Chapter 1
Plane Curves

Algebraic geometry *is* geometry. This sounds like a tautology, but it will be easy to forget once we start learning about sheaves, cohomology, Hodge structures, and so on. So perhaps it is a good idea to keep ourselves grounded by taking a very quick tour of the classical theory of complex algebraic curves in the plane, using only primitive, and occasionally nonrigorous, tools. This will hopefully provide a better sense of where the subject comes from and where we want to go. Once we have laid the proper foundations in later chapters, we will revisit these topics and supply some of the missing details.

The treatment here is very much inspired by Clemens's wonderful book [20] as well as the first chapter of Arbarello, Cornalba, Griffiths, and Harris's treatise [5].

1.1 Conics

A complex *affine algebraic plane curve* is the set of zeros

$$X = V(f) = \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\} \tag{1.1.1}$$

of a nonconstant polynomial $f(x,y) \in \mathbb{C}[x,y]$. Notice that we call this a curve because it has one complex dimension. However, we will be slightly inconsistent and refer to this occasionally as a surface, especially when we want to emphasize its topological aspects. The curve X is called a conic if f is a quadratic polynomial. The study of conics over $\mathbb{R}$ is something one learns in school. The complex case is actually easier, since distinctions between ellipses and hyperbolas disappear. The group of affine transformations

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a_{11}x + a_{12}y + b_1 \\ a_{21}x + a_{22}y + b_2 \end{pmatrix}$$

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with $\det(a_{ij}) \neq 0$ acts on $\mathbb{C}^2$. High-school methods can be used to show that after making a suitable affine transformation, there are three possibilities along with subcases:

1. A union of two (possibly identical, parallel, or incident) lines.
2. A circle $x^2 + y^2 = 1$.
3. A parabola $y = x^2$.

Things become simpler if we add a line at infinity. This can be achieved by passing to the projective plane $\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{C}}$, which is the set of lines in $\mathbb{C}^3$ containing the origin. To any $(x_0, x_1, x_2) \in \mathbb{C}^3 - \{0\}$, there corresponds a unique point $[x_0, x_1, x_2] = \text{span}\{(x_0, x_1, x_2)\} \in \mathbb{P}^2$. We embed $\mathbb{C}^2 \subset \mathbb{P}^2$ as an open set by sending $(x, y) \mapsto [x, y, 1]$. The line at infinity is the complement given by $x_2 = 0$. The x_i are called homogeneous coordinates, although these are not coordinates in the technical sense of the word. The true coordinates are given by the ratios $x_0/x_2, x_1/x_2$ on the chart $\{x_2 \neq 0\}$, $x_0/x_1, x_2/x_1$ on $\{x_1 \neq 0\}$, and $x_1/x_0, x_2/x_0$ on $\{x_0 \neq 0\}$. We identify $x = x_0/x_2$, $y = x_1/x_2$. The closure of an affine plane curve $X = V(f)$ in $\mathbb{P}^2$ is the *projective algebraic plane curve*

$$\overline{X} = \{[x_0, x_1, x_2] \in \mathbb{P}^2 \mid F(x_0, x_1, x_2) = 0\}, \tag{1.1.2}$$

where

$$F(x_0, x_1, x_2) = x_2^{\deg f} f(x_0/x_2, x_1/x_2)$$

is the homogenization of f.

The projective linear group $\mathrm{PGL}_3(\mathbb{C}) = \mathrm{GL}_3(\mathbb{C})/\mathbb{C}^*$ acts on $\mathbb{P}^2$ via the standard $\mathrm{GL}_3(\mathbb{C})$ action on $\mathbb{C}^3$. The game is now to classify the projective conics up to a projective linear transformation. The list simplifies to three cases including all the degenerate cases: a single line, two distinct lines that meet, and the projectivized parabola C given by

$$x_0^2 - x_1 x_2 = 0. \tag{1.1.3}$$

If we allow nonlinear transformations, then things simplify further. The map from the complex projective line to the plane given by $[s, t] \mapsto [st, s^2, t^2]$ gives a bijection of $\mathbb{P}^1$ to C. The inverse can be expressed as

$$[x_0, x_1, x_2] \mapsto \begin{cases} [x_1, x_0] & \text{if } (x_1, x_0) \neq 0, \\ [x_0, x_2] & \text{if } (x_0, x_2) \neq 0. \end{cases}$$

Note that these expressions are consistent by (1.1.3). These formulas show that C is homeomorphic, and in fact isomorphic in a sense to be explained in the next chapter, to $\mathbb{P}^1$. Topologically, this is just the two-sphere S^2.

Exercises

1.1.1. Show that the subgroup of $\mathrm{PGL}_3(\mathbb{C})$ fixing the line at infinity is the group of affine transformations.

1.1.2. Deduce the classification of projective conics from the classification of quadratic forms over $\mathbb{C}$.

1.1.3. Deduce the classification of affine conics from Exercise 1.1.1.

1.2 Singularities

We recall a version of the implicit function theorem:

Theorem 1.2.1. *If $f(x,y)$ is a polynomial such that $f_y(0,0) = \frac{\partial f}{\partial y}(0,0) \neq 0$, then in a neighborhood of $(0,0)$, $V(f)$ is given by the graph of an analytic function $y = \phi(x)$ with $\phi'(0) \neq 0$.*

In outline, we can use Newton's method. Set $\phi_0(x) = 0$, and

$$\phi_{n+1}(x) = \phi_n(x) - \frac{f(x,\phi_n(x))}{f_y(x,\phi_n(x))}.$$

Then ϕ_n will converge to ϕ. Proving this requires some care, of course.

A point (a,b) on an affine curve $X = V(f)$ is a *singular* point if

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0;$$

otherwise, it is *nonsingular*. In a neighborhood of a nonsingular point, we can use the implicit function theorem to write x or y as an analytic function of the other variable. So locally at such a point, X looks like a disk. By contrast, the nodal curve $y^2 = x^2(x+1)$ looks like a union of two disks touching at $(0,0)$ in a small neighborhood of this point given by $|t \pm 1| < \varepsilon$ in the parameterization

$$x = t^2 - 1,$$
$$y = xt.$$

See Figure 1.1 for the real picture.

The two disks are called branches of the singularity. Singularities may have only one branch, as in the case of the cusp $y^2 = x^3$ (Figure 1.2).

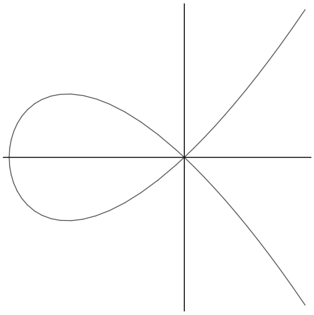

Fig. 1.1 Nodal curve.

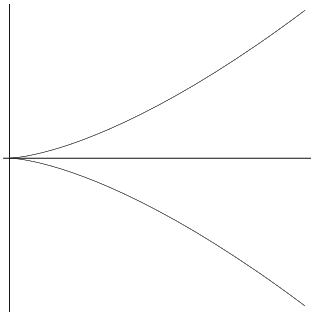

Fig. 1.2 Cuspidal curve.

In order to get a better sense of the topology of a complex singularity, we can intersect the singularity $f(x,y) = 0$ with a small 3-sphere,

$$S^3 = \{(x,y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 = \varepsilon^2\},$$

to get a circle S^1 embedded in S^3 in the case of one branch. The embedded circle is unknotted when this is nonsingular, but it would be knotted otherwise. For the cusp, we would get a trefoil or $(2,3)$ torus knot [87].

The affine plane curve X (1.1.1) is called nonsingular if all its points are nonsingular. The projective curve $\overline{X}$ (1.1.2) is nonsingular if all of its points including points at infinity are nonsingular. In explicit terms, this means that the affine curves $f(x,y) = F(x,y,1) = 0$, $F(1,y,z) = 0$, and $F(x,1,z) = 0$ are all nonsingular. A nonsingular curve is an example of a Riemann surface or a one-dimensional complex manifold.

Exercises

1.2.2. Prove the convergence of Newton's method in the ring of formal power series $\mathbb{C}[[x]]$, where $\phi_n \to 0$ if and only if the degree of its leading term $\to \infty$. Note that this ring is equipped with the x-adic topology, where the ideals (x^N) form a fundamental system of neighborhoods of 0.

1.2.3. Prove that Fermat's curve $x_0^n + x_1^n + x_2^n = 0$ in $\mathbb{P}^2$ is nonsingular.

1.3 Bézout's Theorem

An important feature of the projective plane is that any two lines meet. In fact, it has a much stronger property:

Theorem 1.3.1 (Weak Bézout's theorem). *Any two algebraic curves in $\mathbb{P}^2$ intersect.*

We give an elementary classical proof here using resultants. Given two monic polynomials

$$f(y) = y^n + a_{n-1}y^{n-1} + \cdots + a_0 = \prod_{i=1}^{n}(y - r_i),$$

$$g(y) = y^m + b_{m-1}y^{m-1} + \cdots + b_0 = \prod_{j=1}^{m}(y - s_j),$$

their resultant is the expression

$$\operatorname{Res}(f,g) = \prod_{ij}(r_i - s_j).$$

It is obvious that $\operatorname{Res}(f,g) = 0$ if and only if f and g have a common root. From the way we have written it, it is also clear that $\operatorname{Res}(f,g)$ is a polynomial of degree mn in $r_1,\dots,r_n,s_1,\dots,s_m$ that is symmetric separately in the r's and s's. So it can be rewritten as a polynomial in the elementary symmetric polynomials in the r's and s's. In other words, $\operatorname{Res}(f,g)$ is a polynomial in the coefficients a_i and b_j. Standard formulas for it can be found, for example, in [76].

Proof. Assume that the curves are given by homogeneous polynomials $F(x,y,z)$ and $G(x,y,z)$ respectively. After translating the line at infinity if necessary, we can assume that the polynomials $f(x,y) = F(x,y,1)$ and $g(x,y) = G(x,y,1)$ are both nonconstant in x and y. Treating these as polynomials in y with coefficients in $\mathbb{C}[x]$, the resultant $\operatorname{Res}(f,g)(x)$ can be regarded as a nonconstant polynomial in x. Since $\mathbb{C}$ is algebraically closed, $\operatorname{Res}(f,g)(x)$ must have a root, say a. Then $f(a,y) = 0$ and $g(a,y) = 0$ have a common solution. □

It is worth noting that this argument is entirely algebraic, and therefore applies to any algebraically closed field, such as the field of algebraic numbers $\overline{\mathbb{Q}}$. So as a bonus, we get the following arithmetic consequence.

Corollary 1.3.2. *If the curves are defined by equations with coefficients in $\overline{\mathbb{Q}}$, then the there is a point of intersection with coordinates in $\overline{\mathbb{Q}}$.*

Suppose that the curves $C, D \subset \mathbb{P}^2$ are irreducible and distinct. Then it is not difficult to see that $C \cap D$ is finite. We can ask how many points are in the intersection. To get a more refined answer, we can assign a multiplicity to the points of intersection. If the curves are defined by polynomials $f(x,y)$ and $g(x,y)$ with a common isolated zero at the origin $O = (0,0)$, then define the *intersection multiplicity* at O by

$$i_O(C,D) = \dim \mathbb{C}[[x,y]]/(f,g),$$

where $\mathbb{C}[[x,y]]$ is the ring of formal power series in x and y. The ring of convergent power series can be used instead, and it would lead to the same result. The multiplicities can be defined at other points by a similar procedure. While this definition is concise, it does not give us much geometric insight. Here is an alternative: $i_p(D,E)$ is the number of points close to p in the intersection of small perturbations of these curves. More precisely, we have the following:

Lemma 1.3.3. *$i_p(D,E)$ is the number of points in $\{f(x,y) = \varepsilon\} \cap \{g(x,y) = \eta\} \cap B_\delta(p)$ for small positive $|\varepsilon|, |\eta|, \delta$, where $B_\delta(p)$ is a δ-ball around p.*

Proof. This follows from [42, 1.2.5e]. □

There is another nice interpretation of this number worth mentioning. If $K_1, K_2 \subset S^3$ are disjoint knots, perhaps with several components, their linking number is roughly the number of times one of them passes through the other. A precise definition can be found in any basic book on knot theory (e.g., [98]).

Theorem 1.3.4. *Given a small sphere S^3 about p, $i_p(D,E)$ is the linking number of $D \cap S^3$ and $E \cap S^3$.*

Proof. See [42, 19.2.4]. □

We can now state the strong form of Bézout's theorem. We will revisit this in Corollary 11.2.7.

Theorem 1.3.5 (Bézout's theorem). *Suppose that C and D are algebraic curves with no common components. Then the sum of intersection multiplicities at points of $C \cap D$ equals the product of degrees $\deg C \cdot \deg D$, where $\deg C$ and $\deg D$ are the degrees of the defining polynomials.*

Corollary 1.3.6. *The cardinality $\#C \cap D$ is at most $\deg C \cdot \deg D$.*

Exercises

1.3.7. Show that the vector space $\mathbb{C}[[x,y]]/(f,g)$ considered above is finite-dimensional if $f = 0$ and $g = 0$ have an isolated zero at $(0,0)$.

1.3.8. Suppose that $f = y$. Using the original definition show that $i_O(C,D)$ equals the multiplicity of the root $x = 0$ of $g(x,0)$. Now prove Bézout's theorem when C is a line.

1.4 Cubics

We now turn our attention to the very rich subject of cubic curves. In the degenerate case, the polynomial factors into a product of a linear and quadratic polynomial or three linear polynomials. Then the curve is a union of a line with a conic or three lines. So now assume that X (1.1.2) is defined by an irreducible cubic polynomial. It is called an *elliptic curve* because of its relationship to elliptic functions and integrals.

Lemma 1.4.1. *After a projective linear transformation, an irreducible cubic can be transformed into the projective closure of an affine curve of the form $y^2 = p(x)$, where $p(x)$ is a cubic polynomial. This is nonsingular if and only if $p(x)$ has no multiple roots.*

Proof. See [105, III §1]. □

We note that nonsingular cubics are very different from conics, even topologically.

Proposition 1.4.2. *A nonsingular cubic X is homeomorphic to a torus $S^1 \times S^1$.*

There is a standard way to visualize this (see Figure 1.3). Mark four points $a,b,c,d = \infty$ on $\mathbb{P}^1$, where the first three are the roots of $p(x)$. Join a to b and c to d by nonintersecting arcs α and β. The preimage of the complement $Y = \mathbb{P}^1 - (\alpha \cup \beta)$ in X should fall into two pieces both of which are homeomorphic to Y. So in other words, we can obtain X by first taking two copies of the sphere, slitting them along α and β, and then gluing them along the slits to obtain a torus.

Perhaps that was not very convincing. Instead, we will use a parameterization by elliptic functions to verify Proposition 1.4.2 and more. By applying a further projective linear transformation, we can put our equation for X into Weierstrass form

$$y^2 = 4x^3 - a_2 x - a_3 \tag{1.4.1}$$

with discriminant $a_2^3 - 27a_3^2 \neq 0$. The idea is to parameterize the cubic by the elliptic integral

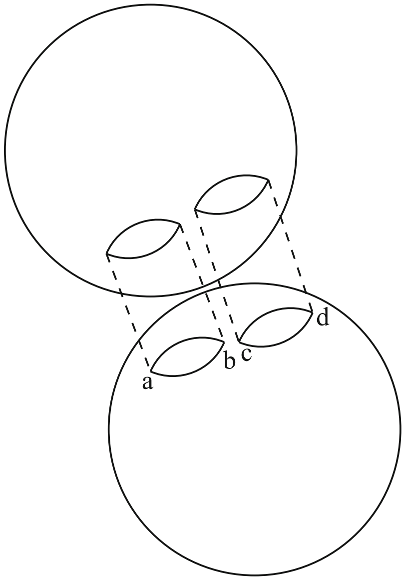

Fig. 1.3 Visualizing the cubic.

$$E(z) = \int_{z_0}^{z} \frac{dx}{y} = \int_{z_0}^{z} \frac{dx}{\sqrt{4x^3 - a_2 x - a_3}}. \tag{1.4.2}$$

While the integrand appears to have singularities at the zeros of $p(x) = 4x^3 - a_2x - a_3$, by differentiating $y^2 = p(x)$ and substituting, we see that

$$\frac{dx}{y} = \frac{2dy}{p'(x)}$$

has no singularities at these points. Thus the integral (1.4.2) should determine a holomorphic function E, but it would be "multivalued" because it depends on the path of integration. We should understand this to mean that E is really a holomorphic function on the universal cover $\tilde{X}$ of X. To understand the multivaluedness more precisely, let us introduce the set of periods $L \subset \mathbb{C}$ as the set of integrals of dx/y around closed loops on X. The set L is actually a subgroup. To see this, let $\mathrm{Loop}(X)$ be the free abelian group consisting of finite formal integer linear combinations of $\sum n_i \gamma_i$ of closed loops on X. The map $\gamma \mapsto \int_\gamma dx/y$ gives a homomorphism of $\mathrm{Loop}(X) \to \mathbb{C}$. The image is exactly L, and it is isomorphic to the first homology group $H_1(X, \mathbb{Z})$, which will discussed in more detail later on. We can see that E descends to a map $X \to \mathbb{C}/L$, which is in fact the homeomorphism alluded to in Proposition 1.4.2.

The above story can be made more explicit by working backward in some sense. First, we characterize the group L in a different way.

Theorem 1.4.3. *There exists a unique lattice $L \subset \mathbb{C}$, i.e., an abelian subgroup generated by two $\mathbb{R}$-linearly independent numbers, such that*

$$a_2 = g_2(L) = 60 \sum_{\lambda \in L, \lambda \neq 0} \lambda^{-4},$$

$$a_4 = g_3(L) = 140 \sum_{\lambda \in L, \lambda \neq 0} \lambda^{-6}.$$

Proof. [106, I 4.3]. □

Fix the period lattice L as above. The Weierstrass $\wp$-function is given by

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in L, \lambda \neq 0} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

This converges to an elliptic function, which means that it is meromorphic on $\mathbb{C}$ and doubly periodic: $\wp(z+\lambda) = \wp(z)$ for $\lambda \in L$ [105]. This function satisfies the Weierstrass differential equation

$$(\wp')^2 = 4\wp^3 - g_2(L)\wp^2 - g_3(L).$$

Thus $\wp$ gives exactly the inverse to the integral E. We get an embedding $\mathbb{C}/L \to \mathbb{P}^2$ given by

$$z \mapsto \begin{cases} [\wp(z), \wp'(z), 1] & \text{if } z \notin L, \\ [0,1,0] & \text{otherwise.} \end{cases}$$

The image is the cubic curve X defined by (1.4.1). This shows that X is a torus topologically as well as analytically. See [105, 106] for further details.

Exercises

1.4.4. Prove that the projective curve defined by $y^2 = p(x)$ is nonsingular if and only if $p(x)$ has no repeated roots.

1.4.5. Prove that the singular projective curve $y^2 = x^3$ is homeomorphic to the sphere.

1.5 Genus 2 and 3

A compact orientable surface is classified up to homeomorphism by a single number called the *genus*. The genus is 0 for a sphere, 1 for a torus, and 2 for the surface depicted in Figure 1.4.

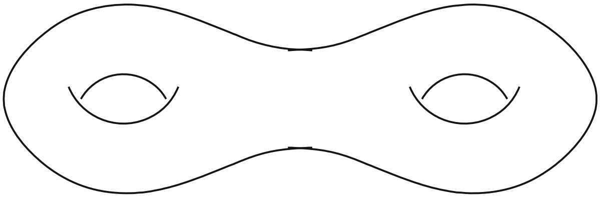

Fig. 1.4 Genus-2 surface.

We claim that a nonsingular quartic in $\mathbb{P}^2$ is a three-holed or genus-3 surface. A heuristic argument is as follows. Let $f \in \mathbb{C}[x,y,z]$ be the defining equation of our nonsingular quartic, and let $g = (x^3+y^3+z^3)x$. The degenerate quartic $g = 0$ is the union of a nonsingular cubic and a line. Topologically this is a torus meeting a sphere in three points (Figure 1.5). Consider the pencil $f_t = tf + (1-t)g$. As t evolves from $t = 0$ to 1, the three points of intersection in $f_t = 0$ open up into circles, resulting in a genus-3 surface (Figure 1.6).

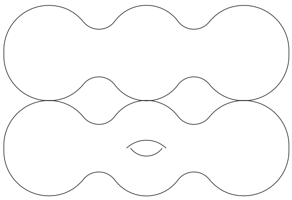

Fig. 1.5 Degenerate quartic.

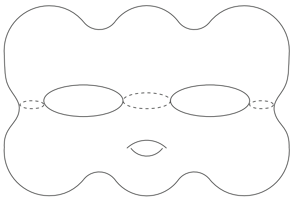

Fig. 1.6 Nonsingular quartic.

In going from degree 3 to 4, we seem to have skipped over genus 2. It is possible to realize such a surface in the plane, but only by allowing singularities. Consider the curve $X \subset \mathbb{P}^2$ given by

$$x_0^2x_2^2 - x_1^2x_2^2 + x_0^2x_1^2 = 0.$$

This has a single singularity at the origin $[0,0,1]$. To analyze this, switch to affine coordinates $x = x_0/x_2, y = x_1/x_2$. Then the polynomial $x^2 - y^2 + x^2y^2$ is irreducible, so it cannot be factored into polynomials, but it can be factored into convergent power series

$$x^2 - y^2 + x^2y^2 = \underbrace{(x + y + \sum a_{ij}x^iy^j)}_{f}\underbrace{(x - y + \sum b_{ij}x^iy^j)}_{g}.$$

By the implicit function theorem, the branches $f = 0$ and $g = 0$ are local analytically equivalent to disks. It follows that in a neighborhood of the origin, the curve looks like two disks touching at a point. We get a genus-2 surface by pulling these apart (Figure 1.7).

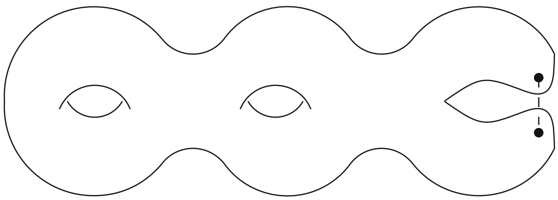

Fig. 1.7 Normalization of singular quartic.

The procedure of pulling apart the points described above can be carried out within algebraic geometry. It is called *normalization*:

Theorem 1.5.1. *Given a curve X, there exist a nonsingular curve $\tilde{X}$ (called the normalization of X) and a proper surjective morphism $H : \tilde{X} \to X$ that is finite-to-one everywhere and one-to-one over all but finitely many points. This is uniquely characterized by these properties.*

The word "morphism" will not be defined precisely until the next chapter. For the present, we should understand it to be a map definable by algebraic expressions such as polynomials. We sketch the construction, which is entirely algebraic. Further details will be given later on. Given an integral domain R with field of fractions K, the *integral closure* of R is the set of elements $x \in K$ such that $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ for some $a_i \in R$. This is closed under addition and multiplication. Therefore it forms a ring [8, Chapter 5]. The basic facts can be summarized as follows:

Theorem 1.5.2. *If $f \in \mathbb{C}[x,y]$ is an irreducible polynomial, then the integral closure $\tilde{R}$ of the domain $R = \mathbb{C}[x,y]/(f)$ is finitely generated as an algebra. If $\mathbb{C}[x_1,\ldots,x_n] \to \tilde{R}$ is a surjection, and $f_1,\ldots,f_N$ generators for the kernel, then*

$$V(f_1,\ldots,f_N) = \{(a_1,\ldots,a_n) \mid f_i(a_1,\ldots,a_n) = 0\}$$

is nonsingular in the sense that the Jacobian matrix has expected rank (§2.5).

Proof. See [8, Proposition 9.2] and [33, Corollary 13.13]. □

Suppose that $X = V(f)$. Then we set $\tilde{X} = V(f_1, \ldots, f_N) \subset \mathbb{C}^n$. We can lift the inclusion $R \subset \tilde{R}$ to a homomorphism of polynomial rings by completing the diagram

$$\begin{array}{ccc} \mathbb{C}[x_1,\ldots,x_n] & \longrightarrow & \tilde{R} \\ \uparrow h & & \uparrow \\ \mathbb{C}[x,y] & \longrightarrow & R \end{array}$$

This determines a pair of polynomials $h(x), h(y) \in \mathbb{C}[x_1, \ldots, x_n]$, which gives a polynomial map $H : \mathbb{C}^n \to \mathbb{C}^2$. By restriction, we get our desired map $H : \tilde{X} \to X$. This is the construction in the affine case. In general, we proceed by gluing these affine normalizations together. The precise construction will be given in §3.4.

Exercises

1.5.3. Verify that $x_0^2 x_2^2 - x_1^2 x_2^2 + x_0^2 x_1^2 = 0$ is irreducible and has exactly one singular point.

1.5.4. Verify that $x^2 - y^2 + x^2 y^2$ can be factored as above using formal power series.

1.5.5. Show that $t = y/x$ lies in the integral closure $\tilde{R}$ of $\mathbb{C}[x,y]/(y^2 - x^3)$. Show that $\tilde{R} \cong \mathbb{C}[t]$.

1.5.6. Show that $t = y/x$ lies in the integral closure $\tilde{R}$ of $\mathbb{C}[x,y]/(x^2 - y^2 + x^2 y^2)$. Show that $\tilde{R} \cong \mathbb{C}[x,t]/(1 - t^2 - x^2 t^2)$.

1.6 Hyperelliptic Curves

An affine hyperelliptic curve is a curve of the form $y^2 = p(x)$, where $p(x)$ has distinct roots. The associated *hyperelliptic* curve X is gotten by taking the closure in $\mathbb{P}^2$ and then normalizing to obtain a nonsingular curve. (We are bending the rules a bit here; usually the term hyperelliptic is applied only when the degree of $p(x)$ is at least 5.) Once again we start by describing the topology.

Proposition 1.6.1. *X is a genus-$g = \lceil \deg p(x)/2 \rceil - 1$ surface, where $\lceil\ \rceil$ means round up to the nearest integer.*

We postpone a rigorous proof. For now, we can see this by using a cut-and-paste construction generalizing what we did for cubics. Let $a_1, \ldots, a_n$ denote the roots of $p(x)$ if $\deg p(x)$ is even, or the roots together with ∞ otherwise. These points, called branch points, are exactly where the map $X \to \mathbb{P}^1$ is one-to-one. Take two copies of

$\mathbb{P}^1$ and make slits from a_1 to a_2, a_3 to a_4, and so on, and then join them along the slits. The genus of the result is $n/2-1$.

Corollary 1.6.2. *Every natural number is the genus of some algebraic curve.*

The original interest in hyperelliptic curves stemmed from the study of integrals of the form $\int q(x)dx/\sqrt{p(x)}$. As with cubics, these are well defined only modulo their periods $L_q = \{\int_\gamma q(x)dx/\sqrt{p(x)} \mid \gamma \text{ closed}\}$. However, this is usually no longer a discrete subgroup, so $\mathbb{C}/L_q$ would be a very strange object. What turns out to be better is to consider all these integrals simultaneously.

Theorem 1.6.3. *The differentials* $\frac{x^i dx}{\sqrt{p(x)}}$, $i = 0,\dots,g-1$, *span the space of holomorphic differentials on* X, *and the set*

$$L = \left\{ \left(\int_\gamma \frac{x^i dx}{\sqrt{p(x)}} \right)_{0\le i<g} \mid \gamma \text{ closed} \right\}$$

is a lattice in $\mathbb{C}^g$, *i.e.,* L *is a discrete subgroup of maximal rank* $2g$.

So it appears that the genus plays a deeper role than one might have initially suspected. That these differentials are holomorphic can be seen by the same sort of calculation we did in Section 1.4. The remaining assertions will follow almost immediately from the Hodge decomposition (see Section 10.3). We thus get a well-defined map from X to the torus $J(X) = \mathbb{C}^g/L$ given by

$$\alpha(x) = \left(\int_{x_0}^{x} \frac{x^i dx}{\sqrt{p(x)}} \right) \mod L. \tag{1.6.1}$$

The torus $J(X)$ is called the *Jacobian* of X, and α is called the *Abel–Jacobi* map. Together these form one of the cornerstones of algebraic curve theory.

We can make this more explicit in an example.

Example 1.6.4. Consider the curve X defined by $y^2 = x^6 - 1$. This has genus two, so that $J(X)$ is a two-dimensional torus. Let E be the elliptic curve given by $v^2 = u^3 - 1$. We have two morphisms $\pi_i : X \to E$ defined by

$$\pi_1 : u = x^2, \quad v = y,$$
$$\pi_2 : u = x^{-2}, \quad v = \sqrt{-1}yx^{-3}.$$

The second map appears to have singularities, but one can appeal to either general theory or explicit calculation to show that it is defined everywhere. We can see that the differential du/v on E pulls back to $2xdx/y$ and $2\sqrt{-1}dx/y$ under π_1 and π_2 respectively. Combining these yields a map $\pi_1 \times \pi_2 : X \to E \times E$ under which the lattice defining $E \times E$ corresponds to a sublattice $L' \subseteq L$. Therefore

$$J(X) = \mathbb{C}^2/L = (\mathbb{C}^2/L')/(L/L') = (E \times E)/(L/L').$$

We express this relation by saying that $J(X)$ is *isogenous* to $E \times E$, which means that it is a quotient of the second by a finite abelian group.

It is worthwhile understanding what is going on at a more abstract level, so as to identify some important players later on in the story. The lattice L can be identified with either the first homology group $H_1(X,\mathbb{Z})$ via the homomorphism $\mathrm{Loop}(X) \to \mathbb{C}^g$ as before, or with the first cohomology group $H^1(X,\mathbb{Z}) = \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z})$ using Poincaré duality. Using the second description, we have a map

$$H^1(E,\mathbb{Z}) \oplus H^1(E,\mathbb{Z}) \cong H^1(E \times E,\mathbb{Z}) \to H^1(X,\mathbb{Z}),$$

and L' is the image.

A hyperelliptic curve with eight branch points has genus 3. We have also encountered genus-3 curves as quartics in $\mathbb{P}^2$. These constructions yield distinct classes of examples:

Proposition 1.6.5. *A genus-3 curve is either a quartic in $\mathbb{P}^2$ or hyperelliptic, and these cases are mutually exclusive.*

We give just the broad outline of the last part. First, note that $\alpha : X \to J(X)$ can be defined for any curve X hyperelliptic or otherwise, by a similar recipe replacing $x^i dx/\sqrt{p(x)}$ with a basis of holomorphic differentials (see §10.3). Let us see how it can used to distinguish these cases. As we shall see later, $\dim J(X) = 3$ because X has genus 3. The set of tangent spaces to any manifold can be assembled into an object called the tangent bundle, and for $J(X)$ it turns out to be trivial. Thus we may sensibly identify all the tangent spaces of $J(X)$ with a fixed $\mathbb{C}^3$. So now to every $x \in X$, we have a line $T_x \subset \mathbb{C}^3$ given by the image of the derivative of the Abel–Jacobi map. In this way, we get a map $\alpha' : X \to \mathbb{P}^2$, sometimes called the Gauss map, which is the key to the proposition. In the hyperelliptic case $y^2 = p(x)$, we can calculate the Gauss map by formally differentiating (1.6.1) and projecting to $\mathbb{P}^2$ to obtain the map

$$\alpha'(x) = \left[\frac{1}{y}, \frac{x}{y}, \frac{x^2}{y}\right] = \left[1, x, x^2\right],$$

which is nothing but the original map $X \to \mathbb{P}^1$, defined by $(x,y) \mapsto x$, followed by the embedding of $\mathbb{P}^1 \to \mathbb{P}^2$ as a conic. In particular, α' would be two-to-one in this case.

In the nonhyperelliptic case, X is defined by a homogeneous quartic polynomial $F(x_0,x_1,x_2)$ in homogeneous coordinates. Setting $x = x_0/x_2, y = x_1/x_2$ as usual, a holomorphic differential on $X \cap \mathbb{C}^2$ is simply given as a restriction of $\omega = g(x,y)dx + h(x,y)dy$ with g and h holomorphic. It can be shown that a nonzero ω would become singular at infinity, and this makes it hard to find forms such that $\omega|_X$ is holomorphic everywhere. Instead, we do an indirect construction using residues. We recall that the wedge product is determined by

$$(gdx + hdy) \wedge (pdx + qdy) = (gq - hp)dx \wedge dy,$$

where $dx \wedge dy \neq 0$ is a symbol. Let $f(x,y) = F(x,y,1)$. We can wedge ω and df/f together to obtain a 2-form

$$\omega' = \omega \wedge \frac{df}{f} = \omega \wedge \left(\frac{f_x}{f} dx + \frac{f_y}{f} dy \right).$$

The inverse $\omega' \mapsto \omega$ is well defined modulo df. So the restriction $\omega|_{X \cap \mathbb{C}^2}$ depends only on ω' and is called the Poincaré residue of ω'. (A more complete abstract treatment of residues will be given in Section 12.6.) Now consider the forms

$$\omega_i = \frac{p_i(x,y) dx \wedge dy}{f(x,y)}, \quad p_i = x, y, 1. \tag{1.6.2}$$

At infinity, we can switch to new coordinates $v = x_2/x_1 = y^{-1}, u = x_0/x_1 = xy^{-1}$ and $t = x_2/x_0 = x^{-1}, s = x_1/x_0 = yx^{-1}$. A direct calculation in these coordinates shows that these forms have poles of order 1 along X and no other singularities (Exercise 1.6.8). Therefore their residues will be holomorphic everywhere along X, and in fact, they give a basis for the space of such differentials. Thus

$$\alpha' = \left[\frac{x}{f}, \frac{y}{f}, \frac{1}{f} \right] = [x, y, 1],$$

and this coincides with the given embedding $X \subset \mathbb{P}^2$.

So we see that the geometry of α' separates the hyperelliptic and nonhyperelliptic cases.

Exercises

1.6.6. The curve X above can be constructed in Example 1.6.4 explicitly by gluing charts defined by $y^2 = x^6 - 1$ and $y_2^2 = 1 - x_2^6$ via $x_2 = x^{-1}$, $y_2 = yx^{-3}$. Check that X is nonsingular and that it maps onto the projective closure of $y^2 = x^6 - 1$.

1.6.7. Using the coordinates from the previous exercise, show that the above formulas for π_i define maps from X to the projective closure of $v^2 = u^3 - 1$.

1.6.8. Set $v = x_2/x_1 = y^{-1}, u = x_0/x_1 = xy^{-1}$. Rewrite the 2-forms given in (1.6.2) in these new coordinates. Verify that these are holomorphic multiples of $du \wedge dv/F(u,1,v)$. Ditto for the coordinates s and t.