2 g_1)dz_1 \wedge dz_2 = 0.$$

Therefore $f_2/f_1 = g_2/g_1$. Call the common value F. Thus $\omega_2 = F\omega_1$. Since the ω_i are globally defined, $F = \omega_2/\omega_1$ defines a global meromorphic function $X \dashrightarrow \mathbb{P}^1$.

By Corollary 11.1.10, there exists $Y \to X$ that is a composition of blowups such that F extends to a holomorphic function $F' : Y \to \mathbb{P}^1$. The fibers of F' need not be connected. Stein's factorization theorem [60] shows that the map can be factored as

$$Y \xrightarrow{\Phi} C \to \mathbb{P}^1,$$

where Φ has connected fibers and $C \to \mathbb{P}^1$ is a finite-to-one map of smooth projective curves. To avoid too much notation, let us denote the pullbacks of ω_i to Y by ω_i as well. We now have a relation $\omega_2 = \Phi\omega_1$.

We claim that the ω_i are pullbacks of holomorphic 1-forms on C. We will check this locally around a general point $p \in Y$. Since the ω_i are harmonic (by Exercise 10.2.9) and therefore closed,

$$d\omega_2 = d\Phi \wedge \omega_1 = 0. \tag{11.5.1}$$

Let t_1 be a local coordinate centered at $\Phi(p) \in C$. Let us also denote the pullback of this function to neighborhood of p by t_1. Then we can choose a function t_2 such that t_1, t_2 give local coordinates at p. Then (11.5.1) becomes $dt_1 \wedge \omega_1 = 0$. Consequently, $\omega_1 = f(t_1,t_2)dt_1$, for some function f. The relation $d\omega_1 = 0$ implies that f is a function of t_1 alone. Thus ω_1 is locally the pullback of a 1-form on C, as claimed. The same reasoning applies to ω_2. This implies that the genus of C is at least two.

The final step is to prove that blowing up was unnecessary. Let

$$Y = Y_1 \xrightarrow{\pi_1} Y_2 \to \cdots X$$

be a composition of blowups at points $p_i \in Y_i$. Then $\pi_1^{-1}(p_2) \cong \mathbb{P}^1$. Any map from $\mathbb{P}^1$ to C is constant, since it has positive genus. So we conclude that $Y \to C$ factors through Y_1, and then likewise through Y_2 and so on until we reach Y. □

An obvious corollary is the following:

Corollary 11.5.4. *If $q \geq 2$ and $p_g = 0$, then X admits a nonconstant map to a curve as above.*

Exercises

11.5.5. Given an elliptic surface X, show that K_X is vertical. Conclude that $K_X^2 \leq 0$.

11.5.6. Show that X maps onto a curve of genus ≥ 2 if $q > p_g + 1$. (This bound can be sharpened to $(p_g + 3)/2$, but the argument is more delicate; cf. [9, IV, 4.2].)