Perturbation Theory for Non-smooth Systems

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Article Outline

Glossary

Non-smooth dynamical system Systems derived from ordinary differential equations when the non-uniqueness of solutions is allowed. In this article we deal with discontinuous vector fields in R^n where the discontinuities are concentrated in a codimension-one surface.

Bifurcation In a k-parameter family of systems, a bifurcation is a parameter value at which the phase portrait is not structurally stable.

Typical singularity Are points on the discontinuity set where the orbits of the system through them must be distinguished.

Definition of the Subject

In this article we survey some qualitative and geometric aspects of non-smooth dynamical systems theory. Our goal is to provide an overview of the state of the art on the theory of contact between a vector field and a manifold, and on discontinuous vector fields and their perturbations. We also establish a bridge between two-dimensional non-smooth systems and the geometric singular perturbation theory. Non-smooth dynamical systems is a subject that has been developing at a very fast pace in recent years due to various factors: its mathematical beauty, its strong relationship with other branches of science and the challenge in establishing reasonable and consistent definitions and conventions. It has become certainly one of the common frontiers between mathematics and physics/engineering. We mention that certain phenomena in control systems, impact in mechanical systems and nonlinear oscillations are the main sources of motivation for our study concerning the dynamics of those systems that emerge from differential equations with discontinuous right-hand sides. We understand that non-smooth systems are driven by applications and they play an intrinsic role in a wide range of technological areas.

Introduction

The purpose of this article is to present some aspects of the geometric theory of a class of non-smooth systems. Our main concern is to bring the theory into the domain of geometry and topology in a comprehensive mathematical manner.

Since this is an impossible task, we do not attempt to touch upon all sides of this subject in one article. We focus on exploring the local behavior of systems around typical singularities. The first task is to describe a generic persistence of a local theory (structural stability and bifurcation) for discontinuous systems mainly in the two- and three-dimensional cases. Afterwards we present some striking features and results of the regularization process of two-dimensional discontinuous systems in the framework developed by Sotomayor and Teixeira in [44] and establish a bridge between those systems and the fundamental role played by the Geometric Singular Perturbation Theory (GSPT). This transition was introduced in [10] and we reproduce here its main features in the two-dimensional case. For an introductory reading on the methods of geometric singular perturbation theory we refer to [16,18,30]. In Sect. "Definition of the Subject" we introduce the setting of this article. In Sect. "Introduction" we survey the state of the art of the contact between a vector field and a manifold. The results contained in this section are crucial for the development of our approach. In Sect. "Preliminaries" we discuss the classification of typical singularities of non-smooth vector fields. The study of non-smooth systems, via GSPT, is presented in Sect. "Vector Fields near the Boundary". In Sect. "Generic Bifurcation" some theoretical open problems are presented.

One aspect of the qualitative point of view is the problem of structural stability, the most comprehensive of many different notions of stability. This theme was studied in 1937 by Andronov–Pontryagin (see [3]). This problem is of obvious importance, since in practice one obtains a lot of qualitative information not only on a fixed system but also on its nearby systems.

We deal with non-smooth vector fields in R^{n+1} having a codimension-one submanifold M as its discontinuity set. The scheme in this work toward a systematic classification of typical singularities of non-smooth systems follows the ideas developed by Sotomayor–Teixeira in [43] where the problem of contact between a vector field and the boundary of a manifold was discussed. Our approach intends to be self-contained and is accompanied by an extensive bibliography. We will try to focus here on areas that are complimentary to some recent reviews made elsewhere.

The concept of structural stability in the space of non-smooth vector fields is based on the following definition:

Definition 1 Two vector fields Z and $\tilde{Z}$ are C^0 equivalent if there is an M-invariant homeomorphism $h\colon R^{n+1} \to R^{n+1}$ that sends orbits of Z to orbits of $\tilde{Z}$.

A general discussion is presented to study certain unstable non-smooth vector fields within a generic context. The framework in which we shall pursue these unstable systems is sometimes called generic bifurcation theory. In [3] the concept of kth-order structural stability is also presented; in a local approach such setting gives rise to the notion of a codimension-k singularity. In studies of classical dynamical systems, normal form theory has been well accepted as a powerful tool in studying the local theory (see [6]). Observe that, so far, bifurcation and normal form theories for non-smooth vector fields have not been extensively studied in a systematic way.

Control Theory is a natural source of mathematical models of these systems (see, for instance, [4,8,20,41,45]). Interesting problems concerning discontinuous systems can be formulated in systems with hysteresis ([41]), economics ([23,25]) and biology ([7]). It is worth mentioning that in [5] a class of relay systems in R^n is discussed. They have the form:

$$X = Ax + \operatorname{sgn}(x_1)k$$

where $x = (x_1, x_2, \ldots, x_n)$, $A \in M_R(n, n)$ and $k = (k_1, k_2, \ldots, k_n)$ is a constant vector in R^n. In [28,29] the generic singularities of reversible relay systems in $4D$ were classified. In [54] some properties of non-smooth dynamics are discussed in order to understand some phenomena that arise in chattering control. We mention the presence of chaotic behavior in some non-smooth systems (see for example [12]). It is worthwhile to cite [17], where the main problem in the classical calculus of variations was carried out to study discontinuous Hamiltonian vector fields. We refer to [14] for a comprehensive text involving non-smooth systems which includes many models and applications. In particular motivating models of several non-smooth dynamical systems arising in the occurrence of impacting motion in mechanical systems, switchings in electronic systems and hybrid dynamics in control systems are presented together with an extensive literature on impact oscillators which we do not attempt to survey here. For further reading on some mathematical aspects of this subject we recommend [11] and references therein. A setting of general aspects of non-smooth systems can be found also in [35] and references therein. Our discussion does not focus on continuous but rather on non-smooth dynamical systems and we are aware that the interest in this subject goes beyond the approach adopted here.

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Preliminaries

Now we introduce some of the terminology, basic concepts and some results that will be used in the sequel.

Definition 2 Two vector fields Z and $\tilde{Z}$ on R^n with $Z(0) = \tilde{Z}(0)$ are *germ-equivalent* if they coincide on some neighborhood V of 0.

The equivalent classes for this equivalence are called germs of vector fields. In the same way as defined above, we may define germs of functions. For simplicity we are considering the germ notation and we will not distinguish a germ of a function and any one of its representatives. So, for example, the notation $h\colon R^n, 0 \to R$ means that the h is a germ of a function defined in a neighborhood of 0 in R^n. Refer to [15] for a brief and nice introduction of the concepts of *germ* and *k-jet* of functions.

Discontinuous Systems

Let $M = h^{-1}(0)$, where h is (a germ of) a smooth function $h\colon R^{n+1}, 0 \longrightarrow R$ having $0 \in R$ as its regular value. We assume that $0 \in M$.

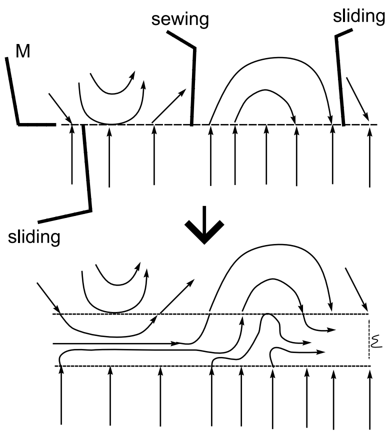
Designate by $\chi(n+1)$ the space of all germs of C^r vector fields on R^{n+1} at 0 endowed with the C^r-topology with $r > 1$ and large enough for our purposes. Call $\Omega(n+1)$ the space of all germs of vector fields Z in $R^{n+1}, 0$ such that

$$Z(q) = \begin{cases} X(q)\,, & \text{for } h(q) > 0\,, \\ Y(q)\,, & \text{for } h(q) < 0\,, \end{cases} \tag{1}$$

The above field is denoted by $Z = (X, Y)$. So we are considering $\Omega(n+1) = \chi(n+1) \times \chi(n+1)$ endowed with the product topology.

Definition 3 We say that $Z \in \Omega(n+1)$ is structurally stable if there exists a neighborhood U of Z in $\Omega(n+1)$ such that every $\tilde{Z} \in U$ is C^0-equivalent with Z.

To define the orbit solutions of Z on the switching surface M we take a pragmatic approach. In a well characterized open set O of M (described below) the solution of Z



Perturbation Theory for Non-smooth Systems, Figure 1
A discontinuous system and its regularization

through a point $p \in O$ obeys the Filippov rules and on $M - O$ we accept it to be multivalued. Roughly speaking, as we are interested in studying the structural stability in $\Omega(n+1)$ it is convenient to take into account all the leaves of the foliation in R^{n+1} generated by the orbits of Z (and also the orbits of X and Y) passing through $p \in M$. (see Fig. 1)

The trajectories of Z are the solutions of the autonomous differential system $\dot{q} = Z(q)$.

In what follows we illustrate our terminology by presenting a simplified model that is found in the classical electromagnetism theory (see for instance [26]):

$$\ddot{x} - \dddot{x} + \alpha \text{sign} x = 0 .$$

with $\alpha > 0$.

So this system can be expressed by the following objects: $h(x,y,z) = x$ and $Z = (X, Y)$ with $X(x,y,z) = (y, z, z+\alpha)$ and $Y(x,y,z) = (y, z, z-\alpha)$.

For each $X \in \chi(n+1)$ we define the smooth function $Xh\colon R^{n+1} \to R$ given by $Xh = X \cdot \nabla h$ where $\cdot$ is the canonical scalar product in R^{n+1}.

We distinguish the following regions on the discontinuity set M:

(i) M_1 is the *sewing region* that is represented by $h = 0$ and $(Xh)(Yh) > 0$;
(ii) M_2 is the *escaping region* that is represented by $h = 0$, $(Xh) > 0$ and $(Yh) < 0$;
(iii) M_3 is the *sliding region* that is represented by $h = 0$, $(Xh) < 0$ and $(Yh) > 0$.

We set $\mathcal{O} = \bigcup_{i=1,2,3} M_i$.

Consider $Z = (X, Y) \in \Omega(n+1)$ and $p \in M_3$. In this case, following Filippov's convention, the solution $\gamma(t)$ of Z through p follows, for $t \geq 0$, the orbit of a vector field tangent to M. Such system is called *sliding vector field* associated with Z and it will be defined below.

Definition 4 The sliding vector field associated to $Z = (X, Y)$ is the smooth vector field Z^s tangent to M and defined at $q \in M_3$ by $Z^s(q) = m - q$ with m being the point where the segment joining $q + X(q)$ and $q + Y(q)$ is tangent to M.

It is clear that if $q \in M_3$ then $q \in M_2$ for $-Z$ and then we define the *escaping vector field* on M_2 associated with Z by $Z^e = -(-Z)^s$. In what follows we use the notation Z^M for both cases.

We recall that sometimes Z^M is defined in an open region U with boundary. In this case it can be C^r extended to a full neighborhood of $p \in \partial U$ in M.

When the vectors $X(p)$ and $Y(p)$, with $p \in M_2 \bigcup M_3$ are linearly dependent then $Z^M(p) = 0$. In this case we say that p is a simple singularity of Z. The other singularities of Z are concentrated outside the set O.

We finish this subsection with a three-*dimensional* example:

Let $Z = (X, Y) \in \Omega(3)$ with $h(x,y,z) = z$, $X = (1, 0, x)$ and $Y = (0, 1, y)$. The system determines four quadrants around 0, bounded by $\tau_X = \{x = 0\}$ and $\tau_Y = \{y = 0\}$. They are: $Q_1^+ = \{x > 0, y > 0\}$, $Q_1^- = \{x < 0, y < 0\}$, $Q_2 = \{x < 0, y > 0\}$ (sliding region) and $Q_3 = \{x > 0, y < 0\}$ (escaping region). Observe that $M_1 = Q_1^+ \bigcup Q_1^-$.

The sliding vector field defined in Q_2 is expressed by:

$$Z^s(x,y,z) = (y-x)^{-1}\left(x + y, \frac{y+x}{8}, 0\right) .$$

Such a system is (in Q_2) equivalent to $G(x,y,z) = (x + y, \frac{y+x}{8}, 0))$. In our terminology we consider G a smooth extension of Z^s, that is defined in a whole neighborhood of 0. It is worthwhile to say that G is in fact a system which is equivalent to the original system in Q_2.

In [50] a generic classification of one-parameter families of sliding vector fields is presented.

Singular Perturbation Problem

A singular perturbation problem is expressed by a differential equation $z' = \alpha(z, \varepsilon)$ (refer to [16,18,30]) where $z \in R^{n+m}$, ε is a small non-negative real number and α is a C^∞ mapping.

Let $z = (x, y) \in R^{n+m}$ and $f\colon R^{m+n} \to R^m$, $g\colon R^{m+n} \to R^n$ be smooth mappings. We deal with equations that may be written in the form

$$\begin{cases} x' = f(x, y, \varepsilon) \\ y' = \varepsilon g(x, y, \varepsilon) \end{cases} \quad x = x(\tau), y = y(\tau) . \tag{2}$$

An interesting model of such systems can be obtained from the singular van der Pol's equation

$$\varepsilon x'' + (x^2 + x)x' + x - a = 0 . \tag{3}$$

The main trick in the geometric singular perturbation (GSP) is to consider the family (2) in addition to the family

$$\begin{cases} \varepsilon \dot{x} = f(x, y, \varepsilon) \\ \dot{y} = g(x, y, \varepsilon) \end{cases} \quad x = x(t), y = y(t) \tag{4}$$

obtained after the time rescaling $t = \varepsilon\tau$.

Equation (2) is called the *fast system* and (4) the *slow system*. Observe that for $\varepsilon > 0$ the phase portrait of fast and slow systems coincide.

For $\varepsilon = 0$, let S be the set of all singular points of (2). We call S the slow manifold of the singular perturbation problem and it is important to notice that Eq. (4) defines a dynamical system on S called the *reduced problem*.

Combining results on the dynamics of these two limiting problems (2) and (4), with $\varepsilon = 0$, one obtains information on the dynamics for small values of ε. In fact, such techniques can be exploited to formally construct approximated solutions on pieces of curves that satisfy some limiting version of the original equation as ε goes to zero.

Definition 5 Let $A, B \subset R^{n+m}$ be compact sets. The Hausdorff distance between A and B is $D(A, B) = \max_{z_1 \in A, z_2 \in B}\{d(z_1, B), d(z_2, A)\}$.

The main question in GSP-theory is to exhibit conditions under which a singular orbit can be approximated by regular orbits for $\varepsilon \downarrow 0$, with respect to the Hausdorff distance.

Regularization Process

An approximation of the discontinuous vector field $Z = (X, Y)$ by a one-parameter family of continuous vector fields will be called a regularization of Z. In [44], Sotomayor and Teixeira introduced the regularization procedure of a discontinuous vector field. A transition function is used to average X and Y in order to get a family of continuous vector fields that approximates the discontinuous one. Figure 1 gives a clear illustration of the regularization process.

Let $Z = (X, Y) \in \Omega(n + 1)$.

Definition 6 A C^∞ function $\varphi\colon R \longrightarrow R$ is a transition function if $\varphi(x) = -1$ for $x \leq -1$, $\varphi(x) = 1$ for $x \geq 1$ and $\varphi'(x) > 0$ if $x \in (-1, 1)$. The ϕ-regularization of $Z = (X, Y)$ is the one-parameter family $X_\varepsilon \in C^r$ given by

$$Z_\varepsilon(q) = \left(\frac{1}{2} + \frac{\varphi_\varepsilon(h(q))}{2}\right) X(q) + \left(\frac{1}{2} - \frac{\varphi_\varepsilon(h(q))}{2}\right) Y(q). \tag{5}$$

with h given in the above Subsect. "Discontinuous Systems" and $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$, for $\varepsilon > 0$.

As already said before, a point in the phase space which moves on an orbit of Z crosses M when it reaches the region M_1. Solutions of Z through points of M_3, will remain in M in forward time. Analogously, solutions of Z through points of M_2 will remain in M in backward time. In [34,44] such conventions are justified by the regularization method in dimensions two and three respectively.

Vector Fields near the Boundary

In this section we discuss the behavior of smooth vector fields in R^{n+1} relative to a codimension-one submanifold (say, the above defined M). We base our approach on the concepts and results contained in [43,53]. The principal advantage of this setting is that the generic contact between a smooth vector field and M can often be easily recognized. As an application the typical singularities of a discontinuous system can be further classified in a straightforward way.

We say that $X, Y \in \chi(n + 1)$ are M-equivalent if there exists an M-preserving homeomorphism $h\colon R^{n+1}, 0 \longrightarrow R^{n+1}, 0$ that sends orbits of X into orbits of Y. In this way we get the concept of M-structural stability in $\chi(n + 1)$.

We call $\Gamma_0(n + 1)$ the set of elements X in $\chi(n + 1)$ satisfying one of the following conditions:

0) $Xh(0) \neq 0$ (0 is a regular point of X). In this case X is transversal to M at 0.
1) $Xh(0) = 0$ and $X^2h(0) \neq 0$ (0 is a 2-*fold* point of X;)
2) $Xh(0) = X^2h(0) = 0$, $X^3h(0) \neq 0$ and the set $\{Dh(0), DXh(0), DX^2h(0)\}$ is linearly independent (0 is a *cusp* point of X;)

...

n) $Xh(0) = X^2h(0) = \cdots = X^n h(0) = 0$ and $X^{n+1}h(0) \neq 0$. Moreover the set $\{Dh(0), DXh(0), DX^2h(0), \ldots, DX^n h(0)\}$ is linearly independent, and 0 is a regular point of the mapping $Xh_{|M}$.

We say that 0 is an M-singularity of X if $h(0) = Xh(0) = 0$. It is a *codimension-zero* M-singularity provided that $X \in \Gamma_0(n + 1)$.

We know that $\Gamma_0(n + 1)$ is an open and dense set in $\chi(n + 1)$ and it coincides with the M-structurally stable vector fields in $\chi(n + 1)$ (see [53]).

Denote by $\tau_X \subset M$ the M-singular set of $X \in \chi(n+1)$; this set is represented by the equations $h = Xh = 0$. It is worthwhile to point out that, generically, all two-folds constitute an open and dense subset of τ_X. Observe that if $X(0) = 0$ then $X \notin \Gamma_0(n+1)$.

The M-bifurcation set is represented by $\chi_1(n+1) = \chi(n+1) - \Gamma_0(n+1)$

Vishik in [53] exhibited the normal forms of a *codimension-zero* M-singularity. They are:

I) Straightened vector field

$$X = (1, 0, \dots, 0)$$

and

$$h(x) = x_1^{k+1} + x_2 x_1^{k-1} + x_3 x_1^{k-2} + \dots + x_{k+1}\,, \quad k = 0, 1, \dots, n$$

or

II) Straightened boundary

$$h(x) = x_1$$

and

$$X(x) = (x_2, x_3, \dots, x_k, 1, 0, 0, \dots, 0)$$

We now discuss an important interaction between vector fields near M and singularities of mapping theory. We discuss how singularity-theoretic techniques help the understanding of the dynamics of our systems.

We outline this setting, which will be very useful in the sequel. The starting point is the following construction.

A Construction

Let $X \in \chi(n+1)$. Consider a coordinate system $x = (x_1, x_2, \dots, x_{n+1})$ in $R^{n+1}, 0$ such that

$$M = \{x_1 = 0\}$$

and

$$X = (X^1, X^2, \dots, X^{n+1})$$

Assume that $X(0) \neq 0$ and $X^1(0) = 0$. Let N_0 be any transversal section to X at 0.

By the implicit function theorem, we derive that:

for each $p \in M, 0$ there exists a unique $t = t(p)$ in $R, 0$ such that the orbit-solution $t \mapsto \gamma(p, t)$ of X through p meets N_0 at a point $\tilde{p} = \gamma(p, t(p))$.

We define the smooth mapping $\rho_X\colon R^n, 0 \longrightarrow R^n, 0$ by $\rho_X(p) = \tilde{p}$. This mapping is a powerful tool in the study of vector fields around the boundary of a manifold (refer to [21,42,43,46,53]). We observe that τ_X coincides with the singular set of ρ_X.

The late construction implements the following method. If we are interested in finding an equivalence between two vector fields which preserve M, then the problem can be sometimes reduced to finding an equivalence between ρ_X and ρ_Y in the sense of singularities of mappings.

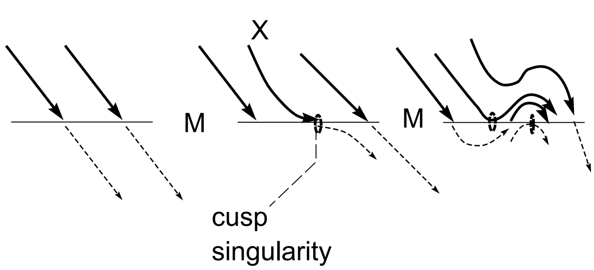
We recall that when 0 is a fold M-singularity of X then associated to the fold mapping ρ_X there is the symmetric diffeomorphism β_X that satisfies $\rho_X \circ \beta_X = \rho_X$.

Given $Z = (X, Y) \in \Omega(n+1)$ such that ρ_X and ρ_Y are fold mappings with $X^2 h(0) < 0$ and $Y^2 h(0) > 0$ then the composition of the associated symmetric mappings β_X and β_Y provides a first return mapping β_Z associated to Z and M. This situation is usually called a *distinguished fold-fold* singularity, and the mapping β_Z plays a fundamental role in the study of the dynamics of Z.

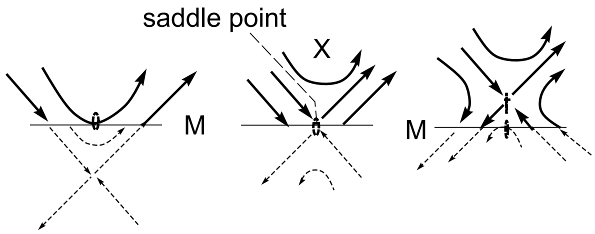
Codimension-one M-Singularity in Dimensions Two and Three

Case $n = 1$ In this case the unique codimension-zero M-singularity is a fold point in $R^2, 0$. The codimension-one M-singularities are represented by the subset $\Gamma_1(2)$ of $\chi_1(2)$ and it is defined as follows.

Definition 7 A codimension-one M-singularity of $X \in \Gamma_1(2)$ is either a *cusp* singularity or an M-hyperbolic critical point p in M of the vector field X. A cusp singularity (illustrated in Fig. 2) is characterized by $Xh(p) = X^2 h(p) = 0$, $X^3 h(p) \neq 0$. In the second case this means that p is a hyperbolic critical point (illustrated in Fig. 3) of X with distinct eigenvalues and with invariant manifolds (stable, unstable and strong stable and strong unstable) transversal to M.



Perturbation Theory for Non-smooth Systems, Figure 2
The cusp singularity and its unfolding



Perturbation Theory for Non-smooth Systems, Figure 3
The saddle point in the boundary and its unfolding

In this subsubsection we consider a coordinate system in $R^2, 0$ such that $h(x, y) = y$.

The next result was proved in [46]. It presents the normal forms of the codimension-one singularities defined above.

Theorem 8 *Let $X \in \chi_1(2)$. The vector field X is M-structurally stable relative to $\chi_1(2)$ if and only if $X \in \Gamma_1(2)$. Moreover, $\Gamma_1(2)$ is an embedded codimension-one sub manifold and dense in $\chi_1(2)$. We still require that any one-parameter family X_λ, ($\lambda \in (-\varepsilon, \varepsilon)$) in $\chi(1)$ transverse to $\Gamma_1(2)$ at X_0, has one of the following normal forms:*

0.1: $X_\lambda(x, y) = (1, 0)$ *(regular point);*
0.2: $X_\lambda(x, y) = (1, x)$ *(fold singularity);*
1.1: $X_\lambda(x, y) = (1, \lambda + x^2)$ *(cusp singularity);*
1.2: $X_\lambda(x, y) = (ax, x + by + \lambda), a = \pm 1, b = \pm 2;$
1.3: $X_\lambda(x, y) = (x, x - y + \lambda);$
1.4: $X_\lambda(x, y) = (x + y, -x + y + \lambda).$

Case $n = 2$

Definition 9 A vector field $X \in \chi(3)$ belongs to the set $\Gamma_1(a)$ if the following conditions hold:

(i) $X(0) = 0$ and 0 is a hyperbolic critical point of X;
(ii) the eigenvalues of $DX(0)$ are pairwise distinct and the corresponding eigenspaces are transversal to M at 0;
(iii) each pair of non complex conjugate eigenvalues of $DX(0)$ has distinct real parts.

Definition 10 A vector field $X \in \chi(3)$ belongs to the set $\Gamma_1(b)$ if $X(0) \neq 0, Xh(0) = 0, X^2h(0) = 0$ and one of the following conditions hold:

(1) $X^3h(0) \neq 0, rank\{Dh(0), DXh(0), DX^2h(0)\} = 2$ and 0 is a non-degenerate critical point of $Xh_{|M}$.
(2) $X^3h(0) = 0, X^4h(0) \neq 0$ and 0 is a regular point of $Xh_{|M}$.

The next results can be found in [43].

Theorem 11 *The following statements hold:*

(i) $\Gamma_1(3) = \Gamma_1(a) \cup \Gamma_2(b)$ *is a codimension-one submanifold of* $\chi(3)$.
(ii) $\Gamma_1(3)$ *is open and dense set in* $\chi_1(3)$ *in the topology induced from* $\chi_1(3)$.
(iv) For a residual set of smooth curves $\gamma: R, 0 \to \chi(3), \gamma$ *meets* $\Gamma_1(3)$ *transversally.*

Throughout this subsubsection we fix the function $h(x, y, z) = z$.

Lemma 12 (Classification Lemma) *The elements of $\Gamma_1(3)$ are classified as follows:*

(a_{11}) ***Nodal M-Singularity:*** *$X(0) = 0$, the eigenvalues of $DX(0), \lambda_1, \lambda_2$, and$\lambda_3$, are real, distinct, $\lambda_1\lambda_j > 0$, $j = 2, 3$ and the eigenspaces are transverse to M at 0;*
(a_{12}) ***Saddle M-Singularity:*** *$X(0) = 0$, the eigenvalues of $DX(0), \lambda_1, \lambda_2$ and λ_3, are real, distinct, $\lambda_1\lambda_j < 0$, $j = 2$ or 3 and the eigenspaces are transverse to M at 0;*
(a_{13}) ***Focal M-Singularity:*** *0 is a hyperbolic critical point of X, the eigenvalues of $DX(0)$ are $\lambda_{12} = a \pm ib, \lambda_3 = c$, with a, b, c distinct from zero and $c \neq a$, and the eigenspaces are transverse to M at 0.*
(b_{11}) ***Lips M-Singularity:*** *presented in* ***Definition 8, item 1****, when $Hess(Fh_{/S}(0)) > 0$:*
(b_{12}) ***Bec to Bec M-Singularity:*** *presented in* ***Definition 8, item 1****, when $Hess(Fh_{/S}(0)) < 0$;*
(b_{13}) ***Dove's Tail M-Singularity:*** *presented in* ***Definition 8, item 2****.*

The next result is proved in [38]. It deals with the normal forms of a codimension-one singularity.

Theorem 13 *i) (Generic Bifurcation and normal forms) Let $X \in \chi(3)$. The vector field X is M-structurally stable relative to $\chi_1(3)$ if and only if $X \in \Gamma_1(3)$. ii) (Versal unfolding) In the space of one-parameter families of vector fields X_α in $\chi(3), \alpha \in (-\varepsilon, \varepsilon)$ an everywhere dense set is formed by generic families such that their normal forms are:*

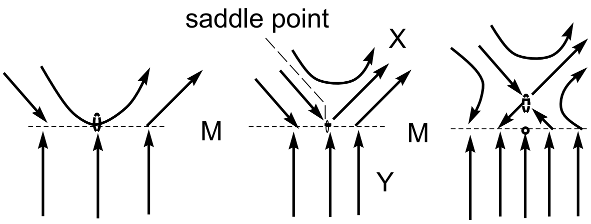
- $X_\alpha \in \Gamma_0(3)$
  0.1: $X_\alpha(x, y, z) = (0, 0, 1)$
  0.2: $X_\alpha(x, y, z) = (z, 0, \pm x)$
  0.3: $X_\alpha(x, y, z) = (z, 0, x^2 + y)$
- $X_0 \in \Gamma_1(3)$
  1.1: $X_\alpha(x, y, z) = (z, 0, \frac{-3x^2+y^2+\alpha}{2})$
  1.2: $X_\alpha(x, y, z) = (z, 0, \frac{-3x^2-y^2+\alpha}{2})$
  1.3: $X_\alpha(x, y, z) = (z, 0, \frac{4\delta x^3+y+\alpha x}{2})$, *with* $\delta = \pm 1$
  1.4: $X_\alpha(x, y, z) = (axz, byz, \frac{ax+by+cz^2+\alpha}{2})$, *with* $(a, b, c) = \delta(3, 2, 1), \delta = \pm 1$

1.5: $X_\alpha(x,y,z) = (axz, byz, \frac{ax+by+cz^2+\alpha}{2})$, *with* $(a,b,c) = \delta(1,3,2)$, $\delta = \pm 1$

1.6: $X_\alpha(x,y,z) = (axz, byz, \frac{ax+by+cz^2+\alpha}{2})$, *with* $(a,b,c) = \delta(1,2,3)$, $\delta = \pm 1$

1.7: $X_\alpha(x,y,z) = (xz, 2yz, \frac{x+2y-cz^2+\alpha}{2})$

1.8: $X_\alpha(x,y,z) = ((-x+y)z, (-x-y)z, \frac{-3x-y+z^2+\alpha}{2})$



Perturbation Theory for Non-smooth Systems, Figure 4
***M*-critical point for *X*, *M*-regular for *Y* and its unfolding**

Generic Bifurcation

Let $Z = (X,Y) \in \Omega^r(n+1)$. Call by $\Sigma_0(n+1)$ (resp. $\Sigma_1(n+1)$) the set of all elements that are structurally stable in $\Omega^r(n+1)$ (resp. $\Omega_1^r(n+1) = \Omega^r(n+1)_{\setminus \Sigma_0(n+1)}$) in $\Omega^r(n+1)$. It is clear that a pre-classification of the generic singularities is immediately reached by:

If $Z = (X,Y) \in \Sigma_0(n+1)$ (resp. $Z = (X,Y) \in \Sigma_1(n+1)$) then X and Y are in $\Gamma_0(n+1)$ (resp. $X \in \Gamma_0(n+1)$ and $Y \in \Gamma_1(n+1)$ or vice versa). Of course, the case when both X and Y are in $\Gamma_1(n+1)$ is a-codimension-two phenomenon.

Two-Dimensional Case

The following result characterizes the structural stability in $\Omega^r(2)$.

Theorem A *(see [31,44]): $\Sigma_0(2)$ is an open and dense set of $\Omega^r(2)$. The vector field $Z = (X,Y)$ is in $\Sigma_0(2)$ if and only one of the following conditions is satisfied:*

i) Both elements X and Y are regular. When $0 \in M$ is a simple singularity of Z then we assume that it is a hyperbolic critical point of Z^M.
ii) X is a fold singularity and Y is regular (and vice-versa).

The following result still deserves a systematic proof. Following the same strategy stipulated in the generic classification of an M-singularity, Theorem 11 could be very useful. It is worthwhile to mention [33] where the problem of generic bifurcation in $2D$ was also addressed.

Theorem B (Generic Bifurcation) *(see [36,43]) $\Sigma_1(2)$ is an open and dense set of $\Omega_1^r(2)$. The vector field $Z = (X,Y)$ is in $\Sigma_1(2)$ provided that one of the following conditions is satisfied:*

i) Both elements X and Y are M-regular. When $0 \in M$ is a simple singularity of Z then we assume that it is a codimension-one critical point (saddle-node or a Bogdanov–Takens singularity) of Z^M.
ii) 0 is a codimension-one M-singularity of X and Y is M-regular. This case includes when 0 is either a cusp M-singularity or a critical point. Figure 4 illustrates the case when 0 is a saddle critical point in the boundary.
iii) Both X and Y are fold M-singularities at 0. In this case we have to impose that 0 is a hyperbolic critical point of the C^r-extension of Z^M provided that it is in the boundary of $M_2 \cup M_3$ (see example below). Moreover when 0 is a distinguished fold-fold singularity of Z then 0 is a hyperbolic fixed point of the first return mapping β_Z.

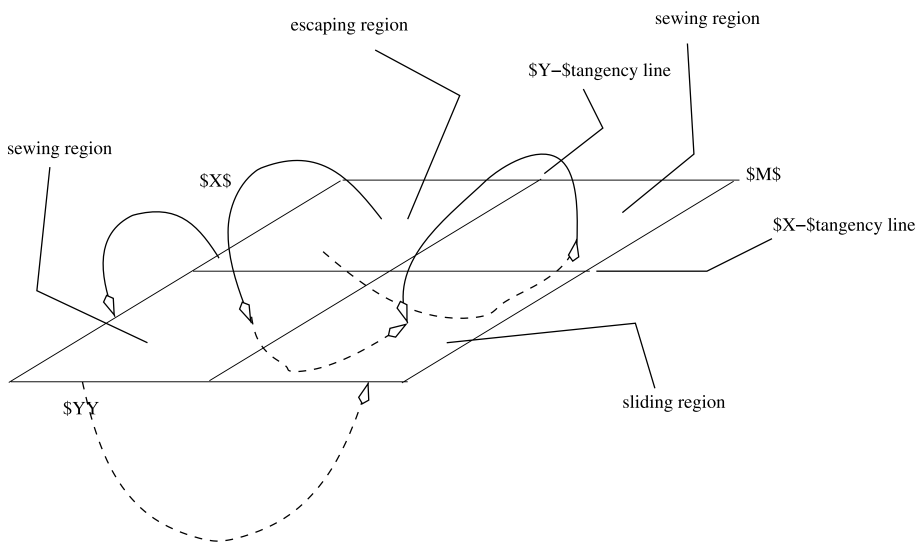
Consider in a small neighborhood of 0 in R^2, the system $Z = (X,Y)$ with $X(x,y) = (1 - x^3 + y^2, x)$, $Y(x,y) = (1 + x + y, -x + x^2)$ and $h(x,y) = y$. The point 0 is a *fold-fold*-singularity of Z with $M_2 = \{x < 0\}$ and $Z^s(x,0) = (2x - x^2)^{-1}(2x - x^4 + x^5)$. Observe that 0 is a hyperbolic critical point of the extended system $G(x,y) = 2x - x^4 + x^5$.

The classification of the codimension-two singularities in $\Omega^r(2)$ is still an open problem. In this direction [51] contains information about the classification of codimension-two M-singularities.

Three-Dimensional Case

Let $Z = (X,Y) \in \Omega^r(3)$.

The most interesting case to be analyzed is when both vector fields, X and Y are fold singularities at 0 and the tangency sets τ_X and τ_Y in M are in general position at 0. In fact they determine (in M) four quadrants, two of them are M_1-regions, one is an M_3-region and the other is an M_2-region (see Fig. 5). We emphasize that the sliding vector field Z^M can be C^r-extended to a full neighborhood of 0 in M. Moreover, $Z^M(0) = 0$. Inside this class the *distinguished fold-fold* singularity (as defined in Subsect. "A Construction") must be taken into account. Denote by A the set of all *distinguished fold-fold singularities* $Z \in \Omega^r(3)$. Moreover, the eigenvalues of $D\beta_Z(0)$ are $\lambda = a \pm \sqrt{(a^2-1)}$. If $\lambda \in R$ we say that Z belongs to A_s. Otherwise Z is in A_e. Recall that β_Z is the first return mapping associated to Z and M at 0 as defined in Subsect. "A Construction".



Perturbation Theory for Non-smooth Systems, Figure 5
The distinguished fold-fold singularity

It is evident that the elements in the open set A_e are structurally unstable in $\Omega^r(3)$. It is worthwhile to mention that in A_e we detect elements which are asymptotically stable at the origin [48]. Concerning A_s few things are known.

We have the following result:

Theorem C *The vector field $Z = (X, Y)$ belongs to $\Sigma_0(3)$ provided that one of the following conditions occurs:*

i) *Both elements X and Y are regular. When $0 \in M$ is a simple singularity of Z then we assume that it is a hyperbolic critical point of Z^M.*
ii) *X is a fold singularity at 0 and Y is regular.*
iii) *X is a cusp singularity at 0 and Y is regular.*
iv) *Both systems X and Y are of fold type at 0. Moreover: a) the tangency sets τ_X and τ_Y are in general position at 0 in M; b) The eigenspaces associated with Z^M are transverse to τ_X and τ_Y at $0 \in M$ and c) Z* ***is not in*** *A. Moreover the real parts of non conjugate eigenvalues are distinct.*

We recall that bifurcation diagrams of sliding vector fields are presented in [50,52].

Singular Perturbation Problem in 2D

Geometric singular perturbation theory is an important tool in the field of continuous dynamical systems. Needless to say that in this area very good surveys are available (refer to [16,18,30]). Here we highlight some results (see [10]) that bridge the space between discontinuous systems in $\Omega^r(2)$ and singularly perturbed smooth systems.

Definition 14 Let $U \subset R^2$ be an open subset and $\varepsilon \geq 0$. A singular perturbation problem in U (SP-Problem) is a differential system which can be written as

$$x' = \frac{dx}{d}\tau = f(x, y, \varepsilon)\,, \quad y' = \frac{dy}{d}\tau = \varepsilon g(x, y, \varepsilon) \quad (6)$$

or equivalently, after the time re-scaling $t = \varepsilon\tau$

$$\varepsilon\dot{x} = \varepsilon\frac{dx}{d}t = f(x, y, \varepsilon)\,, \quad \dot{y} = \frac{dy}{d}t = g(x, y, \varepsilon)\,, \quad (7)$$

with $(x, y) \in U$ and f, g smooth in all variables.

Our first result is concerned with the transition between non-smooth systems and GSPT.

Theorem D *Consider $Z \in \Omega^r(2)$, Z_ε its φ-regularization, and $p \in M$. Suppose that φ is a polynomial of degree k in a small interval $I \subseteq (-1, 1)$ with $0 \in I$. Then the trajectories of Z_ε in $V_\varepsilon = \{q \in R^2, 0\colon h(q)/\varepsilon \in I\}$ are in correspondence with the solutions of an ordinary differential equation $z' = \alpha(z, \varepsilon)$, satisfying that α is smooth in both variables and $\alpha(z, 0) = 0$ for any $z \in M$. Moreover, if $((X - Y)h^k)(p) \neq 0$ then we can take a C^{r-1}-local coordinate system $\{(\partial/\partial x)(p), (\partial/\partial y)(p)\}$ such that this smooth ordinary differential equation is a SP-problem.*

The understanding of the phase portrait of the vector field associated to a SP-problem is the main goal of the *geometric singular perturbation-theory* (GSP-theory). The techniques of GSP-theory can be used to obtain information on the dynamics of (6) for small values of $\varepsilon > 0$, mainly in searching minimal sets.

System (6) is called the *fast system*, and (7) the *slow system* of the SP-problem. Observe that for $\varepsilon > 0$ the phase portraits of the fast and the slow systems coincide.

Theorem D says that we can transform a discontinuous vector field in a SP-problem. In general this transition cannot be done explicitly. Theorem E provides an explicit formula of the SP-problem for a suitable class of vector fields. Before the statement of such a result we need to present some preliminaries.

Consider $C = \{\xi\colon R^2, 0 \to R\}$ with $\xi \in C^r$ and $L(\xi) = 0$ where $L(\xi)$ denotes the linear part of ξ at $(0,0)$.

Let $\Omega_d \subset \Omega^r(2)$ be the set of vector fields $Z = (X, Y)$ in $\Omega^r(2)$ such that there exists $\xi \in C$ that is a solution of

$$\nabla\xi(X - Y) = \Pi_i(X - Y)\,, \tag{8}$$

where $\nabla\xi$ is the gradient of the function and Π_i denote the canonical projections, for $i = 1$ or $i = 2$.

Theorem E *Consider $Z \in \Omega_d$ and Z_ε its φ-regularization. Suppose that φ is a polynomial of degree k in a small interval $I \subset R$ with $0 \in I$. Then the trajectories of Z_ε on $V_\varepsilon = \{q \in R^2, 0\colon h(q)/\varepsilon \in I\}$ are solutions of a SP-problem.*

We remark that the singular problems discussed in the previous theorems, when $\varepsilon \searrow 0$, defines a dynamical system on the discontinuous set of the original problem. This fact can be very useful for problems in Control Theory.

Our third theorem says how the fast and the slow systems approximate the discontinuous vector field. Moreover, we can deduce from the proof that whereas the fast system approximates the discontinuous vector field, the slow system approaches the corresponding sliding vector field.

Consider $Z \in \Omega^r(2)$ and $\rho\colon R^2, 0 \longrightarrow R$ with $\rho(x, y)$ being the distance between (x, y) and M. We denote by $\widehat{Z}$ the vector field given by $\widehat{Z}(x, y) = \rho(x, y)Z(x, y)$.

In what follows we identify $\widehat{Z}_\varepsilon$ and the vector field on $\{\{R^2, 0\} \setminus M \times R\} \subset R^3$ given by $\widehat{Z}(x, y, \varepsilon) = (\widehat{Z}_\varepsilon(x, y), 0)$.

Theorem F *Consider $p = 0 \in M$. Then there exists an open set $U \subset R^2$, $p \in U$, a three-dimensional manifold M, a smooth function $\Phi\colon M \longrightarrow R^3$ and a SP-problem W on M such that Φ sends orbits of $W|_{\Phi^{-1}(U\times(0,+\infty))}$ in orbits of $\widehat{Z}|_{(U\times(0,+\infty))}$.*

Examples

1. Take $X(x, y) = (1, x)$, $Y(x, y) = (-1, -3x)$, and $h(x, y) = y$. The discontinuity set is $\{(x, 0) \mid x \in R\}$. We have $Xh = x$, $Yh = -3x$, and then the unique non-regular point is $(0,0)$. In this case we may apply Theorem E.
2. Let $Z_\varepsilon(x, y) = \big(y/\varepsilon, 2xy/\varepsilon - x\big)$. The associated partial differential equation (refer to Theorem E) with $i = 2$ given above becomes $2(\partial\xi/\partial x) + 4x(\partial\xi/\partial y) = 4x$. We take the coordinate change $\overline{x} = x, \overline{y} = y - x^2$. The trajectories of X_ε in these coordinates are the solutions of the singular system
$$\varepsilon\dot{\overline{x}} = \overline{y} + \overline{x}^2\,, \quad \dot{\overline{y}} = -\overline{x}\,.$$
3. In what follows we try, by means of an example, to present a rough idea on the transition from non-smooth systems to GSPT. Consider $X(x, y) = (3y^2 - y - 2, 1)$, $Y(x, y) = (-3y^2 - y + 2, -1)$ and $h(x, y) = x$. The regularized vector field is
$$Z_\varepsilon(x, y) = \left(\frac{1}{2} + \frac{1}{2}\varphi\left(\frac{x}{\varepsilon}\right)\right)(3y^2 - y - 2, 1) + \left(\frac{1}{2} - \frac{1}{2}\varphi\left(\frac{x}{\varepsilon}\right)\right)(-3y^2 - y + 2, -1)\,.$$

After performing the polar blow up coordinates $\alpha\colon [0, +\infty) \times [0, \pi] \times R \to R^3$ given by $x = r\cos\theta$ and $\varepsilon = r\sin\theta$ the last system is expressed by:

$$r\dot{\theta} = -\sin\theta(-y + \varphi(\cot\theta)(3y^2 - 2))\,, \quad \dot{y} = \varphi(\cot\theta)\,.$$

So the slow manifold is given implicitly by $\varphi(\cot\theta) = y/(3y^2 - 2)$ which defines two functions $y_1(\theta) = (1 + \sqrt{1 + 24\varphi^2(\cot\theta)})/(6\varphi(\cot\theta))$ and $y_2(\theta) = (1 - \sqrt{1 + 24\varphi^2(\cot\theta)})/(6\varphi(\cot\theta))$. The function $y_1(\theta)$ is increasing, $y_1(0) = 1$, $\lim_{\theta\to\pi/2^-} y_1(\theta) = +\infty$, $\lim_{\theta\to\pi/2^+} y_1(\theta) = -\infty$ and $y_1(\pi) = -1$. The function $y_2(\theta)$ is increasing, $y_2(0) = -2/3$, $\lim_{\theta\to\pi/2} y_2(\theta) = 0$ and $y_2(\pi) = 2/3$. We can extend y_2 to $(0, \pi)$ as a differential function with $y_2(\pi/2) = 0$.

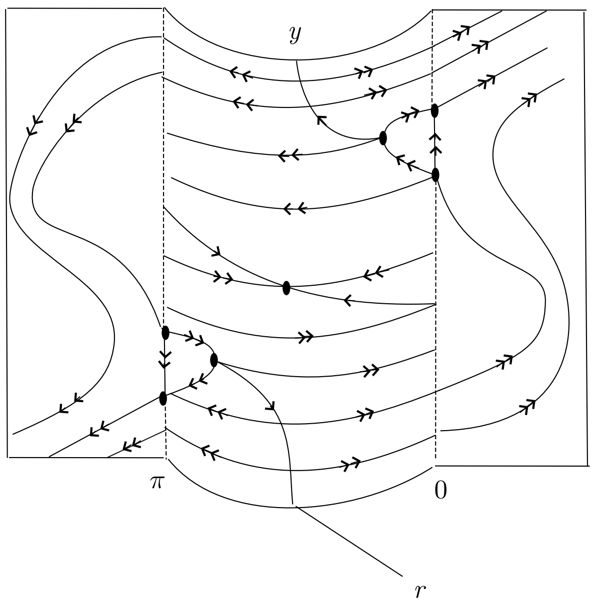
The fast vector field is $(\theta', 0)$ with $\theta' > 0$ if (θ, y) belongs to

$$\left[\left(0, \frac{\pi}{2}\right) \times (y_2(\theta), y_1(\theta)) \bigcup \left(\frac{\pi}{2}, \pi\right) \times (y_2(\theta), +\infty) \bigcup \left(\frac{\pi}{2}, \pi\right) \times (-\infty, y_1(\theta))\right]$$

and with $\theta' < 0$ if (θ, y) belongs to

$$\left[\left(0, \frac{\pi}{2}\right) \times (y_1(\theta), +\infty) \bigcup \left(0, \frac{\pi}{2}\right) \times (-\infty, y_2(\theta)) \bigcup \left(\frac{\pi}{2}, \pi\right) \times (y_1(\theta), y_2(\theta))\right].$$

The reduced flow has one singular point at $(0,0)$ and it takes the positive direction of the y-axis if $y \in (-\frac{2}{3}, 0) \cup (1, \infty)$ and the negative direction of the y-axis if $y \in (-\infty, -1) \cup (0, \frac{2}{3})$.



Perturbation Theory for Non-smooth Systems, Figure 6
Example of fast and slow dynamics of the SP-Problem

One can see that the singularities $(\theta, y, r) = (0, 1, 0)$ and $(\theta, y, r) = (0, -1, 0)$ are not normally hyperbolic points. In this way, as usual, we perform additional blow ups. In Fig. 6 we illustrate the fast and the slow dynamics of the SP-problem. We present a phase portrait on the blowing up locus where a double arrow over a trajectory means that the trajectory belongs to the fast dynamical system, and a simple arrow means that the trajectory belongs to the slow dynamical system.

Future Directions

Our concluding section is devoted to an outlook. Firstly we present some open problems linked with the setting that point out future directions of research. The main task for the future seems to bring the theory of non-smooth dynamical systems to a similar maturity as that of smooth systems. Finally we briefly discuss the main results in this text.

Some Problems

In connection to this present work, some theoretical problems remain open:

1. The description of the bifurcation diagram of the codimension-two singularities in $\Omega(2)$. In this last class we find some models (see [37]) where the following questions can also be addressed. a) When is a typical singularity topologically equivalent to a regular center? b) How about the isochronicity of such a center? c) When does a polynomial perturbation of such a system in $\Omega(2)$ produce limit cycles? The articles [9,13,21,22,47] can be useful auxiliary references.
2. Let $\Omega(N)$ be the set of all non-smooth vector fields on a two-dimensional compact manifold N having a codimension-one compact submanifold M as its discontinuity set. The problem is to study the global generic bifurcation in $\Omega(N)$. The articles [31,33,40,46] can be useful auxiliary references.
3. Study of the bifurcation set in $\Omega^r(3)$. The articles [38,40,43,50] can be useful auxiliary references.
4. Study of the dynamics of the distinguished *fold-fold* singularity in $\Omega^r(n+1)$. The article [48] can be a useful auxiliary reference.
5. In many applications examples of non-smooth systems where the discontinuities are located on algebraic varieties are available. For instance, consider the system $\ddot{x} + x sign(x) + sign(\dot{x}) = 0$. Motivated by such models we present the following problem. Let 0 be a non-degenerate critical point of a smooth mapping $h: R^{n+1}, 0 \to R, 0$. Let $\Phi(n+1)$ be the space of all vector fields Z on $R^{n+1}, 0$ defined in the same way as $\Omega(n+1)$. We propose the following. i) Classify the typical singularities in that space. ii) Analyze the elements of $\Phi(2)$ by means of "regularization processes" and the methods of GSPT, similarly to Sect. "Vector Fields near the Boundary". The articles [1,2] can be very useful auxiliary references.
6. In [27,29] classes of $4D$-relay systems are considered. Conditions for the existence of one-parameter families of periodic orbits terminating at typical singularities are provided. We propose to find conditions for the existence of such families for *n-dimensional* relay systems.

Conclusion

In this paper we have presented a compact survey of the geometric/qualitative theoretical features of non-smooth dynamical systems. We feel that our survey illustrates that this field is still in its early stages but enjoying growing interest. Given the importance and the relevance of such a theme, we have pointed above some open questions and we remark that there is still a wide range of bifurcation problems to be tackled. A brief summary of the main results in the text is given below.

1. We firstly deal with two-*dimensional* non-smooth vector fields $Z = (X, Y)$ defined around the origin in R^2, where the discontinuity set is concentrated on the line $\{y = 0\}$. The first task is to characterize those systems which are structurally stable. This characterization is

a starting point with which to establish a bifurcation theory as indicated by the Thom–Smale program.

2. In higher dimension the problem becomes much more complicated. We have presented here sufficient conditions for the three-*dimensional* local structural stability. Any further investigation on bifurcation in this context must pass through a deep analysis of the so called *fold-fold* singularity.
3. We have established a bridge between discontinuous and singularly perturbed smooth systems. Many similarities between such systems were observed and a comparative study of the two categories is called for.

Bibliography

Primary Literature

1. Alexander JC, Seidman TI (1998) Sliding modes in intersecting switching surfaces I: Blending. Houston J Math 24(3):545–569
2. Alexander JC, Seidman TI (1999) Sliding modes in intersecting switching surfaces II: Hysteresis. Houston J Math 25:185–211
3. Andronov A, Pontryagin S (1937) Structurally stable systems. Dokl Akad Nauk SSSR 14:247–250
4. Andronov AA, Vitt AA, Khaikin SE (1966) Theory of oscillators, Dover, New York
5. Anosov DV (1959) Stability of the equilibrium positions in relay systems. Autom Remote Control XX(2):135–149
6. Arnold VI (1983) Methods in the theory of ordinary differential equations. Springer, New York
7. Bazykin AD (1998) Nonlinear dynamics of interacting populations. World Sc. Publ. Co. Inc., River–Edge, NJ
8. Bonnard B, Chyba M (2000) Singular trajectories and their role in control theory. Mathématiques and Applications, vol 40. Springer, Berlin
9. Broucke ME, Pugh CC, Simić SN (2001) Structural stability of piecewise smooth systems. Comput Appl Math 20(1–2):51–89
10. Buzzi C, Silva PR, Teixeira MA (2006) A singular approach to discontinuous vector fields on the plane. J Differ Equ 23:633–655
11. Chillingworth DR (2002) J.(4-SHMP) Discontinuity geometry for an impact oscillator. Dyn Syst 17(4):389–420
12. Chua LO (1992) The genesis of Chua's circuit. AEU 46:250
13. Coll B, Gasull A, Prohens R (2001) Degenerate Hopf bifurcations in discontinuous planar systems. J Math Anal Appl 253(2):671–690
14. di Bernardo M, Budd C, Champneys AR, Kowalczyk P, Nordmark AB, Olivar G, Piiroinen PT (2005) Bifurcations in non-smooth dynamical systems. Bristol Centre for Applied Nonlinear Mathematics, N. 2005-4, Bristol
15. Dumortier F (1977) Singularities of vector fields. IMPA, Rio de Janeiro
16. Dumortier F, Roussarie R (1996) Canard cycles and center manifolds. Mem Amer Mat Soc 121:x+100
17. Ekeland I (1977) Discontinuités de champs hamiltoniens et existence de solutions optimales en calcul des variations. Inst Hautes Études Sci Publ Math 47:5–32
18. Fenichel N (1979) Geometric singular perturbation theory for ordinary differential equations. J Differ Equ 31:53–98
19. Filippov AF (1988) Differential equations with discontinuous righthand sides. Kluwer Academic, Dordrecht
20. Flúgge-Lotz I (1953) Discontinuous automatic control. Princeton University, Princeton, pp vii+168
21. Garcia R, Teixeira MA (2004) Vector fields in manifolds with boundary and reversibility-an expository account. Qual Theory Dyn Syst 4:311–327
22. Gasull A, Torregrosa J (2003) Center-focus problem for discontinuous planar differential equations. Int J Bifurc Chaos Appl Sci Eng 13(7):1755–1766
23. Henry P (1972) Diff. equations with discontinuous right-hand side for planning procedures. J Econ Theory 4:545–551
24. Hogan S (1989) On the dynamics of rigid-block motion under harmonic forcing. Proc Roy Soc London A 425:441–476
25. Ito T (1979) A Filippov solution of a system of diff. eq. with discontinuous right-hand sides. Econ Lett 4:349–354
26. Jackson JD (1999) Classical electrodynamics, 3rd edn. Wiley, New York, pp xxii+808
27. Jacquemard A, Teixeira MA (2003) Computer analysis of periodic orbits of discontinuous vector fields. J Symbol Comput 35:617–636
28. Jacquemard A, Teixeira MA (2003) On singularities of discontinuous vector fields. Bull Sci Math 127:611–633
29. Jacquemard A, Teixeira MA (2005) Invariant varieties of discontinuous vector fields. Nonlinearity 18:21–43
30. Jones C (1995) Geometric singular perturbation theory. C.I.M.E. Lectures, Montecatini Terme, June 1994, Lecture Notes in Mathematics 1609. Springer, Heidelberg
31. Kozlova VS (1984) Roughness of a discontinuous system. Vestinik Moskovskogo Universiteta, Matematika 5:16–20
32. Kunze M, Kupper T (1997) Qualitative bifurcation analysis of a non-smooth friction oscillator model. Z Angew Math Phys 48:87–101
33. Kuznetsov YA et al (2003) One-parameter bifurcations in planar Filippov systems. Int J Bifurac Chaos 13:2157–2188
34. Llibre J, Teixeira MA (1997) Regularization of discontinuous vector fields in dimension 3. Discret Contin Dyn Syst 3(2):235–241
35. Luo CJ (2006) Singularity and dynamics on discontinuous vector fields. Monograph Series on Nonlinear Science and Complexity. Elsevier, New York, pp i+310
36. Machado AL, Sotomayor J (2002) Structurally stable discontinuous vector fields in the plane. Qual Theory Dyn Syst 3:227–250
37. Manosas F, Torres PJ (2005) Isochronicity of a class of piecewise continuous oscillators. Proc of AMS 133(10):3027–3035
38. Medrado J, Teixeira MA (1998) Symmetric singularities of reversible vector fields in dimension three. Physica D 112:122–131
39. Medrado J, Teixeira MA (2001) Codimension-two singularities of reversible vector fields in 3D. Qualit Theory Dyn Syst J 2(2):399–428
40. Percell PB (1973) Structural stability on manifolds with boundary. Topology 12:123–144
41. Seidman T (2006) Aspects of modeling with discontinuities. In: N'Guerekata G (ed) Advances in Applied and Computational Mathematics Proc Dover Conf, Cambridge. http://www.umbc.edu/~seideman

42. Sotomayor J (1974) Structural stability in manifolds with boundary. In: Global analysis and its applications, vol III. IEAA, Vienna, pp 167–176
43. Sotomayor J, Teixeira MA (1988) Vector fields near the boundary of a 3-manifold. Lect Notes in Math, vol 331. Springer, Berlin-Heidelberg, pp 169–195
44. Sotomayor J, Teixeira MA (1996) Regularization of discontinuous vector fields. International Conference on Differential Equations, Lisboa. World Sc, Singapore, pp 207–223
45. Sussmann H (1979) Subanalytic sets and feedback control. J Differ Equ 31:31–52
46. Teixeira MA (1977) Generic bifurcations in manifolds with boundary. J Differ Equ 25:65–89
47. Teixeira MA (1979) Generic bifurcation of certain singularities. Boll Unione Mat Ital 16-B:238–254
48. Teixeira MA (1990) Stability conditions for discontinuous vector fields. J Differ Equ 88:15–24
49. Teixeira MA (1991) Generic Singularities of Discontinuous Vector Fields. An Ac Bras Cienc 53(2):257–260
50. Teixeira MA (1993) Generic bifurcation of sliding vector fields. J Math Anal Appl 176:436–457
51. Teixeira MA (1997) Singularities of reversible vector fields. Physica D 100:101–118
52. Teixeira MA (1999) Codimension-two singularities of sliding vector fields. Bull Belg Math Soc 6(3):369–381
53. Vishik SM (1972) Vector fields near the boundary of a manifold. Vestnik Moskovskogo Universiteta, Matematika 27(1): 13–19
54. Zelikin MI, Borisov VF (1994) Theory of chattering control with applications to astronautics, robotics, economics, and engineering. Birkhäuser, Boston

Books and Reviews

Agrachev AA, Sachkov YL (2004) Control theory from the geometric viewpoint. Encyclopaedia of Mathematical Sciences, 87. Control theory and optimization, vol II. Springer, Berlin, pp xiv+412

Barbashin EA (1970) Introduction to the theory of stability. Wolters–Noordhoff Publishing, Groningen. Translated from the Russian by Transcripta Service, London. Edited by T. Lukes, pp 223

Brogliato B (ed) (2000) Impacts in mechanical systems. Lect Notes in Phys, vol 551. Springer, Berlin, pp 160

Carmona V, Freire E, Ponce E, Torres F (2005) Bifurcation of invariant cones in piecewise linear homogeneous systems. Int J Bifurc Chaos 15(8):2469–2484

Davydov AA (1994) Qualitative theory of control systems. American Mathematical Society, Providence, RI. (English summary) Translated from the Russian manuscript by V. M. Volosov. Translations of Mathematical Monographs, 141, pp viii+147

Dercole F, Gragnani F, Kuznetsov YA, Rinaldi S (2003) Numerical sliding bifurcation analysis: an application to a relay control system. IEEE Trans Circuit Systems – I: Fund Theory Appl 50:1058–1063

Glocker C (2001) Set-valued force laws: dynamics of non-smooth systems. Lecture Notes in Applied Mechanics, vol 1. Springer, Berlin

Hogan J, Champneys A, Krauskopf B, di Bernardo M, Wilson E, Osinga H, Homer M (eds) Nonlinear dynamics and chaos: Where do we go from here? Institute of Physics Publishing (IOP), Bristol, pp xi+358, 2003

Huertas JL, Chen WK, Madan RN (eds) (1997) Visions of nonlinear science in the 21st century, Part I. Festschrift dedicated to Leon O. Chua on the occasion of his 60th birthday. Papers from the workshop held in Sevilla, June 26, 1996. Int J Bifurc Chaos Appl Sci Eng 7 (1997), no. 9. World Scientific Publishing Co. Pte. Ltd., Singapore, pp i–iv, pp 1907–2173

Komuro M (1990) Periodic bifurcation of continuous piece-wise vector fields. In: Shiraiwa K (ed) Advanced series in dynamical systems, vol 9. World Sc, Singapore

Kunze M (2000) Non-smooth dynamical systems. Lecture Notes in Mathematics, vol 1744. Springer, Berlin, pp x+228

Kunze M, Küpper T (2001) Non-smooth dynamical systems: an overview. In: Fiedler B (ed) Ergodic theory, analysis, and efficient simulation of dynamical systems. Springer, Berlin, pp 431–452

Llibre J, Silva PR, Teixeira MA (2007) Regularization of discontinuous vector fields on R^3 via singular perturbation. J Dyn Differ Equ 19(2):309–331

Martinet J (1974) Singularités des fonctions et applications différentiables. Pontifícia Universidade Catolica do Rio de Janeiro, Rio de Janeiro, pp xiii+181 (French)

Minorsky N (1962) Nonlinear oscillations. D Van Nostrand Co. Inc., Princeton, N.J., Toronto, London, New York, pp xviii+714

Minorsky N (1967) Théorie des oscillations. Mémorial des Sciences Mathématiques Fasc, vol 163. Gauthier-Villars, Paris, pp i+114 (French)

Minorsky N (1969) Theory of nonlinear control systems. McGraw-Hill, New York, London, Sydney, pp xx+331

Ostrowski JP, Burdick JW (1997) Controllability for mechanical systems with symmetries and constraints. Recent developments in robotics. Appl Math Comput Sci 7(2):305–331

Peixoto MC, Peixoto MM (1959) Structural Stability in the plane with enlarged conditions. Anais da Acad Bras Ciências 31:135–160

Rega G, Lenci S (2003) Nonsmooth dynamics, bifurcation and control in an impact system. Syst Anal Model Simul Arch 43(3), Gordon and Breach Science Publishers, Inc. Newark, pp 343–360

Seidman T (1995) Some limit problems for relays. In: Lakshmikantham V (ed) Proc First World Congress of Nonlinear Analysis, vol I. Walter de Gruyter, Berlin, pp 787–796

Sotomayor J (1974) Generic one-parameter families of vector fields on 2-manifolds. Publ Math IHES 43:5–46

Szmolyan P (1991) Transversal heteroclinic and homoclinic orbits in singular perturbation problems. J Differ Equ 92:252–281

Teixeira MA (1985) Topological invariant for discontinuous vector fields. Nonlinear Anal TMA 9(10)1073–1080

Utkin V (1978) Sliding modes and their application in variable structure systems. Mir, Moscow

Utkin V (1992) Sliding Modes in Control and Optimization. Springer, Berlin