4

Expansion of Positivity

1 Time and Space Propagation of Positivity

The *Expansion of Positivity* is a property of nonnegative supersolutions to elliptic and parabolic partial differential equations, that is at the heart of any form of Harnack estimate. Roughly speaking, it asserts that information on the *measure* of the "positivity set" of u at the time level s, over the cube $K_\rho(y)$, translates into an expansion of the positivity set both in space (from a cube $K_\rho(y)$ to $K_{2\rho}(y)$), and in time (from s to $s+\theta\rho^2$, for some suitable θ).

Such an expansion involves some unavoidable technical arguments. To convey the main ideas we will present it first in § 2 in the context of nondegenerate ($p=2$ or $m=1$), homogeneous equations. Then we will present it separately for degenerate ($p>2$ or $m>1$) and singular ($1<p<2$ or $0<m<1$) equations with full quasilinear structure. In all cases one first "propagates" a positivity information at some time level s on a cube $K_\rho(y)$ to further times, within the same cube. Then one expands the positivity set in the space variables from $K_\rho(y)$ to $K_{2\rho}(y)$.

The first step of time propagation of positivity is technically common to all cases and we present it here in a unified fashion.

Henceforth in this section assume that u is a nonnegative, local, weak supersolution in E_T to (1.1)–(1.2) of Chapter 3, for some $p>1$.

Most of our arguments and proofs are based on the energy estimates and DeGiorgi-type lemmas of § 2–4 of Chapter 3. According to the discussion in § 1.3 and Remarks 2.2, 3.1, and 4.3 of Chapter 3, a constant γ depends only on the data if it can be quantitatively determined a priori only in terms of $\{p, N, C_o, C_1\}$. The constant C appearing in the structure conditions (1.2) of Chapter 3, enters in the various statements only via some alternatives.

For $(y,s)\in E_T$ and $n,m\in\mathbb{N}$, introduce the "forward" and "backward" cylinders

$$(y,s)+\mathcal{Q}^+_{n\rho}(m\theta) = K_{n\rho}(y)\times(s, s+m\theta\rho^p]$$
$$(y,s)+\mathcal{Q}^-_{n\rho}(m\theta) = K_{n\rho}(y)\times(s-m\theta\rho^p, s].$$

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These differ from the cylinders $Q_\rho^\pm(\theta)$ introduced in (2.1)–(2.2) of Chapter 3, in that their cross section $K_{n\rho}(y)$ and their height $\theta m\rho^p$ are permitted to vary independently. In what follows it will be assumed that $(y,s)\in E_T$ and $\rho>0$ are such that $(y,s)+\mathcal{Q}_{n\rho}^\pm(m\theta)\subset E_T$.

Lemma 1.1 *Assume that for some $(y,s)\in E_T$ and some $\rho>0$*

$$\big|[u(\cdot,s)\ge M]\cap K_\rho(y)\big|\ge\alpha\big|K_\rho(y)\big|$$

for some $M>0$ and some $\alpha\in(0,1)$. There exist δ and ϵ in $(0,1)$, depending only on the data $\{p,N,C_o,C_1\}$, and α, and independent of M, such that either

$$C\rho>\min\{1\,,\,M\}$$

or

$$|[u(\cdot,t)>\epsilon M]\cap K_\rho(y)|\ge\tfrac12\alpha|K_\rho|\quad \textit{for all}\;\; t\in\Big(s,s+\frac{\delta\rho^p}{M^{p-2}}\Big]. \tag{1.1}$$

Proof Assume $(y,s)=(0,0)$ and for $k>0$ and $t>0$ set

$$A_{k,\rho}(t)=[u(\cdot,t)<k]\cap K_\rho.$$

The assumption implies

$$|A_{M,\rho}(0)|\le(1-\alpha)|K_\rho|. \tag{1.2}$$

Write down the energy inequalities (2.3) of Chapter 3, for the truncated functions $(u-M)_-$, over the cylinder $K_\rho\times(0,\theta\rho^p]$, where $\theta>0$ is to be chosen. The cutoff function ζ is taken independent of t, nonnegative, and such that

$$\zeta=1\quad\text{on}\;\;K_{(1-\sigma)\rho},\qquad\text{and}\qquad |D\zeta|\le\frac{1}{\sigma\rho}$$

where $\sigma\in(0,1)$ is to be chosen. Discarding the nonnegative term containing $D(u-M)_-$ on the left-hand side, these inequalities yield

$$\begin{aligned}\int_{K_{(1-\sigma)\rho}}&(u-M)_-^2(x,t)dx\le\int_{K_\rho}(u-M)_-^2(x,0)dx\\&\quad+\frac{\gamma}{(\sigma\rho)^p}\int_0^{\theta\rho^p}\int_{K_\rho}(u-M)_-^p\,dx\,d\tau\\&\quad+\gamma C^p\int_0^{\theta\rho^p}\int_{K_\rho}\big[\chi_{[u<M]}+(u-M)_-^p\big]dx\,d\tau\\&\le M^2\Big[(1-\alpha)+\gamma\frac{\theta M^{p-2}}{\sigma^p}+\gamma\Big(\frac{C\rho}{\min\{1\,,\,M\}}\Big)^p\theta M^{p-2}\Big]|K_\rho|\\&\le M^2\Big[(1-\alpha)+2\gamma\frac{\theta M^{p-2}}{\sigma^p}\Big]|K_\rho|\end{aligned}$$

for all $t \in (0, \theta\rho^p]$, where we have enforced (1.2), and provided that $C\rho < M$, $C < \rho^{-1}$. The left-hand side is estimated below by

$$\begin{aligned}\int_{K_{(1-\sigma)\rho}} (u-M)_-^2(x,t)dx &\geq \int_{K_{(1-\sigma)\rho}\cap[u<\epsilon M]} (u-M)_-^2(x,t)dx \\ &\geq M^2(1-\epsilon)^2|A_{\epsilon M,(1-\sigma)\rho}(t)|\end{aligned}$$

where $\epsilon \in (0,1)$ is to be chosen. Next estimate

$$\begin{aligned}|A_{\epsilon M,\rho}(t)| &= |A_{\epsilon M,(1-\sigma)\rho}(t) \cup (A_{\epsilon M,\rho}(t) - A_{\epsilon M,(1-\sigma)\rho}(t))| \\ &\leq |A_{\epsilon M,(1-\sigma)\rho}(t)| + |K_\rho - K_{(1-\sigma)\rho}| \\ &\leq |A_{\epsilon M,(1-\sigma)\rho}(t)| + N\sigma|K_\rho|.\end{aligned}$$

Combining these estimates gives

$$\begin{aligned}|A_{\epsilon M,\rho}(t)| &\leq \frac{1}{M^2(1-\epsilon)^2}\int_{K_{(1-\sigma)\rho}} (u-M)_-^2(x,t)dx + N\sigma|K_\rho| \\ &\leq \frac{1}{(1-\epsilon)^2}\Big[(1-\alpha) + \frac{2\gamma}{\sigma^p}\theta M^{p-2} + N\sigma\Big]|K_\rho|.\end{aligned}$$

Choose $\theta = \delta M^{2-p}$ and then choose

$$\sigma = \frac{\alpha}{8N}, \quad \epsilon \leq 1 - \frac{\sqrt{1-\frac{3}{4}\alpha}}{\sqrt{1-\frac{1}{2}\alpha}} \approx \frac{1}{8}\alpha, \quad \delta = \frac{\alpha^{p+1}}{2^{3p+4}\gamma N^p}. \tag{1.3}$$

This proves the lemma. ■

Remark 1.1 The proof is based on the energy inequalities (2.3) of Chapter 3, whose constant dependence is indicated in Remark 2.1. Therefore the constant $\delta = \delta(p)$ deteriorates either as $p \to 1$ or as $p \to \infty$, but it is stable as $p \to 2$, with seamless transition from the singular range $p < 2$ to the degenerate range $p > 2$.

Remark 1.2 If $p = 2$, one takes $\theta = \delta$ and the interval in (1.1) becomes independent of M.

2 The Expansion of Positivity for Nondegenerate, Homogeneous, Quasilinear Parabolic Equations

Let u be a nonnegative, local, weak supersolution in E_T to (1.1)–(1.2) of Chapter 3, with $p = 2$ and $C = 0$.

Proposition 2.1 *Assume that for some* $(y,s) \in E_T$ *and some* $\rho > 0$

$$|[u(\cdot,s) \geq M] \cap K_\rho(y)| \geq \alpha\big|K_\rho(y)\big| \tag{2.1}$$

for some $M > 0$ and some $\alpha \in (0,1)$. There exist constants η and $\delta \in (0,1)$ depending only on the data $\{N, C_o, C_1\}$, and α, such that

$$u \geq \eta M \qquad a.e.\ in\ \ K_{2\rho}(y) \times \big(s + \tfrac{1}{2}\delta\rho^2\,,\, s + \delta\rho^2\big]. \tag{2.2}$$

Proof Assume $(y,s) = (0,0)$. The number $\alpha > 0$ being fixed, let δ and ϵ be the numbers claimed by Lemma 1.1 for $p = 2$. The conclusion of the lemma implies that

$$|[u(\cdot,t) > \epsilon M] \cap K_{4\rho}| > \tfrac{1}{2}\alpha 4^{-N}|K_{4\rho}|, \qquad \text{for all}\ \ t \in (0, \delta\rho^2). \tag{2.3}$$

Lemma 2.1 *For every $\nu \in (0,1)$ there exists ϵ_ν depending only on the data $\{N, C_o, C_1\}$, δ (and hence α), and ν, such that*

$$\big|[u < \epsilon_\nu M] \cap \mathcal{Q}^+_{4\rho}(\delta)\big| < \nu\big|\mathcal{Q}^+_{4\rho}(\delta)\big|. \tag{2.4}$$

Thus the set $[u < \epsilon_\nu M]$ in the cylinder $\mathcal{Q}^+_{4\rho}(\delta)$ can be made arbitrarily small, provided ϵ_ν is chosen accordingly. The main tools of the proof are the estimate (2.3) of the measure of the sets $A_{\epsilon M,4\rho}(t)$ for all $t \in (0, \delta\rho^2)$, and the discrete isoperimetric inequality of Lemma 2.2 of the Preliminaries.

Proof Write down the energy estimates (2.3) of Chapter 3 over the cylinder

$$\mathcal{Q}^+_{8\rho}(\delta) \cup \mathcal{Q}^-_{8\rho}(\delta) = K_{8\rho} \times (-\delta\rho^2, \delta\rho^2)$$

for the truncated functions

$$(u - k_j)_- \quad \text{for the levels} \quad k_j = \frac{1}{2^j}\epsilon M, \qquad \text{for}\ \ j = 0, 1, \dots.$$

The nonnegative, piecewise smooth, test function ζ is chosen so that it vanishes outside $K_{8\rho}$ and for $t \leq -\delta\rho^2$, and

$$\zeta = 1\ \text{ on }\ \mathcal{Q}^+_{4\rho}(\delta), \qquad |D\zeta| \leq \frac{1}{4\rho}, \qquad \text{and}\ \ 0 \leq \zeta_t \leq \frac{1}{\delta\rho^2}.$$

The first term on the left-hand side is discarded since it is nonnegative, and the second vanishes because of our choice of test function. The term involving $|D(u - k_j)_-|$ is minorized by extending the integration over the cylinder $\mathcal{Q}^+_{4\rho}(\delta)$, which is the set where $\zeta = 1$. The terms containing C on the right-hand side are eliminated since $C = 0$. These remarks give the inequalities

$$\begin{aligned}
&\iint_{\mathcal{Q}^+_{4\rho}(\delta)} |D(u - k_j)_-|^2\zeta^2 dx\, d\tau \\
&\qquad \leq \gamma \int_{-\delta\rho^2}^{\delta\rho^2}\int_{K_{8\rho}} (u - k_j)^2_-\big(|D\zeta|^2 + \zeta_\tau\big)dx\, d\tau \\
&\qquad \leq \gamma\delta\rho^2 k_j^2\Big(\frac{1}{\rho^2} + \frac{2}{\delta\rho^2}\Big)|K_{8\rho}| \\
&\qquad \leq \gamma k_j^2|K_{4\rho}|
\end{aligned} \tag{2.5}$$

for a new constant γ depending only on the data $\{N, C_o, C_1\}$.

Apply the discrete isoperimetric inequality of Lemma 2.2 of the Preliminaries to the levels

$$\ell = k_j = \frac{\epsilon h}{2^j} \quad \text{and} \quad k = k_{j+1} = \frac{\epsilon h}{2^{j+1}} \quad \text{for } j = 0, 1, \dots$$

and take into account (2.3) to obtain

$$k_{j+1}\big|A_{k_{j+1},4\rho}(t)\big| \le \frac{8^N \gamma}{\alpha}\rho \int_{K_{4\rho}\cap[k_{j+1}<u<k_j]} |Du(\cdot,t)|dx.$$

Integrate this in dt over $(0, \delta\rho^2)$ and set

$$|A_j| = |[u < k_j] \cap \mathcal{Q}^+_{4\rho}(\delta)| = \int_0^{\delta\rho^2} \big|A_{k_j}(\tau)\big|d\tau.$$

Then the previous inequality yields

$$\begin{aligned} k_{j+1}\big|A_{j+1}\big| &\le \gamma\rho \iint_{\mathcal{Q}^+_{4\rho}(\delta)\cap[k_{j+1}<u<k_j]} |Du|dx\,d\tau \\ &\le \gamma\rho\Big(\iint_{\mathcal{Q}^+_{4\rho}(\delta)} |D(u-k_j)_-|^2 dx\,d\tau\Big)^{\frac{1}{2}} \big|A_j - A_{j+1}\big|^{\frac{1}{2}} \\ &\le \gamma k_j \sqrt{\mathcal{Q}^+_{4\rho}(\delta)}\,\big(|A_j| - |A_{j+1}|\big)^{\frac{1}{2}} \end{aligned}$$

where we have used the energy estimates (2.5). Next divide by $k_{j+1} = \frac{1}{2}k_j$, and square both sides to obtain the recursive inequalities

$$|A_{j+1}|^2 \le (2\gamma)^2 |\mathcal{Q}^+_{4\rho}(\delta)|\,\big(|A_j| - |A_{j+1}|\big) \quad \text{for } j = 0, 1 \dots.$$

Add these inequalities for $j = 0, 1, \dots, j_* - 1$ where j_* is a positive integer to be chosen. Minorize the terms on the left-hand side by their smallest value $|A_{j_*}|^2$ and majorize the right-hand side with the corresponding telescopic series. The indicated estimations yield

$$\begin{aligned} j_*|A_{j_*}|^2 \le \sum_{j=0}^{j_*-1} |A_{j+1}|^2 &\le (2\gamma)^2 |\mathcal{Q}^+_{4\rho}(\delta)| \sum_{j=0}^{\infty} \big(|A_j| - |A_{j+1}|\big) \\ &\le (2\gamma)^2 |\mathcal{Q}^+_{4\rho}(\delta)|^2. \end{aligned}$$

From this

$$|A_{j_*}| \le \frac{2\gamma}{\sqrt{j_*}}\,|\mathcal{Q}^+_{4\rho}(\delta)|. \tag{2.6}$$

Thus having fixed $\nu \in (0,1)$, one can choose j_* so large that

$$\frac{\big|[u < \epsilon_\nu M] \cap \mathcal{Q}^+_{4\rho}(\delta)\big|}{|\mathcal{Q}^+_{4\rho}(\delta)|} < \nu, \quad \text{for } \frac{2\gamma}{\sqrt{j_*}} \le \nu, \quad \text{and} \quad \epsilon_\nu = \frac{\epsilon}{2^{j_*}}. \qquad \blacksquare$$

Proof (of Proposition 2.1, Concluded) Apply Lemma 3.1 of Chapter 3 over the cylinder $\mathcal{Q}^+_{4\rho}(\delta)$ in the version of (3.1)–(3.3), with $\mu_- = 0$ and $\xi\omega = \epsilon_\nu M$ and $a = \frac{1}{2}$. Choose ν from (3.12) of Chapter 3 and observe that since $p = 2$ (nondegenerate equations), the number ν is independent of $\epsilon_\nu M$. It only depends on the data $\{N, C_o, C_1\}$ and δ, which itself has been determined and fixed in terms of the data $\{N, C_o, C_1\}$ and α. Such a ν being fixed a priori only in terms of the data, choose $j_* \in \mathbb{N}$ by the indicated procedure, so that the assumptions of Lemma 2.1 are verified. Then Lemma 3.1 of Chapter 3 implies that

$$u(x,t) > \tfrac{1}{2}\epsilon_\nu M \qquad \text{a.e. in } K_{2\rho} \times (\tfrac{1}{2}\delta\rho^2, \delta\rho^2).$$

Thus the conclusion holds with $\eta = \frac{1}{2}\epsilon_\nu$. ■

Remark 2.1 If in (2.1) one has $\alpha = 1$, the condition reads

$$u(\cdot, s) \geq M \qquad \text{a.e. in } K_\rho(s) \tag{2.7}$$

which is of the same form as the "initial datum" of (4.1) of Chapter 3. Lemma 4.1 of Chapter 3 then translates that bound below to later times over *smaller cubes.* Proposition 2.1, however, is stronger, as it translates such "initial conditions" into a positivity information at later times and over a *larger cube.*

3 Some Counterexamples for Degenerate and Singular Equations

Let now u be a nonnegative, local, weak supersolution to the prototype equation (1.3) of Chapter 3 in some cylindrical domain E_T, for some $p \neq 2$. If u is bounded below on some cube $K_\rho(y)$, say for example as in (2.7), then the analog of Proposition 2.1 would be that

$$u(\cdot, s + \delta\rho^p) \geq \eta M \qquad \text{a.e. in } K_{2\rho}(y) \tag{3.1}$$

for constants $\delta > 0$ and $\eta \in (0,1)$ depending only on the data $\{p, N, C_o, C_1\}$, and independent of u. It turns out that if $p \neq 2$, no constants δ and η can be determined a priori only in terms of N and p for which (2.7) would imply (3.1).

3.1 A First Counterexample for $p > 2$

Consider the one-parameter family of nonnegative functions defined in the whole $\mathbb{R} \times \mathbb{R}$

$$u(x,t;c) = A(1 - x + ct)_+^{\frac{p-1}{p-2}} \qquad \text{where } A = c^{\frac{1}{p-2}} \left(\frac{p-2}{p-1}\right)^{\frac{p-1}{p-2}}. \tag{3.2}$$

One verifies that such a $u(\cdot,\cdot;c)$ is a weak solution to the homogeneous prototype p-Laplacian equation in the whole $\mathbb{R}\times\mathbb{R}$, for all $c>0$, and is constructed by seeking solutions in the form of traveling waves. Fix

$$(y,s)=\big(\tfrac{1}{2}(1-\varepsilon),0\big),\qquad \rho=\tfrac{1}{2}(1-\varepsilon)$$

and let

$$K_\rho(y)=\big\{|x-\tfrac{1}{2}(1-\varepsilon)|<\tfrac{1}{2}(1-\varepsilon)\big\}.$$

At time $\delta\rho^p$ the bound below (3.1) is possible for some $\eta>0$, however small, only if

$$\delta>\frac{2^p}{c}\,\frac{1-3\varepsilon}{(1-\varepsilon)^p}.$$

Thus for (3.1) to hold for some η, the constant δ must depend on the parameter c, and hence on the solution $u(\cdot,\cdot;c)$.

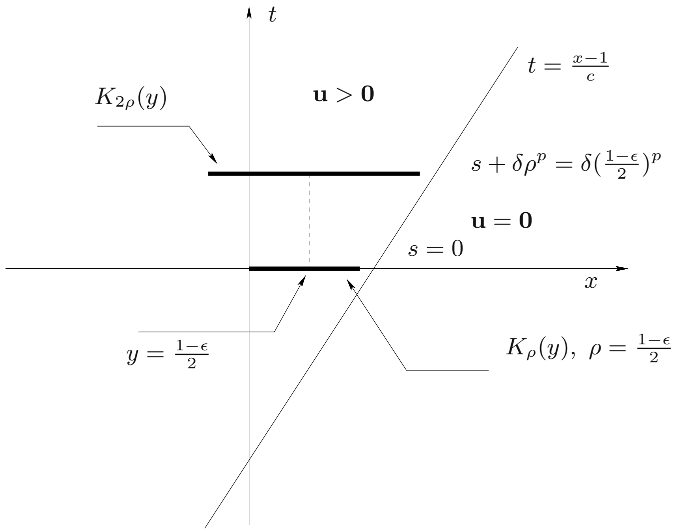

Fig. 3.1. The Traveling Wave Solution

3.2 A Second Counterexample for $p>2$

Consider the Barenblatt solution to the parabolic p-Laplacian equation for $p>2$ in $\mathbb{R}^N\times\mathbb{R}^+$ ([13]):

$$\varGamma_p(x;t)=\frac{1}{t^{N/\lambda}}\left[1-\gamma_p\left(\frac{|x|}{t^{1/\lambda}}\right)^{\frac{p}{p-1}}\right]_+^{\frac{p-1}{p-2}}\qquad t>0 \tag{3.3}$$

where

$$\gamma_p = \left(\frac{1}{\lambda}\right)^{\frac{1}{p-1}} \frac{p-2}{p}, \qquad \lambda = N(p-2) + p. \tag{3.4}$$

The moving boundary is the sphere centered at the origin and radius $R_m(t)$ given by

$$R_m(t) = \gamma_p^{\frac{1-p}{p}} t^{\frac{1}{\lambda}}.$$

For fixed $\varepsilon > 0$ and $s > 0$ let

$$\rho_1 = \left(\frac{1-3\varepsilon}{\gamma_p}\right)^{\frac{p-1}{p}} s^{\frac{1}{\lambda}}, \qquad \rho_2 = \left(\frac{1-\varepsilon}{\gamma_p}\right)^{\frac{p-1}{p}} s^{\frac{1}{\lambda}}$$

and set

$$\rho = \frac{\rho_2 - \rho_1}{2} = \frac{(1-\varepsilon)^{\frac{p-1}{p}} - (1-3\varepsilon)^{\frac{p-1}{p}}}{2\gamma_p^{\frac{p-1}{p}}} s^{\frac{1}{\lambda}},$$

$$|y| = \frac{\rho_2 + \rho_1}{2} = \frac{(1-\varepsilon)^{\frac{p-1}{p}} + (1-3\varepsilon)^{\frac{p-1}{p}}}{2\gamma_p^{\frac{p-1}{p}}} s^{\frac{1}{\lambda}}.$$

One verifies that

$$u(\cdot, s) \geq \frac{1}{s^{\frac{N}{\lambda}}} \varepsilon^{\frac{p-1}{p-2}} \quad \text{in } B_\rho(y).$$

If the expansion of positivity (3.1) were to hold for some $\delta > 0$ depending only on N and p, then points on the ball $B_{2\rho}(y)$, at time $s + \delta\rho^p$ should be within the support of $u(\cdot, s + \delta\rho^p)$. That is,

$$|y| + 2\rho < R_m(s + \delta\rho^p).$$

From this and the expression of $R_m(\cdot)$ one computes

$$\delta > \frac{1}{2^{N(p-2)}\gamma_p^{p-1}} \frac{3(1-\varepsilon)^{\frac{p-1}{p}} - (1-3\varepsilon)^{\frac{p-1}{p}}}{(1-\varepsilon)^{\frac{p-1}{p}} - (1-3\varepsilon)^{\frac{p-1}{p}}} s^{\frac{N(p-2)}{\lambda}}.$$

If ε is sufficiently small, the right-hand side is a positive factor of $s^{N(p-2)/\lambda}$, and hence δ grows with s.

3.3 A Family of Counterexamples for $1 < p < 2$

When $1 < p < 2$, nonnegative solutions to the prototype equation (1.3) of Chapter 3 in some cylindrical domain E_T, might vanish identically in finite time. That is, there might exist a finite $T > 0$ such that

$$u(\cdot, t) = 0 \quad \text{in } E \quad \text{for all } t \geq T.$$

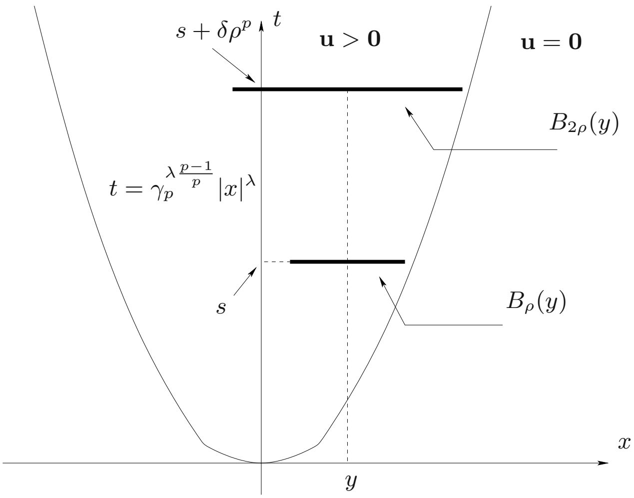

Fig. 3.2. The Barenblatt Solution

If E is a bounded domain with smooth boundary ∂E and u is the solution to the initial-boundary value problem, with bounded initial data and homogeneous Dirichlet data on ∂E, this extinction phenomenon occurs for all $1<p<2$ and the extinction time T can be estimated in terms of the initial datum ([41], Chapter VII § 2, and also [60]).

If $E=\mathbb{R}^N$ and u is the solution to the Cauchy problem with smooth and compactly supported initial datum, this phenomenon occurs for $1<p<\frac{2N}{N+1}$ ([41], Chapter VII § 3, and also [60]).

It is apparent that for a cylinder $K_\rho(y)\times(s,s+\delta\rho^p)$ such that $u(\cdot,s)>0$ on $K_\rho(y)$, the expansion of (3.1) does not hold true if $s+\delta\rho^p$ exceeds the extinction time T.

If $N=1$, a family of such solutions can be constructed semi-explicitly, by separation of variables. Consider the boundary value problem

$$\begin{aligned}&u_t-(|u_x|^{p-2}u_x)_x=0 \quad \text{in } [|x|<1]\times[t>0]\\ &u(-1,t)=u(1,t)=0\\ &u(\cdot,0)=T^{\frac{1}{2-p}}X(\cdot;\mu)\end{aligned} \tag{3.5}$$

where $X(\cdot)$ is a nonnegative solution to

$$\begin{aligned}&-(|X'|^{p-2}X')'=\mu X \quad \text{in } (0,1)\\ &X(-1)=X(1)=0,\end{aligned} \tag{3.6}$$

for some $\mu>0$. Whence such an $X(\cdot)$ is constructed, a solution to (3.5) is

$$u(x,t)=\big[T-(2-p)\mu t\big]^{\frac{1}{2-p}}X(x;\mu).$$

A construction procedure for nonnegative solutions to (3.6) is in § 8.1.

3.4 The Expansion of Positivity in Some Intrinsic Geometry

These examples raise the natural question, whether a version of the expansion of positivity still holds, in some form, for supersolutions to equations (1.1)–(1.2), and (5.1)–(5.2) of Chapter 3 for $p \neq 2$, or for $m \neq 1$. Such a result would pave the way to a Harnack inequality when $p \neq 2$, or $m \neq 1$.

It turns out that the expansion of positivity continues to hold for these degenerate and singular equations, but in a time-intrinsic geometry.

In the next sections we make precise the notion of *intrinsic geometry* and state and prove the expansion of positivity in such a geometry, respectively for degenerate equations ($p > 2$ or $m > 1$) and singular equations ($1 < p < 2$ or $0 < m < 1$).

4 The Expansion of Positivity for Degenerate Quasilinear Parabolic Equations ($p > 2$)

Throughout this section let u be a nonnegative, local, weak supersolution to (1.1)–(1.2) of Chapter 3 in E_T, for $p > 2$. For $(y, s) \in E_T$, and some given positive number M, consider the cylinder

$$K_{8\rho}(y) \times (s, s + \frac{b^{p-2}}{(\eta M)^{p-2}}\delta\rho^p], \tag{4.1}$$

where b, η, δ are the constants given by Proposition 4.1, and $\rho > 0$ is so small that it is included in E_T.

Proposition 4.1 *Assume that for some* $(y, s) \in E_T$ *and some* $\rho > 0$

$$\big|[u(\cdot, s) \geq M] \cap K_\rho(y)\big| \geq \alpha\big|K_\rho(y)\big| \tag{4.2}$$

for some $M > 0$ *and some* $\alpha \in (0, 1)$. *There exist constants* η *and* δ *in* $(0, 1)$ *and* $\gamma, b > 1$ *depending only on the data* $\{p, N, C_o, C_1\}$, *and* α, *such that either* $\gamma C\rho > \min\{1\,, M\}$, *or*

$$u(\cdot, t) \geq \eta M \qquad a.e.\ in\ \ K_{2\rho}(y) \tag{4.3}$$

for all times

$$s + \frac{b^{p-2}}{(\eta M)^{p-2}}\tfrac{1}{2}\delta\rho^p \leq t \leq s + \frac{b^{p-2}}{(\eta M)^{p-2}}\delta\rho^p. \tag{4.4}$$

Remark 4.1 The cylinder in (4.1) is “intrinsic” to the supersolution itself, since its height depends on the lower bound M in (4.2). The conclusion (4.3)–(4.4) is analogous to the conclusion (2.2) of Proposition 2.1, except that the time is rescaled by a factor $(\eta M)^{2-p}$. In this sense Proposition 4.1 is an “intrinsic” expansion of positivity.

Remark 4.2 The constants η, δ, γ, and b are stable as $p \to 2$ and therefore the statement of Proposition 2.1, valid for the nondegenerate case $p = 2$, can be recovered from Proposition 4.1 by letting $p \to 2$. This stability of γ, η, and b will be established in § 6.

Remark 4.3 The proposition transforms the measure-theoretical information (4.2) into the pointwise expansion of positivity (4.3). The proof below shows that the functional dependence of η on the measure-theoretical parameter α is of the form

$$\eta = \eta_o \alpha B^{-\frac{1}{\alpha^d}} \tag{4.5}$$

for parameters η_o, B, d depending only on the data $\{p, N, C_o, C_1\}$. Such a dependence will be improved in Proposition 7.1 of Chapter 5.

4.1 Structure of the Proof

Assume $(y, s) = (0, 0)$ and let ϵ and δ be the numbers claimed by Lemma 1.1.

Following the proof for the nondegenerate case $p = 2$, one seeks to convert the information (1.1) originating from Lemma 1.1, into an estimate of the type of (2.4) of Lemma 2.1. The proof could then be concluded, as in the nondegenerate case, by an application of Lemma 3.1 of Chapter 3. The conclusion of this lemma holds, provided the number ν can be chosen so small as in (3.12) of Chapter 3 with ω replaced by $\epsilon_\nu M$. If $p = 2$, such a choice can be made independent of $(\epsilon_\nu M)$. If $p > 2$, the number ν can be determined in terms only of the data if θ is chosen to satisfy $\theta(\epsilon_\nu M)^{p-2} = 1$. Thus the smaller is ϵ_ν the longer is the cylinder $Q^+_{4\rho}(\theta)$. Therefore an information of the form of (1.1) would need to be derived over a large cylinder.

This is precisely the main difficulty of the proof. It is overcome by introducing a suitable change of the time variable, and the function u for which a version of (1.1) continues to hold over "large times."

4.2 Changing the Time Variable

We may assume $(y, s) = (0, 0)$. The assumption (4.2) implies

$$|[u(\cdot, 0) \geq \sigma M] \cap K_\rho| \geq \alpha |K_\rho| \quad \text{for all } \sigma \leq 1. \tag{4.2$'$}$$

The conclusion of Lemma 1.1 continues to hold, with the same parameters ϵ and δ, if one replaces M by σM, and yields

$$\Big| \Big[u\Big(\cdot, \frac{\delta \rho^p}{(\sigma M)^{p-2}}\Big) \geq \epsilon \sigma M \Big] \bigcap K_\rho \Big| \geq \tfrac{1}{2} \alpha |K_\rho| \quad \text{for all } \sigma \leq 1.$$

For $\tau \geq 0$ set

$$\sigma_\tau = e^{-\frac{\tau}{p-2}} \tag{4.6}$$

and

$$w(x,\tau) \stackrel{\text{def}}{=} \frac{e^{\frac{\tau}{p-2}}}{M}(\delta\rho^p)^{\frac{1}{p-2}}\, u\Big(x, \frac{e^\tau}{M^{p-2}}\delta\rho^p\Big). \tag{4.7}$$

Then for all $\tau \ge 0$

$$\Big|\Big[u\Big(\cdot, \frac{e^\tau}{M^{p-2}}\,\delta\rho^p\Big) \ge \epsilon\frac{M}{e^{\frac{\tau}{p-2}}}\Big] \bigcap K_\rho\Big| \ge \tfrac{1}{2}\alpha|K_\rho|$$

which, in terms of $w(\cdot,\tau)$, means

$$|[w(\cdot,\tau) \ge k_o] \cap K_\rho| \ge \tfrac{1}{2}\alpha|K_\rho| \quad \text{for all } \ \tau > 0,$$

where

$$k_o \stackrel{\text{def}}{=} \epsilon(\delta\rho^p)^{\frac{1}{p-2}}. \tag{4.8}$$

From this

$$|K_{4\rho} - [w(\cdot,\tau) < k_o]| \ge \tfrac{1}{2}\alpha 4^{-N}|K_{4\rho}| \quad \text{for all } \ \tau > 0. \tag{4.9}$$

4.2.1 Relating w to the Evolution Equation

Since $u \ge 0$, by formal calculations

$$\begin{aligned} w_\tau &= \Big(\frac{e^{\frac{\tau}{p-2}}}{M}(\delta\rho^p)^{\frac{1}{p-2}}\Big)^{p-1} u_t + \frac{1}{p-2}\,\frac{e^{\frac{\tau}{p-2}}}{M}(\delta\rho^p)^{\frac{1}{p-2}}\, u \\ &\ge \Big(\frac{e^{\frac{\tau}{p-2}}}{M}(\delta\rho^p)^{\frac{1}{p-2}}\Big)^{p-1}\big[\operatorname{div}\mathbf{A}(x,t,u,Du) + B(x,t,u,Du)\big] \\ &= \operatorname{div}\tilde{\mathbf{A}}(x,\tau,w,Dw) + \tilde{B}(x,\tau,w,Dw) \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} &\tilde{\mathbf{A}} : (E \times \mathbb{R}^+) \times \mathbb{R}^{N+1} \to \mathbb{R}^N \\ &\tilde{B} : (E \times \mathbb{R}^+) \times \mathbb{R}^{N+1} \to \mathbb{R} \end{aligned}$$

satisfy the structure conditions

$$\begin{cases} \tilde{\mathbf{A}}(x,\tau,w,Dw)\cdot Dw \ge C_o|Dw|^p - \tilde{C}^p \\ |\tilde{\mathbf{A}}(x,\tau,w,Dw)| \le C_1|Dw|^{p-1} + \tilde{C}^{p-1} \\ |\tilde{B}(x,\tau,w,Dw)| \le C|Dw|^{p-1} + C\tilde{C}^{p-1} \end{cases} \quad \text{a.e. in } \ E \times \mathbb{R}^+,$$

where C_o, C_1, and C are the constants appearing in the structure conditions (1.2) of Chapter 3, and

$$\tilde{C}(\tau) = C\,\frac{e^{\frac{\tau}{p-2}}}{M}(\delta\rho^p)^{\frac{1}{p-2}}. \tag{4.11}$$

The formal differential inequality (4.10) can be made rigorous by starting from the weak formulation (1.4)–(1.7) of Chapter 3, by operating the corresponding change of variables from t into τ, and by taking testing functions $\varphi \ge 0$. We will use (4.10) in space-time domains contained in $K_{8\rho} \times \mathbb{R}^+$.

Write the energy estimates for $(w-k)_-$, of the type of (2.3) of Chapter 3, over cylinders $Q^+_{8\rho}(\theta) \subset E \times \mathbb{R}^+$, as defined in (2.1)–(2.2) of Chapter 3, with bottom center at $(0,0)$, and in the new variables (x,τ). Precisely

$$\begin{aligned}
\operatorname*{ess\,sup}_{0<\tau<\theta(8\rho)^p} \int_{K_{8\rho}} (w-k)^2_- \zeta^p(x,\tau)dx + \iint_{Q^+_{8\rho}(\theta)} |D(w-k)_-\zeta|^p dx\,d\tau \\
\le \gamma \iint_{Q^+_{8\rho}(\theta)} \big[(w-k)^p_- |D\zeta|^p + (w-k)^2_- |\zeta_\tau|\big]\,dx\,d\tau \\
+\gamma\{\tilde{C}[\theta(8\rho)^p]\}^p \iint_{Q^+_{8\rho}(\theta)} \chi_{[(w-k)_->0]}\zeta^p dx\,d\tau + \gamma C^p \iint_{Q^+_{8\rho}(\theta)} (w-k)^p_- \zeta^p dx\,d\tau
\end{aligned}$$

for a nonnegative, piecewise smooth cutoff function that vanishes on the parabolic boundary of $Q^+_{8\rho}(\theta)$. Choose ζ to be one on the cylinder

$$\mathcal{Q}_{4\rho}(\theta) = K_{4\rho} \times \big((4\rho)^p\theta, (8\rho)^p\theta\big]$$

and such that

$$|D\zeta| \le \frac{1}{4\rho} \quad \text{and} \quad |\zeta_\tau| \le \frac{1}{\theta(4\rho)^p}.$$

With these choices, the previous energy inequalities yield

$$\begin{aligned}
&\iint_{\mathcal{Q}_{4\rho}(\theta)} |D(w-k)_-|^p dx\,d\tau \\
&\le \frac{\gamma k^p}{(4\rho)^p} |\mathcal{Q}_{4\rho}(\theta)| \Big(1 + \frac{1}{\theta k^{p-2}} + (C\rho)^p + \frac{\big\{\tilde{C}\big[\theta(8\rho)^p\big]\big\}^p (4\rho)^p}{k^p}\Big).
\end{aligned} \tag{4.12}$$

4.3 The Set Where w Is Small Can Be Made Small Within $\mathcal{Q}_{4\rho}(\theta)$ for Large θ

Lemma 4.1 *Let (4.2) hold and let k_o be defined by (4.8). For every $\nu > 0$, there exist $\epsilon_\nu \in (0,1)$ depending only on the data $\{p, N, C_o, C_1\}$ and α, and $\theta = \theta(k_o, \epsilon_\nu)$ depending only on k_o, ϵ_ν and the data, and $\gamma = \gamma(\theta)$ depending only on θ and the data, such that either*

$$\gamma(\theta)C\rho > \min\{1\,,\, M\}$$

or

$$|[w < \epsilon_\nu k_o] \cap \mathcal{Q}_{4\rho}(\theta)]| \le \nu\, |\mathcal{Q}_{4\rho}(\theta)|.$$

Proof Write down the energy inequalities (4.12) for the level k_j and the parameter θ given by

$$k_j = \frac{1}{2^j} k_o \;\text{ for }\; j = 0, 1, \ldots, j_* \quad \text{and} \quad \theta = k_{j_*}^{2-p} = \Big(\frac{2^{j_*}}{k_o}\Big)^{p-2},$$

where $j_* \in \mathbb{N}$ is to be chosen depending only on the data $\{p, N, C_o, C_1\}$. The term involving $\tilde{C}$ is estimated by the definition (4.11) of $\tilde{C}(\tau)$ and the definition (4.8) of k_o. Thus

$$\frac{\{\tilde{C}[\theta(8\rho)^p]\}^p(4\rho)^p}{k_j^p} \leq \bar{\gamma}(j_*, \text{data})^p\Big(\frac{\rho C}{M}\Big)^p.$$

Therefore, if

$$M > \bar{\gamma}(j_*, \text{data})C\rho,$$

the last term is majorized by an absolute constant depending only on the data $\{p, N, C_o, C_1\}$ and the previous inequality yields

$$\iint_{\mathcal{Q}_{4\rho}(\theta)} |D(w-k_j)_-|^p dx\, d\tau \leq \frac{\gamma k_j^p}{(4\rho)^p}|\mathcal{Q}_{4\rho}(\theta)| \tag{4.13}$$

for a constant γ depending only on the data $\{p, N, C_o, C_1\}$, and independent of j_*. Set

$$A_j(\tau) = [w(\cdot,\tau) < k_j] \cap K_{4\rho}, \qquad A_j = [w < k_j] \cap \mathcal{Q}_{4\rho}(\theta)$$

so that

$$|A_j| = \int_{\theta(4\rho)^p}^{\theta(8\rho)^p} |A_j(\tau)| d\tau.$$

By Lemma 2.2 of the Preliminaries

$$(k_j - k_{j+1})|A_{j+1}(\tau)| \leq \frac{\gamma\rho^{N+1}}{|K_{4\rho} - A_j(\tau)|}\int_{K_{4\rho}\cap[k_{j+1}<w(\cdot,\tau)<k_j]} |Dw| dx$$

for all $\tau \in (\theta(4\rho)^p, \theta(8\rho)^p]$. For all such τ, applying (4.9)

$$\tfrac{1}{2}k_j|A_{j+1}(\tau)| \leq \frac{2\gamma 4^N\rho}{\alpha}\int_{K_{4\rho}\cap[k_{j+1}<w(\cdot,\tau)<k_j]} |Dw| dx.$$

Integrate this in $d\tau$ over $\big(\theta(4\rho)^p, \theta(8\rho)^p\big)$ and majorize the resulting integral on the right-hand side by Hölder's inequality, and by means of (4.13), to obtain

$$\begin{aligned}
\tfrac{1}{2}k_j|A_{j+1}| &\leq \gamma\rho\Big(\iint_{A_j - A_{j+1}} |Dw|^p dx\, d\tau\Big)^{\frac{1}{p}}|A_j - A_{j+1}|^{\frac{p-1}{p}} \\
&\leq \gamma\rho\Big(\iint_{\mathcal{Q}_{4\rho}(\theta)} |D(w-k_j)_-|^p dx\, d\tau\Big)^{\frac{1}{p}}|A_j - A_{j+1}|^{\frac{p-1}{p}} \\
&\leq \gamma k_j|\mathcal{Q}_{4\rho}(\theta)|^{\frac{1}{p}}|A_j - A_{j+1}|^{\frac{p-1}{p}}.
\end{aligned}$$

From this, by taking the $\frac{p}{p-1}$ power of both sides, we arrive at the recursive inequalities

$$|A_{j+1}|^{\frac{p}{p-1}} \le \gamma |\mathcal{Q}_{4\rho}(\theta)|^{\frac{1}{p-1}} |A_j - A_{j+1}|$$

for a quantitative constant γ depending only on the data $\{p, N, C_o, C_1\}$ and α, and independent of j_*. Now add these for $j = 0, 1, \dots, j_* - 1$, and majorize the sum on the right-hand side by the corresponding telescopic series. This gives

$$(j_* - 1)|A_{j_*}|^{\frac{p}{p-1}} \le \gamma |\mathcal{Q}_{4\rho}(\theta)|^{\frac{p}{p-1}}.$$

Rewriting this as

$$|A_{j_*}| \le \left(\frac{\gamma}{j_*}\right)^{\frac{p-1}{p}} |\mathcal{Q}_{4\rho}(\theta)|,$$

proves the proposition for the choices

$$\epsilon_\nu = \frac{1}{2^{j_*}} \qquad \text{and} \qquad \nu = \left(\frac{\gamma}{j_*}\right)^{\frac{p-1}{p}}. \tag{4.14}$$

4.4 Expanding the Positivity of w

The measure-theoretical information in (4.9), valid for all $\tau > 0$, will be expanded in the space variables over the cube $K_{2\rho}$ for "times" τ sufficiently large.

Lemma 4.2 *Let (4.2) hold. There exist $\nu \in (0,1)$ and $\gamma(\nu) > 1$, that can be determined a priori only in terms of the data $\{p, N, C_o, C_1\}$ and α, such that either*

$$\gamma(\nu) C\rho > \min\{1\,,\, M\}$$

or

$$w(\cdot, \tau) \ge \tfrac{1}{2}\epsilon_\nu k_o \qquad \textit{a.e. in } K_{2\rho} \times \left(\frac{(6\rho)^p}{(\epsilon_\nu k_o)^{p-2}}, \frac{(8\rho)^p}{(\epsilon_\nu k_o)^{p-2}}\right] \tag{4.15}$$

where ϵ_ν is the number claimed by Lemma 4.1 corresponding to ν.

Proof Apply (3.1)–(3.3) of Lemma 3.1 of Chapter 3 to w over the cylinder

$$\mathcal{Q}_{4\rho}(\theta) = (0, \tau_*) + Q^-_{4\rho}(\theta) \quad \text{ for } \tau_* = \theta(8\rho)^p.$$

The parameter $\xi\omega$ is replaced by $\epsilon_\nu k_o$ and $\mu_- \ge 0$ is neglected. Taking into account (3.12) of Chapter 3, and choosing $a = \frac{1}{2}$ gives

$$w(x,\tau) \ge \tfrac{1}{2}\epsilon_\nu k_o \qquad \text{for a.e. } (x,\tau) \in [(0,\tau_*) + Q^-_{2\rho}(\theta)]$$

provided $M > \gamma(\epsilon_\nu) C\rho$ and

$$\frac{\big|[w < \epsilon_\nu k_o] \cap \mathcal{Q}_{4\rho}(\theta)\big|}{\big|\mathcal{Q}_{4\rho}(\theta)\big|} \le \gamma^{-1}\left(\frac{1}{2}\right)^{N+2} \frac{[\theta(\epsilon_\nu k_o)^{p-2}]^{\frac{N}{p}}}{[1 + \theta(\epsilon_\nu k_o)^{p-2}]^{\frac{p+N}{p}}} = \nu.$$

Choosing now ν from (4.14) determines ϵ_ν and therefore θ quantitatively.

4.5 Expanding the Positivity of u

Return to the definitions (4.6)–(4.8) of τ, w, and k_o. As τ ranges over the interval in (4.15), $e^{\frac{\tau}{p-2}}$ ranges over

$$b_1 \stackrel{\text{def}}{=} \exp\left\{\frac{6^p}{(p-2)[\epsilon_\nu \epsilon \delta^{\frac{1}{p-2}}]^{p-2}}\right\} \le f(\tau) \le \exp\left\{\frac{8^p}{(p-2)[\epsilon_\nu \epsilon \delta^{\frac{1}{p-2}}]^{p-2}}\right\} \stackrel{\text{def}}{=} b_2$$

where b_1 and b_2 are constants that can be determined a priori only in terms of the data $\{p, N, C_o, C_1\}$, and are independent of ρ, M, and u. Translating Lemma 4.2 in terms of u and t gives

$$u(x,t) \ge \frac{\epsilon_\nu \epsilon M}{2b_2} \stackrel{\text{def}}{=} \eta M \qquad \text{for a.e. } x \in K_{2\rho}$$

for all times

$$\frac{b^{p-2}}{(\eta M)^{p-2}} \tfrac{1}{2} \delta \rho^p \le t \le \frac{b^{p-2}}{(\eta M)^{p-2}} \delta \rho^p$$

for a suitable b depending only on the data $\{p, N, C_o, C_1\}$. ∎

5 The Expansion of Positivity for Singular Quasilinear Parabolic Equations ($1 < p < 2$)

Throughout this section we let u be a nonnegative, local, weak supersolution to (1.1)–(1.2) of Chapter 3 with $1 < p < 2$, and let the cylinder

$$(y,s) + Q_{16\rho}(\delta M^{2-p}) = K_{16\rho}(y) \times (s, s + \delta M^{2-p} \rho^p]$$

be contained in E_T.

Proposition 5.1 *Assume that for some $(y,s) \in E_T$ and some $\rho > 0$*

$$|[u(\cdot, s) \ge M] \cap K_\rho(y)| \ge \alpha |K_\rho(y)| \tag{5.1}$$

for some $M > 0$ and some $\alpha \in (0,1)$. There exist constants η, δ, and ε in $(0,1)$ and $\gamma > 1$ depending only on the data $\{p, N, C_o, C_1\}$, and α, such that either

$$\gamma C \rho > \min\{1\,,\, M\}$$

or

$$u(\cdot, t) \ge \eta M \qquad \textit{a.e. in } K_{2\rho}(y) \tag{5.2}$$

for all times

$$s + (1-\varepsilon)\delta M^{2-p} \rho^p \le t \le s + \delta M^{2-p} \rho^p. \tag{5.3}$$

Remark 5.1 The proposition transforms the measure-theoretical information (5.1) into the pointwise expansion of positivity (5.2). The proof below shows that the functional dependence of η on the measure-theoretical parameter α is of the form

$$\eta = \eta_o \alpha\, 2^{-\gamma_1/\alpha^{p+2}} \exp(-\gamma_2 \alpha^p 2^{\gamma_1/\alpha^{p+2}}), \tag{5.4}$$

for parameters $\eta_o, \gamma_1, \gamma_2$ depending only on the data $\{p, N, C_o, C_1\}$. It is not known whether the dependence can be improved to be power-like, as in the degenerate case $p > 2$, for the general *singular* equations (1.1)–(1.2) of Chapter 3.

Proof Assume $(y, s) = (0, 0)$, and let δ and ϵ in $(0, 1)$ be the numbers claimed by Lemma 1.1 depending only on the data $\{p, N, C_o, C_1\}$ and α. The conclusion of the lemma is that either $\gamma C\rho > \min\{1\,,\, M\}$, or

$$|[u(\cdot, t) > \epsilon M] \cap K_\rho| \geq \tfrac{1}{2}\alpha |K_\rho| \quad \text{for all } t \in \big(0, \delta M^{2-p}\rho^p\big]. \tag{5.5}$$

5.1 Transforming the Variables and the Equation

Let $\rho > 0$ be so that

$$Q_{16\rho}(\delta M^{2-p}) = K_{16\rho} \times (0, \delta M^{2-p}\rho^p] \subset E_T. \tag{5.6}$$

Introduce the change of variables and the new unknown function

$$z = \frac{x}{\rho}, \quad -e^{-\tau} = \frac{t - \delta M^{2-p}\rho^p}{\delta M^{2-p}\rho^p}, \quad v(z, \tau) = \frac{1}{M} u(x, t) e^{\frac{\tau}{2-p}}. \tag{5.7}$$

This maps the cylinder in (5.6) into $K_{16} \times (0, \infty)$ and transforms the equations (1.1)–(1.2) of Chapter 3 into

$$v_\tau - \operatorname{div}_z \bar{\mathbf{A}}(z, \tau, v, D_z v) = \bar{B}(z, \tau, v, D_z v) + \frac{1}{2-p} v \tag{5.8}$$

weakly in $K_{16} \times (0, \infty)$, where $\bar{\mathbf{A}}$, and $\bar{B}$ are measurable functions of their arguments, satisfying the structure conditions

$$\begin{cases} \bar{\mathbf{A}}(z, \tau, v, D_z v) \cdot D_z v \geq \delta C_o |D_z v|^p - \delta \bar{C}^p \\ |\bar{\mathbf{A}}(z, \tau, v, D_z v)| \leq \delta C_1 |D_z v|^{p-1} + \delta \bar{C}^{p-1} \\ |\bar{B}(z, \tau, v, D_z v)| \leq \delta\rho C |D_z v|^{p-1} + \delta\rho C \bar{C}^{p-1} \end{cases} \tag{5.9}$$

a.e. in $K_{16} \times (0, \infty)$, where C_o and C_1 are the constants in the structure conditions (1.2) of Chapter 3, δ is the number claimed by Lemma 1.1, and

$$\bar{C} = \bar{C}(\tau) = \rho \frac{C}{M}\, e^{\frac{\tau}{2-p}}.$$

In this setting, the information (5.5) becomes

$$\left|[v(\cdot,\tau) \geq \epsilon\, e^{\frac{\tau}{2-p}}] \cap K_1\right| \geq \tfrac{1}{2}\alpha|K_1| \quad \text{for all } \tau \in (0,+\infty). \tag{5.10}$$

Let $\tau_o > 0$ to be chosen and set

$$k_o = \epsilon\, e^{\frac{\tau_o}{2-p}}, \qquad \text{and} \qquad k_j = \frac{1}{2^j} k_o \quad \text{for } j = 0,1,\dots,j_*,$$

where j_* is to be chosen. With this symbolism (5.10) implies

$$|[v(\cdot,\tau) \geq k_j] \cap K_8| \geq \tfrac{1}{2}\alpha 8^{-N}|K_8| \quad \text{for all } \tau \in (\tau_o,+\infty) \tag{5.11}$$

and for all $j \in \mathbb{N}$. Introduce the cylinders

$$\begin{aligned} Q_{\tau_o} &= K_8 \times \left(\tau_o + k_o^{2-p}, \tau_o + 2k_o^{2-p}\right) \\ Q'_{\tau_o} &= K_{16} \times \left(\tau_o, \tau_o + 2k_o^{2-p}\right) \end{aligned}$$

and a nonnegative, piecewise smooth, cutoff function in Q'_{τ_o} of the form $\zeta(z,\tau) = \zeta_1(z)\zeta_2(\tau)$, where

$$\zeta_1 = \begin{cases} 1 & \text{in } K_8 \\ 0 & \text{in } \mathbb{R}^N - K_{16} \end{cases} \qquad |D\zeta_1| \leq \frac{1}{8},$$

$$\zeta_2 = \begin{cases} 0 & \text{for } \tau < \tau_o \\ 1 & \text{for } \tau \geq \tau_o + k_o^{2-p} \end{cases} \qquad 0 \leq \zeta_{2,\tau} \leq \frac{1}{k_o^{2-p}}.$$

Write down the energy estimates (2.3) of Chapter 3, for $(v-k_j)_-$ over Q'_{τ_o}, and for the indicated choice of cutoff function ζ. These are derived by taking $-(v-k_j)_-\zeta^p$ as a testing function in the weak formulation of (5.8). Discarding the nonpositive contribution of the right-hand side, coming from the nonnegative term $\frac{1}{2-p}v$, standard calculations give

$$\begin{aligned} &\iint_{Q'_{\tau_o}} |D(v-k_j)_-\zeta|^p dz\, d\tau \\ &\qquad \leq \gamma \iint_{Q'_{\tau_o}} \left[(v-k_j)_-^p |D\zeta|^p + (v-k_j)_-^2 \zeta_t\right] dz\, d\tau \\ &\qquad\quad + \gamma \bar{C}^p(\tau_o + 2k_o^{2-p}) \iint_{Q'_{\tau_o}} \chi_{[(v-k_j)_->0]} dz\, d\tau \\ &\qquad\quad + \gamma C^p \rho^p \iint_{Q'_{\tau_o}} (v-k_j)_-^p dz\, d\tau, \end{aligned}$$

where $\gamma = \tilde{\gamma}/\delta$, the constant $\tilde{\gamma}$ depends only on $\{p, N, C_o, C_1\}$, and δ is the parameter claimed by Lemma 1.1, and appearing in the transformed structure conditions (5.9). From this

$$\iint_{Q_{\tau_o}} |D(v-k_j)_-|^p dz\, d\tau \leq \gamma k_j^p |Q_{\tau_o}| \Big[2 + \frac{\bar{C}^p(\tau_o + 2k_o^{2-p})}{k_j^p} + C^p\rho^p\Big].$$

Taking into account the expressions of $\bar{C}$ and k_o, estimate

$$\frac{\bar{C}^p(\tau_o + 2k_o^{2-p})}{k_j^p} \le 2^{j_* p} \frac{C^p}{M^p} \rho^p e^{\frac{2p}{2-p} k_o^{2-p}}.$$

Suppose for the moment that j_* and k_o have been chosen, and set

$$\gamma(j_*, \tau_o) = 2^{j_*} e^{\frac{2}{2-p} k_o^{2-p}}. \tag{5.12}$$

Therefore either $M < \gamma(j_*, \tau_o) C\rho$, or the previous inequality yields

$$\iint_{Q_{\tau_o}} |D(v - k_j)_-|^p dz\, d\tau \le 4\gamma k_j^p |Q_{\tau_o}| \tag{5.13}$$

for a constant γ depending only on the data $\{p, N, C_o, C_1\}$, and δ.

5.2 Estimating the Measure of the Set $[v < k_j]$ Within Q_{τ_o}

Set

$$A_j(\tau) = [v(\cdot, \tau) < k_j] \cap K_8, \qquad A_j = [v < k_j] \cap Q_{\tau_o}.$$

By Lemma 2.2 of the Preliminaries, and (5.11)

$$\begin{aligned}(k_j - k_{j+1})|A_{j+1}(\tau)| &\le \frac{\gamma(N)}{|K_8 - A_j(\tau)|} \int_{K_8 \cap [k_{j+1} < v(\cdot,\tau) < k_j]} |Dv| dz \\ &\le \frac{\gamma(N)}{\alpha} \int_{K_8 \cap [k_{j+1} < v(\cdot,\tau) < k_j]} |Dv| dz\end{aligned}$$

for all $\tau \ge \tau_o$. Integrate this in $d\tau$ over $(\tau_o + k_o^{2-p}, \tau_o + 2k_o^{2-p})$, majorize the resulting integral on the right-hand side by the Hölder inequality, and use (5.13) to get

$$\begin{aligned}\frac{k_j}{2}|A_{j+1}| &\le \gamma(\text{data}, \alpha) \iint_{A_j - A_{j+1}} |Dv| dz\, d\tau \\ &\le \gamma(\text{data}, \alpha) \Big(\iint_{A_j - A_{j+1}} |Dv|^p dz\, d\tau \Big)^{\frac{1}{p}} |A_j - A_{j+1}|^{\frac{p-1}{p}} \\ &\le \gamma(\text{data}, \alpha) \Big(\iint_{Q_{\tau_o}} |D(v - k_j)_-|^p dz\, d\tau \Big)^{\frac{1}{p}} |A_j - A_{j+1}|^{\frac{p-1}{p}} \\ &\le \gamma(\text{data}, \alpha, \delta) k_j |Q_{\tau_o}|^{\frac{1}{p}} |A_j - A_{j+1}|^{\frac{p-1}{p}}.\end{aligned}$$

Taking the $\frac{p}{p-1}$ power yields the recursive inequalities

$$|A_{j+1}|^{\frac{p}{p-1}} \le \gamma(\text{data}, \alpha, \delta) |Q_{\tau_o}|^{\frac{1}{p-1}} |A_j - A_{j+1}|.$$

Add these inequalities for $j = 0, 1, \dots, j_* - 1$, where j_* is an integer to be chosen, and majorize the sum on the right-hand side by the corresponding telescopic series. This gives

$$(j_* - 1)|A_{j_*}|^{\frac{p}{p-1}} \le \gamma(\text{data}, \alpha, \delta)|Q_{\tau_o}|^{\frac{p}{p-1}}.$$

Equivalently

$$|[v < k_{j_*}] \cap Q_{\tau_o}| \le \nu |Q_{\tau_o}| \quad \text{where} \quad \nu = \Big(\frac{\gamma(\text{data}, \alpha, \delta)}{j_*}\Big)^{\frac{p-1}{p}}. \tag{5.14}$$

Taking into account (1.3), the constant γ in (5.14) can be traced to be of the form $\gamma = \dfrac{\tilde{\gamma}(\text{data})}{\alpha^{p+2}}$.

5.3 Segmenting Q_{τ_o}

Assume momentarily that j_* and hence ν have been determined. By possibly increasing j_* to be not necessarily integer, without loss of generality we may assume that $(2^{j_*})^{2-p}$ is an integer. Then subdivide Q_{τ_o} into $(2^{j_*})^{2-p}$ cylinders, each of length $k_{j_*}^{2-p}$, by setting

$$\begin{aligned} &Q_n = K_8 \times \big(\tau_o + k_o^{2-p} + n k_{j_*}^{2-p}\,,\, \tau_o + k_o^{2-p} + (n+1)k_{j_*}^{2-p}\big) \\ &\text{for } \; n = 0, 1, \dots, (2^{j_*})^{2-p} - 1. \end{aligned}$$

For at least one of these, say Q_n, there must hold

$$|[v < k_{j_*}] \cap Q_n| \le \nu |Q_n|.$$

Apply Lemma 3.1 of Chapter 3 to v over Q_n with

$$\mu_- = 0, \quad \xi\omega = k_{j_*}, \quad a = \tfrac{1}{2}, \quad \theta = k_{j_*}^{2-p}.$$

It gives

$$v\big(z, \tau_o + k_o^{2-p} + (n+1)k_{j_*}^{2-p}\big) \ge \tfrac{1}{2} k_{j_*} \qquad \text{a.e. in } K_4$$

provided

$$\frac{|[v < k_{j_*}] \cap Q_n|}{|Q_n|} \le 2^{-\frac{N+p}{p}} \bar{\gamma}_o(\text{data}) = \nu.$$

Choose now j_*, and hence ν, from this and (5.14). Summarizing, for such a choice of j_*, and hence ν, there exists a time level τ_1 in the range

$$\tau_o + k_o^{2-p} < \tau_1 < \tau_o + 2k_o^{2-p} \tag{5.15}$$

such that

$$v(z, \tau_1) \ge \sigma_o e^{\frac{\tau_o}{2-p}} \qquad \text{where} \qquad \sigma_o = \epsilon\, 2^{-(j_*+1)}.$$

Remark 5.2 Notice that j_* and hence ν are determined only in terms of the data and are independent of the parameter τ_o, which is still to be chosen.

5.4 Returning to the Original Coordinates

In terms of the original coordinates and the original function $u(x,t)$ this implies

$$u(\cdot,t_1) \ge \sigma_o M e^{-\frac{\tau_1-\tau_o}{2-p}} \stackrel{\text{def}}{=} M_o \qquad \text{in } K_{4\rho},$$

where the time t_1 corresponding to τ_1 is computed from (5.7) and (5.15). Apply now Lemma 4.1 of Chapter 3 with M replaced by M_o and $\xi = 1$ over the cylinder

$$(t_1,0) + Q^+_{4\rho}(\theta) = K_{4\rho} \times \big(t_1, t_1 + \theta(4\rho)^p\big].$$

By choosing

$$\theta = \nu_o M_o^{2-p} \qquad \text{where} \qquad \nu_o = \nu_o(\text{data})$$

the assumption (4.2) of Chapter 3 is satisfied, and the lemma yields

$$\begin{aligned} u(\cdot,t) &\ge \tfrac{1}{2} M_o = \tfrac{1}{2}\sigma_o M e^{-\frac{\tau_1-\tau_o}{2-p}} \\ &\ge \tfrac{1}{2}\sigma_o e^{-\frac{2}{2-p}e^{\tau_o}} M \end{aligned} \qquad \text{in} \quad K_{2\rho} \tag{5.16}$$

for all times

$$t_1 \le t \le t_1 + \nu_o M_o^{2-p}(4\rho)^p. \tag{5.17}$$

If the right-hand side equals $\delta M^{2-p}\rho^p$, then (5.16) and the conclusion (5.2) will hold for the time $t = \delta M^{2-p}\rho^p$. The transformed τ_o level is still undetermined, and it will be so chosen as to verify such a requirement. Precisely, taking into account the change of variables (5.7)

$$\delta M^{2-p}\rho^p e^{-\tau_1} = \delta M^{2-p}\rho^p - t_1 = \nu_o \sigma_o^{2-p} M^{2-p}(4\rho)^p e^{-(\tau_1-\tau_o)}$$

which implies

$$e^{\tau_o} = \frac{\delta}{4^p \nu_o \sigma_o^{2-p}}.$$

This determines quantitatively $\tau_o = \tau_o(\text{data})$. The proof of Proposition 5.1 is now completed by inserting such a τ_o on the right-hand side of (5.16) and in (5.17). In particular (5.16) holds for all times

$$t_1 = \delta M^{2-p}\rho^p - \nu_o M_o^{2-p}(4\rho)^p \le t \le \delta M^{2-p}\rho^p.$$

From the previous definitions and transformations one estimates

$$t_1 \le (1-\varepsilon)\delta M^{2-p}\rho^p, \qquad \text{where} \qquad \varepsilon = e^{-\tau_o - 2e^{\tau_o}}.$$

Notice that once j_* and τ_o are fixed, then the constant γ in (5.12) is also defined, only in terms of the data $\{p, N, C_o, C_1\}$ and α.

Remark 5.3 As it will be apparent in the next chapters, the Harnack inequality has different formulations, respectively when $\frac{2N}{N+1} < p < 2$ and $1 < p \le \frac{2N}{N+1}$. It is remarkable, however, that the expansion of positivity holds with the same statement in the full singular range $1 < p < 2$.

Remark 5.4 It might seem that two approaches for the degenerate case $p > 2$ and the singular case $1 < p < 2$ are similar, based as they are on an exponential-type change of variable, respectively (4.6)–(4.7) and (5.7). The two phenomena, however, are markedly different.

In the degenerate case, starting at time level s, the transformation itself chooses the final time level, as indicated in (4.4), in terms of the lower bound M. In the singular case, the final time level $\delta M^{2-p}\rho^p$ is fixed in terms of M, as indicated in (5.3). The structural constants only determine how the original time interval shrinks, about the upper limit, which remains fixed.

6 Stability of the Expansion of Positivity for $p \to 2$

The proof of Proposition 4.1 for the degenerate case $p > 2$ shows that the constants b and η in (4.3)–(4.4) depend on p as (see § 4.5)

$$b \approx \exp\Big(\gamma_b \frac{h^{p-2}}{p-2}\Big), \qquad \eta \approx \exp\Big(-\gamma_\eta \frac{k^{p-2}}{p-2}\Big)$$

for constants γ_b, γ_η, h, k all larger than 1, depending only on the data $\{N, C_o, C_1\}$, and independent of p. Thus the ratio $(b/\eta)^{p-2}$ that determines the "waiting time" needed to preserve and expand the positivity, deteriorates as $p \to \infty$. However, it is stable as $p \to 2$ and (4.4) remains meaningful for p near 2. On the other hand, $\eta(p) \to 0$ as $p \to 2$ and (4.3) becomes vacuous.

Likewise, in the proof of Proposition 5.1, for the singular case $1 < p < 2$, the change of variables (5.7) and the subsequent arguments, yield constants that deteriorate as $p \to 2$.

Nevertheless the conclusions of both Proposition 4.1, for $p > 2$, and Proposition 5.1 for $1 < p < 2$, continue to hold with constants that are stable as $p \to 2$, in the sense of (1.9) of Chapter 3. This is the content of the next proposition.

Proposition 6.1 *Let u be a nonnegative, local, weak solution to (1.1)–(1.2) of Chapter 3 for $p > 1$ in E_T. Let*

$$K_{8\rho}(y) \times (s, s + \frac{\delta\rho^p}{M^{p-2}}] \subset E_T$$

and assume that for some $(y,s) \in E_T$ and some $\rho > 0$

$$|[u(\cdot, s) \ge M] \cap K_\rho(y)| \ge \alpha |K_\rho(y)|$$

for some $M > 0$ and some $\alpha \in (0,1)$. There exist constants $\gamma_* > 1$, δ, σ_*, η_* in $(0,1)$, depending only on the data $\{N, C_o, C_1\}$ and α, and independent of (y,s), ρ, M, and p, such that if $|p-2| < \sigma_*$, then either

$$\gamma_* C\rho > \min\{1\,,\,M\}$$

or

$$u(x,t) \geq \eta_* M \qquad \textit{for all } \; x \in K_{2\rho}(y)$$

for all

$$s + \frac{\frac{1}{2}\delta\rho^p}{M^{p-2}} \leq t \leq s + \frac{\delta\rho^p}{M^{p-2}}.$$

Remark 6.1 The constants γ_*, δ, σ_*, and η_* are stable as $p \to 2$, in the sense of (1.9) of Chapter 3.

6.1 Proof of Proposition 6.1

Assume that $(y,s) = (0,0)$ and let $\epsilon(p)$ and $\delta(p)$ be the constants corresponding to α, claimed by Lemma 1.1. The lemma does not distinguish between $p > 2$ and $1 < p < 2$ and it implies

$$|[u(\cdot,t) < \epsilon M] \cap K_{4\rho}| > \tfrac{1}{2}\alpha 4^{-N}|K_{4\rho}|, \qquad \text{for all } \; t \in (0, \delta M^{2-p}\rho^p). \tag{6.1}$$

By Remark 1.1 the constants $\epsilon(p)$ and $\delta(p)$ are stable as $p \to 2$. The proof now proceeds for p near 2 irrespective of the degeneracy ($p > 2$) or singularity ($1 < p < 2$) of the partial differential equation. For this reason we denote by $|p-2|$ the proximity of p to 2 from either side.

Lemma 6.1 *For every $\nu^* \in (0,1)$ there exist constants σ^*, $\epsilon_{\nu^*} \in (0,1)$ and $\gamma_* > 1$, depending only on the data $\{N, C_o, C_1\}$ and α and independent of u, M, p, and ρ, such that for all $|p-2| \leq \sigma_*$, either*

$$\gamma_* C\rho > \min\{1\,,\, M\}$$

or

$$|[u < \epsilon_{\nu^*} M] \cap \mathcal{Q}^+_{4\rho}(\delta M^{2-p})| \leq \nu_* |\mathcal{Q}^+_{4\rho}(\delta M^{2-p})|.$$

Proof Write down the energy inequalities in (2.3) of Chapter 3, for $(u-k_j)_-$, over the cylinder

$$\mathcal{Q}^+_{8\rho}(\delta M^{2-p})$$

for a nonnegative, piecewise smooth, cutoff function ζ that equals one on $\mathcal{Q}^+_{4\rho}(\delta M^{2-p})$, and such that

$$|D\zeta| \leq \frac{1}{4\rho} \qquad \text{and} \qquad |\zeta_t| \leq \frac{1}{\delta M^{2-p}\rho^p}.$$

The levels k_j are taken as

$$k_j = \frac{\epsilon M}{2^j} \qquad \text{for } \; j = 0, 1, \ldots, j_* \quad \text{where } j_* \in \mathbb{N} \text{ is to be chosen.}$$

The first term on the left-hand side is discarded and the integral involving $D(u-k_j)_-$ is minorized by extending it over $\mathcal{Q}^+_{4\rho}(\delta M^{2-p})$, which is the set where $\zeta = 1$. The right-hand side is majorized in a standard fashion and gives

$$\iint_{\mathcal{Q}^+_{4\rho}(\delta M^{2-p})} |D(u-k_j)_-|^p dx\,dt \le \gamma \frac{k_j^p}{\delta\rho^p} 2^{j_*|p-2|}|\mathcal{Q}^+_{4\rho}|\Big[1+\frac{C^p\rho^p}{k_j^p}+C^p\rho^p\Big].$$

Assume momentarily that j_* has been chosen in terms only of the data and α. Then either $M < C2^{j_*}\rho$, or the previous inequality yields

$$\iint_{\mathcal{Q}^+_{4\rho}(\delta M^{2-p})} |D(u-k_j)_-|^p dx\,dt \le \gamma \frac{k_j^p}{\delta\rho^p} 2^{j_*|p-2|}|\mathcal{Q}^+_{4\rho}|.$$

The number j_* will be chosen shortly depending only on the data $\{N, C_o, C_1\}$ and α, and independent of u, M, ρ, and p. Assuming momentarily that such a choice has been made, choose $\sigma_* \in (0,1)$ so that $j_*|p-2| \le 1$ for all $|p-2| < \sigma_*$. This yields the energy estimates

$$\iint_{\mathcal{Q}^+_{4\rho}(\delta M^{2-p})} |D(u-k_j)_-|^p dx\,dt \le \frac{\gamma k_j^p}{\rho^p}|\mathcal{Q}^+_{4\rho}(\delta M^{2-p})| \tag{6.2}$$

for a constant γ depending only on the data $\{N, C_o, C_1\}$ and independent of u, M, ρ, and p, provided $M > C\gamma_*\rho$ for $\gamma_* = 2^{j_*}$.

Starting from these energy estimates, the proof can now be concluded as in the proof of Lemma 2.1 valid for nondegenerate equations. Precisely, set

$$A_j = [u < k_j] \cap \mathcal{Q}^+_{4\rho}(\delta M^{2-p})$$

and proceed as in that context by making use of (6.1) and (6.2), to arrive at the analog of (2.6)

$$|A_{j_*}| \le \Big(\frac{\gamma}{j_*}\Big)^{\frac{p-1}{p}} |\mathcal{Q}^+_{4\rho}(\delta M^{2-p})| \tag{6.3}$$

for a constant γ depending only on the data $\{N, C_o, C_1\}$ and independent of u, M, ρ, and p. Choosing

$$\epsilon_{\nu^*} = \frac{\epsilon}{2^{j_*}} \qquad \text{and} \qquad \nu^* = \Big(\frac{\gamma}{j_*}\Big)^{\frac{p-1}{p}} \tag{6.4}$$

proves the lemma. ∎

To conclude the proof of Proposition 6.1, apply Lemma 3.1 of Chapter 3, with $\mu_- = 0$, $\xi = \epsilon_{\nu^*}$, $a = \frac{1}{2}$, $\omega = M$, $\theta = \delta M^{2-p}$ and ρ replaced by 2ρ. The lemma yields

$$u > \tfrac{1}{2}\epsilon_{\nu^*} M \quad \text{ in } K_{2\rho} \times (\tfrac{1}{2}\delta\rho^p, \delta\rho^p),$$

provided

$$Y_o = \frac{|[u < \epsilon_{\nu^*}] \cap \mathcal{Q}^+_{4\rho}(\delta M^{2-p})|}{|\mathcal{Q}^+_{4\rho}(\delta M^{2-p})|} = \frac{|A_{j_*}|}{|\mathcal{Q}^+_{4\rho}(\delta M^{2-p})|} = \nu^*.$$

Here the number ν^* is chosen from (3.12) of Chapter 3 for $p > 1$. For $p > 2$ compute

$$
\begin{aligned}
Y_o &\le \frac{1}{\bar\gamma(\text{data})}\frac{[\delta M^{2-p}(\epsilon_{\nu^*}M)^{p-2}]^{\frac{N}{p}}}{[1+\delta M^{2-p}(\epsilon_{\nu^*}M)^{p-2}]^{\frac{N+p}{p}}}\\
&= \frac{1}{\bar\gamma(\text{data})}\frac{[\delta\epsilon^{p-2}2^{j_*(2-p)}]^{\frac{N}{p}}}{[1+\delta\epsilon^{p-2}2^{j_*(2-p)}]^{\frac{N+p}{p}}} = \nu^*.
\end{aligned}
$$

Stipulate to choose $|p-2|\le\sigma_*$ and then σ_* so small that $2^{j_*|p-2|}\in(1,2)$. Then, from (6.3)–(6.4) choose j_* so large as to satisfy this requirement. The calculations for $1<p<2$ are identical starting once more from (3.12) of Chapter 3. ∎

The argument is a hybrid between the nondegenerate case of § 2 and the degenerate case of § 4 and the singular case of § 5. It mimics the degenerate or singular case in that the length of the cylinders is of the order of M^{2-p} thereby abiding to the notion of intrinsic geometry. If a lower bound of the type $\epsilon_{\nu^*}M=\epsilon 2^{-j_*}M$ is sought, then the intrinsic geometry required by Lemma 3.1 of Chapter 3 would require a cylinder of length $(\epsilon_{\nu^*}M)^{2-p}$, relative to ρ^p. However, because of the indicated choices $\epsilon_{\nu^*}^{p-2}\approx 1$ if $p\approx 2$. Roughly speaking the partial differential equation, while degenerate or singular, for $p\approx 2$ is "mildly degenerate or singular," and it transitions from its nondegenerate regime $p=2$ to its degenerate regime $p>2$ or singular regime $1<p<2$, in a stable manner.

7 The Expansion of Positivity for Porous Medium Type Equations

Throughout this section let u be a nonnegative, local, weak supersolution to (5.1)–(5.2) of Chapter 3 in E_T, for $m>0$. For $(y,s)\in E_T$, and some given positive number M, consider the cylinders

$$
\begin{aligned}
&K_{8\rho}(y)\times\Big(s,s+\frac{b^{m-1}}{(\eta M)^{m-1}}\delta\rho^2\Big] \quad\text{for}\quad m>1\\
&K_{16\rho}(y)\times(s,s+\delta M^{1-m}\rho^2] \quad\text{for}\quad 0<m<1
\end{aligned}
$$

where b, δ, η are the constants given by Propositions 7.1 and 7.2, and $\rho>0$ is so small that they are both included in E_T. The results of the previous sections are based solely on the following technical tools: (i) Lemmas 3.1 and 4.1 of Chapter 3, (ii) the discrete isoperimetric inequality of Lemma 2.2 and the embedding Proposition 4.1 of the Preliminaries, and (iii) the change of variables introduced respectively in (4.6)–(4.7) for the degenerate case $p>2$ and in (5.7) for the singular case $1<p<2$.

For porous medium type equations Lemmas 3.1 and 4.1 of Chapter 3 have their exact counterpart respectively in Lemmas 7.1 and 8.1 for $m>1$, and in Lemmas 10.1 and 11.1 for $0<m<1$ of Chapter 3.

The discrete isoperimetric inequality and the embeddings of the Preliminaries are facts of Classical Analysis, independent of partial differential equations. Therefore the expansion of positivity effect continues to hold for these equations, by essentially the same proof, whence one introduces changes of variables analogous to (4.6)–(4.7) for the degenerate case $m > 1$ and to (5.7) for the singular case $0 < m < 1$. Below we outline the main differences in the proofs by distinguishing the degenerate case $m > 1$ from the singular case $0 < m < 1$.

7.1 Expansion of Positivity When $m > 1$

The starting point is a time propagation of positivity similar to Lemma 1.1.

Lemma 7.1 *Assume that for some $(y, s) \in E_T$ and some $\rho > 0$*

$$|[u(\cdot, s) \geq M] \cap K_\rho(y)| \geq \alpha |K_\rho(y)|$$

for some $M > 0$ and some $\alpha \in (0, 1)$. There exist δ and ϵ in $(0, 1)$, depending only on the data $\{m, N, C_o, C_1\}$ and α, and independent of M, such that either $C\rho > 1$, or

$$|[u(\cdot, t) > \epsilon M] \cap K_\rho(y)| \geq \tfrac{1}{2}\alpha |K_\rho(y)| \quad \textit{for all } t \in \Big(s, s + \frac{\delta \rho^2}{M^{m-1}}\Big].$$

Proof Same as in Lemma 1.1 by minor changes. We may assume

$$\delta = \frac{\alpha^3}{\gamma 2^{10} N^2},$$

with ϵ as in Lemma 1.1. ■

Proposition 7.1 *Assume that for some $(y, s) \in E_T$ and some $\rho > 0$*

$$|[u(\cdot, s) \geq M] \cap K_\rho(y)| \geq \alpha |K_\rho(y)|$$

for some $M > 0$ and some $\alpha \in (0, 1)$. There exist constants $b > 1$, $\delta, \eta \in (0, 1)$, depending only on the data $\{m, N, C_o, C_1\}$ and α, and independent of (y, s), ρ, M, such that either $C\rho > 1$, or

$$u(\cdot, t) \geq \eta M \quad \textit{in } K_{2\rho}(y)$$

for all times

$$s + \frac{b^{m-1}}{(\eta M)^{m-1}} \tfrac{1}{2} \delta \rho^2 \leq t \leq s + \frac{b^{m-1}}{(\eta M)^{m-1}} \delta \rho^2.$$

The constants b, δ, η deteriorate as $m \to \infty$, but they are stable as $m \to 1$.

Proof Assume $(y, s) = (0, 0)$ and let ϵ and δ be determined as in Lemma 7.1. The proof is almost identical to that of § 4 by means of the change of variables

$$w(x, \tau) \overset{\text{def}}{=} \frac{e^{\frac{\tau}{m-1}}}{M} (\delta \rho^2)^{\frac{1}{m-1}} u\Big(x, s + \frac{e^\tau}{M^{m-1}} \delta \rho^2\Big),$$

modulo the obvious changes in symbolism. The stability analysis of the constants for $m \approx 1$ is carried out as in § 6. ■

7.2 Expansion of Positivity When $0 < m < 1$

The starting point is a time propagation of positivity similar to Lemma 1.1.

Lemma 7.2 *Let* $0 < m < 1$ *and assume that for some* $(y, s) \in E_T$ *and some* $\rho > 0$

$$|[u(\cdot, s) \geq M] \cap K_\rho(y)| \geq \alpha |K_\rho(y)|$$

for some $M > 0$ *and some* $\alpha \in (0, 1)$. *There exist* δ *and* ϵ *in* $(0, 1)$, *depending only on the data* $\{m, N, C_o, C_1\}$ *and* α, *and independent of* M, *such that either* $C\rho > 1$, *or*

$$|[u(\cdot, t) > \epsilon M] \cap K_\rho(y)| \geq \tfrac{1}{2}\alpha |K_\rho(y)| \quad \textit{for all } \ t \in \Big(s, s + \frac{\delta \rho^2}{M^{m-1}}\Big].$$

Proof Assume $(y, s) = (0, 0)$, and consider the cylinder

$$Q_\rho^+(\delta M^{1-m}) = K_\rho \times (0, \delta M^{1-m} \rho^2]$$

where $\delta \in (0, 1)$ is to be chosen. In the weak formulation (5.5) of Chapter 3, take the test function

$$\varphi = -(u^m - M^m)_- \zeta^2$$

where $x \to \zeta(x)$ is a nonnegative, piecewise smooth cutoff function in K_ρ which equals one on $K_{(1-\sigma)\rho}$ and such that $|D\zeta| \leq (\sigma\rho)^{-1}$. Proceeding as in § 9 of Chapter 3 and enforcing the condition $C\rho \leq 1$ gives

$$\begin{aligned} \int_{K_\rho} \int_{u(x,t)}^{M} (M^m - s^m)_+ ds \zeta^2 dx \leq & \int_{K_\rho} \int_{u(x,0)}^{M} (M^m - s^m)_+ ds \zeta^2 dx \\ & + \gamma |K_\rho| \frac{\delta M^{m+1}}{\sigma^2} \end{aligned}$$

for all times $0 < t < \delta M^{1-m} \rho^2$. Enforcing the assumptions of the lemma, estimate

$$\begin{aligned} \int_{K_\rho} \int_{u(x,0)}^{M} (M^m - s^m)_+ ds \zeta^2 dx &\leq \frac{m}{m+1} M^{m+1} (1 - \alpha) |K_\rho| \\ \int_{K_\rho} \int_{u(x,t)}^{M} (M^m - s^m)_+ ds \zeta^2 dx &\geq \int_{K_{(1-\sigma)\rho} \cap [u < \epsilon M]} \int_{u(x,t)}^{M} (M^m - s^m)_+ ds \, dx \\ &\geq \frac{m}{m+1} (1 - \frac{m+1}{m} \epsilon) M^{m+1} |A_{\epsilon M, (1-\sigma)\rho}(t)|. \end{aligned}$$

Therefore proceeding as in the proof of Lemma 1.1

$$|A_{\epsilon M, \rho}(t)| \leq \frac{1}{1 - \epsilon \frac{m+1}{m}} \Big[(1 - \alpha) + \gamma \frac{m+1}{m} \frac{\delta}{\sigma^2} + N\sigma \Big] |K_\rho|.$$

From here on, conclude as in the proof of Lemma 1.1. ■

Proposition 7.2 *Let $0 < m < 1$ and assume that for some $(y, s) \in E_T$ and some $\rho > 0$*

$$|[u(\cdot, s) \geq M] \cap K_\rho(y)| \geq \alpha |K_\rho(y)| \tag{7.1}$$

for some $M > 0$ and some $\alpha \in (0, 1)$. There exist constants $\varepsilon, \delta, \eta \in (0, 1)$, depending only on the data $\{m, N, C_o, C_1\}$ and α, and independent of (y, s), ρ, M, such that either $C\rho > 1$, or

$$u(\cdot, t) \geq \eta M \qquad in \;\; K_{2\rho}(y) \tag{7.2}$$

for all times

$$s + (1 - \varepsilon)\delta M^{1-m}\rho^2 \leq t \leq s + \delta M^{1-m}\rho^2.$$

The constants ε, δ, η deteriorate as $m \to 0$, but they are stable as $m \to 1$.

Proof The proof is similar to that of § 5. Nevertheless, since the particular structure of the energy estimates of § 9 of Chapter 3 brings about some differences, here we present the full proof. The arguments below show that the functional dependence of η on the measure-theoretical parameter α is of the form

$$\eta = \eta_o \alpha \, 2^{-\gamma_1/\alpha^4} \exp(-\gamma_2 \alpha^2 2^{\gamma_1/\alpha^4}), \tag{7.3}$$

for parameters $\eta_o, \gamma_1, \gamma_2$ depending only on the data $\{m, N, C_o, C_1\}$. It is not known whether the dependence can be improved to be power-like, for the general *singular* equations (5.1)–(5.2) of Chapter 3. ■

7.2.1 Transforming the Variables and the Equation

Assume $(y, s) = (0, 0)$, let δ and ϵ be as determined in Lemma 7.2, and let $\rho > 0$ be so that

$$Q_{16\rho}(\delta M^{1-m}) = K_{16\rho} \times (0, \delta M^{1-m}\rho^2] \subset E_T.$$

Introduce the change of variables and the new unknown function

$$z = \frac{x}{\rho}, \qquad -e^{-\tau} = \frac{t - \delta M^{1-m}\rho^2}{\delta M^{1-m}\rho^2}, \qquad v(z, \tau) = \frac{1}{M} u(x, t) e^{\frac{\tau}{1-m}}. \tag{7.4}$$

This maps the cylinder $Q_{16\rho}(\delta M^{1-m})$ into $K_{16} \times (0, \infty)$ and transforms the equations (5.1)–(5.2) of Chapter 3 into

$$v_\tau - \operatorname{div}_z \bar{\mathbf{A}}(z, \tau, v, D_z v) = \bar{B}(z, \tau, v, D_z v) + \frac{1}{1-m} v \tag{7.5}$$

weakly in $K_{16} \times (0, \infty)$, where $\bar{\mathbf{A}}$, and $\bar{B}$ are measurable functions of their arguments, satisfying the structure conditions

$$\begin{cases} \bar{\mathbf{A}}(z, \tau, v, D_z v) \cdot D_z v \geq m\delta C_o v^{m-1} |D_z v|^2 - \delta \bar{C}^2 v^{m+1} \\ |\bar{\mathbf{A}}(z, \tau, v, D_z v)| \leq m\delta C_1 v^{m-1} |D_z v| + \delta \bar{C} v^m \\ |\bar{B}(z, \tau, v, D_z v)| \leq m\delta \bar{C} v^{m-1} |D_z v| + \delta \bar{C}^2 v^m \end{cases}$$

a.e. in $K_{16}\times(0,\infty)$. Here C_o and C_1 are the constants in the structure conditions (5.2) of Chapter 3, δ is the number claimed by Lemma 7.2, and $\bar{C}=\rho C$. In this setting, the information of Lemma 7.2 reads

$$\big|[v(\cdot,\tau)\ge \epsilon\, e^{\frac{\tau}{1-m}}]\cap K_1\big|\ge \tfrac12\alpha|K_1|\quad \text{for all }\ \tau\in(0,+\infty).$$

Let $\tau_o>0$ to be chosen and set

$$k_o=\epsilon\, e^{\frac{\tau_o}{1-m}},\qquad \text{and}\qquad k_j=\frac{1}{2^j}k_o\quad \text{for }\ j=0,1,\dots,j_*,$$

where j_* is to be chosen. With this symbolism

$$|[v(\cdot,\tau)\ge k_j]\cap K_8|\ge \tfrac12\alpha 8^{-N}|K_8|\quad \text{for all }\ \tau\in(\tau_o,+\infty) \tag{7.6}$$

and for all $j\in\mathbb{N}$. Introduce the cylinders

$$\begin{aligned} Q_{\tau_o}&=K_8\times\big(\tau_o+k_o^{1-m},\tau_o+2k_o^{1-m}\big)\\ Q'_{\tau_o}&=K_{16}\times\big(\tau_o,\tau_o+2k_o^{1-m}\big)\end{aligned}$$

and a nonnegative, piecewise smooth, cutoff function in Q'_{τ_o} of the form $\zeta(z,\tau)=\zeta_1(z)\zeta_2(\tau)$, where

$$\zeta_1=\begin{cases}1 & \text{in }\ K_8\\ 0 & \text{in }\ \mathbb{R}^N-K_{16}\end{cases}\qquad\qquad |D\zeta_1|\le\tfrac18,$$

$$\zeta_2=\begin{cases}0 & \text{for }\ \tau<\tau_o\\ 1 & \text{for }\ \tau\ge\tau_o+k_o^{1-m}\end{cases}\qquad 0\le\zeta_{2,\tau}\le\frac{1}{k_o^{1-m}}.$$

In the weak formulation of (7.5), analogous to (5.5) of Chapter 3, take as test function

$$-(v^m-k_j^m)_-\zeta^2\qquad\text{over}\qquad Q'_{\tau_o},$$

for the indicated choice of cutoff function ζ. Performing calculations in all analogous to the ones of § 9 and 10 of Chapter 3, yields

$$\iint_{Q_{\tau_o}}|D(v-k_j)_-|^2dz\,d\tau\le 2\gamma k_j^2|Q_{\tau_o}| \tag{7.7}$$

for a constant γ depending only on the data $\{m,N,C_o,C_1\}$, and δ.

7.2.2 Estimating the Measure of the Set $[v<k_j]$ Within Q_{τ_o}

Set

$$A_j(\tau)=[v(\cdot,\tau)<k_j]\cap K_8\qquad\text{and}\qquad A_j=[v<k_j]\cap Q_{\tau_o}.$$

By Lemma 2.2 of the Preliminaries, and (7.6)

$$(k_j - k_{j+1})|A_{j+1}(\tau)| \le \frac{\gamma(N)}{|K_8 - A_j(\tau)|} \int_{K_8 \cap [k_{j+1} < v(\cdot,\tau) < k_j]} |Dv| dz$$
$$\le \frac{\gamma(N)}{\alpha} \int_{K_8 \cap [k_{j+1} < v(\cdot,\tau) < k_j]} |Dv| dz$$

for all $\tau \ge \tau_o$. Integrate this in $d\tau$ over $(\tau_o + k_o^{1-m}, \tau_o + 2k_o^{1-m})$, majorize the resulting integral on the right-hand side by the Hölder inequality, and use (7.7) to get

$$\begin{aligned}
\frac{k_j}{2}|A_{j+1}| &\le \gamma(\text{data}, \alpha) \iint_{A_j - A_{j+1}} |Dv| dz\, d\tau \\
&\le \gamma(\text{data}, \alpha) \Big(\iint_{A_j - A_{j+1}} |Dv|^2 dz\, d\tau \Big)^{\frac{1}{2}} |A_j - A_{j+1}|^{\frac{1}{2}} \\
&\le \gamma(\text{data}, \alpha) \Big(\iint_{Q_{\tau_o}} |D(v - k_j)_-|^2 dz\, d\tau \Big)^{\frac{1}{2}} |A_j - A_{j+1}|^{\frac{1}{2}} \\
&\le \gamma(\text{data}, \alpha, \delta) k_j |Q_{\tau_o}|^{\frac{1}{2}} |A_j - A_{j+1}|^{\frac{1}{2}}.
\end{aligned}$$

Taking the square yields the recursive inequalities

$$|A_{j+1}|^2 \le \gamma(\text{data}, \alpha, \delta)|Q_{\tau_o}||A_j - A_{j+1}|.$$

Add these inequalities for $j = 0, 1, \ldots, j_* - 1$, where j_* is an integer to be chosen, and majorize the sum on the right-hand side by the corresponding telescopic series. This gives

$$(j_* - 1)|A_{j_*}|^2 \le \gamma(\text{data}, \alpha, \delta)|Q_{\tau_o}|^2.$$

Equivalently

$$|[v < k_{j_*}] \cap Q_{\tau_o}| \le \nu |Q_{\tau_o}| \quad \text{where} \quad \nu = \Big(\frac{\gamma(\text{data}, \alpha, \delta)}{j_*} \Big)^{\frac{1}{2}}. \tag{7.8}$$

7.2.3 Segmenting Q_{τ_o}

Assume momentarily that j_* and hence ν have been determined. By possibly increasing j_* to be not necessarily integer, without loss of generality we may assume that $(2^{j_*})^{1-m}$ is an integer. Then subdivide Q_{τ_o} into $(2^{j_*})^{1-m}$ cylinders, each of length $k_{j_*}^{1-m}$, by setting

$$\begin{aligned}
&Q_n = K_8 \times \big(\tau_o + k_o^{1-m} + n k_{j_*}^{1-m}\, , \, \tau_o + k_o^{1-m} + (n+1) k_{j_*}^{1-m} \big) \\
&\text{for } \; n = 0, 1, \ldots, (2^{j_*})^{1-m} - 1.
\end{aligned}$$

For at least one of these, say Q_n, there must hold

$$|[v < k_{j_*}] \cap Q_n| \le \nu |Q_n|.$$

Apply Lemma 10.1 of Chapter 3 to v over Q_n with $\xi\omega = k_{j_*}$, $a = \frac{1}{2}$, and $\theta = k_{j_*}^{1-m}$. This gives

$$v\big(z, \tau_o + k_o^{1-m} + (n+1)k_{j_*}^{1-m}\big) \geq \tfrac{1}{2}k_{j_*} \qquad \text{a.e. in } K_4,$$

provided $C\rho < 1$ and

$$\frac{|[v < k_{j_*}] \cap Q_n|}{|Q_n|} \leq 2^{-(N+2)^2}\bar{\gamma}_o(\text{data}) = \nu.$$

Choose now j_*, and hence ν, from this and (7.8). Summarizing, for such a choice of j_*, and hence ν, there exists a time level τ_1 in the range

$$\tau_o + k_o^{1-m} < \tau_1 < \tau_o + 2k_o^{1-m} \tag{7.9}$$

such that

$$v(z, \tau_1) \geq \sigma_o e^{\frac{\tau_o}{1-m}} \qquad \text{where} \qquad \sigma_o = \epsilon\, 2^{-(j_*+1)}.$$

Remark 7.1 Notice that j_* and hence ν are determined only in terms of the data and are independent of the parameter τ_o, which is still to be chosen.

7.2.4 Returning to the Original Coordinates

In terms of the original coordinates and the original function $u(x,t)$ this implies

$$u(\cdot, t_1) \geq \sigma_o M e^{-\frac{\tau_1 - \tau_o}{1-m}} \overset{\text{def}}{=} M_o \qquad \text{in } K_{4\rho}$$

where the time t_1 corresponding to τ_1 is computed from (7.4) and (7.9). Apply now Lemma 11.1 of Chapter 3 with M replaced by M_o and $\xi = 1$ over the cylinder

$$(0, t_1) + Q_{4\rho}^+(\theta) = K_{4\rho} \times \big(t_1, t_1 + \theta(4\rho)^2\big].$$

By choosing

$$\theta = \nu_o M_o^{1-m} \qquad \text{where} \qquad \nu_o = \nu_o(\text{data}),$$

the assumption (11.1) of Chapter 3 is satisfied, and the lemma yields

$$\begin{aligned} u(\cdot, t) &\geq \tfrac{1}{2}M_o = \tfrac{1}{2}\sigma_o M e^{-\frac{\tau_1 - \tau_o}{1-m}} \\ &\geq \tfrac{1}{2}\sigma_o e^{-\frac{2}{1-m}e^{\tau_o}} M \end{aligned} \qquad \text{in} \quad K_{2\rho} \tag{7.10}$$

for all times

$$t_1 \leq t \leq t_1 + \nu_o M_o^{1-m}(4\rho)^2. \tag{7.11}$$

If the right-hand side equals $\delta M^{1-m}\rho^2$, then (7.10) and the conclusion of Proposition 7.2 hold for the time $t = \delta M^{1-m}\rho^2$. The transformed τ_o level is still undetermined, and it will be so chosen as to verify such a requirement. Precisely, taking into account the change of variables (7.4)

$$\delta M^{1-m}\rho^2 e^{-\tau_1} = \delta M^{1-m}\rho^2 - t_1 = \nu_o \sigma_o^{1-m} M^{1-m}(4\rho)^2 e^{-(\tau_1-\tau_o)}$$

which implies

$$e^{\tau_o} = \frac{\delta}{16\nu_o\sigma_o^{1-m}}.$$

This determines quantitatively $\tau_o = \tau_o(\text{data})$. The proof of Proposition 7.2 with $0 < m < 1$ is now completed by inserting such a τ_o on the right-hand side of (7.10) and in (7.11). In particular (7.10) holds for all times

$$t_1 = \delta M^{1-m}\rho^2 - \nu_o M_o^{1-m}(4\rho)^2 \le t \le \delta M^{1-m}\rho^2.$$

From the previous definitions and transformations one estimates

$$t_1 \le (1-\varepsilon)\delta M^{1-m}\rho^2, \quad \text{where} \quad \varepsilon = e^{-\tau_o - 2e^{\tau_o}}.$$

8 Remarks and Bibliographical Notes

Proposition 2.1 was first established in [40]. The main idea is in realizing that the classical theorems of [36, 101] can be read in an "expanding fashion," instead of a "shrinking one," as originally conceived by DeGiorgi.

The notion of expansion of positivity is related to the so-called growth lemmas, introduced by Landis ([104]). Based on these lemmas, Landis gave alternative proofs of the results by DeGiorgi ([36]) and Moser ([120]), on the Hölder regularity and Harnack inequalities for solutions to second-order elliptic equations in divergence form. This approach is flexible enough as to adapt to equations in nondivergence form ([93, 94, 135], see also [3]).

For the homogeneous prototype degenerate equations (1.3) and (5.3) of Chapter 3, the expansion of positivity was realized in [39, 59] by means of comparison with suitable subsolutions.

In the full generality of Proposition 4.1, this expansion effect was established in [49], including the analysis of stability of the various parameters, either as $p \to 2$ or as $m \to 1$. The proof we present here is a simpler, more streamlined version of that in [49]. Some measure-theoretical lemmas are avoided, and the statements are shown to hold in the more general assumptions (2.1), (4.2), (5.2), with any $\alpha \in (0,1]$ instead of $\alpha = 1$ as established in [49, 51, 54].

In the context of singular equations ($1 < p < 2$ or $0 < m < 1$) the proof of Proposition 5.1 was first given in [31] and reported in [41], Chapter IV, § 5. The proof is rather involved and not intuitive. The proof we present here follows an idea of [51] and [54]; it is more direct, being based on geometrical ideas. Both proofs require $p > 1$ and $m > 0$. The restriction is not only technical in view of the geometrical significance of the homogeneous, prototype equation (1.3) of Chapter 3 for $p = 1$ even in the elliptic case ([117]), and the homogeneous equation (5.3) of Chapter 3 for $m \to 0$ ([35, 44, 45]).

8.1 Solving (3.6)

Seek convex, symmetric about $x = 0$, smooth solutions. For these (3.6) is transformed into

$$\begin{aligned} X' &= -\Big(\frac{p}{p-1}\Big)^{\frac{1}{p}} \big(C - \tfrac{1}{2}\mu X^2\big)^{\frac{1}{p}}, \quad \text{in } (0,1) \\ X'(0) &= 0, \quad X(1) = 0 \end{aligned} \tag{8.1}$$

for positive parameters C and μ. In general this problem cannot be solved explicitly. However, one can show that solutions actually exist, by studying their qualitative behavior. By setting

$$\begin{aligned} K &= \frac{2C}{\mu} \\ y &= \alpha x, \quad \alpha = \left(\frac{K^{2-p}\mu p}{2(p-1)}\right)^{\frac{1}{p}} = \left(\frac{C^{2-p}\mu^{p-1}p}{2^{p-1}(p-1)}\right)^{\frac{1}{p}} \\ Y(y) &= \frac{X(x)}{K}, \end{aligned}$$

problem (8.1) can be rewritten as

$$\begin{aligned} Y' &= -\big(1 - Y^2\big)^{\frac{1}{p}}, \quad \text{in } (0,\alpha) \\ Y'(0) &= 0, \quad Y(\alpha) = 0. \end{aligned}$$

The parameter dependence is now transferred into α. Because of the two-point condition for a first-order differential equation, the problem may appear overdetermined. By standard ODE's theory, the family of solutions to the Cauchy problem

$$Y' = -\big(1 - Y^2\big)^{\frac{1}{p}}, \qquad Y'(0) = 0 \tag{8.2}$$

behave as in Figure 8.3 below. Such a Cauchy problem does not have a unique solution, and Y_{min}, Y_{max} represent the minimal and the maximal solutions, respectively. Now Y_{min} intersects the $Y = 0$ axis at y_{int}. By properly choosing the pair (C, μ), one may realize $\alpha = y_{int}$. There exist ∞^1 possible choices for (C, μ), that realize such a condition, and hence there exists an infinite number of functions X that solve the original boundary value problem. This is expected, as (3.6) is the 1–D case of the nonlinear eigenvalue problem

$$\begin{aligned} -\operatorname{div}(|Dw|^{p-2}Dw) &= \mu w, \quad \text{in } E \\ w|_{\partial E} &= 0. \end{aligned}$$

By the results of [73], such a problem admits infinitely many solutions.

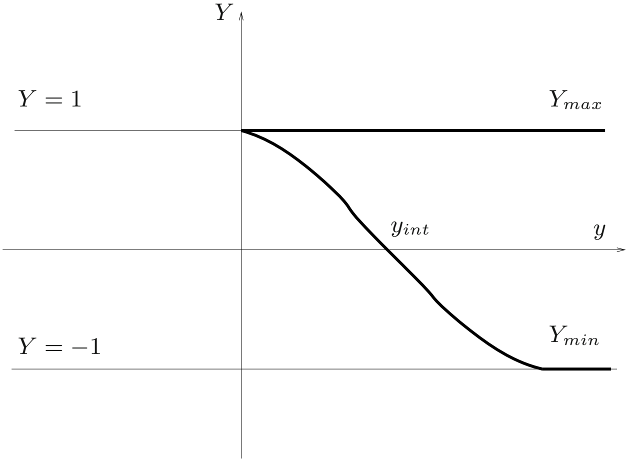

Fig. 8.3. Qualitative Behavior of the Solution to (8.2)