Springer Monographs in Mathematics

Emmanuele DiBenedetto
Ugo Gianazza
Vincenzo Vespri

Harnack's Inequality for Degenerate and Singular Parabolic Equations

Springer

S*pringer* **M***onographs in* **M***athematics*

For further volumes:
http://www.springer.com/series/3733

Emmanuele DiBenedetto
Ugo Gianazza
Vincenzo Vespri

Harnack's Inequality for Degenerate and Singular Parabolic Equations

Springer

Emmanuele DiBenedetto
Department of Mathematics
Vanderbilt University
Nashville, 37240 TN
USA
em.diben@vanderbilt.edu

Ugo Gianazza
Dipartimento di Matematica "F. Casorati"
Università degli Studi di Pavia
27100 Pavia
Italy
gianazza@imati.cnr.it

Vincenzo Vespri
Dipartimento di Matematica "U. Dini"
Università degli Studi di Firenze
50134 Firenze
Italy
vespri@math.unifi.it

ISSN 1439-7382
ISBN 978-1-4614-1583-1 e-ISBN 978-1-4614-1584-8
DOI 10.1007/978-1-4614-1584-8
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2011941123

Mathematics Subject Classification (2010): 35-XX, 35K65, 35K67, 35J62

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Contents

Preface xiii

1 Introduction 1
1 The Classical Harnack Inequality 1
2 Quasilinear Coercive Elliptic and Parabolic Equations 4
3 Degenerate and Singular Parabolic Equations 5
3.1 Quasilinear Equations of p-Laplacian Type 5
3.2 Quasilinear Equations of Porous Medium Type 6
3.3 Aim of the Monograph 6
4 Parabolic Harnack Estimates. The Role of the Structure 7
4.1 Degenerate Equations of the p-Laplacian Type for $p > 2$ 8
4.2 Singular Equations of the p-Laplacian Type for $\frac{2N}{N+1} < p < 2$ 9
4.3 Outstanding Issues 10

2 Preliminaries 13
1 Poincaré and Sobolev Inequalities 13
2 Cuts and Truncations of Functions in $W^{1,p}(E)$ and Their Embeddings 15
3 A Measure-Theoretical Lemma ([48]) 16
4 Parabolic Spaces and Embeddings 18
5 Some Technical Facts 20
5.1 A Lemma on Fast Geometric Convergence 20
5.2 An Interpolation Lemma 21
5.3 Steklov Averages 21
6 Remarks and Bibliographical Notes 21

3 Degenerate and Singular Parabolic Equations 23
1 Quasilinear Equations of p-Laplacian Type 23
1.1 An Alternate Formulation in Terms of Steklov Averages 24
1.2 On the Notion of Parabolicity 24

1.3 Dependence on the Parameters $\{p, N, C_o, C_1\}$ and Stability ... 26
2 Energy Estimates for $(u-k)_\pm$ on Cylinders $(y,s)+Q_\rho^\pm(\theta) \subset E_T$ 26
3 A DeGiorgi-Type Lemma ... 29
4 A Variant of DeGiorgi-Type Lemma Involving "Initial Data" .. 32
5 Quasilinear Equations of the Porous Medium Type ... 33
5.1 An Alternate Formulation in Terms of Steklov Averages 34
5.2 On the Notion of Parabolicity ... 35
5.3 Dependence on the Parameters $\{m, N, C_o, C_1\}$ and Stability ... 36
6 Energy Estimates for $(u-k)_\pm$ on Cylinders $(y,s)+Q_\rho^\pm(\theta) \subset E_T$ for Degenerate Equations $(m>1)$ ... 36
7 A DeGiorgi-Type Lemma for Nonnegative Sub(Super)-Solutions to Degenerate Equations $(m>1)$ ... 39
7.1 Proof of (7.1)–(7.3) ... 40
7.2 Proof of (7.4)–(7.5) ... 42
8 A Variant of DeGiorgi-Type Lemma, for Nonnegative Supersolutions to Degenerate Equations $(m>1)$, Involving "Initial Data" ... 45
9 Energy Estimates for $(u-k)_-$ on Cylinders $(y,s)+Q_\rho^\pm(\theta) \subset E_T$ for Singular Equations $(0<m<1)$ ... 46
10 A DeGiorgi-Type Lemma for Nonnegative Supersolutions to Singular Equations $(0<m<1)$ ... 50
11 A Variant of DeGiorgi-Type Lemma, for Nonnegative Supersolutions to Singular Equations $(0<m<1)$, Involving "Initial Data" ... 52
12 Remarks and Bibliographical Notes ... 54

4 Expansion of Positivity ... 57
1 Time and Space Propagation of Positivity ... 57
2 The Expansion of Positivity for Nondegenerate, Homogeneous, Quasilinear Parabolic Equations ... 59
3 Some Counterexamples for Degenerate and Singular Equations 62
3.1 A First Counterexample for $p>2$ ... 62
3.2 A Second Counterexample for $p>2$ ... 63
3.3 A Family of Counterexamples for $1<p<2$ ... 64
3.4 The Expansion of Positivity in Some Intrinsic Geometry 66
4 The Expansion of Positivity for Degenerate Quasilinear Parabolic Equations $(p>2)$ ... 66
4.1 Structure of the Proof ... 67
4.2 Changing the Time Variable ... 67
4.3 The Set Where w Is Small Can Be Made Small Within $\mathcal{Q}_{4\rho}(\theta)$ for Large θ ... 69
4.4 Expanding the Positivity of w ... 71

4.5 Expanding the Positivity of u 72
5 The Expansion of Positivity for Singular Quasilinear Parabolic Equations $(1 < p < 2)$ 72
5.1 Transforming the Variables and the Equation 73
5.2 Estimating the Measure of the Set $[v < k_j]$ Within Q_{τ_o} . 75
5.3 Segmenting Q_{τ_o} 76
5.4 Returning to the Original Coordinates 77
6 Stability of the Expansion of Positivity for $p \to 2$ 78
6.1 Proof of Proposition 6.1 79
7 The Expansion of Positivity for Porous Medium Type Equations 81
7.1 Expansion of Positivity When $m > 1$ 82
7.2 Expansion of Positivity When $0 < m < 1$ 83
8 Remarks and Bibliographical Notes 88
8.1 Solving (3.6) 89

5 The Harnack Inequality for Degenerate Equations 91
1 The Intrinsic Harnack Inequality 91
1.1 The Mean Value Form of Moser's Harnack Inequality . . 91
1.2 The Intrinsic Mean Value Harnack Inequality for Degenerate Equations 92
1.3 Significance of Theorem 1.1 94
2 Proof of the Intrinsic, Forward Harnack Inequality in Theorem 1.1 95
2.1 Proof of Proposition 2.1 96
2.2 Expanding the Positivity of v 97
2.3 Proof of Proposition 2.1 Concluded 99
2.4 On the Connection Between γ and c 100
3 Proof of the Intrinsic, Backward Harnack Inequality in Theorem 1.1 100
3.1 There Exists $t < t_o$ Satisfying (3.1) 101
4 The Intrinsic Harnack Inequality Implies the Hölder Continuity . . 102
5 Liouville-Type Results 105
5.1 Two-Sided Bounds and Liouville-Type Results 106
5.2 Two-Sided Bound at One Point, as $t \to \infty$ 107
6 Subpotential Lower Bounds 108
7 A Weak Harnack Inequality for Positive Supersolutions 112
8 Proof of Proposition 7.1 113
9 Proof of Theorem 7.1 by Alternatives 115
10 A Reverse Hölder Inequality for Supersolutions 116
11 A Uniform Bound Above on the $L^q_{\rm loc}$-Integral of Supersolutions . 120
12 An Integral Bound Below for Supersolutions 121
13 Proof of Theorem 7.1 123
14 A Consequence of Theorem 7.1 125
15 Equations of the Porous Medium Type 127

15.1 The Intrinsic Harnack Inequality 127
16 Some Consequences of the Harnack Inequality 130
16.1 Hölder Continuity 130
16.2 Liouville-Type Results 131
16.3 Subpotential Lower Bounds 132
17 A Weak Harnack Inequality for Positive Supersolutions 133
18 Remarks and Bibliographical Notes 134
18.1 Weak Harnack Estimates 136

6 The Harnack Inequality for Singular Equations 137
1 Supercritical, Singular Equations 137
1.1 The Intrinsic, Mean Value, Harnack Inequality 137
1.2 Time-Insensitive, Intrinsic, Mean Value, Harnack Inequalities 139
1.3 On the Range (1.1) of p 140
2 Main Components in the Proof of Theorems 1.1 and 1.2 142
3 The Right-Hand-Side Harnack Estimate of Theorem 1.2 143
4 Locating the Supremum of v in K_1 144
5 Estimating the Sup of v in Some Intrinsic Neighborhood About $(\bar{x}, 0)$ 144
6 Expanding the Positivity of v 146
7 Proof of the Right-Hand-Side Harnack Inequality of Theorem 1.1 148
7.1 On the Functional Relation $\gamma = \gamma(\epsilon)$ 150
8 Proof of the Left-Hand-Side Harnack Inequality in Theorem 1.2 151
9 Proof of the Left-Hand-Side Harnack Inequality in Theorem 1.1 151
10 Some Consequences of the Harnack Inequality 153
10.1 Local Hölder Continuity 153
10.2 A Liouville-Type Result 153
11 Critical and Subcritical Singular Equations 155
12 Components of the Proof of Theorem 11.1 156
12.1 L^r_{loc}–L^∞_{loc} Harnack-Type Estimates for $r \geq 1$ Such That $\lambda_r > 0$ 156
12.2 L^r_{loc} Estimates Backward in Time 157
13 Estimating the Positivity Set of the Solutions 157
13.1 A First Form of the Harnack Inequality 159
14 The First Form of the Harnack Inequality Implies the Hölder Continuity of u 160
15 Proof of Theorem 11.1 Concluded 163
16 Supercritical, Singular Equations of the Porous Medium Type 165
16.1 The Intrinsic, Mean Value, Harnack Inequality 165
16.2 Time-Insensitive, Intrinsic, Mean Value, Harnack Inequalities 167
16.3 On the Range (16.1) of m 168
17 On the Proof of Theorems 16.1 and 16.2 170

17.1 An L^1_{loc}–L^∞_{loc} Harnack-Type Estimate ... 170
17.2 The Right-Hand-Side Harnack Estimate of Theorem 16.2 170
18 Some Consequences of the Harnack Inequality ... 171
18.1 Analyticity of Nonnegative Solutions ... 171
18.2 Hölder Continuity ... 172
18.3 A Liouville-Type Result ... 175
19 Critical and Subcritical Singular Equations of the Porous Medium Type ... 175
20 On the Proof of Theorem 19.1 ... 177
20.1 L^r_{loc}–L^∞_{loc} Harnack-Type Estimates for $r \geq 1$ Such That $\lambda_r > 0$ ... 177
20.2 L^r_{loc} Estimates Backward in Time ... 177
20.3 Main Points of the Proof ... 178
21 Remarks and Bibliographical Notes ... 179
21.1 About Theorem 11.1 ... 180
21.2 On the Weak Harnack Inequality ... 181
21.3 On the Boundedness of Weak Solutions ... 181
21.4 On the Two Critical Values $p_{**} = \frac{2N}{N+2} < \frac{2N}{N+1} = p_*$ ... 182
21.5 Singular Equations of the Porous Medium Type ... 183
21.6 On the Two Critical Values $0 \leq m_{**} \leq m_* < 1$ ... 184

7 Homogeneous Monotone Singular Equations ... 187
1 Monotone Structure Conditions ... 187
1.1 A Less Constrained Harnack Inequality ... 189
2 The Intrinsic Harnack Inequality ... 190
2.1 The Right-Hand-Side Harnack Estimate of Theorem 2.2 191
3 Proof of Proposition 2.1 ... 191
3.1 Largeness of v in Q_1 ... 192
3.2 Expanding the Positivity of v ... 193
4 Subpotential Lower Bounds ... 196
5 Monotone Structures for Singular Porous Medium Type Equations ... 199
5.1 A Less Constrained Harnack Inequality ... 201
6 The Intrinsic Harnack Inequality ... 202
6.1 The Right-Hand-Side Harnack Estimate of Theorem 2.2 203
6.2 Subpotential Lower Bounds ... 203
7 Remarks and Bibliographical Notes ... 204

8 Appendix A ... 205
A.1 An L^1_{loc}-Form of the Harnack Inequality for All $p \in (1, 2)$ ... 205
A.1.1 Auxiliary Lemmas ... 205
A.1.2 Proof of Proposition A.1.1 ... 208
A.1.3 An Estimate for Supersolutions ... 210
A.2 L^r_{loc}–L^∞_{loc} Estimates ... 210

A.2.1 Proof of Proposition A.2.1 for p in the Range $\max\{1, \frac{2N}{N+2}\} < p < 2$ . . . 211
A.2.2 Proof of Proposition A.2.1 for p in the Range $1 < p \leq \max\{1, \frac{2N}{N+2}\}$ . . . 213
A.3 L^r_{loc} Estimates Backward in Time . . . 215
A.3.1 Proof of Proposition A.3.1 . . . 215
A.4 Remarks and Bibliographical Notes . . . 219

9 Appendix B . . . 221
B.1 An L^1_{loc}-Form of the Harnack Inequality for All $m \in (0,1)$ . . . 221
B.1.1 An Auxiliary Lemma . . . 221
B.1.2 Proof of Proposition B.1.1 . . . 223
B.1.3 An Estimate for Supersolutions . . . 224
B.2 Energy Estimates for Sub(Super)-Solutions When $0 < m < 1$ . . . 225
B.3 A Different Type of Energy Estimates for Sub(Super)-Solutions When $0 < m < 1$ . . . 227
B.4 L^r_{loc}–L^∞_{loc} Estimates . . . 230
B.4.1 Proof of Proposition B.4.1 for $\frac{(N-2)_+}{N+2} < m < 1$ . . . 231
B.4.2 Proof of Proposition B.4.1 for $0 < m \leq \frac{(N-2)_+}{N+2}$ . . . 234
B.5 L^r_{loc} Estimates Backward in Time . . . 235
B.6 A DeGiorgi-Type Lemma for Nonnegative Subsolutions to Singular Porous Medium Type Equations . . . 238
B.7 A Logarithmic Estimate for Nonnegative Subsolutions to Singular Porous Medium Type Equations . . . 240
B.8 Hölder Continuity . . . 243
B.8.1 Proof of Theorem B.8.1. Preliminaries . . . 243
B.9 The Induction Argument. Two Alternatives . . . 244
B.10 A Uniform Time Control on the Measure of the Level Sets . . . 245
B.11 The Second Alternative Continued . . . 248
B.12 The Induction Argument Concluded-(i) . . . 250
B.13 The Induction Argument Concluded-(ii) . . . 251
B.14 Remarks and Bibliographical Notes . . . 252

10 Appendix C . . . 255
C.1 More General Structures . . . 255
C.2 Energy Estimates for $(u-k)_\pm$ on Cylinders $(y,s)+Q^\pm_\rho(\theta) \subset E_T$ 256
C.3 DeGiorgi-Type Lemmas . . . 257
C.3.1 A Variant of DeGiorgi-Type Lemma, Involving "Initial Data" . . . 258
C.3.2 A Technical Lemma . . . 258
C.4 Expansion of Positivity for Nonnegative Solutions to (C.1.1)–(C.1.4) . . . 259
C.5 Equations of Porous Medium Type . . . 261
C.6 Energy Estimates for $(u-k)_\pm$ on Cylinders $(y,s)+Q^\pm_\rho(\theta) \subset E_T$ 261
C.7 DeGiorgi-Type Lemmas . . . 263

C.7.1 A Variant of DeGiorgi-Type Lemma, Involving "Initial Data" . . . 264
C.8 Expansion of Positivity for Solutions to (C.5.1)–(C.5.3) . . . 265
C.9 Remarks and Bibliographical Notes . . . 266

References . . . 269

Index . . . 277

Preface

Degenerate and singular parabolic equations have been the subject of extensive research for the last 25 years but the issue of the Harnack inequality has remained basically open. Recently considerable progress has been made in this area to the point that the theory is reasonably complete, — except for the singular subcritical range — both for the p-Laplacian and the porous medium equations.

This monograph represents a comprehensive treatment of the Harnack inequality for nonnegative solutions to p-Laplace and porous medium type equations, starting from the notion of solution and building all the necessary technical tools. The work is solely mathematical in nature, highlights the main issues and the problems that still remain open, and its aim is to introduce a novel set of tools and techniques that provide a better understanding of the notion of degeneracy and/or singularity in partial differential equations.

The book is self-contained. The readership is aimed at all professionals active in the field, and also at advanced graduate students who are interested in understanding the main issues of this fascinating research field.

We have benefited from the interest and input of several people. An incomplete list includes John Lewis, Peter Lindqvist, Juan Manfredi, Sandro Salsa, Giuseppe Mingione, Mikhail Surnachev, and Vitali Liskevich. We are also thankful to Simona Fornaro, Andrea Fugazzola, Naian Liao, Simona Puglisi, and Maria Sosio, who have carefully read various portions of the manuscript, and helped us in correcting some misprints, and imprecise statements.

This work was partially supported by NSF grant DMS-0652385, and by FAR-funds of the University of Pavia. The monograph was written while U.G. was visiting Vanderbilt University during his sabbatical leave: the warm hospitality of the Department of Mathematics of VU is gratefully acknowledged.

Nashville, USA — Emmanuele DiBenedetto
Pavia, Italy — Ugo Gianazza
Florence, Italy — Vincenzo Vespri

July 2011

1

Introduction

1 The Classical Harnack Inequality

In [82], paragraph 19, page 62 the German mathematician C.-G. Axel von Harnack formulates and proves the following theorem in the case $N = 2$.

Theorem 1.1 (Harnack Inequality) *Let u be a nonnegative harmonic function in an open set $E \subset \mathbb{R}^N$. Then for all $x \in B_r(x_o) \subset B_R(x_o) \subset E$*

$$\left(\frac{R}{R+r}\right)^{N-2} \frac{R-r}{R+r} u(x_o) \le u(x) \le \left(\frac{R}{R-r}\right)^{N-2} \frac{R+r}{R-r} u(x_o), \tag{1.1}$$

where $B_R(x_o) = \{x \in \mathbb{R}^N : \ |x - x_o| < R\}$.

The estimate above is scale invariant in the sense that it does not change for various choices of R, when $r = cR$, $c \in (0,1)$ are fixed. Moreover it depends neither on the position of the ball $B_R(x_o)$, nor on u.

The proof is so simple that it is worth reporting it here. Set $\rho = |x - x_o|$, and choose $R' \in (r, R)$. Since u is continuous in $\overline{B_{R'}(x_o)}$, the Poisson representation formula for harmonic functions can be applied, yielding

$$u(x) = \frac{R'^2 - \rho^2}{\omega_N R'} \int_{\partial B_{R'}(x_o)} u(y)|x-y|^{-N} d\sigma(y), \tag{1.2}$$

where ω_N is the area of the unit sphere in $\mathbb{R}^N$, and $d\sigma$ denotes the surface measure on $\partial B_{R'}(x_o)$. Since

$$\frac{R'^2 - \rho^2}{(R'+\rho)^N} \le \frac{R'^2 - \rho^2}{|x-y|^N} \le \frac{R'^2 - \rho^2}{(R'-\rho)^N}, \tag{1.3}$$

combining (1.2)–(1.3), and using the mean value characterization of harmonic functions, we have

$$\left(\frac{R'}{R'+\rho}\right)^{N-2} \frac{R'-\rho}{R'+\rho} u(x_o) \le u(x) \le \left(\frac{R'}{R'-\rho}\right)^{N-2} \frac{R'+\rho}{R'-\rho} u(x_o). \tag{1.4}$$

E. DiBenedetto et al., *Harnack's Inequality for Degenerate and Singular Parabolic Equations*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-1584-8_1,

Letting $R' \to R$ and realizing that the bounds are monotone in ρ, proves (1.1).

In its modern version, currently used in the theory of partial differential equations, the Harnack inequality for harmonic functions is given the following "mean value form."

Theorem 1.2 *Let $N \geq 2$ and let $E \subset \mathbb{R}^N$ be an open set. Then there exists a constant $\gamma > 1$, dependent only on the dimension N, such that*

$$\gamma^{-1} \sup_{B_r(x_o)} u \leq u(x_o) \leq \gamma \inf_{B_r(x_o)} u, \tag{1.5}$$

for every nonnegative harmonic function $u : E \to \mathbb{R}$, and for every ball $B_r(x_o)$, such that $B_{2r}(x_o)$ is contained in E.

Although the proof seems to indicate that the Harnack inequality is an almost trivial consequence of the Poisson representation formula, such an estimate, in either of its two forms (1.1) and (1.5), has a whole host of important consequences, and we list the main ones here.

- A nonnegative harmonic function in $\mathbb{R}^N$ is constant (Liouville Theorem).
- If $u : B_R(0)\backslash\{0\} \to \mathbb{R}$ is harmonic, and satisfies $u(x) = o(|x|^{2-N})$ for $x \to 0$, then $u(0)$ can be defined in such a way that $u : B_R(0) \to \mathbb{R}$ is harmonic (Removable Singularity Theorem).
- If a sequence of functions which are harmonic in a bounded domain $E \subset \mathbb{R}^N$ and continuous on $\overline{E}$ converges uniformly on the boundary ∂E, then it also converges uniformly in E to a harmonic function (Harnack's First Convergence Theorem).
- Let $\{u_n\}$ be a sequence of harmonic functions in a connected open set $E \subset \mathbb{R}^N$. If $\{u_n\}$ is monotone increasing and there exists a point $x_o \in E$ such that the sequence $\{u_n(x_o)\}$ is convergent, then $\{u_n\}$ is uniformly convergent on every compact subset of E to a harmonic function u (Harnack's Second Convergence Theorem).
- The Harnack inequality plays a fundamental role in the construction of solutions to the Dirichlet Problem for the Laplace equation using Perron's method (for more details see, for example, [43], Chapter 2, § 6).

Things become more involved when we move from the elliptic to the parabolic setting. In the following by a *caloric* function, we mean a **nonnegative** solution u to the heat equation

$$u_t - \Delta u = 0$$

in some open set $\Omega \subset \mathbb{R}^{N+1}$.

It is not obvious what the analog of (1.1)–(1.5) for caloric functions could be. If we consider the function

$$\Gamma(x,t) = \begin{cases} \dfrac{1}{\sqrt{4\pi t}} \exp\Big(-\dfrac{x^2}{4t}\Big) & \text{for } t > 0 \\ 0 & \text{for } \ x \neq 0 \ \text{ and } \ t \leq 0, \end{cases}$$

it is nonnegative and caloric in $\mathbb{R}^2 - \{(0,0)\}$, but there is no positive constant γ satisfying

$$\gamma^{-1} \sup_B \Gamma \le \Gamma(1,0) \le \gamma \inf_B \Gamma$$

where

$$B = \{(x,t) \in \mathbb{R}^2 : \ (x-1)^2 + t^2 < \tfrac{1}{4}\}.$$

Indeed $\Gamma(1,0) = 0$, whereas $\sup_B \Gamma > 0$.

The first parabolic Harnack-type estimate was established independently by Hadamard ([80]) and Pini ([127]).

Theorem 1.3 *Let E be an open set in $\mathbb{R}^N$ and for $T > 0$ let E_T denote the cylindrical domain $E \times (0,T]$. Let u be a caloric function in E_T and for $(x_o, t_o) \in E_T$ consider a cylinder*

$$B_{2\rho}(x_o) \times \{t_o - 4\rho^2, t_o + \rho^2\} \subset E_T.$$

Then there exists a constant γ depending only upon N, such that

$$\gamma^{-1} \sup_{B_\rho(x_o)} u(\cdot, t_o - \rho^2) \le u(x_o, t_o). \tag{1.6}$$

Both Hadamard's and Pini's proofs are based on local representations of solutions in terms of heat potentials, thereby paralleling the initial approach of Harnack.

As for nonnegative harmonic functions, the original statement of Hadamard's and Pini's inequality can be given the equivalent "mean value form" ([52])

$$\gamma^{-1} \sup_{B_\rho(x_o)} u(\cdot, t_o - \rho^2) \le u(x_o, t_o) \le \gamma \inf_{B_\rho(x_o)} u(\cdot, t_o + \rho^2). \tag{1.7}$$

A counterexample of Moser ([121]) shows that the "waiting time" from $t_o - \rho^2$ to t_o and from t_o to $t_o + \rho^2$ is needed for (1.7) to hold. Indeed such an inequality "at the same time level t_o" would be false for caloric functions.

It is immediate to see that, as a consequence of the Harnack inequality, a caloric function which is strictly positive in a point cannot vanish.

Also a *bounded* caloric function in $t < 0$ is constant ([152], p. 236). However, unlike nonnegative harmonic functions in $\mathbb{R}^N$, caloric functions in $t < 0$ as such (i.e., nonnegative) need not be constant. As a counterexample, the function

$$u(x,t) = \exp(x + t)$$

is caloric in $\mathbb{R}^2$ and nonconstant. This shows that a single bound (either from above or from below) is not sufficient for a Liouville-type theorem to hold.

For a more thorough discussion of the Harnack inequality for caloric functions, and in particular its relevance in the context of Parabolic Potential Theory, we refer the reader to the recent paper [103], whence the previous example with Γ is taken.

2 Quasilinear Coercive Elliptic and Parabolic Equations

The results of the previous sections are not limited to the prototype examples, respectively of the Laplace and the heat equation.

In a series of seminal papers, Moser ([121, 122]) considers nonnegative solutions to linear, parabolic equations in divergence form, with measurable and bounded coefficients, namely,

$$u_t - \operatorname{div}(a_{ij}(x,t)D_j u) = 0, \tag{2.1}$$

where

$$\lambda|\xi|^2 \le a_{ij}(x,t)\xi_i\xi_j \le \Lambda|\xi|^2 \tag{2.2}$$

for almost all $(x,t) \in E_T$ and all $\xi \in \mathbb{R}^N$ for two constants

$$0 < \lambda \le \Lambda.$$

In ([121, 122]) it is shown that nonnegative, weak solutions to these equations satisfy the Harnack estimate (1.7).

The positive parameter λ is defined as the *modulus of ellipticity* of (2.1).

Moser's methods are only measure-theoretical, and based on the notion of "ellipticity" and "parabolicity," more than possible local representations. The analysis of his proofs reveals that the linearity is immaterial, and that essentially the same arguments apply as well to nonnegative weak solutions to quasilinear equations of the form

$$u_t - \operatorname{div} \mathbf{A}(x,t,u,Du) = 0 \qquad \text{weakly in } E_T \tag{2.3}$$

where the function $\mathbf{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ is only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x,t,u,Du) \cdot Du \ge \lambda|Du|^2 \\ |\mathbf{A}(x,t,u,Du)| \le \Lambda, \end{cases} \tag{2.4}$$

where λ and Λ are given positive constants ([10, 148]).

In the context of parabolic equations (2.3), with a full quasilinear structure not necessarily given by (2.4) (see for example (3.3) below), the function $\mathbf{A}$ is called the *principal part* of the partial differential equation.

The constants λ and Λ in (2.4) could be replaced by nonlinear functions of (x,t,u,Du). The partial differential equation in (2.3) is coercive and non-singular, if there exist two positive constants $\lambda_o \le \Lambda_1$ such that

$$\lambda_o \le \lambda(x,t,u,Du) \quad \text{and} \quad \Lambda(x,t,u,Du) \le \Lambda_1.$$

In such a case, the nonlinear function $\lambda(x,t,u,Du)$ is still referred to as the *modulus of ellipticity* of the partial differential equation in (2.3).

Harnack estimates have been at the core of understanding the local behavior of solutions to coercive, nonsingular elliptic and parabolic equations of second order. In particular they imply the local Hölder continuity of solutions and constrain their asymptotic behavior through Liouville-type theorems ([78, 79, 91]). In addition they play a central role in investigating the local structure of free boundaries ([69, 25]), and the boundary behavior of solutions in terms of Wiener-type criteria ([72, 110, 83, 89, 65, 114, 115]). For further details on the subject, we refer to the survey paper [88] and its long list of references.

3 Degenerate and Singular Parabolic Equations

A parabolic partial differential equation of the type (2.3) is termed *degenerate* if the modulus of ellipticity $\lambda(x,t,u,Du)$ tends to zero at points of its domain of definition; whereas it is termed *singular* if $\Lambda(x,t,u,\eta)$ tends to infinity at points of $E_T \times \mathbb{R}^{N+1}$. Such a behavior is either *prescribed* or *intrinsic.*

It is *prescribed* if the coefficients of the equation exhibit such a degenerate or singular behavior. In such a case the set of degeneracy or singularity is known a priori and the issue is that of investigating the behavior of possible solutions, near those given points, in terms of the behavior of the coefficients. Examples are in [95, 96, 97]. This is a very interesting, active research field, but we will not deal with it here.

It is *intrinsic* if the vanishing or blowing up of the modulus of ellipticity occurs through the solution or its gradient. In this case the degeneracy/singularity set on one side, and the behavior of the solutions on the other, are mutually intertwined. This monograph investigates this class of equations by focusing on local solutions to equations of p-Laplacian type and of porous medium type.

3.1 Quasilinear Equations of p-Laplacian Type

Let E be an open set in $\mathbb{R}^N$ and for $T>0$ let E_T denote the cylindrical domain $E\times(0,T]$.

The quasilinear equation (2.3) is of p-Laplacian type if it is subject to the structure conditions

$$\begin{cases} \mathbf{A}(x,t,u,Du)\cdot Du \ge C_o|Du|^p \\ |\mathbf{A}(x,t,u,Du)| \le C_1|Du|^{p-1} \end{cases} \qquad p>1, \tag{3.1}$$

where C_o and C_1 are given positive constants.

No regularity is assumed on the function $\mathbf{A}: E_T\times\mathbb{R}^{N+1}\to\mathbb{R}^N$ other than being measurable and subject to the structure conditions (3.1). A solution is meant weakly in the sense

$$u\in C_{\mathrm{loc}}\big(0,T;L^2_{\mathrm{loc}}(E)\big)\cap L^p_{\mathrm{loc}}\big(0,T;W^{1,p}_{\mathrm{loc}}(E)\big).$$

A precise definition of the functional spaces in (3.1) is in § 4 of the Preliminaries, and the notion of weak solution is in § 1 of Chapter 3. The prototype parabolic p-Laplace equation is

$$u_t - \operatorname{div} |Du|^{p-2} Du = 0 \quad \text{weakly in } E_T, \quad p > 1. \tag{3.2}$$

The modulus of ellipticity of this class of equations is $\lambda = |Du|^{p-2}$. For $p > 2$ such a modulus vanishes if the gradient Du vanishes, and the equation is *degenerate* on the set $[|Du| = 0]$. For $p \in (1, 2)$, the modulus of ellipticity $\lambda = \Lambda = |Du|^{p-2} \to \infty$ when $|Du| \to 0$, and the equation is *singular* on the set $[|Du| = 0]$.

We are interested only in *local* solutions, with no reference to possible boundary or initial data.

3.2 Quasilinear Equations of Porous Medium Type

The quasilinear equation (2.3) is of porous medium type if it is subject to the structure conditions

$$\begin{cases} \mathbf{A}(x,t,u,Du) \cdot Du \geq C_o m |u|^{m-1} |Du|^2 \\ |\mathbf{A}(x,t,u,Du)| \leq C_1 m |u|^{m-1} |Du| \end{cases} \qquad m > 0, \tag{3.3}$$

where C_o and C_1 are given positive constants.

No regularity is assumed on the function $\mathbf{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ other than being measurable and subject to the structure conditions (3.3). A solution is meant weakly in the sense

$$u \in C_{\text{loc}}\big(0, T; L^2_{\text{loc}}(E)\big) \text{ and } |u|^m \in L^2_{\text{loc}}\big(0, T; W^{1,2}_{\text{loc}}(E)\big).$$

The prototype of this class of equations is the porous medium equation

$$u_t - m \operatorname{div} |u|^{m-1} Du = 0 \quad \text{weakly in } E_T, \quad m > 0. \tag{3.4}$$

The modulus of ellipticity of this class of equations is $\lambda = |u|^{m-1}$. For $m > 1$ such a modulus vanishes if u vanishes, and the equation is *degenerate* on the set $[|u| = 0]$. For $0 < m < 1$, the modulus of ellipticity $\lambda = \Lambda = |u|^{m-1} \to \infty$ when $|u| \to 0$, and the equation is *singular* on the set $[|u| = 0]$.

We are interested only in *local* solutions in the indicated functional spaces, and irrespective of possible boundary or initial data.

3.3 Aim of the Monograph

The importance of these classes of degenerate and/or singular partial differential equations stems both from their intrinsic mathematical interest, and their central role in the modeling of a host of nonlinear phenomena, such as thin film dynamics ([35, 44, 46]), non-Newtonian fluid mechanics and flow in porous media ([8, 13, 18, 19, 100, 20, 112, 113, 125, 105, 15, 16, 34, 118, 140]), elasticity and science of "smart materials"([81, 33, 87, 23, 118, 134, 85]), and emerging issues in biomathematics and biophysics related to degenerate and/or singular diffusion of molecules on cell surfaces ([1, 4, 12, 14, 24, 111, 132, 139, 138]).

Notwithstanding the modeling, this monograph is solely mathematical in nature. Its aim is to introduce a novel set of tools and techniques in Analysis, for a better understanding of the notion of degeneracy and/or singularity in partial differential equations.

Perhaps the most crucial property is the "expansion of positivity" of non-negative solutions to these degenerate or singular parabolic equations (Chapter 4). If one such solution is positive at some time level, in a measure-theoretical sense, then the positivity expands in space at some further time, driven by the intrinsic geometry of these equations.

The monograph builds on some recent advances of the theory ([48, 49, 50, 51, 55]), that are suggesting novel ways of interpreting the notion of degeneracy and singularity.

The degeneracy and/or singularity limits the degree of regularity of solutions to such equations. For example, local solutions to (3.2) are no more regular than $C^{1,\alpha}_{\rm loc}$, whereas local solutions to (3.4) are no more than $C^{\alpha}_{\rm loc}$. Similar results hold for solutions to the quasilinear equation (2.3) with either the structure conditions (3.1) or (3.3). The theory in this respect is mature and we refer to [41, 62, 149] for an account.

However, a more refined insight is afforded by a possible Harnack inequality satisfied by nonnegative solutions to such equations. While a Harnack inequality for linear equations implies Hölder continuity, the converse is in general false.

For degenerate and singular parabolic equations the theory is at its inception, due to the inherent measure-theoretical difficulties generated by their lack of coercivity: as we said above, λ can either tend to zero or to infinity. However, while only a handful of such estimates are known ([39, 59, 60, 49]), whenever available they bear the same wealth of consequences as for nondegenerate equations. For example, for parabolic equations of p-Laplacian type, with full quasilinear structure, they imply the local and boundary Hölder continuity of solutions and permit one to precisely describe their behavior in the vicinity of their zero set (moving boundary).

4 Parabolic Harnack Estimates. The Role of the Structure

The Harnack inequality of Moser for nonnegative solutions to elliptic equations (2.1) for $u_t = 0$ was extended almost verbatim to nonnegative solutions to *elliptic* equations (2.3) for $u_t = 0$, even for a principal part $\mathbf{A}$ satisfying the structure conditions (3.1) for all $p > 1$ ([141, 147]). However, an extension to nonnegative solutions to *parabolic* equations (2.3) with the structure (3.1) was possible only for $p = 2$ ([10, 148]). The full structure (3.1) revealed serious difficulties and it remained open.

At almost the same time as Moser proved the Harnack inequality for solutions to (2.1)–(2.2), Ladyzhenskaya, Solonnikov and Ural'tzeva established, by means of DeGiorgi-type measure-theoretical arguments, that weak solutions to the *elliptic* versions of (2.3)–(2.4) with $u_t = 0$, are Hölder continuous for all $p > 1$, whereas the analogous result for *parabolic* equations was possible *only for* $p = 2$ ([101, 102]).

Neither Moser's nor DeGiorgi's ideas, nor Nash's approach [123] seemed to apply when $p \neq 2$.

Thus, vis-à-vis the Harnack inequality, there was a disconnect between the elliptic and parabolic theory.

It turns out that the parabolic theory is markedly different. Indeed the Harnack inequality (1.7) is false even for nonnegative solutions to the prototype equation (3.2). A series of counterexamples is in § 3 of Chapter 4, and § 1.3 of Chapter 5.

Notwithstanding some partial results ([39, 59, 40]), the issue of the Harnack inequality for nonnegative solutions to equations of the type (2.3), with the full quasilinear structure (3.1), and for all $p > 1$, while raised by several authors ([10, 148, 101, 41]), remained open.

Recently considerable progress has been made on this issue ([49, 51, 52, 98]) to the point that for all $p > \frac{2N}{N+1}$ the theory is reasonably complete.

This monograph is a systematic and self-contained account of this new theory, whose main points we outline below.

4.1 Degenerate Equations of the *p*-Laplacian Type for $p > 2$

Nonnegative, local weak solutions to (2.3)–(3.1) for $p > 2$, satisfy the parabolic Harnack inequality in some *intrinsic form.* Precisely, there exist positive constants γ and c depending only on N and $p > 2$, and the structure conditions in (3.1), such that (Chapter 5)

$$\gamma^{-1} \sup_{B_\rho(x_o)} u(x, t_o - \theta\rho^p) \le u(x_o, t_o) \le \gamma \inf_{B_\rho(x_o)} u(x, t_o + \theta\rho^p), \tag{4.1}$$

for all *intrinsic* parabolic cylinders

$$B_{4\rho}(x_o) \times \big(t_o - \theta(4\rho)^p, t_o + \theta(4\rho)^p\big) \qquad \theta = \left(\frac{c}{u(x_o, t_o)}\right)^{p-2} \tag{4.2}$$

contained within the domain of definition of the solution. Notice that the "waiting time" $\theta\rho^p$ is "intrinsic" to the solution itself. The constants γ and c tend to infinity as $p \to \infty$ and they are "stable" as $p \to 2$. Thus by formally letting $p \to 2$ one recovers the classical Harnack inequality (1.7) for caloric functions. The proof is based only on measure-theoretical arguments, bypassing any notion of maximum principle and parabolic potentials as in [39, 59]. Its significance is in paralleling Moser's measure-theoretical approach, in dispensing with Hadamard's and Pini's potential representations.

The approach of [49, 52] differs substantially from the classical ideas of Moser ([121, 122]) in that no properties of BMO spaces are used, nor covering arguments, nor cross-over estimates. Moreover the Harnack inequality is shown to imply the Hölder continuity of the solution.

The techniques are flexible enough to apply by minor changes to local, weak solutions to equations of the porous medium type (2.3) with the structure condition (3.3) for $m > 1$. Moreover the arguments, being only measure-theoretical, hold the promise of a wider applicability.

Of special notice is the measure-theoretical Lemma 3.1 of the Preliminaries. While disconnected from partial differential equations, it affords power-like dependences on the measure of the positivity set of a solution to degenerate parabolic equations (Proposition 7.1 of Chapter 5). This is a vast improvement with respect to the DeGiorgi method where the dependence is exponential.

4.2 Singular Equations of the p-Laplacian Type for $\frac{2N}{N+1} < p < 2$

Let u be a nonnegative, weak solution to (2.3)–(3.1) for $\frac{2N}{N+1} < p < 2$. The value $p_* = \frac{2N}{N+1}$ is called *critical* and we refer to $p_* < p < 2$ as the supercritical range of the parameter p.

Any such nonnegative solution satisfies an *intrinsic* Harnack estimate in the "mean value form" (4.1)–(4.2) (Chapter 6).

The positive constant γ depends only on p, N and the structural constants appearing in (3.1), whereas c can be taken to be 1. The constant γ can be precisely traced as a function of p, and it is "stable" as $p \to 2$. As a consequence, letting $p_* < p \to 2$ in (4.1), one recovers the classical Harnack estimate for caloric functions in the "mean value" form (1.7).

Thus (4.1) holds seamlessly as the parameter p goes from $p < 2$ (singular equations) to $p = 2$ (heat equation) to $p > 2$ (degenerate equations).

The constant, however, deteriorates as $p \to p_*$; specifically $\gamma(p) \to \infty$, and (4.1)–(4.2) lose meaning for $p = \frac{2N}{N+1}$.

A more intriguing fact, however, is that any such solution satisfies, in addition, a family of Harnack inequalities that are simultaneously *forward* in time, *backward* in time, and *elliptic*. Precisely, there exist positive constants γ and δ, depending only on p, N and the structural constants in (3.1), such that

$$\gamma^{-1} \sup_{B_\rho(x_o)} u(\cdot, \tau) \le u(x_o, t_o) \le \gamma \inf_{B_\rho(x_o)} u(\cdot, \sigma) \tag{4.3}$$

for all

$$t_o - \delta[u(x_o, t_o)]^{2-p} \rho^p \le \tau, \sigma \le t_o + \delta[u(x_o, t_o)]^{2-p} \rho^p \tag{4.4}$$

and for all parabolic cylinders of the form (4.2) contained in the space-time domain of definition of the solution.

When $\tau = \sigma = t_o$ the inequality takes the "*elliptic*" form (1.5) of nonnegative harmonic functions. Thus such an estimate defies the role of time, in parabolic Harnack estimates, as identified by Moser [121]. In particular, it is false for caloric functions ($p = 2$), and accordingly the constants $\gamma(p)$ and $\delta(p)$ deteriorate as $p_* < p \to 2$.

It turns out that these constants also deteriorate as $p \to p_* = \frac{2N}{N+1}$. Therefore (4.3)–(4.4) cease to hold for $p = p_*$ and it raises the question of what is causing such a behavior.

From a technical point of view, the value $p_* = \frac{2N}{N+1}$ enters as the limiting Sobolev exponent associated with the prototype equation (3.2). Nevertheless the range $\frac{2N}{N+1} < p < 2$ is optimal for a Harnack estimate to hold. A series of counterexamples are in § 1.3 of Chapter 6, showing that if $1 < p \le \frac{2N}{N+1}$, no Harnack estimate is possible in any of the forms (4.1)–(4.2), nor (4.3) for *any* value of τ, σ in the range (4.4).

The proofs are measure-theoretical, and potentials do not play any role. As such they are flexible enough to apply to a large class of singular equations including porous medium type equations (2.3) with the structure conditions (3.3) for $\frac{(N-2)_+}{N} < m < 1$.

While building on the results of [41], this monograph explores the issue of Harnack inequality for nonnegative solutions to these degenerate or singular parabolic PDEs, which in [41] was established only for the prototype equations, satisfying some sort of a comparison principle.

An effort has been made to make it self-contained and, at the same time, not to duplicate known facts. Whenever possible, the results of [41] are recalled without proof or new proofs have been given. In the interest of completeness we have also reproduced proofs of known facts, for which a clear trace to the literature could not be made. Examples include the local Hölder continuity of nonnegative, weak solutions to singular porous medium type equations, the forward-backward $L^1_{\rm loc}$ estimates, and the $L^r_{\rm loc}$-$L^\infty_{\rm loc}$ estimates for these solutions (Appendices A and B). These facts, widely used, do not have a precise trace in the literature.

4.3 Outstanding Issues

The main issues emerging from these investigations regard the singular case $1 < p < 2$ and more specifically the critical and subcritical range $1 < p \le \frac{2N}{N+1}$ of the parameter p. While some possible forms of Harnack-type inequalities are in Chapter 6, the picture is not entirely understood.

First one might ask what is the "structural" reason for a diminished role of the time when $p < 2$, as evidenced by the forward-backward-elliptic Harnack inequality (4.3)–(4.4).

It might appear as though the PDE is exhibiting a more "elliptic" nature, since the modulus of ellipticity $|Du|^{p-2} \to \infty$ as $|Du| \to 0$. On the other hand, the principal part of the equation is coercive in its own topology of $W^{1,p}_{\rm loc}$, that is,

$$\mathbf{A}(x, t, u, \eta) \cdot \eta \ge C_o |\eta|^p,$$

for a.e. $(x,t) \in E_T$, and for all $(u,\eta) \in \mathbb{R}^{N+1}$, with C_o a positive parameter: this suggests a diffusion phenomenon occurring within a topological setting intrinsic to the PDE.

This is the main guiding idea of this monograph; that is, these degenerate/singular diffusion processes, in terms of a Harnack estimate, behave like the heat equation, provided one reads them in their own time-intrinsic geometry, which in turn generates a "pointwise metric." A recent account of the various directions and applications of this idea is in [149].

This point of view, however, might not be sufficient to understand the local behavior of solutions to these PDEs, especially as p transitions from the supercritical range $p_* < p < 2$ to its critical and subcritical values $1 < p \le p_*$.

For critical and subcritical $1 < p \le p_*$, the behavior of solutions is rather intriguing. First, unlike the case $p > p_*$, weak solutions need not be bounded ([41], Chapter V). However, *locally bounded* solutions are locally Hölder continuous, irrespective of $1 < p < 2$ ([41], Chapter IV), whereas nonnegative solutions, *even if bounded*, do not satisfy a Harnack inequality in *any* of the forms indicated above.

The issue is deeper than a Harnack estimate per se. As an example observe that the Barenblatt similarity solution of the prototype equation (3.2) is well defined for all $\frac{2N}{N+1} < p$ and ceases to exist for $p = \frac{2N}{N+1}$ ((21.1)–(21.2) of Chapter 6). The Barenblatt solution can be regarded a p-parabolic potential.

Thus the critical value p_* which enters in the proofs in a more or less technical fashion, seems to be the dividing line between existence and nonexistence of the p-potentials. On the other hand, a second value $p_{**} = \frac{2N}{N+2}$, naturally linked with the critical Sobolev exponent, discriminates between locally bounded and unbounded solutions.

Understanding the structural reasons for this behavior remains a major challenge, and it might shed light on unexplored mathematical structures and physical behavior of systems modeled by these equations.

2

Preliminaries

1 Poincaré and Sobolev Inequalities

Let E be a bounded domain in $\mathbb{R}^N$ with boundary ∂E. If $f \in L^q(E)$ for some $1 \le q \le \infty$, denote by $\|f\|_{q,E}$ the $L^q(E)$-norm of f over E. We also write $\|f\|_q$ whenever the specification of the domain E is unambiguous from the context. The function $f \in L^q_{\mathrm{loc}}(E)$ if $\|f\|_{q,K} < \infty$, for all compact subsets $K \subset E$. For $f \in C^1(E)$ denote by $Df = (f_{x_1}, \dots, f_{x_N})$ its gradient and set

$$\|f\|_{1,p;E} = \|f\|_{p,E} + \|Df\|_{p,E}.$$

The spaces $W^{1,p}(E)$ and $W^{1,p}_o(E)$ for $p \ge 1$ are defined as

$$\begin{aligned} &W^{1,p}(E) \quad \text{the completion of } C^\infty(E) \text{ under } \|\cdot\|_{1,p;E} \\ &W^{1,p}_o(E) \quad \text{the completion of } C^\infty_o(E) \text{ under } \|\cdot\|_{1,p;E}. \end{aligned}$$

Equivalently $W^{1,p}(E)$ is the Banach space of functions $f \in L^p(E)$ whose generalized derivatives f_{x_i} belong to $L^p(E)$ for all $i = 1, \dots, N$.

A function $f \in W^{1,p}_{\mathrm{loc}}(E)$ if $\|f\|_{1,p;K} < \infty$ for every compact subset $K \subset E$.

Let $W^{1,\infty}(E)$ denote the Banach space of functions $f \in L^\infty(E)$ whose distributional derivatives $f_{x_i} \in L^\infty(E)$, for $i = 1, \dots, N$.

The space $W^{1,\infty}_{\mathrm{loc}}(E)$ is defined analogously.

Theorem 1.1 (Gagliardo–Nirenberg [71, 124]) *Let $v \in W^{1,p}_o(E)$ for some $p \ge 1$. For every $s \ge 1$ there exists a constant C depending only on N, p, q, and s, and independent of E, such that*

$$\|v\|_{q,E} \le C \|Dv\|^{\alpha}_{p,E} \|v\|^{1-\alpha}_{s,E} \tag{1.1}$$

where $\alpha \in [0,1]$ and $p, q \ge 1$, are linked by

$$\alpha = \left(\frac{1}{s} - \frac{1}{q}\right)\left(\frac{1}{N} - \frac{1}{p} + \frac{1}{s}\right)^{-1}$$

E. DiBenedetto et al., *Harnack's Inequality for Degenerate and Singular Parabolic Equations*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-1584-8_2,

and their admissible range is

$$\text{if } N=1, \qquad \alpha\in[0,\tfrac{p}{p+s(p-1)}], \qquad q\in[s,\infty];$$

$$\text{if } 1\le p<N, \qquad \alpha\in[0,1], \qquad \begin{cases} q\in\big[s,\frac{Np}{N-p}\big] & \text{if } s\le\frac{Np}{N-p},\\ q\in\big[\frac{Np}{N-p},s\big] & \text{if } s\ge\frac{Np}{N-p};\end{cases}$$

$$\text{if } 1<N\le p, \qquad \alpha\in\big[0,\tfrac{Np}{Np+s(p-N)}\big), \qquad q\in[s,\infty).$$

Corollary 1.1 *Let $v\in W_o^{1,p}(E)$, and assume $p\in[1,N)$. There exists a constant $\gamma=\gamma(N,p)$ such that*

$$\|v\|_{q,E}\le\gamma\|Dv\|_{p,E}, \qquad \text{where} \qquad q=\frac{Np}{N-p}. \tag{1.1$'$}$$

The boundary ∂E is *piecewise smooth* if it is the union of finitely many portions of $(N-1)$-dimensional hypersurfaces of class $C^{1,\lambda}$, for some $\lambda\in(0,1)$.

If ∂E is piecewise smooth, functions v in $W^{1,p}(E)$ are defined up to ∂E via their traces denoted by $v\,\big|_{\partial E}$.

Theorem 1.2 *Let ∂E be piecewise smooth. There exists a constant C depending only on N,p and the structure of ∂E such that*

$$\|v\|_{q,\partial E}\le C\|v\|_{W^{1,p}(E)},$$

where

$$q\in\big[1,\tfrac{(N-1)p}{N-p}\big], \qquad \text{if } 1<p<N$$

$$q\in[1,\infty), \qquad \text{if } p=N.$$

If ∂E is piecewise smooth, the space $W_o^{1,p}(E)$ can be defined equivalently as the set of functions $v\in W^{1,p}(E)$ whose trace on ∂E is zero.

Remark 1.1 The embedding inequalities of Theorem 1.1 and Corollary 1.1 continue to hold for functions v in $W^{1,p}(E)$, not necessarily vanishing on ∂E in the sense of the traces, provided ∂E is piecewise smooth and

$$\int_E v(x)dx=0.$$

In such a case the constant C depends on s,p,q,N and the structure of ∂E. However, it does not depend on the *size* of E, and in particular it does not change by dilations of E.

2 Cuts and Truncations of Functions in $W^{1,p}(E)$ and Their Embeddings

Let k be any real number and for a function $v \in W^{1,p}(E)$ consider the truncations of v given by

$$(v-k)_+ = \max\{(v-k);0\}$$
$$(v-k)_- = \max\{-(v-k);0\}.$$

Lemma 2.1 (Stampacchia [144]) *Let $v \in W^{1,p}(E)$. Then $(v-k)_\pm \in W^{1,p}(E)$ for all $k \in \mathbb{R}$. If in addition the trace of v on ∂E is essentially bounded and*

$$\|v\|_{\infty,\partial E} \le M \qquad \text{for some } M > 0,$$

then $(v-k)_\pm \in W_o^{1,p}(E)$ for all $k \ge M$.

Corollary 2.1 *Let $v_i \in W^{1,p}(E)$ for $i = 1,\dots,n \in \mathbb{N}$. Then*

$$w = \min\{v_1,\dots,v_n\} \in W^{1,p}(E).$$

For a function v defined in E and real numbers $k < \ell$, set

$$[v > \ell] = \{x \in E \big| v(x) > \ell\}$$
$$[v < k] = \{x \in E \big| v(x) < k\}$$
$$[k < v < \ell] = \{x \in E \big| k < v(x) < \ell\}.$$

For $\rho > 0$ and $y \in \mathbb{R}^N$, denote by $B_\rho(y)$ the ball of radius ρ centered at y, and by $K_\rho(y)$ the cube of edge ρ, centered at y and with faces parallel to the coordinate planes. If y is the origin, let $B_\rho(0) = B_\rho$, and $K_\rho(0) = K_\rho$.

For a Lebesgue measurable set $A \subset \mathbb{R}^N$ denote by $|A|$ its measure.

Lemma 2.2 (DeGiorgi [36]) *Let $v \in W^{1,1}(K_\rho(y))$, and let $k < \ell$ be real numbers. There exists a constant γ depending only on N,p and independent of k,ℓ,v,y,ρ, such that*

$$(\ell-k)|[v>\ell]| \le \gamma \frac{\rho^{N+1}}{|[v<k]|} \int_{[k<v<\ell]} |Dv|dx. \tag{2.1}$$

Remark 2.1 The conclusion of the lemma continues to hold for functions $v \in W^{1,1}(E)$ provided E is *convex*. It can be used for balls $B_\rho(y)$.

The embedding $(1.1)'$ of Corollary 1.1 gives a majorization of the $L^q(E)$-norm of u solely in terms of the $L^p(E)$-norm of its gradient. This is possible because u vanishes on ∂E in the sense of the traces.

A Poincaré-type inequality bounds some integral norm of a function $u \in W^{1,p}(E)$ in terms *only* of some integral norm of its gradient, provided some information is available on the set where u vanishes.

Proposition 2.1 *Let $E \subset \mathbb{R}^N$ be bounded and convex and let $\varphi \in C(\bar{E})$ satisfy*

$$0 \le \varphi \le 1, \qquad \textit{and the sets } [\varphi > k] \textit{ are convex for all } k \in \mathbb{R}_+.$$

Let $v \in W^{1,p}(E)$ and assume that the set

$$\mathcal{E} = [v = 0] \cap [\varphi = 1]$$

has positive measure. There exists a constant C depending only on N and p and independent of v and φ, such that

$$\Big(\int_E \varphi |v|^p dx \Big)^{\frac{1}{p}} \le C \frac{(\operatorname{diam} E)^N}{|\mathcal{E}|^{\frac{N-1}{N}}} \Big(\int_E \varphi |Dv|^p dx \Big)^{\frac{1}{p}}. \tag{2.2}$$

Remark 2.2 Inequality (2.1) follows from this by applying (2.2) with $\varphi = 1$ and $p = 1$ to the function

$$w = \begin{cases} \min\{v\,,\,\ell\} - k & \text{if } \ v > k \\ 0 & \text{if } \ v \le k. \end{cases}$$

By Lemma 2.1 such a function is in $W^{1,1}(E)$.

3 A Measure-Theoretical Lemma ([48])

If $u \in C(E)$ and $u(y) = 1$ for some $y \in E$, for every $\sigma \in (0,1)$ there exists a ball $B_\rho(y) \subset E$, such that $u \ge 1-\sigma$ in $B_\rho(y)$, with ρ being determined by σ and the modulus of continuity of u. A similar statement valid for measurable functions follows from the Severini–Egorov theorem ([142], [64]), where, however, one cannot, in general, quantify the size and shape of the neighborhood of y where, roughly speaking, u is near 1. The following measure-theoretical lemma can be regarded as a quantitative version of the Severini–Egorov theorem, for functions $u \in W^{1,1}_{\mathrm{loc}}(E)$.

Lemma 3.1 *Let $u \in W^{1,1}(K_\rho)$ satisfy*

$$\|u\|_{W^{1,1}(K_\rho)} \le \gamma \rho^{N-1} \qquad \textit{and} \qquad |[u > 1]| \ge \alpha |K_\rho|$$

for some $\gamma > 0$ and $\alpha \in (0,1)$. Then, for every $\delta \in (0,1)$ and $0 < \lambda < 1$ there exist $y \in K_\rho$ and $\varepsilon = \varepsilon(\alpha, \delta, \gamma, \lambda, N) \in (0,1)$, such that

$$|[u > \lambda] \cap K_{\varepsilon\rho}(y)| > (1-\delta)|K_{\varepsilon\rho}(y)|.$$

Roughly speaking the lemma asserts that if the set where u is bounded away from zero occupies a sizable portion of K_ρ, then there exist at least one point y and a neighborhood $K_{\varepsilon\rho}(y)$ where u remains large in a large portion of $K_{\varepsilon\rho}(y)$. Thus the set where u is positive clusters about at least one point $y \in K_\rho$.

Proof It suffices to establish the lemma for u smooth and $\rho = 1$. For $n \in \mathbb{N}$ partition K_1 into n^N cubes, with pairwise disjoint interior and each of edge $1/n$. Divide these cubes into two finite subcollections $\mathbf{Q}^+$ and $\mathbf{Q}^-$ by

$$\begin{aligned} Q_j \in \mathbf{Q}^+ \quad &\Longleftrightarrow \quad |[u > 1] \cap Q_j| > \tfrac{1}{2}\alpha|Q_j| \\ Q_i \in \mathbf{Q}^- \quad &\Longleftrightarrow \quad |[u > 1] \cap Q_i| \le \tfrac{1}{2}\alpha|Q_i| \end{aligned}$$

and denote by $\#(\mathbf{Q}^+)$ the number of cubes in $\mathbf{Q}^+$. By the assumption

$$\sum_{Q_j \in \mathbf{Q}^+} |[u > 1] \cap Q_j| + \sum_{Q_i \in \mathbf{Q}^-} |[u > 1] \cap Q_i| > \alpha |K_1| = \alpha n^N |Q|$$

where $|Q|$ is the common measure of the Q_ℓ. From the definition of $\mathbf{Q}^\pm$

$$\begin{aligned} \alpha n^N &< \sum_{Q_j \in \mathbf{Q}^+} \frac{|[u > 1] \cap Q_j|}{|Q_j|} + \sum_{Q_i \in \mathbf{Q}^-} \frac{|[u > 1] \cap Q_i|}{|Q_i|} \\ &< \#(\mathbf{Q}^+) + \frac{\alpha}{2}(n^N - \#(\mathbf{Q}^+)). \end{aligned}$$

Therefore

$$\#(\mathbf{Q}^+) > \frac{\alpha}{2 - \alpha} n^N. \tag{3.1}$$

Fix $\delta, \lambda \in (0, 1)$. The integer n can be chosen depending on $\alpha, \delta, \lambda, \gamma$, and N, such that

$$|[u > \lambda] \cap Q_j| \ge (1 - \delta)|Q_j| \quad \text{for some } \; Q_j \in \mathbf{Q}^+. \tag{3.2}$$

This would establish the lemma for $\varepsilon = 1/n$. Let $Q \in \mathbf{Q}^+$ satisfy

$$|[u > \lambda] \cap Q| < (1 - \delta)|Q|. \tag{3.3}$$

We will show that for such a cube, there exists a constant $c = c(\delta, \lambda, N)$ such that

$$\|u\|_{W^{1,1}(Q)} \ge \alpha c(\delta, \lambda, N) \frac{1}{n^{N-1}}. \tag{3.4}$$

From the assumptions

$$|[u \le \lambda] \cap Q| \ge \delta |Q| \qquad \text{and} \qquad \left| \left[u > \frac{1 + \lambda}{2} \right] \cap Q \right| > \frac{\alpha}{2} |Q|.$$

For fixed $x \in [u \le \lambda] \cap Q$ and $y \in [u > (1 + \lambda)/2] \cap Q$,

$$\frac{1 - \lambda}{2} \le u(y) - u(x) = \int_0^{|y - x|} Du(x + t\mathbf{n}) \cdot \mathbf{n} \, dt$$

where

$$\mathbf{n} = \frac{y - x}{|x - y|}, \qquad \text{for} \qquad x \ne y.$$

Let $R(x,\omega)$ be the polar representation of ∂Q with pole at x, for the solid angle ω. Integrate the previous relation in dy over $[u > (1+\lambda)/2] \cap Q$. Minorize the resulting left-hand side, by using the lower bound on the measure of such a set, and majorize the resulting integral on the right-hand side by extending the integration over Q. Expressing such integration in polar coordinates with pole at $x \in [u \le \lambda] \cap Q$ gives

$$\begin{aligned}\frac{\alpha(1-\lambda)}{4}|Q| &\le \int_{|\mathbf{n}|=1} \int_0^{R(x,\mathbf{n})} r^{N-1} \int_0^{|y-x|} |Du(x+t\mathbf{n})| dt\, dr\, d\mathbf{n} \\ &\le N^{N/2}|Q| \int_{|\mathbf{n}|=1} \int_0^{R(x,\mathbf{n})} |Du(x+t\mathbf{n})| dt\, d\mathbf{n} \\ &= N^{N/2}|Q| \int_Q \frac{|Du(z)|}{|z-x|^{N-1}} dz.\end{aligned}$$

Integrate now in dx over $[u \le \lambda] \cap Q$. Minorize the resulting left-hand side by using the lower bound on the measure of such a set, and majorize the resulting right-hand side, by extending the integration to Q. This gives

$$\begin{aligned}\frac{\alpha\delta(1-\lambda)}{4N^{N/2}}|Q| &\le \|u\|_{W^{1,1}(Q)} \sup_{z\in Q} \int_Q \frac{1}{|z-x|^{N-1}} dx \\ &\le C(N)|Q|^{1/N} \|u\|_{W^{1,1}(Q)}\end{aligned}$$

for a constant $C(N)$ depending only on N, thereby establishing (3.4).

If (3.2) does not hold for any cube $Q_j \in \mathbf{Q}^+$, then (3.3) and hence (3.4) is verified for all such Q_j. Adding over such cubes and taking into account (3.1),

$$\frac{\alpha^2}{2-\alpha} c(\delta,\lambda,N)n \le \|u\|_{W^{1,1}(K_1)} \le \gamma. \qquad \blacksquare$$

Remark 3.1 Following the various steps of the proof, the dependence of the reducing parameter ε on the measure-theoretical parameter α, and on the constant γ appearing in the assumptions of the lemma, can be traced to be of the form

$$\varepsilon = B^{-1}\frac{\alpha^2}{\gamma} \tag{3.5}$$

for a constant $B > 1$ depending on δ, N, and λ and independent of α.

4 Parabolic Spaces and Embeddings

For $0 < T < \infty$ let E_T denote the cylindrical domain $E \times (0,T]$. The space $L^{r,q}(E_T)$ for $q, r \ge 1$ is the collection of functions f defined and measurable in E_T such that

$$\|f\|_{q,r;E_T} = \Big(\int_0^T \Big(\int_E |f|^q dx \Big)^{\frac{r}{q}} d\tau \Big)^{\frac{1}{r}} < \infty.$$

Also $f \in L^{q,r}_{\rm loc}(E_T)$, if for every compact subset $K \subset E$ and every subinterval $[t_1, t_2] \subset (0, T]$

$$\int_{t_1}^{t_2} \Big(\int_K |f|^q dx \Big)^{\frac{r}{q}} d\tau < \infty.$$

Whenever $q = r$ we set $L^{q,q}(E_T) = L^q(E_T)$. These definitions are extended in the obvious way when either q or r is infinity.

We introduce spaces of functions, depending on $(x, t) \in E_T$, that exhibit different behavior in the space and time variables. These are spaces where typically solutions to parabolic equations in divergence form are found.

Let $m, p \geq 1$ and consider the Banach spaces

$$V^{m,p}(E_T) = L^\infty\big(0, T; L^m(E)\big) \cap L^p\big(0, T; W^{1,p}(E)\big)$$
$$V_o^{m,p}(E_T) = L^\infty\big(0, T; L^m(E)\big) \cap L^p\big(0, T; W_o^{1,p}(E)\big)$$

both equipped with the norm

$$\|v\|_{V^{m,p}(E_T)} = \operatorname*{ess\,sup}_{0<t<T} \|v(\cdot, t)\|_{m,E} + \|Dv\|_{p,E_T}.$$

When $m = p$, set $V_o^{p,p}(E_T) = V_o^p(E_T)$ and $V^{p,p}(E_T) = V^p(E_T)$. Both spaces are embedded in $L^q(E_T)$ for some $q > p$. In a precise way we have

Proposition 4.1 *There exists a constant γ depending only on N, p, m such that for every $v \in V_o^{m,p}(E_T)$*

$$\iint_{E_T} |v(x,t)|^q dx\, dt \leq \gamma^q \Big(\iint_{E_T} |Dv(x,t)|^p dx\, dt \Big) \times \Big(\operatorname*{ess\,sup}_{0<t<T} \int_E |v(x,t)|^m dx \Big)^{\frac{p}{N}} \tag{4.1}$$

where

$$q = p \frac{N+m}{N}.$$

Moreover

$$\|v\|_{q,E_T} \leq \gamma \|v\|_{V^{m,p}(E_T)}. \tag{4.2}$$

Remark 4.1 The multiplicative inequality (4.1) and the embedding (4.2) continue to hold for functions $v \in V^{m,p}(E_T)$ such that

$$\int_E v(x,t) dx = 0 \qquad \text{for a.e. } t \in (0,T)$$

provided ∂E is piecewise smooth. In such a case the constant γ depends also on the structure of ∂E, but not on its size.

The next corollary follows from Proposition 4.1 by taking $m = p$ and by applying Hölder's inequality.

Corollary 4.1 *Let $p > 1$. There exists a constant γ depending only on N and p, such that for every $v \in V_o^p(E_T)$,*

$$\|v\|_{p,E_T}^p \leq \gamma \big|[|v| > 0]\big|^{\frac{p}{N+p}} \|v\|_{V^p(E_T)}^p.$$

When $m = p$, Proposition 4.1 takes the form

Proposition 4.2 *There exists a constant γ depending only on N and p such that for every $v \in V_o^p(E_T)$,*

$$\|v\|_{q,r;E_T} \leq \gamma \|v\|_{V^p(E_T)},$$

where the numbers $q, r \geq 1$ are linked by

$$\frac{1}{r} + \frac{N}{pq} = \frac{N}{p^2},$$

and their admissible range is

$$\begin{aligned} &\textit{if } \ N = 1, && q \in (p, \infty], && r \in [p^2, \infty); \\ &\textit{if } \ 1 \leq p < N, && q \in \big[p, \tfrac{Np}{N-p}\big], && r \in [p, \infty]; \\ &\textit{if } \ 1 < N \leq p, && q \in [p, \infty), && r \in \big(\tfrac{p^2}{N}, \infty\big]. \end{aligned}$$

We conclude this section by stating a parabolic version of Lemma 2.1 and Corollary 2.1 concerning the truncated functions $(v - k)_\pm$.

Lemma 4.1 *Let $v \in V^{m,p}(E_T)$. Then $(v - k)_\pm \in V^{m,p}(E_T)$ for all $k \in \mathbb{R}$. Assume in addition that ∂E is piecewise smooth and that the trace of $v(\cdot, t)$ on ∂E is essentially bounded and*

$$\operatorname*{ess\,sup}_{0<t<T} \|v(\cdot, t)\|_{\infty,\partial E} \leq M \qquad \textit{for some } \ M > 0.$$

Then $(v - k)_\pm \in V_o^{m,p}(E_T)$ for all $k \geq M$.

5 Some Technical Facts

5.1 A Lemma on Fast Geometric Convergence

Lemma 5.1 *Let $\{Y_n\}$ for $n = 0, 1, \ldots,$ be a sequence of positive numbers, satisfying the recursive inequalities*

$$Y_{n+1} \leq C b^n Y_n^{1+\alpha},$$

where $C, b > 1$ and $\alpha > 0$ are given numbers. If

$$Y_o \leq C^{-1/\alpha} b^{-1/\alpha^2},$$

then $\{Y_n\} \to 0$ as $n \to \infty$.

5.2 An Interpolation Lemma

Lemma 5.2 *Let $\{Y_n\}$ for $n = 0, 1, \ldots,$ be a sequence of equi-bounded positive numbers satisfying the recursive inequalities*

$$Y_n \le C b^n Y_{n+1}^{1-\alpha},$$

where $C, b > 1$ and $\alpha \in (0, 1)$ are given constants. Then

$$Y_o \le \left(\frac{2C}{b^{1-\frac{1}{\alpha}}} \right)^{\frac{1}{\alpha}}.$$

Remark 5.1 The lemma turns the *qualitative* information of equi-boundedness of the sequence $\{Y_n\}$ into a *quantitative* a priori estimate for Y_o.

5.3 Steklov Averages

Let $v \in L^1(E_T)$ and let $0 < h < T$. The Steklov averages $v_h(\cdot, t)$ and $v_{\bar{h}}(\cdot, t)$ are defined by

$$v_h = \begin{cases} \dfrac{1}{h} \displaystyle\int_t^{t+h} v(\cdot, \tau) d\tau & \text{for } \ t \in (0, T-h], \\[2ex] 0, & \text{for } \ t > T - h. \end{cases}$$

$$v_{\bar{h}} = \begin{cases} \dfrac{1}{h} \displaystyle\int_{t-h}^{t} v(\cdot, \tau) d\tau & \text{for } \ t \in (h, T], \\[2ex] 0, & \text{for } \ t < h. \end{cases}$$

Lemma 5.3 *Let $v \in L^{q,r}(E_T)$. Then, as $h \to 0$, $v_h \to v$ in $L^{q,r}(E_{T-\varepsilon})$ for every $\varepsilon \in (0, T)$. If $v \in C(0, T; L^q(E))$, then $v_h(\cdot, t) \to v(\cdot, t)$ in $L^q(E)$ for every $t \in (0, T - \varepsilon)$ for all $\varepsilon \in (0, T)$.*

A similar statement holds for $v_{\bar{h}}$. The proof of the lemma is straightforward from the theory of L^p spaces.

6 Remarks and Bibliographical Notes

The proofs of the multiplicative embedding of Theorem 1.1 and the embeddings of Theorem 1.2, are in a number of monographs ([102, 114, 5, 42]).

The best constants in (1.1) are traced by Talenti [146].

Theorem 1.2 is due to Sobolev and Nikol'ski [143]. The dependence of the constant C of the structure of ∂E is traced in [42].

Poincaré first stated and proved the inequality that was later named after him in [128]; he then gave a second and more refined proof in [129].

A thorough treatment of Sobolev inequalities and their connection with Harnack-type estimates is in [136].

In the context of partial differential equations, the truncations $(v-k)_\pm$ were introduced by Bernstein ([17]) and effectively used by Stampacchia ([144]) and DeGiorgi ([36]).

The proof of Lemma 2.1, on the truncations $(v-k)_\pm$, is in [144]. A simpler proof is reported in [76].

Inequality (2.1) is due to DeGiorgi [36] and it is referred to as a "discrete" isoperimetric inequality. A continuous version is in [67].

The proof of Proposition 2.1 is in [101] and it follows essentially DeGiorgi's proof of Lemma 2.2.

A version of the measure-theoretical Lemma 3.1 was first established in [63] for $u \in W^{1,p}(K_\rho)$ and $p > 1$. Such a limitation on p was essential to the proof. The proof presented here, taken from [48], removes such a restriction and is simpler.

The parabolic spaces $V^{m,p}(E_T)$ and $V_o^{m,p}(E_T)$ are generalizations of the spaces $V^2(E_T)$ introduced in [101]. The generalizations are introduced to track down the notion of degenerate and singular parabolic equations. The embeddings of § 4 of these spaces are established in [41].

The proof of Lemma 5.1 on fast geometric convergence is in [36] and reported in [102, 101]. A simpler proof is in [41].

The interpolation Lemma 5.2 is taken from [26, 27].

In a series of papers published at the beginning of the 20th century, Steklov studied completeness problems, making a large use of integral averaging of functions. This later prompted the use of the term Steklov averages.

3

Degenerate and Singular Parabolic Equations

1 Quasilinear Equations of p-Laplacian Type

Let E be an open set in $\mathbb{R}^N$ and for $T > 0$ let E_T denote the cylindrical domain $E \times (0, T]$. Consider quasilinear, degenerate or singular parabolic partial differential equations of the form

$$u_t - \operatorname{div} \mathbf{A}(x, t, u, Du) = B(x, t, u, Du) \quad \text{weakly in } E_T \tag{1.1}$$

where the functions $\mathbf{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ and $B : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}$ are only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x, t, u, Du) \cdot Du \geq C_o |Du|^p - C^p \\ |\mathbf{A}(x, t, u, Du)| \leq C_1 |Du|^{p-1} + C^{p-1} \\ |B(x, t, u, Du)| \leq C |Du|^{p-1} + C^p \end{cases} \quad \text{a.e. in } E_T \tag{1.2}$$

where $p > 1$, C_o and C_1 are given positive constants, and C is a given nonnegative constant. When $C = 0$ the equation is *homogeneous*.

The homogeneous prototype of such a class of parabolic equations is

$$u_t - \operatorname{div}(|Du|^{p-2} Du) = 0, \quad p > 1, \quad \text{weakly in } E_T. \tag{1.3}$$

A function

$$u \in C_{\text{loc}}\big(0, T; L^2_{\text{loc}}(E)\big) \cap L^p_{\text{loc}}\big(0, T; W^{1,p}_{\text{loc}}(E)\big) \tag{1.4}$$

is a local, weak sub(super)-solution to (1.1) if for every compact set $K \subset E$ and every subinterval $[t_1, t_2] \subset (0, T]$

$$\begin{aligned} \int_K u\varphi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K \big[-u\varphi_t + \mathbf{A}(x, t, u, Du) \cdot D\varphi \big] dx \, dt \\ \leq (\geq) \int_{t_1}^{t_2} \int_K B(x, t, u, Du) \varphi \, dx \, dt \end{aligned} \tag{1.5}$$

for all nonnegative testing functions

E. DiBenedetto et al., *Harnack's Inequality for Degenerate and Singular Parabolic Equations*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-1584-8_3,

$$\varphi \in W^{1,2}_{\rm loc}\big(0,T;L^2(K)\big) \cap L^p_{\rm loc}\big(0,T;W^{1,p}_o(K)\big). \tag{1.6}$$

This guarantees that all the integrals in (1.5) are convergent.

Our focus will be on Harnack estimates satisfied by nonnegative weak solutions in the interior of E_T. For this reason we will only be interested in *local* solutions irrespective of any boundary or initial data.

The partial differential equation (1.1) is degenerate when $p > 2$ and singular when $1 < p < 2$, since the modulus of ellipticity $|Du|^{p-2}$ respectively tends to 0 or to $+\infty$ as $|Du| \to 0$.

When $p = 2$, the equation is nondegenerate, and its theory is reasonably complete ([101]). In the following particular care will be devoted to the stability of the various statements and estimates for $p \approx 2$.

1.1 An Alternate Formulation in Terms of Steklov Averages

It would be technically convenient to have a formulation of weak solutions that involves u_t. Unfortunately weak solutions to (1.1)–(1.2), whenever they exist, possess a modest degree of regularity in the time variable and, in general, u_t has a meaning only in the sense of distributions.

The following notion of local weak sub(super)-solution involves the discrete time derivative of u and is equivalent to (1.5)–(1.6).

Fix $t \in (0,T)$ and let h be a positive number such that $0 < t < t+h < T$. In (1.5) take $t_1 = t$, $t_2 = t+h$ and choose a testing function φ independent of the variable $\tau \in (t, t+h)$. Dividing by h and recalling the definition of Steklov averages we obtain

$$\begin{aligned}\int_{K\times\{t\}} \big(u_{h,t}\varphi + [\mathbf{A}(x,\tau,u,Du)]_h \cdot D\varphi\big)\,dx \qquad\qquad \\ \le (\ge) \int_{K\times\{t\}} [B(x,\tau,u,Du)]_h \varphi\,dx\end{aligned} \tag{1.7}$$

for all $0 < t < T-h$ and all nonnegative $\varphi \in W^{1,p}_o(K)$. To recover (1.5), fix a subinterval $0 < t_1 < t_2 < T$, choose h so small that $t_2 + h \le T$, and in (1.7) take a testing function as in (1.6). Such a choice is admissible, since the testing functions in (1.7) are independent of the variable $\tau \in (t, t+h)$ but may be dependent on t. Integrating over $[t_1, t_2]$ and letting $h \to 0$ with the aid of Lemma 5.3 of the Preliminaries gives (1.5).

1.2 On the Notion of Parabolicity

The structure conditions (1.2) are not sufficient to characterize parabolic partial differential equations. For example, the principal part

$$\mathbf{A}(x,t,u,Du) = |Du|^{p-2}\Big(Du - \frac{Du}{|Du|}\Big)$$

satisfies the first of (1.2) for all $p > 1$. However, its modulus of ellipticity changes signum at $|Du| = 1$, and the equation transitions from "forward" parabolic to "backward" in time.

Definition:

The partial differential equation in (1.1) is *parabolic* if it satisfies the structure conditions (1.2) and in addition, for every weak, local sub(super)-solution u, the truncations $+(u-k)_+$, $-(u-k)_-$, for all $k \in \mathbb{R}$, are weak, local sub(super)-solutions to (1.1), in the sense of (1.5)–(1.6), with $\mathbf{A}(x,t,u,Du)$ and $B(x,t,u,Du)$ replaced respectively by

$$\mathbf{A}(x,t,k \pm (u-k)_\pm, \pm D(u-k)_\pm),$$
$$B(x,t,k \pm (u-k)_\pm, \pm D(u-k)_\pm).$$

To clarify the connection between sub(super)-solutions and parabolic structures, we derive some sufficient conditions on $\mathbf{A}(x,t,u,Du)$ for such a notion of parabolicity to be enforced.

Lemma 1.1 *Assume that for all* $(x,t,u) \in E_T \times \mathbb{R}$

$$\mathbf{A}(x,t,u,\eta) \cdot \eta \geq 0 \quad \text{for all } \eta \in \mathbb{R}^N. \tag{1.8}$$

Then (1.1)–(1.2) is parabolic.

Proof Let u be a local weak subsolution to (1.1), and in (1.7) take the testing function

$$\frac{(u_h - k)_+}{(u_h-k)_+ + \varepsilon}\,\varphi, \qquad \text{where } \varepsilon > 0, \text{ and } \varphi \geq 0 \text{ satisfies (1.6).}$$

Integrate in dt over $[t_1,t_2] \subset (0,T)$ and let first $h \to 0$ and then $\varepsilon \to 0$ to obtain

$$\begin{aligned}
&\int_K (u-k)_+\varphi(\cdot,t)dx\Big|_{t_1}^{t_2} \\
&\quad + \int_{t_1}^{t_2}\int_K [-(u-k)_+\varphi_t + \mathbf{A}(x,t,k+(u-k)_+, D(u-k)_+)\cdot D\varphi]dx\,dt \\
&\leq \int_{t_1}^{t_2}\int_K B(x,t,k+(u-k)_+, D(u-k)_+)\varphi\,dx\,dt \\
&\quad - \liminf_{\varepsilon\to 0}\varepsilon \int_{t_1}^{t_2}\int_K \frac{\mathbf{A}(x,t,k+(u-k)_+, D(u-k)_+)\cdot D(u-k)_+}{((u-k)_+ + \varepsilon)^2}\varphi\,dx\,dt.
\end{aligned}$$

Thus $(u-k)_+$ is a weak, local subsolution. If u is a weak, local supersolution, the same argument shows that $-(u-k)_-$ is a weak, local supersolution. ∎

Henceforth we will assume that the principal part $\mathbf{A}(x,t,u,Du)$ satisfies (1.8) so that (1.1)–(1.2) is parabolic.

One checks that the assumptions of the lemma are verified for example by equations with principal part

$$\operatorname{div}\mathbf{A}(x,t,u,Du) = \Big[|Du|^{p-2}\Big(a_{ij}(x,t)u_{x_i} + f(x,t)\frac{u_{x_j}}{|Du|}\Big)\Big]_{x_j}$$

where f is bounded and nonnegative, and the matrix (a_{ij}) is only measurable and locally positive definite in E_T.

1.3 Dependence on the Parameters $\{p, N, C_o, C_1\}$ and Stability

The set of parameters $\{p, N, C_o, C_1\}$ are the *data*, and we say that a generic positive constant $\gamma = \gamma(p, N, C_o, C_1)$ depends only on the data, if it can be quantitatively determined a priori, only in terms of these parameters.

The constant $C \geq 0$ is also a *datum* of the equations. However, all our estimates will only involve $\{p, N, C_o, C_1\}$, while C will appear as an alternative. This is further illustrated in the energy estimates of Proposition 2.1 and Remark 2.2 following it, and in Lemmas 3.1 and 4.1, and Remarks 3.1 and 4.3 following them.

A positive constant γ depending only on the data is "stable" as $p \to 2$ if there exists a positive constant $\gamma(2, N, C_o, C_1)$ such that

$$\lim_{p\to 2}\gamma(p, N, C_o, C_1) = \gamma(2, N, C_o, C_1). \tag{1.9}$$

We will show that all our estimates are stable as $p \to 2$. As a consequence, the classical theory for nondegenerate equations can be recovered from these degenerate and singular equations, by letting $p \to 2$.

2 Energy Estimates for $(u-k)_\pm$ on Cylinders $(y,s) + Q^\pm_\rho(\theta) \subset E_T$

In the following $Q^\pm_\rho(\theta)$ denote the "forward" and "backward" parabolic cylinders of the form

$$Q^-_\rho(\theta) = K_\rho \times (-\theta\rho^p, 0], \quad Q^+_\rho(\theta) = K_\rho \times (0, \theta\rho^p], \quad p > 1 \tag{2.1}$$

where θ is a positive parameter that determines their length relative to ρ^p. The origin $(0,0)$ of $\mathbb{R}^{N+1}$ is the "upper vertex" of Q^-_ρ and the "lower vertex" of $Q^+_\rho(\theta)$. If $\theta = 1$, write $Q^\pm_\rho(1) = Q^\pm_\rho$. For a fixed $(y,s) \in \mathbb{R}^{N+1}$ denote by

$$\begin{aligned}(y,s) + Q^-_\rho(\theta) &= K_\rho(y) \times (s - \theta\rho^p, s] \\ (y,s) + Q^+_\rho(\theta) &= K_\rho(y) \times (s, s + \theta\rho^p]\end{aligned} \tag{2.2}$$

cylinders congruent to $Q^\pm_\rho(\theta)$ and with "upper vertex" and "lower vertex" respectively at (y,s).

Proposition 2.1 *Let u be a local, weak sub(super)-solution to (1.1)–(1.2) in E_T, in the sense of (1.5)–(1.6).*

There exists a positive constant $\gamma=\gamma(p,N,C_o,C_1)$, such that for every cylinder $(y,s)+Q_\rho^-(\theta)\subset E_T$, every $k\in\mathbb{R}$ and every piecewise smooth, cutoff function ζ vanishing on $\partial K_\rho(y)$, and such that $0\le\zeta\le 1$

$$\begin{aligned}\operatorname*{ess\,sup}_{s-\theta\rho^p<t<s}&\int_{K_\rho(y)}(u-k)_\pm^2\zeta^p(x,t)dx-\int_{K_\rho(y)}(u-k)_\pm^2\zeta^p(x,s-\theta\rho^p)dx\\&+C_o\iint_{(y,s)+Q_\rho^-(\theta)}|D(u-k)_\pm\zeta|^p dx\,d\tau\\&\le\gamma\iint_{(y,s)+Q_\rho^-(\theta)}\left[(u-k)_\pm^p|D\zeta|^p+(u-k)_\pm^2|\zeta_\tau|\right]dx\,d\tau\\&+\gamma C^p\iint_{(y,s)+Q_\rho^-(\theta)}\left[\chi_{[(u-k)_\pm>0]}+(u-k)_\pm^p\right]\zeta^p dx\,d\tau\end{aligned}\tag{2.3}$$

where C_o and C are the constants appearing in the structure conditions (1.2).

Analogous energy estimates hold for "forward" cylinders $(y,s)+Q_\rho^+(\theta)\subset E_T$.

Proof After a translation we may assume $(y,s)=(0,0)$. In (1.5) take the testing function $\varphi=\pm(u-k)_\pm\zeta^p$ and integrate over $K_\rho\times(-\theta\rho^p,t]$, with $t\in(-\theta\rho^p,0]$. The use of such a test function is justified, modulus a standard Steklov averaging process, by making use of the alternate weak formulation (1.7):

$$\begin{aligned}&\pm\iint_{K_\rho\times(-\theta\rho^p,t]}u_\tau(u-k)_\pm\zeta^p dx\,d\tau\\&\pm\iint_{K_\rho\times(-\theta\rho^p,t]}\mathbf{A}(x,\tau,u,Du)D[(u-k)_\pm\zeta^p]dx\,d\tau\\&\le\pm\iint_{K_\rho\times(-\theta\rho^p,t]}B(x,\tau,u,Du)(u-k)_\pm\zeta^p\,dx\,d\tau.\end{aligned}$$

As for the first term

$$\begin{aligned}\pm\iint_{K_\rho\times(-\theta\rho^p,t]}&u_\tau(u-k)_\pm\zeta^p dx\,d\tau=\frac12\iint_{K_\rho\times(-\theta\rho^p,t]}[(u-k)_\pm^2]_\tau\zeta^p dx\,d\tau\\&=\frac12\int_{-\theta\rho^p}^t\frac{d}{d\tau}\int_{K_\rho}(u-k)_-^2\zeta^p dx\,d\tau-\frac p2\iint_{K_\rho\times(-\theta\rho^p,t]}(u-k)_\pm^2\zeta^{p-1}\zeta_\tau dx\,d\tau\\&\ge\frac12\int_{K_\rho}(u-k)_\pm^2\zeta^p(x,t)dx-\frac12\int_{K_\rho}(u-k)_\pm^2\zeta^p(x,-\theta\rho^p)dx\\&-\frac p2\iint_{Q_\rho^-(\theta)}(u-k)_\pm^2\zeta^{p-1}|\zeta_\tau|dx\,d\tau.\end{aligned}$$

The second integral is transformed and estimated as

$$
\begin{aligned}
\pm &\iint_{K_\rho\times(-\theta\rho^p,t]} \mathbf{A}(x,\tau,u,Du)\cdot D[(u-k)_\pm\zeta^p]dx\,d\tau \\
&= \iint_{K_\rho\times(-\theta\rho^p,t]} \pm\mathbf{A}(x,\tau,u,Du)\cdot D(u-k)_\pm\zeta^p dx\,d\tau \\
&\quad \pm p\iint_{K_\rho\times(-\theta\rho^p,t]} (u-k)_\pm\zeta^{p-1}\mathbf{A}(x,\tau,u,Du)\cdot D\zeta\,dx\,d\tau \\
&\geq C_o\iint_{K_\rho\times(-\theta\rho^p,t]} |D(u-k)_\pm|^p\zeta^p dx\,d\tau \\
&\quad - C^p\iint_{Q_\rho^-(\theta)} \chi_{[(u-k)_\pm>0]}\zeta^p dx\,d\tau \\
&\quad - pC_1\iint_{K_\rho\times(-\theta\rho^p,t]} |D(u-k)_\pm|^{p-1}(u-k)_\pm\zeta^{p-1}|D\zeta|dx\,d\tau \\
&\quad - pC^{p-1}\iint_{Q_\rho^-(\theta)} (u-k)_\pm\zeta^{p-1}|D\zeta|dx\,d\tau.
\end{aligned}
$$

By Young's inequality

$$
\begin{aligned}
pC_1 &\iint_{K_\rho\times(-\theta\rho^p,t]} |D(u-k)_\pm|^{p-1}(u-k)_\pm\zeta^{p-1}|D\zeta|dx\,d\tau \\
&\leq \frac{C_o}{4}\iint_{K_\rho\times(-\theta\rho^p,t]} |D(u-k)_\pm|^p\zeta^p dx\,d\tau \\
&\quad + \gamma(C_o)\iint_{Q_\rho^-(\theta)} (u-k)_\pm^p|D\zeta|^p dx\,d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
pC^{p-1}&\iint_{Q_\rho^-(\theta)} (u-k)_\pm\zeta^{p-1}|D\zeta|dx\,d\tau \\
&\leq \gamma\iint_{Q_\rho^-(\theta)} (u-k)_\pm^p|D\zeta|^p dx\,d\tau + C^p\iint_{Q_\rho^-(\theta)} \chi_{[(u-k)_\pm>0]}\zeta^p dx\,d\tau.
\end{aligned}
$$

Combining these terms

$$
\begin{aligned}
\pm &\iint_{K_\rho\times(-\theta\rho^p,t]} \mathbf{A}(x,\tau,u,Du)\cdot D[(u-k)_\pm\zeta^p]dx\,d\tau \\
&\geq \frac{3}{4}C_o\iint_{K_\rho\times(-\theta\rho^p,t]} |D(u-k)_\pm|^p\zeta^p dx\,d\tau \\
&\quad - \gamma\iint_{Q_\rho^-(\theta)} (u-k)_\pm^p|D\zeta|^p dx\,d\tau - \gamma C^p\iint_{Q_\rho^-(\theta)} \chi_{[(u-k)_\pm>0]}\zeta^p dx\,d\tau.
\end{aligned}
$$

Finally

$$
\begin{aligned}
&\pm \iint_{K_\rho \times(-\theta\rho^p,t]} B(x,\tau,u,Du)(u-k)_\pm \zeta^p dx\, d\tau \\
&\quad \le C \iint_{K_\rho \times(-\theta\rho^p,t]} |D(u-k)_\pm|^{p-1}(u-k)_\pm \zeta^p dx\, d\tau \\
&\qquad + C^p \iint_{K_\rho \times(-\theta\rho^p,t]} (u-k)_\pm \zeta^p dx\, d\tau \\
&\quad \le \frac{C_o}{4} \iint_{K_\rho \times(-\theta\rho^p,t]} |D(u-k)_\pm|^p \zeta^p dx\, d\tau \\
&\qquad + \gamma C^p \iint_{Q_\rho^-(\theta)} (u-k)_\pm^p \zeta^p dx\, d\tau + C^p \iint_{Q_\rho^-(\theta)} \chi_{[(u-k)_\pm>0]} \zeta^p dx\, d\tau .
\end{aligned}
$$

Combining the previous estimates and recalling that $t \in (-\theta\rho^p, 0]$ is arbitrary establishes the proposition. ■

Remark 2.1 Because of the application of Young's inequality, the constant $\gamma = \gamma(p) \to \infty$, either as $p \to \infty$ or as $p \to 1$. However, γ is stable as $p \to 2$.

Remark 2.2 The proof traces the dependence of the constant γ on the parameters $\{p, N, C_o, C_1\}$ and leaves explicit the dependence on C.

3 A DeGiorgi-Type Lemma

Local, weak sub(super)-solutions to (1.1)–(1.2) in E_T are locally bounded above(below) in E_T ([41], Chapter V). For a fixed cylinder $(y,s) + Q_{2\rho}^-(\theta) \subset E_T$, denote by $\mu_\pm$ and ω, nonnegative numbers such that

$$
\mu_+ \ge \operatorname*{ess\,sup}_{(y,s)+Q_{2\rho}^-(\theta)} u, \qquad \mu_- \le \operatorname*{ess\,inf}_{(y,s)+Q_{2\rho}^-(\theta)} u, \qquad \omega \ge \mu_+ - \mu_- .
$$

Denote by $\xi \in (0,1]$ and $a \in (0,1)$ fixed numbers.

Lemma 3.1 *Let u be a locally bounded, local, weak supersolution to (1.1)–(1.2) in E_T. There exists a number ν_- depending on the data $\{p, N, C_o, C_1\}$ and the parameters θ, ξ, ω, a, such that if*

$$
\big|[u \le \mu_- + \xi\omega] \cap [(y,s) + Q_{2\rho}^-(\theta)]\big| \le \nu_- |Q_{2\rho}^-(\theta)|, \tag{3.1}
$$

then either

$$
C\rho > \min\{1\,,\,\xi\omega\} \tag{3.2}
$$

or

$$
u \ge \mu_- + a\xi\omega \qquad a.e.\ in\ \big[(y,s) + Q_\rho^-(\theta)\big]. \tag{3.3}
$$

Likewise, if u is a locally bounded, local, weak subsolution to (1.1)–(1.2) in E_T, there exists a number ν_+ depending on the data $\{p, N, C_o, C_1\}$ and the parameters θ, ξ, ω, a, such that if

$$\big|[u \geq \mu_+ - \xi\omega] \cap [(y,s) + Q_{2\rho}^-(\theta)]\big| \leq \nu_+ |Q_{2\rho}^-(\theta)|, \tag{3.4}$$

then either (3.2) holds true, or

$$u \leq \mu_+ - a\xi\omega \qquad a.e.\ in\ \big[(y,s) + Q_\rho^-(\theta)\big]. \tag{3.5}$$

Remark 3.1 The constants $\nu_\pm$ are independent of C, and the latter enters into the statement only via the alternative (3.2).

Proof We prove first (3.1)–(3.3). We may assume $(y,s) = (0,0)$ and for $n = 0, 1, \dots,$ set

$$\rho_n = \rho + \frac{\rho}{2^n}, \qquad K_n = K_{\rho_n}, \qquad Q_n = K_n \times (-\theta\rho_n^p, 0]. \tag{3.6}$$

Apply (2.3) over K_n and Q_n to $(u - k_n)_-$, for the levels

$$k_n = \mu_- + \xi_n \omega \qquad \text{where} \qquad \xi_n = a\xi + \frac{1-a}{2^n}\xi. \tag{3.7}$$

The cutoff function ζ is taken of the form $\zeta(x,t) = \zeta_1(x)\zeta_2(t)$, where

$$\begin{aligned} &\zeta_1 = \begin{cases} 1 & \text{in } K_{n+1} \\ 0 & \text{in } \mathbb{R}^N - K_n \end{cases} \qquad |D\zeta_1| \leq \frac{1}{\rho_n - \rho_{n+1}} = \frac{2^{n+1}}{\rho} \\ &\zeta_2 = \begin{cases} 0 & \text{for } t < -\theta\rho_n^p \\ 1 & \text{for } t \geq -\theta\rho_{n+1}^p \end{cases} \qquad 0 \leq \zeta_{2,t} \leq \frac{1}{\theta(\rho_n^p - \rho_{n+1}^p)} \leq \frac{2^{p(n+1)}}{\theta\rho^p}. \end{aligned} \tag{3.8}$$

The energy inequality (2.3) with these stipulations yields

$$\begin{aligned} &\operatorname*{ess\,sup}_{-\theta\rho_n^p < t < 0} \int_{K_n} (u-k_n)_-^2 \zeta^p(x,t)dx + \iint_{Q_n} |D(u-k_n)_-\zeta|^p dx\,d\tau \\ &\leq \gamma \frac{2^{np}}{\rho^p} \left(\iint_{Q_n} (u-k_n)_-^p dx\,d\tau + \frac{1}{\theta} \iint_{Q_n} (u-k_n)_-^2 dx\,d\tau \right) \\ &\qquad + \gamma C^p \iint_{Q_n} \big(\chi_{[u<k_n]} + (u-k_n)_-^p\big)\, dx\,d\tau \\ &\leq \gamma \frac{2^{np}(\xi\omega)^p}{\rho^p} \left(1 + \frac{1}{\theta(\xi\omega)^{p-2}} + \left(\frac{C\rho}{\xi\omega}\right)^p + (C\rho)^p \right) |[u<k_n] \cap Q_n| \\ &\leq \gamma \frac{2^{np}(\xi\omega)^p}{\rho^p} \left(1 + \frac{1}{\theta(\xi\omega)^{p-2}} \right) |[u<k_n] \cap Q_n|, \end{aligned} \tag{3.9}$$

provided $\xi\omega \geq C\rho$, and $\rho < C^{-1}$, which we assume. By the embedding Proposition 4.1 of the Preliminaries

$$\iint_{Q_n} [(u-k_n)_-\zeta]^{p\frac{N+2}{N}} dx\, d\tau \le \iint_{Q_n} |D[(u-k_n)_-\zeta]|^p dx\, d\tau$$
$$\times \left(\operatorname*{ess\,sup}_{-\theta\rho_n^p<t<0} \int_{K_n} [(u-k_n)_-\zeta(x,t)]^2 dx \right)^{\frac{p}{N}}$$
$$\le \gamma \left[\frac{2^{np}}{\rho^p} \left((\xi\omega)^p + \frac{(\xi\omega)^2}{\theta} \right) \right]^{\frac{N+p}{N}} |[u<k_n]\cap Q_n|^{\frac{N+p}{N}}.$$

Estimate below

$$\iint_{Q_n} [(u-k_n)_-\zeta]^{p\frac{N+2}{N}} dx\, d\tau \ge \left(\frac{(1-a)\xi\omega}{2^{n+1}} \right)^{p\frac{N+2}{N}} |[u<k_{n+1}]\cap Q_{n+1}|$$

and set

$$Y_n = \frac{|[u<k_n]\cap Q_n|}{|Q_n|}.$$

Then

$$Y_{n+1} \le \frac{\gamma b^n}{(1-a)^{(N+2)\frac{p}{N}}} \left(\frac{\theta}{(\xi\omega)^{2-p}} \right)^{\frac{p}{N}} \left(1 + \frac{(\xi\omega)^{2-p}}{\theta} \right)^{\frac{N+p}{N}} Y_n^{1+\frac{p}{N}}$$

where

$$b = 2^{\frac{p}{N}[2(N+1)+p]}. \tag{3.10}$$

By Lemma 5.1 of the Preliminaries, $\{Y_n\} \to 0$ as $n \to \infty$, provided

$$Y_o \le \gamma^{-\frac{N}{p}} b^{-\left(\frac{N}{p}\right)^2} (1-a)^{N+2} \frac{\dfrac{(\xi\omega)^{2-p}}{\theta}}{\left(1+\dfrac{(\xi\omega)^{2-p}}{\theta}\right)^{\frac{N+p}{p}}} \stackrel{\text{def}}{=} \nu_-. \tag{3.11}$$

The proof of (3.4)–(3.5) is almost identical. One starts from the energy inequalities (2.3) written for the truncated functions

$$(u-k_n)_+ \qquad \text{with} \qquad k_n = \mu_+ - \xi_n\omega$$

for the same choice of ξ_n as in (3.7). By the definition of μ_+ one estimates

$$(u-k_n)_+ \le \xi\omega.$$

This validates estimates in all analogous to (3.9) with the same functional dependence on $\xi\omega$. The same arguments, with the proper changes in the meaning of the symbols, lead to (3.11) written for ν_+, and conclude the proof. Thus ν_+ depends on the parameters ξ, ω, and a the same way as ν_- does. ∎

For later use we rewrite the expression of $\nu_\pm$ to serve for all $p>1$, in a form that traces the functional dependence on the indicated parameters

$$\nu_\pm = \nu = \gamma^{-1}(1-a)^{N+2}\frac{[\theta(\xi\omega)^{p-2}]^{\frac{N}{p}}}{[1+\theta(\xi\omega)^{p-2}]^{\frac{N+p}{p}}} \tag{3.12}$$

for a quantitative constant $\gamma = \gamma(p, N, C_o, C_1) > 1$, independent of a and ξ.

Remark 3.2 In Lemma 3.1 the statement relative to (3.1)–(3.3) is given in terms of μ_- and $\xi\omega$. As a matter of fact, as the proof clearly shows, when dealing with the lower truncations $(u-k)_-$ for *nonnegative* functions, all the estimates depend only on $k \ge 0$, without any further assumption on it. Correspondingly in (3.12) the quantity ν_- will depend on θk^{p-2}.

4 A Variant of DeGiorgi-Type Lemma Involving "Initial Data"

Assume now that u is a *nonnegative*, local, weak supersolution to (1.1)–(1.2) in E_T. Assume in addition that some information is available on the "initial data" relative to the cylinder $(y,s)+Q^+_{2\rho}(\theta)$, say for example

$$u(x,s) \ge \xi M \qquad \text{for a.e. } x \in K_{2\rho}(y) \tag{4.1}$$

for some $M>0$ and $\xi \in (0,1]$. Then, writing the energy inequalities (2.3) for $(u-k)_-$, for $k \le \xi M$, over the cylinder $[(y,s)+Q^+_{2\rho}(\theta)]$, the integral extended over $K_{2\rho}$ at the time level $t=s$, vanishes in view of (4.1). Moreover, by taking cutoff functions $\zeta(x,t)=\zeta_1(x)$ independent of t, also the integral involving ζ_t, on the right-hand side of (2.3), vanishes. We may now repeat the same arguments as in the previous proof for $(u-\xi_n M)_-$, over the cylinders Q^+_n, where

$$\xi_n = a\xi + \frac{1-a}{2^n}\xi, \qquad Q^+_n = K_n \times (0, \theta(2\rho)^p].$$

We are led to an analog of (3.9) without the term in $(\cdots)$ on the right-hand side, with Q_n replaced by Q^+_n, and with Y_n replaced by

$$\tilde{Y}_n = \frac{|[u<\xi_n M]\cap Q^+_n|}{|Q^+_n|}$$

provided $\xi M > C\rho$ and $\rho < C^{-1}$. Proceeding as before

$$\tilde{Y}_{n+1} \le \frac{\gamma b^n}{(1-a)^{\frac{p(N+2)}{N}}}\left(\frac{\theta}{(\xi M)^{2-p}}\right)^{\frac{p}{N}} \tilde{Y}_n^{1+\frac{p}{N}}$$

for the same value of the parameter b as in (3.10). This in turn implies that $\{\tilde{Y}_n\} \to 0$ as $n\to\infty$, provided

$$\tilde{Y}_o \le \nu_o \frac{(\xi M)^{2-p}}{\theta} \tag{4.2}$$

for a constant $\nu_o \in (0,1)$ depending only on the data and a, and independent of ξ, M, ρ, and θ. We summarize.

Lemma 4.1 *Let u be a nonnegative, local, weak supersolution to (1.1)–(1.2) in E_T. Let M and ξ be positive numbers such that both (4.1) and (4.2) hold. Then either*

$$C\rho > \min\{1\,,\,\xi M\} \tag{4.3}$$

or

$$u \ge a\xi M \qquad \textit{a.e. in } \; K_\rho(y) \times (s, s+\theta(2\rho)^p].$$

Remark 4.1 Both Lemmas 3.1 and 4.1 continue to hold for cylinders whose cross sections are balls.

Remark 4.2 Both Lemmas 3.1 and 4.1 are based on the embedding Proposition 4.1 of the Preliminaries, and the energy estimates (2.3), whose constant dependence is indicated in Remark 2.1. Therefore these lemmas hold for all $p > 1$, including a seamless transition from the singular range $p < 2$ to the degenerate range $p > 2$, but with constants that deteriorate as either $p \to \infty$ or $p \to 1$.

Remark 4.3 The constant ν_o in (4.2) is independent of C, and the latter enters into the statement only via the alternative (4.3).

5 Quasilinear Equations of the Porous Medium Type

Let E be an open set in $\mathbb{R}^N$ and for $T > 0$ let E_T denote the cylindrical domain $E \times (0, T]$. Consider quasilinear, degenerate or singular parabolic partial differential equations of the form

$$u_t - \operatorname{div} \mathbf{A}(x,t,u,Du) = B(x,t,u,Du) \quad \text{ weakly in } \; E_T \tag{5.1}$$

where the functions $\mathbf{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ and $B : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}$ are only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x,t,u,Du) \cdot Du \ge C_o m|u|^{m-1}|Du|^2 - C^2|u|^{m+1} \\ |\mathbf{A}(x,t,u,Du)| \le C_1 m|u|^{m-1}|Du| + C|u|^m \\ |B(x,t,u,Du)| \le Cm|u|^{m-1}|Du| + C^2|u|^m \end{cases} \quad \text{a.e. in } \; E_T \tag{5.2}$$

where $m > 0$, C_o and C_1 are given positive constants, and C is a given nonnegative constant. When $C = 0$ the equation is *homogeneous*.

The homogeneous prototype of this class of parabolic equations is

$$u_t - \operatorname{div} m|u|^{m-1} Du = 0, \quad m > 0, \quad \text{ weakly in } \; E_T. \tag{5.3}$$

A function

$$
\begin{aligned}
&u \in C_{\mathrm{loc}}\big(0,T;L^2_{\mathrm{loc}}(E)\big), \quad |u|^{\frac{m+1}{2}} \in L^2_{\mathrm{loc}}\big(0,T;W^{1,2}_{\mathrm{loc}}(E)\big) \;\text{ if }\; m>1 \\
&u \in C_{\mathrm{loc}}\big(0,T;L^{m+1}_{\mathrm{loc}}(E)\big), \quad |u|^{m} \in L^2_{\mathrm{loc}}\big(0,T;W^{1,2}_{\mathrm{loc}}(E)\big) \;\text{ if }\; 0<m<1
\end{aligned}
\tag{5.4}
$$

is a local, weak sub(super)-solution to (5.1) if for every compact set $K\subset E$ and every subinterval $[t_1,t_2]\subset(0,T]$

$$
\begin{aligned}
\int_K u\varphi\,dx\bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \big[-u\varphi_t + \mathbf{A}(x,t,u,Du)\cdot D\varphi\big]\,dx\,dt \\
\le (\ge) \int_{t_1}^{t_2}\int_K B(x,t,u,Du)\varphi\,dx\,dt
\end{aligned}
\tag{5.5}
$$

for all nonnegative testing functions

$$
\varphi \in W^{1,2}_{\mathrm{loc}}\big(0,T;L^2(K)\big)\cap L^2_{\mathrm{loc}}\big(0,T;W^{1,2}_o(K)\big). \tag{5.6}
$$

This guarantees that all the integrals in (5.5) are convergent.

Our focus will be on Harnack estimates satisfied by nonnegative weak solutions in the interior of E_T. For this reason we will only be interested in *local* solutions irrespective of any boundary or initial data.

The partial differential equation (5.1) is degenerate when $m>1$ and singular when $m<1$, since the modulus of ellipticity $|u|^{m-1}$ respectively tends to 0 or to $+\infty$ as $|u|\to 0$.

When $m=1$, the equation is nondegenerate, and its theory is reasonably complete ([101]). In the following particular care will be devoted to the stability of the various statements and estimates for $m\approx 1$.

Remark 5.1 Further discriminants on the structure conditions between the degenerate case $m>1$ and the singular case $0<m<1$ are in § C.5 and C.9 of Appendix C. Here we have preferred to present a unified treatment.

5.1 An Alternate Formulation in Terms of Steklov Averages

It would be technically convenient to have a formulation of weak solutions that involves u_t. Unfortunately weak solutions to (5.1)–(5.2), whenever they exist, possess a modest degree of regularity in the time variable and, in general, u_t has a meaning only in the sense of distributions.

The following notion of local weak sub(super)-solution involves the discrete time derivative of u and is equivalent to (5.5)–(5.6).

Fix $t\in(0,T)$ and let h be a positive number such that $0<t<t+h<T$. In (5.5) take $t_1=t$, $t_2=t+h$ and choose a nonnegative testing function φ independent of the variable $\tau\in(t,t+h)$. Dividing by h and recalling the definition of Steklov averages we obtain

$$\int_{K\times\{t\}} \big(u_{h,t}\varphi + [\mathbf{A}(x,\tau,u,Du)]_h \cdot D\varphi\big)\,dx \le (\ge) \int_{K\times\{t\}} [B(x,\tau,u,Du)]_h \varphi\,dx \tag{5.7}$$

for all $0 < t < T - h$ and all nonnegative $\varphi \in W_o^{1,2}(K)$. To recover (5.5), fix a subinterval $0 < t_1 < t_2 < T$, choose h so small that $t_2 + h \le T$ and in (5.7) take a testing function as in (5.6). Such a choice is admissible, since the testing functions in (5.7) are independent of the variable $\tau \in (t, t+h)$ but may be dependent on t. Integrating over $[t_1, t_2]$ and letting $h \to 0$ with the aid of Lemma 5.3 of the Preliminaries gives (5.5).

5.2 On the Notion of Parabolicity

The structure conditions (5.2) are not sufficient to characterize parabolic partial differential equations. For example, the principal part

$$\mathbf{A}(x,t,u,Du) = m|u|^{m-1}Du - |u|^m \frac{Du}{|Du|}$$

satisfies the first of (5.2) for all $m > 0$. However, its modulus of ellipticity changes signum at $|Du| = \frac{1}{m}|u|$, and the equation transitions from "forward" parabolic to "backward" in time.

Definition:

The partial differential equation in (5.1) is *parabolic* if it satisfies the structure conditions (5.2) and in addition, for every weak, local sub(super)-solution u, the truncations $+(u-k)_+$, $-(u-k)_-$, for all $k \in \mathbb{R}$, are weak, local sub(super)-solutions to (5.1), in the sense of (5.5)–(5.6), with $\mathbf{A}(x,t,u,Du)$ and $B(x,t,u,Du)$ replaced respectively by

$$\begin{aligned}&\mathbf{A}(x,t,k \pm (u-k)_\pm, \pm D(u-k)_\pm),\\ &B(x,t,k \pm (u-k)_\pm, \pm D(u-k)_\pm).\end{aligned}$$

Lemma 5.1 *Assume that for all $(x,t,u) \in E_T \times \mathbb{R}$*

$$\mathbf{A}(x,t,u,\eta)\cdot\eta \ge 0 \qquad \textit{for all } \eta \in \mathbb{R}^N. \tag{5.8}$$

Then (5.1)–(5.2) is parabolic.

Proof Same as in Lemma 1.1. ∎

Henceforth we will assume that the principal part $\mathbf{A}(x,t,u,Du)$ satisfies (5.8) so that (5.1)–(5.2) is parabolic.

One verifies that the assumptions of the lemma are verified for example by equations with principal part

$$\operatorname{div}\mathbf{A}(x,t,u,Du) = \Big(|u|^{m-1}a_{ij}(x,t)u_{x_i} + f(x,t)|u|^m \frac{u_{x_j}}{|Du|}\Big)_{x_j},$$

where f is bounded and nonnegative, and the matrix (a_{ij}) is only measurable and locally positive definite in E_T.

5.3 Dependence on the Parameters $\{m, N, C_o, C_1\}$ and Stability

The set of parameters $\{m, N, C_o, C_1\}$ are the *data*, and we say that a generic positive constant $\gamma = \gamma(m, N, C_o, C_1)$ depends only on the data, if it can be quantitatively determined a priori, only in terms of these parameters.

The constant $C \geq 0$ is also a *datum* of the equations. However, all our estimates will only involve $\{m, N, C_o, C_1\}$, while C will appear as an alternative. This is further illustrated in the energy estimates of Propositions 6.1, 9.1 and Remarks 6.2, 9.2 following them and in Lemmas 7.1, 8.1, 10.1, 11.1 and Remarks 7.1, 8.3, 10.1, 11.3 following them.

A positive constant γ depending only on the data is stable as $m \to 1$ if there exists a positive constant $\gamma(1, N, C_o, C_1)$ such that

$$\lim_{m\to 1} \gamma(m, N, C_o, C_1) = \gamma(1, N, C_o, C_1). \tag{5.9}$$

We will show that all our estimates are stable as $m \to 1$. As a consequence, the classical theory for nondegenerate equations can be recovered from these degenerate and singular equations, by letting $m \to 1$.

6 Energy Estimates for $(u-k)_\pm$ on Cylinders $(y, s) + Q_\rho^\pm(\theta) \subset E_T$ for Degenerate Equations $(m > 1)$

Introduce cylinders $Q_\rho^\pm(\theta)$ and their translated $(y, s) + Q_\rho^\pm(\theta)$ as in (2.1) and (2.2) with $p = 2$.

Proposition 6.1 *Let u be a local, weak solution to (5.1)–(5.2) for $m > 1$, in E_T, in the sense of (5.5)–(5.6). There exists a positive constant $\gamma = \gamma(m, N, C_o, C_1)$, such that for every cylinder $(y, s) + Q_\rho^-(\theta) \subset E_T$, every $k \in \mathbb{R}$, and every piecewise smooth cutoff function ζ vanishing on $\partial K_\rho(y)$, and such that* $0 \leq \zeta \leq 1$

$$
\begin{aligned}
\operatorname*{ess\,sup}_{s-\theta\rho^2<t\le s} \int_{K_\rho(y)} & (u-k)_\pm^2 \zeta^2(x,t)dx \\
& - \int_{K_\rho(y)} (u-k)_\pm^2 \zeta^2(x,s-\theta\rho^2)dx \\
& + C_o \iint_{(y,s)+Q_\rho^-(\theta)} |u|^{m-1}|D(u-k)_\pm|^2\zeta^2 dx\,dt \\
& \le \gamma \iint_{(y,s)+Q_\rho^-(\theta)} (u-k)_\pm^2 \zeta|\zeta_t| dx\,dt \\
& + \gamma \iint_{(y,s)+Q_\rho^-(\theta)} |u|^{m-1}(u-k)_\pm^2|D\zeta|^2 dx\,dt \\
& + \gamma C^2 \iint_{(y,s)+Q_\rho^-(\theta)} |u|^{m+1}\chi_{[(u-k)_\pm>0]}\zeta^2 dx\,dt.
\end{aligned}
\tag{6.1}
$$

Analogous estimates hold in the "forward" cylinder $(y,s)+Q_\rho^+(\theta)\subset E_T$.

Proof After a translation we may assume $(y,s)=(0,0)$. In (5.5) take the testing function

$$\varphi_\pm = \pm(u-k)_\pm\,\zeta^2$$

over $K_\rho\times(-\theta\rho^2,t]$, where $-\theta\rho^2<t\le 0$. The use of $\pm(u-k)_\pm$ in this testing function is justified, modulus a standard Steklov averaging process, by making use of the alternate weak formulation (5.7). This gives

$$
\begin{aligned}
& \pm \iint_{K_\rho\times(-\theta\rho^2,t]} u_\tau(u-k)_\pm\zeta^2 dx\,d\tau \\
& \pm \iint_{K_\rho\times(-\theta\rho^2,t]} \mathbf{A}(x,\tau,u,Du)\cdot D(u-k)_\pm\zeta^2 dx\,d\tau \\
& \pm 2 \iint_{K_\rho\times(-\theta\rho^2,t]} (u-k)_\pm\mathbf{A}(x,\tau,u,Du)\cdot D\zeta\,\zeta\,dx\,d\tau \\
& = \pm \iint_{K_\rho\times(-\theta\rho^2,t]} B(x,\tau,u,Du)(u-k)_\pm\,\zeta^2 dx\,d\tau.
\end{aligned}
$$

Transform and estimate these integrals, to get

$$
\begin{aligned}
\pm & \iint_{K_\rho\times(-\theta\rho^2,t]} u_\tau(u-k)_\pm\,\zeta^2 dx\,d\tau \\
& = \frac{1}{2}\int_{K_\rho}(u-k)_\pm^2\zeta^2(x,t)dx - \frac{1}{2}\int_{K_\rho}(u-k)_\pm^2\zeta^2(x,-\theta\rho^2)dx \\
& \quad - \iint_{Q_\rho^-(\theta)} (u-k)_\pm^2\zeta|\zeta_\tau| dx\,d\tau.
\end{aligned}
$$

From the first structure condition (5.2) it follows that

$$
\begin{aligned}
\pm \iint_{K_\rho\times(-\theta\rho^2,t]} & \mathbf{A}(x,\tau,u,Du)\cdot D(u-k)_\pm\,\zeta^2 dx\,d\tau \\
&\ge C_o m \iint_{K_\rho\times(-\theta\rho^2,t]} |u|^{m-1}|D(u-k)_\pm|^2\zeta^2 dx\,d\tau \\
&\quad - C^2 \iint_{Q_\rho^-(\theta)} |u|^{m+1}\zeta^2\,\chi_{[(u-k)_\pm>0]} dx\,d\tau,
\end{aligned}
$$

and from the second condition in (5.2) and Young's inequality it follows that

$$
\begin{aligned}
2\Big|\iint_{K_\rho\times(-\theta\rho^2,t]} & (u-k)_\pm \mathbf{A}(x,\tau,u,Du)\cdot D\zeta\,\zeta\,dx\,d\tau\Big| \\
&\le 2C_1 m \iint_{K_\rho\times(-\theta\rho^2,t]} |u|^{m-1}(u-k)_\pm|D(u-k)_\pm||\zeta|D\zeta|dx\,d\tau \\
&\quad + 2C \iint_{Q_\rho^-(\theta)} |u|^m(u-k)_\pm\zeta|D\zeta|\chi_{[(u-k)_\pm>0]} dx\,d\tau \\
&\le \frac{C_o m}{4}\iint_{K_\rho\times(-\theta\rho^2,t]} |u|^{m-1}|D(u-k)_\pm|^2\zeta^2 dx\,d\tau \\
&\quad + \gamma(C_o)\iint_{Q_\rho^-(\theta)} |u|^{m-1}(u-k)_\pm^2|D\zeta|^2 dx\,d\tau \\
&\quad + C^2\iint_{Q_\rho^-(\theta)} |u|^{m+1}\zeta^2\chi_{[(u-k)_\pm>0]} dx\,d\tau.
\end{aligned}
$$

Finally, the third condition of (5.2) implies

$$
\begin{aligned}
\Big|\iint_{K_\rho\times(-\theta\rho^2,t]} & B(x,\tau,u,Du)(u-k)_\pm\,\zeta^2 dx\,d\tau\Big| \\
&\le \frac{C_o m}{4}\iint_{K_\rho\times(-\theta\rho^2,t]} |u|^{m-1}|D(u-k)_\pm|^2\zeta^2 dx\,d\tau \\
&\quad + \gamma(C_o)\,C^2\iint_{Q_\rho^-(\theta)} |u|^{m-1}(u-k)_\pm^2 dx\,d\tau \\
&\quad + \bar\gamma(C_o)C^2\iint_{Q_\rho^-(\theta)} |u|^{m+1}\chi_{[(u-k)_\pm>0]}\zeta^2 dx\,d\tau.
\end{aligned}
$$

Combining these estimates, and taking the supremum over $t\in(-\theta\rho^2,0]$ proves the proposition. ■

Remark 6.1 The constant $\gamma=\gamma(m,N,C_o,C_1)\to\infty$, as $m\to\infty$, but it is stable as $m\to 1$.

Remark 6.2 The proof traces the dependence of the constant γ on the parameters $\{m,N,C_o,C_1\}$ and leaves explicit the dependence on C.

Remark 6.3 The inequalities (6.1) continue to hold for the truncations $(u-k)_+$ (resp. for $(u-k)_-$), if u is a *nonnegative*, local, weak, sub(super)-solution to (5.1)–(5.2) in E_T. It suffices to observe that, in such a case, the test function $+(-)\varphi_\pm$ is nonnegative (nonpositive), and it can be used in the corresponding formulation (5.5) of sub(super)-solutions.

7 A DeGiorgi-Type Lemma for Nonnegative Sub(Super)-Solutions to Degenerate Equations ($m > 1$)

Local, weak sub(super)-solutions to (5.1)–(5.2) in E_T are locally bounded above(below) in E_T ([7]). For a cylinder $(y,s)+Q^-_{2\rho}(\theta) \subset E_T$ denote by $\mu_\pm$ and ω, numbers satisfying

$$\mu_+ \geq \operatorname*{ess\,sup}_{[(y,s)+Q^-_{2\rho}(\theta)]} u, \qquad \mu_- \leq \operatorname*{ess\,inf}_{[(y,s)+Q^-_{2\rho}(\theta)]} u, \qquad \omega = \mu_+ - \mu_-.$$

Since the degeneracy occurs at $u = 0$, we will assume at the outset that

$$\mu_- = \operatorname*{ess\,inf}_{(y,s)+Q^-_{2\rho}(\theta)} u = 0 \quad \text{so that} \quad \omega = \mu_+.$$

Denote by ξ and a fixed numbers in $(0,1)$.

Lemma 7.1 *Let u be a nonnegative, locally bounded, local, weak supersolution to (5.1)–(5.2) for $m > 1$, in E_T. There exists a positive number ν_-, depending on θ, ω, ξ, a and the data $\{m, N, C_o, C_1\}$, such that if*

$$|[u \leq \xi\omega] \cap [(y,s)+Q^-_{2\rho}(\theta)]| \leq \nu_- |Q^-_{2\rho}(\theta)|, \tag{7.1}$$

then either

$$C\rho > 1 \tag{7.2}$$

or

$$u \geq a\xi\omega \qquad a.e.\ in\ \ (y,s)+Q^-_\rho(\theta). \tag{7.3}$$

Likewise, if u is a nonnegative, locally bounded, local, weak subsolution to (5.1)–(5.2) for $m > 1$ in E_T, there exists a positive number ν_+, depending on ω, θ, ξ, a and the data $\{m, N, C_o, C_1\}$, such that if

$$\big|[u \geq \mu_+ - \xi\omega] \cap [(y,s)+Q^-_{2\rho}(\theta)]\big| \leq \nu_+ |Q^-_{2\rho}(\theta)|, \tag{7.4}$$

then either (7.2) holds, or

$$u \leq \mu_+ - a\xi\omega \qquad a.e.\ in\ \ (y,s)+Q^-_\rho(\theta). \tag{7.5}$$

Remark 7.1 The constants $\nu_\pm$ are independent of C, and the latter enters into the statement only via the alternative (7.2).

Proof Assume $(y,s) = (0,0)$ and introduce the sequence of cubes $\{K_n\}$ and cylinders $\{Q_n\}$ as in (3.6) with $p = 2$ and a cutoff function on Q_n of the form $\zeta(x,t) = \zeta_1(x)\zeta_2(t)$ defined as in (3.8) with $p = 2$.

7.1 Proof of (7.1)–(7.3)

Introduce the sequence of truncating levels $\{k_n\}$ defined as in (3.7) with $\mu_- = 0$. The energy estimates (6.1) on Q_n, for $(u-k_n)_-$, give

$$\begin{aligned}\sup_{-\theta\rho_n^2<t\le 0}&\int_{K_n}(u-k_n)_-^2\zeta^2(x,t)dx\\&+C_o\iint_{Q_n}u^{m-1}|D[(u-k_n)_-\zeta]|^2dx\,d\tau\\&\le\gamma\frac{2^{2n}}{\rho^2}\iint_{Q_n}\Big(k_n^{m+1}+\frac{1}{\theta}k_n^2\Big)\chi_{[u<k_n]}dx\,d\tau\\&+\gamma C^2\iint_{Q_n}u^{m+1}\chi_{[u<k_n]}dx\,d\tau\\&\le\gamma\frac{2^{2n}}{\rho^2}(\xi\omega)^{m+1}\Big(1+\frac{1}{\theta(\xi\omega)^{m-1}}\Big)\big|[u<k_n]\cap Q_n\big|\\&+\gamma\frac{2^{2n}}{\rho^2}(\xi\omega)^{m+1}(C\rho)^2\big|[u<k_n]\cap Q_n\big|.\end{aligned}$$

Therefore, if (7.2) is violated,

$$\begin{aligned}\sup_{-\theta\rho_n^2<t\le 0}&\int_{K_n}(u-k_n)_-^2\zeta^2(x,t)dx\\&+C_o\iint_{Q_n}u^{m-1}|D[(u-k_n)_-\zeta]|^2dx\,d\tau\\&\le\gamma\frac{2^{2n}}{\rho^2}(\xi\omega)^{m+1}\Big(1+\frac{1}{\theta(\xi\omega)^{m-1}}\Big)\big|[u<k_n]\cap Q_n\big|.\end{aligned}$$

To estimate the left-hand side from below we keep u away from zero by introducing the function

$$v=\max\big\{u\,;\,\tfrac{1}{2}a\xi\omega\big\}. \tag{7.6}$$

Then estimate below

$$\int_{K_n}(u-k_n)_-^2\zeta^2(x,t)dx\ge\int_{K_n}(v-k_n)_-^2\zeta^2(x,t)dx.$$

Also we estimate below

$$
\begin{aligned}
(\tfrac{1}{2}a)^{m-1}(\xi\omega)^{m-1} & \iint_{Q_n} |D[(v-k_n)_-\zeta]|^2 dx\, d\tau \\
&\le \iint_{Q_n} v^{m-1}|D[(v-k_n)_-\zeta]|^2 dx\, d\tau \\
&= \iint_{Q_n\cap[u=v]} u^{m-1}|D[(u-k_n)_-\zeta]|^2 dx\, d\tau \\
&\quad + (\xi\omega)^{m-1} \iint_{Q_n\cap[u<v]} (v-k_n)_-^2 |D\zeta|^2 dx\, d\tau \\
&\le \iint_{Q_n} u^{m-1}|D[(u-k_n)_-\zeta]|^2 dx\, d\tau \\
&\quad + \gamma\frac{2^{2n}}{\rho^2}(\xi\omega)^{m+1}\big|[u<k_n]\cap Q_n\big|.
\end{aligned}
$$

Observe that, by the definition (7.6) of the truncated function v, the two sets $[v < k_n]$ and $[u < k_n]$ coincide. Then setting

$$
A_n = [v < k_n] \cap Q_n \quad \text{and} \quad Y_n = \frac{|A_n|}{|Q_n|}
$$

and combining these estimates gives

$$
\begin{aligned}
\sup_{-\theta\rho_n^2<t\le 0} \int_{K_n} (v-k_n)_-^2 \zeta^2(x,t) dx & \\
+ (\xi\omega)^{m-1} \iint_{Q_n} |D[(v-k_n)_-\zeta]|^2 dx\, d\tau & \\
\le \gamma\Lambda(a)\frac{2^{2n}}{\rho^2}(\xi\omega)^{m+1}\Big(1+\frac{1}{\theta(\xi\omega)^{m-1}}\Big)|A_n| &
\end{aligned}
\tag{7.7}
$$

where

$$
\Lambda(a) = \frac{1}{(\frac{1}{2}a)^{m-1}}. \tag{7.8}
$$

Apply Hölder's inequality and the embedding Proposition 4.1 of the Preliminaries, and recall that $\zeta = 1$ on Q_{n+1}, to get

$$
\begin{aligned}
\Big(\frac{1-a}{2^{n+1}}\Big)^2(\xi\omega)^2|A_{n+1}| &\le \iint_{Q_{n+1}} (v-k_n)_-^2\, dx\, d\tau \\
&\le \Big(\iint_{Q_n} [(v-k_n)_-\zeta]^{2\frac{N+2}{N}} dx\, d\tau\Big)^{\frac{N}{N+2}} |A_n|^{\frac{2}{N+2}} \\
&\le \gamma\Big(\iint_{Q_n} |D[(v-k_n)_-\zeta]|^2 dx\, d\tau\Big)^{\frac{N}{N+2}} \\
&\quad \times \Big(\sup_{-\theta\rho_n^2<t\le 0} \int_{K_n} [(v-k_n)_-\zeta]^2(x,t) dx\Big)^{\frac{2}{N+2}} |A_n|^{\frac{2}{N+2}}
\end{aligned}
$$

for a constant γ depending only on N. Combining this with (7.7) gives

$$|A_{n+1}| \le \gamma \frac{2^{4n}\Lambda(a)}{(1-a)^2\rho^2}(\xi\omega)^{\frac{2(m-1)}{N+2}}\Big(1+\frac{1}{\theta(\xi\omega)^{m-1}}\Big)|A_n|^{1+\frac{2}{N+2}}.$$

In terms of $Y_n = \frac{|A_n|}{|Q_n|}$ this can be rewritten as

$$Y_{n+1} \le \gamma \frac{2^{4n}\Lambda(a)}{(1-a)^2}\frac{\big(1+\theta(\xi\omega)^{m-1}\big)}{\big(\theta(\xi\omega)^{m-1}\big)^{\frac{N}{N+2}}}\,Y_n^{1+\frac{2}{N+2}}.$$

By Lemma 5.1 of the Preliminaries, $\{Y_n\} \to 0$ as $n \to \infty$, provided

$$Y_o = \frac{|A_o|}{|Q_o|} \le \left[\frac{(1-a)^2}{\gamma\Lambda(a)}\right]^{\frac{N+2}{2}} 2^{-(N+2)^2}\frac{\big(\theta(\xi\omega)^{m-1}\big)^{\frac{N}{2}}}{\big(1+\theta(\xi\omega)^{m-1}\big)^{\frac{N+2}{2}}} \stackrel{\text{def}}{=} \nu_-.$$

For later use we rewrite the expression of ν_- to serve for all $m > 1$, in a form that traces the functional dependence of ν_-, on the indicated parameters

$$\nu_- = \gamma^{-1}a^{(m-1)\frac{N+2}{2}}(1-a)^{N+2}\frac{[\theta(\xi\omega)^{m-1}]^{\frac{N}{2}}}{[1+\theta(\xi\omega)^{m-1}]^{\frac{N+2}{2}}} \tag{7.9}$$

for a quantitative constant $\gamma = \gamma(m, N, C_o, C_1) > 1$, independent of a and ξ.

Remark 7.2 In Lemma 7.1 the statement relative to (7.1)–(7.3) is given in terms of $\xi\omega$, assuming $\mu_- = 0$. As a matter of fact, as the proof clearly shows, when dealing with the lower truncations $(u-k)_-$ for *nonnegative* functions, all the estimates depend only on $k \ge 0$, without any further assumption on it. Correspondingly in (7.9) the quantity ν_- will depend on θk^{m-1}.

7.2 Proof of (7.4)–(7.5)

Introduce the sequence of truncating levels

$$k_n = \mu_+ - \xi_n\omega \qquad \text{with } \xi_n \text{ as in (3.7)},$$

and write down the energy estimates (6.1) on Q_n, for $(u-k_n)_+$, to get

$$
\begin{aligned}
\sup_{-\theta\rho_n^2<t\le 0} \int_{K_n} & (u-k_n)_+^2\zeta^2(x,t)dx \\
& + C_o \iint_{Q_n} u^{m-1}|D[(u-k_n)_+\zeta]|^2 dx\, d\tau \\
& \le \gamma\frac{2^{2n}}{\rho^2}\iint_{Q_n}\Big(\mu_+^{m+1}+\frac{1}{\theta}(\xi\omega)^2\Big)\chi_{[u>k_n]}dx\, d\tau \\
& + \gamma C^2 \iint_{Q_n} u^{m+1}\chi_{[u>k_n]}dx\, d\tau \\
& \le \gamma\frac{2^{2n}}{\rho^2}\mu_+^{m+1}\Big(1+\frac{1}{\theta(\xi\omega)^{m-1}}\Big)\big|[u>k_n]\cap Q_n| \\
& + \gamma\frac{2^{2n}}{\rho^2}\mu_+^{m+1}(C\rho)^2\big|[u<k_n]\cap Q_n|.
\end{aligned}
$$

Therefore, if (7.2) is violated,

$$
\begin{aligned}
\sup_{-\theta\rho_n^2<t\le 0} \int_{K_n} & (u-k_n)_+^2\zeta^2(x,t)dx \\
& + C_o \iint_{Q_n} u^{m-1}|D[(u-k_n)_+\zeta]|^2 dx\, d\tau \\
& \le \gamma\frac{2^{2n}}{\rho^2}\mu_+^{m+1}\Big(1+\frac{1}{\theta(\xi\omega)^{m-1}}\Big)\big|[u<k_n]\cap Q_n|.
\end{aligned}
$$

To estimate below the second integral on the left-hand side, take into account the domain of integration $[u > k_n] \cap Q_n$. This gives

$$
\begin{aligned}
\iint_{Q_n} & u^{m-1}|D[(u-k_n)_+\zeta]|^2 dx\, d\tau \\
& \ge (1-\xi)^{m-1}\mu_+^{m-1}\iint_{Q_n}|D[(u-k_n)_+\zeta]|^2 dx\, d\tau.
\end{aligned}
$$

Setting

$$
A_n = [u > k_n]\cap Q_n \quad \text{and} \quad Y_n = \frac{|A_n|}{|Q_n|},
$$

and combining these estimates gives

$$
\begin{aligned}
\sup_{-\theta\rho_n^2<t\le 0} \int_{K_n} & (u-k_n)_+^2\zeta^2(x,t)dx \\
& + \mu_+^{m-1}\iint_{Q_n}|D[(u-k_n)_+\zeta]|^2 dx\, d\tau \qquad (7.10)\\
& \le \gamma\Lambda(\xi)\frac{2^{2n}}{\rho^2}\mu_+^{m+1}\Big(1+\frac{1}{\theta(\xi\omega)^{m-1}}\Big)|A_n|
\end{aligned}
$$

where

$$\Lambda(\xi) = \frac{1}{(1-\xi)^{m-1}}.$$

Apply Hölder's inequality and the embedding Proposition 4.1 of the Preliminaries, and recall that $\zeta = 1$ on Q_{n+1}, to get

$$\begin{aligned}\left(\frac{1-a}{2^{n+1}}\right)^2 (\xi\omega)^2 |A_{n+1}| &\le \iint_{Q_{n+1}} (u-k_n)_+^2 \, dx\, d\tau \\ &\le \left(\iint_{Q_n} [(u-k_n)_+\zeta]^{2\frac{N+2}{N}} dx\, d\tau\right)^{\frac{N}{N+2}} |A_n|^{\frac{2}{N+2}} \\ &\le \gamma\left(\iint_{Q_n} |D[(u-k_n)_+\zeta]|^2 dx\, d\tau\right)^{\frac{N}{N+2}} \\ &\quad \times \left(\sup_{-\theta\rho_n^2<t\le 0} \int_{K_n} [(u-k_n)_+\zeta]^2(x,t) dx\right)^{\frac{2}{N+2}} |A_n|^{\frac{2}{N+2}}\end{aligned}$$

for a constant γ depending only on N. Combine this with (7.10) to get

$$\begin{aligned}|A_{n+1}| \le{}& \gamma \frac{2^{4n}\Lambda(\xi)}{(1-a)^2\rho^2}\left(\frac{\mu_+}{\xi\omega}\right)^{\frac{2(N+m+1)}{N+2}} \\ &\times (\xi\omega)^{\frac{2(m-1)}{N+2}}\left(1+\frac{1}{\theta(\xi\omega)^{m-1}}\right)|A_n|^{1+\frac{2}{N+2}}.\end{aligned}$$

In terms of $Y_n = \frac{|A_n|}{|Q_n|}$ this can be rewritten as

$$Y_{n+1} \le \gamma \frac{2^{4n}\Lambda(\xi)}{(1-a)^2}\left(\frac{\mu_+}{\xi\omega}\right)^{\frac{2(N+m+1)}{N+2}} \frac{\left(1+\theta(\xi\omega)^{m-1}\right)}{\left(\theta(\xi\omega)^{m-1}\right)^{\frac{N}{N+2}}} Y_n^{1+\frac{2}{N+2}}.$$

By Lemma 5.1 of the Preliminaries, $\{Y_n\} \to 0$ as $n \to \infty$, provided

$$\begin{aligned}Y_o = \frac{|A_o|}{|Q_o|} \le \left[\frac{(1-a)^2}{\gamma\Lambda(\xi)}\right]^{\frac{N+2}{2}} & 2^{-(N+2)^2} \\ \left(\frac{\xi\omega}{\mu_+}\right)^{N+m+1} & \frac{\left(\theta(\xi\omega)^{m-1}\right)^{\frac{N}{2}}}{\left(1+\theta(\xi\omega)^{m-1}\right)^{\frac{N+2}{2}}} \stackrel{\text{def}}{=} \nu_+.\end{aligned}$$

For later use we rewrite the expression of ν_+ in the special case when

$$\theta = (\xi\omega)^{1-m},$$

that is, the relative length of the cylinders $(y,s) + Q_\rho^\pm(\theta)$ is of the order of $(\xi\omega)^{1-m}$.

Then for all $m > 1$, the functional dependence of ν_+, on ξ and a is

$$\nu_+ = \gamma^{-1}\left[\frac{(1-a)^2}{\Lambda(\xi)}\right]^{\frac{N+2}{2}} \xi^{N+m+1}$$

for a quantitative constant $\gamma = \gamma(m, N, C_o, C_1) > 1$, independent of a and ξ.

8 A Variant of DeGiorgi-Type Lemma, for Nonnegative Supersolutions to Degenerate Equations ($m > 1$), Involving "Initial Data"

Continue to denote by $(y,s)+Q_\rho^+(\theta)$ "forward" cylinders with bottom center at (y,s) as defined in (2.1)–(2.2) with $p=2$.

Assume now that some information is available on the "initial data" relative to the cylinder $(y,s)+Q_{2\rho}^+(\theta)\subset E_T$, say for example

$$u(x,s)\ge \xi M \qquad \text{for a.e. } x\in K_{2\rho}(y) \tag{8.1}$$

for some $M>0$ and $\xi\in(0,1]$. Then

Lemma 8.1 *Let u be a nonnegative, locally bounded, local, weak supersolution to (5.1)–(5.2) for $m>1$, in E_T. Let $a\in(0,1)$ and suppose that (8.1) holds true. Then there exists $\nu_o\in(0,1)$, depending only on a and the data, such that, if*

$$|[u\le \xi M]\cap Q_{2\rho}^+(\theta)|\le \frac{\nu_o}{\theta(\xi M)^{m-1}}|Q_{2\rho}^+(\theta)|, \tag{8.2}$$

then either

$$C\rho>1 \tag{8.3}$$

or

$$u\ge a\xi M \qquad \text{in } K_\rho(y)\times(s,s+\theta(2\rho)^p].$$

Proof Assume $(y,s)=(0,0)$ and for $n=0,1,\dots,$ construct sequences of cubes $\{K_n\}$ as in (3.6), and "forward" cylinders $\{Q_n^+\}$, and levels $\{\xi_n\}$ by

$$Q_n^+=K_n\times(0,\theta(2\rho)^2], \qquad \xi_n=a\xi+\frac{1-a}{2^n}\xi.$$

Let also $\zeta(x,t)=\zeta(x)$ be a cutoff function independent of t, vanishing outside K_n and satisfying the first of (3.8). Finally let

$$v=\max\left\{u\,;\,\tfrac12 a\xi M\right\}.$$

Thus v is defined as in (7.6) with M replacing ω. Apply the energy estimates (6.1) for $(u-k_n)_-$ with $k_n=\xi_n M$, over the "forward" cylinders Q_n^+ and the indicated choice of ζ. Observe that the second integral on the left-hand side of (6.1), extended over the "bottom" of Q_n^+, vanishes in view of (8.1). Also, the integral involving ζ_t vanishes, because of our choice of cutoff function ζ. The various terms can now be transformed and estimated exactly as in the proof of Lemma 7.1 with the obvious changes in the symbolism. The most noticeable change is that, due to the vanishing of ζ_t, all the terms containing the factor θ^{-1} are not present. This leads exactly to (7.7) over the cylinder Q_n^+, with ω replaced by M, with the same value of $\Lambda(a)$ as given in (7.8), and with the term in $(\cdots)$ containing the factor θ^{-1}, replaced by one. Setting

$$A_n^+ = [v < \xi_n M] \cap Q_n^+ \quad \text{and} \quad Y_n = \frac{|A_n^+|}{|Q_n^+|},$$

the estimate (7.7) with the indicated changes and in the current context, takes the form

$$\begin{aligned}\sup_{0<t<\theta(2\rho)^2} \int_{K_n} &(v - \xi_n M)_-^2(x,t)\zeta^2(x)dx \\ &+ (\xi M)^{m-1} \iint_{Q_n^+} |D[(v-k_n)_-\zeta]|^2 dx\, d\tau \\ &\le \gamma \Lambda(a) \frac{2^{2n}}{\rho^2} (\xi M)^{m+1} |A_n^+|.\end{aligned}$$

Starting from this inequality, proceed now exactly as in the proof of Lemma 7.1 following (7.7), to arrive at

$$|A_{n+1}^+| \le \gamma \frac{2^{4n}\Lambda(a)}{(1-a)^2\rho^2} (\xi M)^{\frac{2(m-1)}{N+2}} |A_n^+|^{1+\frac{2}{N+2}}.$$

In terms of $Y_n = |A_n^+|/|Q_n^+|$ this can be rewritten as

$$Y_{n+1} \le \gamma \frac{2^{4n}\Lambda(a)}{(1-a)^2} \left[\theta(\xi M)^{m-1}\right]^{\frac{2}{N+2}} Y_n^{1+\frac{2}{N+2}}.$$

By Lemma 5.1 of the Preliminaries, $\{Y_n\} \to 0$ as $n \to \infty$, provided

$$\begin{aligned}Y_o = \frac{|A_o^+|}{|Q_o^+|} &= \frac{|[u < \xi M] \cap Q_{2\rho}^+(\theta)|}{|Q_{2\rho}^+(\theta)|} \\ &\le \left[\frac{(1-a)^2}{\gamma\Lambda(a)}\right]^{\frac{N+2}{2}} 2^{-(N+2)^2} \frac{1}{\theta(\xi M)^{m-1}} \\ &\stackrel{\text{def}}{=} \frac{\nu_o}{\theta(\xi M)^{m-1}}.\end{aligned}$$

■

Remark 8.1 Both Lemmas 7.1 and 8.1 continue to hold for cylinders whose cross sections are balls.

Remark 8.2 Both Lemmas 7.1 and 8.1 are based on the energy estimates (6.1) and the embedding Proposition 4.1 of the Preliminaries, which continue to hold in a stable manner for $m \to 1$. Therefore these results are valid for all $m > 1$, including a seamless transition to $m = 1$.

Remark 8.3 The constant ν_o in (8.2) is independent of C, and the latter enters into the statement only via the alternative (8.3).

9 Energy Estimates for $(u-k)_-$ on Cylinders $(y,s) + Q_\rho^\pm(\theta) \subset E_T$ for Singular Equations $(0 < m < 1)$

Introduce cylinders $Q_\rho^\pm(\theta)$ and their translated $(y,s) + Q_\rho^\pm(\theta)$ as in (2.1) and (2.2) with $p = 2$.

Proposition 9.1 *Let u be a nonnegative, local, weak supersolution to (5.1)–(5.2) for $0<m<1$, in E_T, in the sense of (5.5)–(5.6). There exists a positive constant $\gamma=\gamma(m,N,C_o,C_1)$, such that for every cylinder $(y,s)+Q_\rho^-(\theta)\subset E_T$, every $k>0$, and every nonnegative, piecewise smooth cutoff function ζ vanishing on $\partial K_\rho(y)$,*

$$
\begin{aligned}
\operatorname*{ess\,sup}_{s-\theta\rho^2\le t\le s}&\int_{K_\rho(y)}(u-k)_-^2\zeta^2(x,t)dx\\
&+C_ok^{m-1}\iint_{(y,s)+Q_\rho^-(\theta)}|D[(u-k)_-\zeta]|^2dx\,dt\\
&\le\gamma k\int_{K_\rho(y)}(u-k)_-\zeta^2(x,s-\theta\rho^2)dx\\
&+\gamma k^2\iint_{(y,s)+Q_\rho^-(\theta)}\chi_{[u<k]}\zeta|\zeta_t|dx\,dt\\
&+\gamma k^{m+1}\iint_{(y,s)+Q_\rho^-(\theta)}\chi_{[u<k]}\Phi(C_o,C_1,C,\zeta,D\zeta)dx\,dt
\end{aligned}\tag{9.1}
$$

where

$$\Phi(C_o,C_1,C,\zeta,D\zeta)=(C^2\zeta^2+C_1^2|D\zeta|^2)C_o^{-1}+C_o|D\zeta|^2+mC^2\zeta^2+C\zeta|D\zeta|.$$

The constant $\gamma(m,N,C_o,C_1)\to\infty$ as $m\to 0$, but is stable as $m\to 1$. Analogous estimates hold in the "forward" cylinder $(y,s)+Q_\rho^+(\theta)\subset E_T$.

Proof After a translation we may assume $(y,s)=(0,0)$. Having fixed $k>0$, in the weak formulation (5.5) take the test function

$$\varphi=-(u^m-k^m)_-\,\zeta^2$$

over the cylinder

$$Q_t=K_\rho\times(-\theta\rho^2,t]\qquad\text{for}\qquad-\theta\rho^2<t\le 0.$$

The use of such a φ as testing function is justified, modulus a standard Steklov averaging process, and in view of the notion (5.4) of weak supersolution. Estimating the various terms separately we have

$$
\begin{aligned}
\iint_{Q_t}&-(u^m-k^m)_-u_\tau\zeta^2dx\,d\tau\\
&=\int_{K_\rho}\int_u^k(k^m-s^m)_+ds\zeta^2(x,t)dx\\
&\quad-\int_{K_\rho}\int_u^k(k^m-s^m)_+ds\zeta^2(x,-\theta\rho^2)dx\\
&\quad-2\iint_{Q_t}\int_u^k(k^m-s^m)_+ds\zeta\zeta_\tau dx\,d\tau.
\end{aligned}
$$

Since $m \in (0,1)$ estimate

$$\int_u^k (k^m - s^m)_+ ds \ge \tfrac{1}{2} m k^{m-1} (u-k)_-^2 .$$

Also

$$\int_u^k (k^m - s^m)_+ ds \le k^m (u-k)_- .$$

Therefore

$$\begin{aligned}
\iint_{Q_t} &- (u^m - k^m)_- u_\tau \zeta^2 dx\, d\tau \\
&\ge \tfrac{1}{2} m k^{m-1} \int_{K_\rho} (u-k)_-^2 (x,t) dx \\
&\quad - k^m \int_{K_\rho} (u-k)_- \zeta^2 (x, -\theta\rho^2) dx \\
&\quad - 2k^m \iint_{Q_t} (u-k)_- \zeta |\zeta_\tau| dx\, d\tau .
\end{aligned}$$

Next

$$\begin{aligned}
\iint_{Q_t} & \mathbf{A}(x,\tau,u,Du) \cdot D[-(u^m - k^m)_- \zeta^2] dx\, d\tau \\
&\ge C_o \iint \big| D(u^m - k^m)_- \big|^2 \zeta^2 dx\, d\tau \\
&\quad - m C^2 \iint_{Q_t} u^{2m} \chi_{[u<k]} \zeta^2 dx\, d\tau \\
&\quad - 2C_1 \iint_{Q_t} |Du^m| (u^m - k^m)_- \zeta |D\zeta| dx\, d\tau \\
&\quad - 2C \iint_{Q_t} u^m (u^m - k^m)_- \zeta |D\zeta| dx\, d\tau \\
&\ge \frac{C_o}{2} \iint_{Q_t} |D(u^m - k^m)_-|^2 \zeta^2 dx\, d\tau \\
&\quad - k^{2m} \iint_{Q_t} \chi_{[u<k]} \Phi_a(C_o, C_1, C, \zeta, D\zeta) dx\, d\tau
\end{aligned}$$

where

$$\Phi_a(C_o, C_1, C, \zeta, D\zeta) = C_1^2 |D\zeta|^2 C_o^{-1} + C\zeta|D\zeta| + mC^2\zeta^2 .$$

Finally

$$\begin{aligned}
\iint_{Q_t} &|B(x,\tau,u,Du)|(u^m-k^m)_-\zeta^2 dx\,d\tau \\
&\le C\iint_{Q_t}|Du^m|(u^m-k^m)_-\zeta^2 dx\,d\tau \\
&\quad + C^2\iint_{Q_t} u^m(u^m-k^m)_-\zeta^2 dx\,d\tau \\
&\le \frac{C_o}{4}\iint_{Q_t}|D(u^m-k^m)_-|^2\zeta^2 dx\,d\tau \\
&\quad + \quad k^{2m}\iint_{Q_t}\chi_{[u<k]}\Phi_b(C_o,C,\zeta)dx\,d\tau
\end{aligned}$$

where

$$\Phi_b(C_o,C,\zeta)=C^2(C_o^{-1}+1)\zeta^2.$$

Combining these estimates and taking into account that $t\in(-\theta\rho^2,0]$ is arbitrary, gives

$$\begin{aligned}
\tfrac{1}{2}mk^{m-1}\operatorname*{ess\,sup}_{-\theta\rho^2\le t\le 0}\int_{K_\rho}&(u-k)_-^2\zeta^2(x,t)dx \\
&+\frac{C_o}{4}\iint_{Q_\rho^-(\theta)}|D(u^m-k^m)_-|^2\zeta^2 dx\,dt \\
&\le k^m\int_{K_\rho}(u-k)_-\zeta^2(x,-\theta\rho^2)dx \\
&\quad + k^{2m}\iint_{Q_\rho^-(\theta)}\chi_{[u<k]}(\Phi_a+\Phi_b)dx\,dt \\
&\quad + 2k^{m+1}\iint_{Q_\rho^-(\theta)}\chi_{[u<k]}\zeta|\zeta_t|dx\,dt.
\end{aligned}$$

Since $m\in(0,1)$

$$\begin{aligned}
\iint_{Q_\rho^-(\theta)}&|D(u^m-k^m)_-|^2\zeta^2 dx\,dt \\
&\ge m^2k^{2(m-1)}\iint_{Q_\rho^-(\theta)}|D(u-k)_-|^2\zeta^2 dx\,dt.
\end{aligned}$$

Combining these estimates and dividing by mk^{m-1} yields

$$
\begin{aligned}
\operatorname*{ess\,sup}_{-\theta\rho^2\le t\le 0} \int_{K_\rho} &(u-k)_-^2\zeta^2(x,t)dx \\
&+ \frac{C_o m}{4} k^{m-1} \iint_{Q_\rho^-(\theta)} |D(u-k)_-|^2\zeta^2 dx\,dt \\
&\le \frac{1}{m} k \int_{K_\rho} (u-k)_-\zeta^2(x,-\theta\rho^2)dx \\
&+ \frac{1}{m} k^{m+1} \iint_{Q_\rho^-(\theta)} \chi_{[u<k]}(\Phi_a+\Phi_b)dx\,dt \\
&+ \frac{2}{m} k^2 \iint_{Q_\rho^-(\theta)} \chi_{[u<k]}\zeta|\zeta_t|dx\,dt.
\end{aligned}
$$

Recalling the definition of Φ_a and Φ_b proves the proposition. ■

Remark 9.1 The constant $\gamma = \gamma(m, N, C_o, C_1)$ is stable as $m \to 1$, but deteriorates as $m \to 0$.

Remark 9.2 The proof traces the dependence of the constant γ on the parameters $\{m, N, C_o, C_1\}$ and leaves explicit the dependence on C through the explicit expression of Φ.

10 A DeGiorgi-Type Lemma for Nonnegative Supersolutions to Singular Equations ($0 < m < 1$)

Local, weak sub(super)-solutions to the singular equations (5.1)–(5.2) for $0 < m < 1$, in E_T, are locally bounded above(below) in E_T (Proposition B.4.1 of Appendix B). For a cylinder $(y,s) + Q_{2\rho}^-(\theta) \subset E_T$ denote by $\mu_\pm$ and ω, numbers satisfying

$$
\mu_+ \ge \operatorname*{ess\,sup}_{[(y,s)+Q_{2\rho}^-(\theta)]} u, \qquad \mu_- \le \operatorname*{ess\,inf}_{[(y,s)+Q_{2\rho}^-(\theta)]} u, \qquad \omega = \mu_+ - \mu_-.
$$

Since the singularity occurs at $u = 0$, we will assume at the outset that

$$
\mu_- = \operatorname*{ess\,inf}_{(y,s)+Q_{2\rho}^-(\theta)} u = 0 \quad \text{so that} \quad \omega = \mu_+.
$$

Denote by ξ and a fixed numbers in $(0, 1)$.

Lemma 10.1 *Let u be a nonnegative, locally bounded, local, weak supersolution to the singular equation (5.1)–(5.2) for $0 < m < 1$, in E_T. There exists a positive number ν_-, depending on θ, ω, ξ, a and the data $\{m, N, C_o, C_1\}$, such that if*

$$
|[u \le \xi\omega] \cap [(y,s) + Q_{2\rho}^-(\theta)]| \le \nu_- |Q_{2\rho}^-(\theta)|, \tag{10.1}
$$

then either

$$C\rho > 1 \tag{10.2}$$

or

$$u \geq a\xi\omega \qquad a.e.\ in\ \ (y,s) + Q_\rho^-(\theta). \tag{10.3}$$

Remark 10.1 The constant ν_- is independent of C, and the latter enters into the statement only via the alternative (10.2).

Proof Assume $(y,s) = (0,0)$. Introduce the sequence of cubes $\{K_n\}$ and cylinders $\{Q_n\}$ as in (3.6) with $p = 2$ and a cutoff function on Q_n of the form $\zeta(x,t) = \zeta_1(x)\zeta_2(t)$ defined as in (3.8) for $p = 2$, and the sequence of truncating levels $\{k_n\}$ defined as in (3.7) with $\mu_- = 0$. Write down the energy estimates (9.1) on Q_n, for $(u - k_n)_-$. The first term on the right-hand side vanishes because of the choice of cutoff function ζ. Set

$$A_n = [u < k_n] \cap Q_n \qquad \text{and} \qquad Y_n = \frac{|A_n|}{|Q_n|}.$$

With this notation, and by virtue of the structure of the cutoff function ζ as defined in (3.8), the second term on the right-hand side of (9.1) is majorized by

$$\gamma \frac{2^n}{\theta\rho^2} k_n^2 |A_n|.$$

If (10.2) is violated, then

$$\Phi(C_o, C_1, C, \zeta, |D\zeta|) \leq \gamma \frac{2^{2n}}{\rho^2}.$$

Combining these remarks in (9.1), and taking into account that $k_n \leq \xi\omega$, gives

$$\begin{aligned} &\sup_{-\theta\rho_n^2 < t \leq 0} \int_{K_n} (u - k_n)_-^2 \zeta^2(x,t)dx \\ &\qquad + C_o(\xi\omega)^{m-1} \iint_{Q_n} |D[(u - k_n)_-\zeta]|^2 dx\, d\tau \\ &\qquad \leq \gamma \frac{2^{2n}}{\rho^2} (\xi\omega)^{m+1} \Big(1 + \frac{1}{\theta(\xi\omega)^{m-1}}\Big) |A_n|. \end{aligned} \tag{10.4}$$

Apply Hölder's inequality and the embedding Proposition 4.1 of the Preliminaries, and recall that $\zeta = 1$ on Q_{n+1}, to get

$$\begin{aligned} \Big(\frac{1-a}{2^{n+1}}\Big)^2 (\xi\omega)^2 |A_{n+1}| &\leq \iint_{Q_{n+1}} (u - k_n)_-^2\, dx\, d\tau \\ &\leq \Big(\iint_{Q_n} [(u - k_n)_-\zeta]^{2\frac{N+2}{N}} dx\, d\tau\Big)^{\frac{N}{N+2}} |A_n|^{\frac{2}{N+2}} \\ &\leq \gamma \Big(\iint_{Q_n} |D[(u - k_n)_-\zeta]|^2 dx\, d\tau\Big)^{\frac{N}{N+2}} \\ &\quad \times \Big(\sup_{-\theta\rho_n^2 < t \leq 0} \int_{K_n} [(u - k_n)_-\zeta]^2(x,t)dx\Big)^{\frac{2}{N+2}} |A_n|^{\frac{2}{N+2}} \end{aligned}$$

for a constant γ depending only on N. Combine this with (10.4) to get

$$|A_{n+1}| \le \gamma \frac{2^{4n}}{(1-a)^2\rho^2} (\xi\omega)^{\frac{2(m-1)}{N+2}} \Big(1 + \frac{1}{\theta(\xi\omega)^{m-1}}\Big) |A_n|^{1+\frac{2}{N+2}}.$$

In terms of Y_n this can be rewritten as

$$Y_{n+1} \le \gamma \frac{2^{4n}}{(1-a)^2} \frac{\big(1+\theta(\xi\omega)^{m-1}\big)}{\big(\theta(\xi\omega)^{m-1}\big)^{\frac{N}{N+2}}} Y_n^{1+\frac{2}{N+2}}.$$

By Lemma 5.1 of the Preliminaries, $\{Y_n\} \to 0$ as $n \to \infty$, provided

$$Y_o = \frac{|A_o|}{|Q_o|} \le \Big[\frac{(1-a)^2}{\gamma}\Big]^{\frac{N+2}{2}} 2^{-(N+2)^2} \frac{\big(\theta(\xi\omega)^{m-1}\big)^{\frac{N}{2}}}{\big(1+\theta(\xi\omega)^{m-1}\big)^{\frac{N+2}{2}}} \stackrel{\text{def}}{=} \nu_-. \qquad \blacksquare$$

For later use we rewrite the expression of ν_- for $0 < m < 1$, in a form that traces the functional dependence on the indicated parameters

$$\nu_- = \gamma^{-1}(1-a)^{N+2} \frac{[\theta(\xi\omega)^{m-1}]^{\frac{N}{2}}}{[1+\theta(\xi\omega)^{m-1}]^{\frac{N+2}{2}}} \tag{10.5}$$

for a quantitative constant $\gamma = \gamma(m, N, C_o, C_1) > 1$, independent of a and ξ.

Remark 10.2 In Lemma 10.1 the statement relative to (10.1)–(10.3) is given in terms of $\xi\omega$, assuming $\mu_- = 0$. As a matter of fact, as the proof clearly shows, when dealing with the lower truncations $(u-k)_-$ for *nonnegative* functions, all the estimates depend only on $k \ge 0$, without any further assumption on it. Correspondingly in (10.5) the quantity ν_- will depend on θk^{m-1}.

11 A Variant of DeGiorgi-Type Lemma, for Nonnegative Supersolutions to Singular Equations ($0 < m < 1$), Involving "Initial Data"

Continue to denote by $(y,s) + Q_\rho^+(\theta)$ "forward" cylinders with bottom center at (y,s) as defined in (2.1)–(2.2) with $p = 2$.

Assume now that some information is available on the "initial data" relative to the cylinder $(y,s) + Q_{2\rho}^+(\theta) \subset E_T$, say for example

$$u(x,s) \ge \xi M \qquad \text{for a.e. } x \in K_{2\rho}(y) \tag{11.1}$$

for some $M > 0$ and $\xi \in (0,1]$. Then

Lemma 11.1 *Let u be a nonnegative, locally bounded, local, weak supersolution to the singular equations (5.1)–(5.2) for $0 < m < 1$, in E_T. Let $a \in (0,1)$ and suppose that (11.1) holds true. Then there exists $\nu_o \in (0,1)$, depending only on a and the data, such that, if*

$$|[u \le \xi M] \cap Q^+_{2\rho}(\theta)| \le \frac{\nu_o}{\theta(\xi M)^{m-1}} |Q^+_{2\rho}(\theta)|, \tag{11.2}$$

then either

$$C\rho > 1 \tag{11.3}$$

or

$$u \ge a\xi M \qquad in \;\; K_\rho(y) \times (s, s + \theta(2\rho)^2].$$

Proof Assume $(y,s) = (0,0)$ and for $n = 0, 1, \ldots,$ construct sequences of cubes $\{K_n\}$ as in (3.6), and "forward" cylinders $\{Q^+_n\}$, and levels $\{\xi_n\}$ by

$$Q^+_n = K_n \times (0, \theta(2\rho)^2], \qquad \xi_n = a\xi + \frac{1-a}{2^n}\xi.$$

Let also $\zeta(x,t) = \zeta(x)$ be a nonnegative, piecewise smooth, cutoff function independent of t, vanishing outside K_n and satisfying the first of (3.8). Apply the energy estimates (9.1) for

$$(u - k_n)_- \qquad \text{with} \qquad k_n = \xi_n M,$$

over the "forward" cylinders Q^+_n and the indicated choice of ζ. Observe that the first integral on the right-hand side of (9.1) is extended over the "bottom" of Q^+_n, and it vanishes in view of (11.1). Also, the integral involving ζ_t vanishes, because of our choice of cutoff function ζ. The various terms can now be transformed and estimated exactly as in the proof of Lemma 10.1 with the obvious changes in the symbolism. The most noticeable change is that, due to the vanishing of ζ_t, all the terms containing the factor θ^{-1} are not present. This leads exactly to (10.4) over the cylinder Q^+_n, with ω replaced by M, and *without* the term in $(\cdots)$ containing the factor θ^{-1}. Setting

$$A^+_n = [u < \xi_n M] \cap Q^+_n \qquad \text{and} \qquad Y_n = \frac{|A^+_n|}{|Q^+_n|},$$

the estimate (10.4) with the indicated changes and in the current context, takes the form

$$\begin{aligned} \sup_{0<t<\theta(2\rho)^2} \int_{K_n} (u-k_n)^2_-(x,t)\zeta^2(x)dx & \\ + C_o(\xi M)^{m-1} \iint_{Q^+_n} |D[(u-k_n)_-\zeta]|^2 dx\, d\tau & \\ \le \gamma \frac{2^{2n}}{\rho^2} (\xi M)^{m+1} |A^+_n|. & \end{aligned}$$

Starting from this inequality, proceed now exactly as in the proof of Lemma 10.1 following (10.4), to arrive at

$$|A^+_{n+1}| \le \gamma \frac{2^{4n}}{(1-a)^2\rho^2} (\xi M)^{\frac{2(m-1)}{N+2}} |A^+_n|^{1+\frac{2}{N+2}}.$$

In terms of $Y_n = |A_n^+|/|Q_n^+|$ this can be rewritten as

$$Y_{n+1} \le \gamma \frac{2^{4n}}{(1-a)^2} \left[\theta(\xi M)^{m-1}\right]^{\frac{2}{N+2}} Y_n^{1+\frac{2}{N+2}}.$$

By Lemma 5.1 of the Preliminaries, $\{Y_n\} \to 0$ as $n \to \infty$, provided

$$\begin{aligned} Y_o = \frac{|A_o^+|}{|Q_o^+|} &= \frac{|[u < \xi M] \cap Q_{2\rho}^+(\theta)|}{|Q_{2\rho}^+(\theta)|} \\ &\le \left[\frac{(1-a)^2}{\gamma}\right]^{\frac{N+2}{2}} 2^{-(N+2)^2} \frac{1}{\theta(\xi M)^{m-1}} \\ &\stackrel{\text{def}}{=} \frac{\nu_o}{\theta(\xi M)^{m-1}}. \end{aligned}$$ ■

Remark 11.1 Both Lemmas 10.1 and 11.1 continue to hold for cylinders whose cross sections are balls.

Remark 11.2 Both Lemmas 10.1 and 11.1 are based on the energy estimates (9.1) and the embedding Proposition 4.1 of the Preliminaries, which continue to hold in a stable manner for $m \to 1$. Therefore these results are valid for all $0 < m < 1$, including a seamless transition from the singular range $m < 1$ to the nonsingular range $m = 1$. The various constants deteriorate as $m \to 0$.

Remark 11.3 The constant ν_o in (11.2) is independent of C, and the latter enters into the statement only via the alternative (11.3).

Remark 11.4 A result analogous to (7.4)–(7.5) holds for nonnegative subsolutions to these singular equations. The statement with the full proof will be given in § B.6 of Appendix B.

12 Remarks and Bibliographical Notes

Weak formulations such as (1.7) and (5.8) in terms of Steklov averages are in [101] for nondegenerate versions of (1.1)–(1.2) for $p = 2$, or (5.1)–(5.2) for $m = 1$. In the generality afforded by these equations, it is not expected that $u_t \in L^1_{\rm loc}(E_T)$. This, however, might occur for the homogeneous prototype equations (1.3) or (5.3). For local weak solutions to the homogeneous p-Laplacian equation (1.3) for $p > 2$, it is shown in [107] that $u_t \in L^{\frac{p}{p-1}}_{\rm loc}(E_T)$. In [28] it is shown that $u_t \in L^2_{\rm loc}(E_T)$ for all $\max\{\frac{3}{2}; \frac{2N}{N+2}\} < p \le 2$, extended in [2] for all $p > \frac{2N}{N+2}$ and in [22] for all $1 < p < 2$.

For the homogeneous, prototype porous medium equation (5.3), for $m > 1$, it is shown in [9] that for nonnegative solutions to the Cauchy problem $u_t \in L^1_{\rm loc}(\mathbb{R}^N \times \mathbb{R}^+)$. For $\frac{(N-2)_+}{N+2} < m < 1$ it is shown in [60] that nonnegative solutions to boundary value problems are locally analytic in the space variables and as a consequence $u_t \in L^\infty_{\rm loc}(E_T)$.

A more general notion of parabolicity can be given in terms of the monotonicity of the principal part with respect to the gradient Du. The function $\mathbb{R}^N \ni \eta \to \mathbf{A}(x,t,u,\eta)$ is monotone at $(x,t,u) \in E_T \times \mathbb{R}$, if

$$\big\langle \mathbf{A}(x,t,u,\eta_1) - \mathbf{A}(x,t,u,\eta_2), \eta_1 - \eta_2 \big\rangle \ge 0 \quad \text{for all } \ \eta_1, \eta_2 \in \mathbb{R}^N. \tag{12.1}$$

Then (1.1)–(1.2) is parabolic if $\mathbf{A}(x,t,u,0) = 0$ and $\eta \to \mathbf{A}(x,t,u,\eta)$ is monotone at all $(x,t,u) \in E_T \times \mathbb{R}$. The condition (1.8) requires only the monotonicity at $\eta = 0$. The monotonicity requirement (12.1) is natural in the existence theory. It permits one to apply Minty's lemma [116] to identify the weak limit of the principal part when (1.1) is approximated by a sequence of regularized problems ([101]). We will consider singular homogeneous equations with monotone principal part in Chapter 7.

The energy inequalities of § 2 and § 6 for the truncations $(u-k)_\pm$ are modeled after analogous ones for nondegenerate equations appearing in [101]. These in turn are parabolic version of analogous "elliptic" energy estimates for such truncations introduced by DeGiorgi [36], following Bernstein [17]. The main difference is in a careful tracking of the space-time geometry to be accounted for in the method of intrinsic geometry introduced in [41] and [149].

Analogous considerations hold for Lemmas 3.1–4.1, 7.1–8.1, and 10.1–11.1. In these a careful analysis is effected to trace the connection between degeneracy or singularity and the geometry of the cylinders $Q^\pm_\rho(\theta)$. The parameter θ will track the degeneracy and/or singularity of these equations. Versions of these lemmas appear in [41, 37, 47].

4

Expansion of Positivity

1 Time and Space Propagation of Positivity

The *Expansion of Positivity* is a property of nonnegative supersolutions to elliptic and parabolic partial differential equations, that is at the heart of any form of Harnack estimate. Roughly speaking, it asserts that information on the *measure* of the "positivity set" of u at the time level s, over the cube $K_\rho(y)$, translates into an expansion of the positivity set both in space (from a cube $K_\rho(y)$ to $K_{2\rho}(y)$), and in time (from s to $s+\theta\rho^2$, for some suitable θ).

Such an expansion involves some unavoidable technical arguments. To convey the main ideas we will present it first in § 2 in the context of nondegenerate ($p=2$ or $m=1$), homogeneous equations. Then we will present it separately for degenerate ($p>2$ or $m>1$) and singular ($1<p<2$ or $0<m<1$) equations with full quasilinear structure. In all cases one first "propagates" a positivity information at some time level s on a cube $K_\rho(y)$ to further times, within the same cube. Then one expands the positivity set in the space variables from $K_\rho(y)$ to $K_{2\rho}(y)$.

The first step of time propagation of positivity is technically common to all cases and we present it here in a unified fashion.

Henceforth in this section assume that u is a nonnegative, local, weak supersolution in E_T to (1.1)–(1.2) of Chapter 3, for some $p>1$.

Most of our arguments and proofs are based on the energy estimates and DeGiorgi-type lemmas of § 2–4 of Chapter 3. According to the discussion in § 1.3 and Remarks 2.2, 3.1, and 4.3 of Chapter 3, a constant γ depends only on the data if it can be quantitatively determined a priori only in terms of $\{p,N,C_o,C_1\}$. The constant C appearing in the structure conditions (1.2) of Chapter 3, enters in the various statements only via some alternatives.

For $(y,s)\in E_T$ and $n,m\in\mathbb{N}$, introduce the "forward" and "backward" cylinders

$$(y,s)+\mathcal{Q}^+_{n\rho}(m\theta)=K_{n\rho}(y)\times(s,s+m\theta\rho^p]$$
$$(y,s)+\mathcal{Q}^-_{n\rho}(m\theta)=K_{n\rho}(y)\times(s-m\theta\rho^p,s].$$

E. DiBenedetto et al., *Harnack's Inequality for Degenerate and Singular Parabolic Equations*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-1584-8_4,

These differ from the cylinders $Q_\rho^\pm(\theta)$ introduced in (2.1)–(2.2) of Chapter 3, in that their cross section $K_{n\rho}(y)$ and their height $\theta m\rho^p$ are permitted to vary independently. In what follows it will be assumed that $(y,s)\in E_T$ and $\rho>0$ are such that $(y,s)+\mathcal{Q}_{n\rho}^\pm(m\theta)\subset E_T$.

Lemma 1.1 *Assume that for some $(y,s)\in E_T$ and some $\rho>0$*

$$\big|[u(\cdot,s)\ge M]\cap K_\rho(y)\big|\ge\alpha\big|K_\rho(y)\big|$$

for some $M>0$ and some $\alpha\in(0,1)$. There exist δ and ϵ in $(0,1)$, depending only on the data $\{p,N,C_o,C_1\}$, and α, and independent of M, such that either

$$C\rho>\min\{1\,,\,M\}$$

or

$$|[u(\cdot,t)>\epsilon M]\cap K_\rho(y)|\ge\tfrac{1}{2}\alpha|K_\rho|\quad \textit{for all}\ \ t\in\Big(s,s+\frac{\delta\rho^p}{M^{p-2}}\Big]. \tag{1.1}$$

Proof Assume $(y,s)=(0,0)$ and for $k>0$ and $t>0$ set

$$A_{k,\rho}(t)=[u(\cdot,t)<k]\cap K_\rho.$$

The assumption implies

$$|A_{M,\rho}(0)|\le(1-\alpha)|K_\rho|. \tag{1.2}$$

Write down the energy inequalities (2.3) of Chapter 3, for the truncated functions $(u-M)_-$, over the cylinder $K_\rho\times(0,\theta\rho^p]$, where $\theta>0$ is to be chosen. The cutoff function ζ is taken independent of t, nonnegative, and such that

$$\zeta=1\quad\text{on}\ \ K_{(1-\sigma)\rho},\qquad\text{and}\qquad|D\zeta|\le\frac{1}{\sigma\rho}$$

where $\sigma\in(0,1)$ is to be chosen. Discarding the nonnegative term containing $D(u-M)_-$ on the left-hand side, these inequalities yield

$$\begin{aligned}\int_{K_{(1-\sigma)\rho}}(u-M)_-^2(x,t)dx&\le\int_{K_\rho}(u-M)_-^2(x,0)dx\\&\quad+\frac{\gamma}{(\sigma\rho)^p}\int_0^{\theta\rho^p}\!\!\int_{K_\rho}(u-M)_-^p\,dx\,d\tau\\&\quad+\gamma C^p\int_0^{\theta\rho^p}\!\!\int_{K_\rho}\big[\chi_{[u<M]}+(u-M)_-^p\big]dx\,d\tau\\&\le M^2\Big[(1-\alpha)+\gamma\frac{\theta M^{p-2}}{\sigma^p}+\gamma\Big(\frac{C\rho}{\min\{1\,,\,M\}}\Big)^p\theta M^{p-2}\Big]|K_\rho|\\&\le M^2\Big[(1-\alpha)+2\gamma\frac{\theta M^{p-2}}{\sigma^p}\Big]|K_\rho|\end{aligned}$$

for all $t \in (0, \theta\rho^p]$, where we have enforced (1.2), and provided that $C\rho < M$, $C < \rho^{-1}$. The left-hand side is estimated below by

$$\begin{aligned}\int_{K_{(1-\sigma)\rho}} (u-M)_-^2(x,t)dx &\ge \int_{K_{(1-\sigma)\rho}\cap[u<\epsilon M]} (u-M)_-^2(x,t)dx \\ &\ge M^2(1-\epsilon)^2|A_{\epsilon M,(1-\sigma)\rho}(t)|\end{aligned}$$

where $\epsilon \in (0,1)$ is to be chosen. Next estimate

$$\begin{aligned}|A_{\epsilon M,\rho}(t)| &= |A_{\epsilon M,(1-\sigma)\rho}(t) \cup (A_{\epsilon M,\rho}(t) - A_{\epsilon M,(1-\sigma)\rho}(t))| \\ &\le |A_{\epsilon M,(1-\sigma)\rho}(t)| + |K_\rho - K_{(1-\sigma)\rho}| \\ &\le |A_{\epsilon M,(1-\sigma)\rho}(t)| + N\sigma|K_\rho|.\end{aligned}$$

Combining these estimates gives

$$\begin{aligned}|A_{\epsilon M,\rho}(t)| &\le \frac{1}{M^2(1-\epsilon)^2}\int_{K_{(1-\sigma)\rho}} (u-M)_-^2(x,t)dx + N\sigma|K_\rho| \\ &\le \frac{1}{(1-\epsilon)^2}\Big[(1-\alpha) + \frac{2\gamma}{\sigma^p}\theta M^{p-2} + N\sigma\Big]|K_\rho|.\end{aligned}$$

Choose $\theta = \delta M^{2-p}$ and then choose

$$\sigma = \frac{\alpha}{8N}, \qquad \epsilon \le 1 - \frac{\sqrt{1-\frac{3}{4}\alpha}}{\sqrt{1-\frac{1}{2}\alpha}} \approx \frac{1}{8}\alpha, \qquad \delta = \frac{\alpha^{p+1}}{2^{3p+4}\gamma N^p}. \tag{1.3}$$

This proves the lemma. ■

Remark 1.1 The proof is based on the energy inequalities (2.3) of Chapter 3, whose constant dependence is indicated in Remark 2.1. Therefore the constant $\delta = \delta(p)$ deteriorates either as $p \to 1$ or as $p \to \infty$, but it is stable as $p \to 2$, with seamless transition from the singular range $p < 2$ to the degenerate range $p > 2$.

Remark 1.2 If $p = 2$, one takes $\theta = \delta$ and the interval in (1.1) becomes independent of M.

2 The Expansion of Positivity for Nondegenerate, Homogeneous, Quasilinear Parabolic Equations

Let u be a nonnegative, local, weak supersolution in E_T to (1.1)–(1.2) of Chapter 3, with $p = 2$ and $C = 0$.

Proposition 2.1 *Assume that for some* $(y,s) \in E_T$ *and some* $\rho > 0$

$$|[u(\cdot,s) \ge M] \cap K_\rho(y)| \ge \alpha\big|K_\rho(y)\big| \tag{2.1}$$

for some $M > 0$ and some $\alpha \in (0,1)$. There exist constants η and $\delta \in (0,1)$ depending only on the data $\{N, C_o, C_1\}$, and α, such that

$$u \geq \eta M \qquad a.e.\ in\ \ K_{2\rho}(y) \times \big(s + \tfrac{1}{2}\delta\rho^2\,,\, s + \delta\rho^2\big]. \tag{2.2}$$

Proof Assume $(y,s) = (0,0)$. The number $\alpha > 0$ being fixed, let δ and ϵ be the numbers claimed by Lemma 1.1 for $p = 2$. The conclusion of the lemma implies that

$$|[u(\cdot,t) > \epsilon M] \cap K_{4\rho}| > \tfrac{1}{2}\alpha 4^{-N}|K_{4\rho}|, \qquad \text{for all}\ \ t \in (0, \delta\rho^2). \tag{2.3}$$

Lemma 2.1 *For every $\nu \in (0,1)$ there exists ϵ_ν depending only on the data $\{N, C_o, C_1\}$, δ (and hence α), and ν, such that*

$$\big|[u < \epsilon_\nu M] \cap \mathcal{Q}^+_{4\rho}(\delta)\big| < \nu\big|\mathcal{Q}^+_{4\rho}(\delta)\big|. \tag{2.4}$$

Thus the set $[u < \epsilon_\nu M]$ in the cylinder $\mathcal{Q}^+_{4\rho}(\delta)$ can be made arbitrarily small, provided ϵ_ν is chosen accordingly. The main tools of the proof are the estimate (2.3) of the measure of the sets $A_{\epsilon M, 4\rho}(t)$ for all $t \in (0, \delta\rho^2)$, and the discrete isoperimetric inequality of Lemma 2.2 of the Preliminaries.

Proof Write down the energy estimates (2.3) of Chapter 3 over the cylinder

$$\mathcal{Q}^+_{8\rho}(\delta) \cup \mathcal{Q}^-_{8\rho}(\delta) = K_{8\rho} \times (-\delta\rho^2, \delta\rho^2)$$

for the truncated functions

$$(u - k_j)_- \qquad \text{for the levels} \qquad k_j = \frac{1}{2^j}\epsilon M, \qquad \text{for}\ \ j = 0, 1, \dots.$$

The nonnegative, piecewise smooth, test function ζ is chosen so that it vanishes outside $K_{8\rho}$ and for $t \leq -\delta\rho^2$, and

$$\zeta = 1\ \ \text{on}\ \ \mathcal{Q}^+_{4\rho}(\delta), \qquad |D\zeta| \leq \frac{1}{4\rho}, \qquad \text{and}\ \ 0 \leq \zeta_t \leq \frac{1}{\delta\rho^2}.$$

The first term on the left-hand side is discarded since it is nonnegative, and the second vanishes because of our choice of test function. The term involving $|D(u - k_j)_-|$ is minorized by extending the integration over the cylinder $\mathcal{Q}^+_{4\rho}(\delta)$, which is the set where $\zeta = 1$. The terms containing C on the right-hand side are eliminated since $C = 0$. These remarks give the inequalities

$$\begin{aligned}
&\iint_{\mathcal{Q}^+_{4\rho}(\delta)} |D(u - k_j)_-|^2 \zeta^2 dx\, d\tau \\
&\qquad \leq \gamma \int_{-\delta\rho^2}^{\delta\rho^2} \int_{K_{8\rho}} (u - k_j)^2_- \big(|D\zeta|^2 + \zeta_\tau\big) dx\, d\tau \\
&\qquad \leq \gamma\delta\rho^2 k_j^2 \Big(\frac{1}{\rho^2} + \frac{2}{\delta\rho^2}\Big)|K_{8\rho}| \\
&\qquad \leq \gamma k_j^2 |K_{4\rho}|
\end{aligned} \tag{2.5}$$

for a new constant γ depending only on the data $\{N, C_o, C_1\}$.

Apply the discrete isoperimetric inequality of Lemma 2.2 of the Preliminaries to the levels

$$\ell = k_j = \frac{\epsilon h}{2^j} \quad \text{and} \quad k = k_{j+1} = \frac{\epsilon h}{2^{j+1}} \quad \text{for } j = 0, 1, \dots$$

and take into account (2.3) to obtain

$$k_{j+1}\big|A_{k_{j+1},4\rho}(t)\big| \le \frac{8^N\gamma}{\alpha}\rho \int_{K_{4\rho}\cap[k_{j+1}<u<k_j]} |Du(\cdot,t)|dx.$$

Integrate this in dt over $(0, \delta\rho^2)$ and set

$$|A_j| = |[u < k_j] \cap \mathcal{Q}^+_{4\rho}(\delta)| = \int_0^{\delta\rho^2} \big|A_{k_j}(\tau)\big|d\tau.$$

Then the previous inequality yields

$$\begin{aligned} k_{j+1}\big|A_{j+1}\big| &\le \gamma\rho \iint_{\mathcal{Q}^+_{4\rho}(\delta)\cap[k_{j+1}<u<k_j]} |Du|dx\,d\tau \\ &\le \gamma\rho\Big(\iint_{\mathcal{Q}^+_{4\rho}(\delta)} |D(u-k_j)_-|^2 dx\,d\tau\Big)^{\frac12}\big|A_j - A_{j+1}\big|^{\frac12} \\ &\le \gamma k_j\sqrt{\mathcal{Q}^+_{4\rho}(\delta)}\,\big(|A_j| - |A_{j+1}|\big)^{\frac12} \end{aligned}$$

where we have used the energy estimates (2.5). Next divide by $k_{j+1} = \frac12 k_j$, and square both sides to obtain the recursive inequalities

$$|A_{j+1}|^2 \le (2\gamma)^2|\mathcal{Q}^+_{4\rho}(\delta)|\,\big(|A_j| - |A_{j+1}|\big) \quad \text{for } j = 0, 1 \dots.$$

Add these inequalities for $j = 0, 1, \dots, j_*-1$ where j_* is a positive integer to be chosen. Minorize the terms on the left-hand side by their smallest value $|A_{j_*}|^2$ and majorize the right-hand side with the corresponding telescopic series. The indicated estimations yield

$$\begin{aligned} j_*|A_{j_*}|^2 \le \sum_{j=0}^{j_*-1}|A_{j+1}|^2 &\le (2\gamma)^2|\mathcal{Q}^+_{4\rho}(\delta)|\sum_{j=0}^{\infty}\big(|A_j| - |A_{j+1}|\big) \\ &\le (2\gamma)^2|\mathcal{Q}^+_{4\rho}(\delta)|^2. \end{aligned}$$

From this

$$|A_{j_*}| \le \frac{2\gamma}{\sqrt{j_*}}\,|\mathcal{Q}^+_{4\rho}(\delta)|. \tag{2.6}$$

Thus having fixed $\nu \in (0,1)$, one can choose j_* so large that

$$\frac{\big|[u < \epsilon_\nu M] \cap \mathcal{Q}^+_{4\rho}(\delta)\big|}{|\mathcal{Q}^+_{4\rho}(\delta)|} < \nu, \quad \text{for } \frac{2\gamma}{\sqrt{j_*}} \le \nu, \quad \text{and} \quad \epsilon_\nu = \frac{\epsilon}{2^{j_*}}. \qquad \blacksquare$$

Proof (of Proposition 2.1, Concluded) Apply Lemma 3.1 of Chapter 3 over the cylinder $\mathcal{Q}_{4\rho}^{+}(\delta)$ in the version of (3.1)–(3.3), with $\mu_{-}=0$ and $\xi\omega=\epsilon_{\nu}M$ and $a=\frac{1}{2}$. Choose ν from (3.12) of Chapter 3 and observe that since $p=2$ (nondegenerate equations), the number ν is independent of $\epsilon_{\nu}M$. It only depends on the data $\{N,C_o,C_1\}$ and δ, which itself has been determined and fixed in terms of the data $\{N,C_o,C_1\}$ and α. Such a ν being fixed a priori only in terms of the data, choose $j_*\in\mathbb{N}$ by the indicated procedure, so that the assumptions of Lemma 2.1 are verified. Then Lemma 3.1 of Chapter 3 implies that

$$u(x,t)>\tfrac{1}{2}\epsilon_{\nu}M \qquad \text{a.e. in } K_{2\rho}\times(\tfrac{1}{2}\delta\rho^2,\delta\rho^2).$$

Thus the conclusion holds with $\eta=\frac{1}{2}\epsilon_{\nu}$. ■

Remark 2.1 If in (2.1) one has $\alpha=1$, the condition reads

$$u(\cdot,s)\ge M \qquad \text{a.e. in } K_{\rho}(s) \tag{2.7}$$

which is of the same form as the "initial datum" of (4.1) of Chapter 3. Lemma 4.1 of Chapter 3 then translates that bound below to later times over *smaller cubes.* Proposition 2.1, however, is stronger, as it translates such "initial conditions" into a positivity information at later times and over a *larger cube.*

3 Some Counterexamples for Degenerate and Singular Equations

Let now u be a nonnegative, local, weak supersolution to the prototype equation (1.3) of Chapter 3 in some cylindrical domain E_T, for some $p\neq 2$. If u is bounded below on some cube $K_\rho(y)$, say for example as in (2.7), then the analog of Proposition 2.1 would be that

$$u(\cdot,s+\delta\rho^p)\ge\eta M \qquad \text{a.e. in } K_{2\rho}(y) \tag{3.1}$$

for constants $\delta>0$ and $\eta\in(0,1)$ depending only on the data $\{p,N,C_o,C_1\}$, and independent of u. It turns out that if $p\neq 2$, no constants δ and η can be determined a priori only in terms of N and p for which (2.7) would imply (3.1).

3.1 A First Counterexample for $p>2$

Consider the one-parameter family of nonnegative functions defined in the whole $\mathbb{R}\times\mathbb{R}$

$$u(x,t;c)=A(1-x+ct)_+^{\frac{p-1}{p-2}} \qquad \text{where } A=c^{\frac{1}{p-2}}\Big(\frac{p-2}{p-1}\Big)^{\frac{p-1}{p-2}}. \tag{3.2}$$

One verifies that such a $u(\cdot,\cdot;c)$ is a weak solution to the homogeneous prototype p-Laplacian equation in the whole $\mathbb{R}\times\mathbb{R}$, for all $c>0$, and is constructed by seeking solutions in the form of traveling waves. Fix

$$(y,s)=\big(\tfrac{1}{2}(1-\varepsilon),0\big),\qquad \rho=\tfrac{1}{2}(1-\varepsilon)$$

and let

$$K_\rho(y)=\big\{|x-\tfrac{1}{2}(1-\varepsilon)|<\tfrac{1}{2}(1-\varepsilon)\big\}.$$

At time $\delta\rho^p$ the bound below (3.1) is possible for some $\eta>0$, however small, only if

$$\delta>\frac{2^p}{c}\,\frac{1-3\varepsilon}{(1-\varepsilon)^p}.$$

Thus for (3.1) to hold for some η, the constant δ must depend on the parameter c, and hence on the solution $u(\cdot,\cdot;c)$.

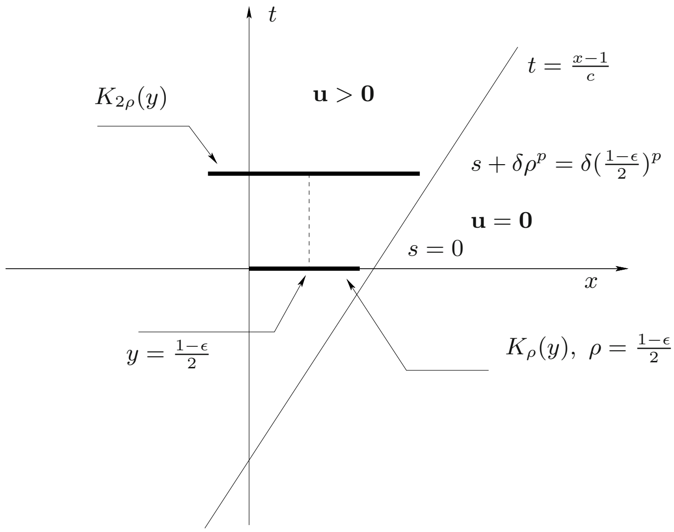

Fig. 3.1. The Traveling Wave Solution

3.2 A Second Counterexample for $p>2$

Consider the Barenblatt solution to the parabolic p-Laplacian equation for $p>2$ in $\mathbb{R}^N\times\mathbb{R}^+$ ([13]):

$$\Gamma_p(x;t)=\frac{1}{t^{N/\lambda}}\Big[1-\gamma_p\Big(\frac{|x|}{t^{1/\lambda}}\Big)^{\frac{p}{p-1}}\Big]_+^{\frac{p-1}{p-2}}\qquad t>0 \tag{3.3}$$

where

$$\gamma_p = \left(\frac{1}{\lambda}\right)^{\frac{1}{p-1}} \frac{p-2}{p}, \qquad \lambda = N(p-2) + p. \tag{3.4}$$

The moving boundary is the sphere centered at the origin and radius $R_m(t)$ given by

$$R_m(t) = \gamma_p^{\frac{1-p}{p}} t^{\frac{1}{\lambda}}.$$

For fixed $\varepsilon > 0$ and $s > 0$ let

$$\rho_1 = \left(\frac{1-3\varepsilon}{\gamma_p}\right)^{\frac{p-1}{p}} s^{\frac{1}{\lambda}}, \qquad \rho_2 = \left(\frac{1-\varepsilon}{\gamma_p}\right)^{\frac{p-1}{p}} s^{\frac{1}{\lambda}}$$

and set

$$\rho = \frac{\rho_2 - \rho_1}{2} = \frac{(1-\varepsilon)^{\frac{p-1}{p}} - (1-3\varepsilon)^{\frac{p-1}{p}}}{2\gamma_p^{\frac{p-1}{p}}} s^{\frac{1}{\lambda}},$$

$$|y| = \frac{\rho_2 + \rho_1}{2} = \frac{(1-\varepsilon)^{\frac{p-1}{p}} + (1-3\varepsilon)^{\frac{p-1}{p}}}{2\gamma_p^{\frac{p-1}{p}}} s^{\frac{1}{\lambda}}.$$

One verifies that

$$u(\cdot, s) \ge \frac{1}{s^{\frac{N}{\lambda}}} \varepsilon^{\frac{p-1}{p-2}} \quad \text{in } B_\rho(y).$$

If the expansion of positivity (3.1) were to hold for some $\delta > 0$ depending only on N and p, then points on the ball $B_{2\rho}(y)$, at time $s + \delta\rho^p$ should be within the support of $u(\cdot, s + \delta\rho^p)$. That is,

$$|y| + 2\rho < R_m(s + \delta\rho^p).$$

From this and the expression of $R_m(\cdot)$ one computes

$$\delta > \frac{1}{2^{N(p-2)}\gamma_p^{p-1}} \frac{3(1-\varepsilon)^{\frac{p-1}{p}} - (1-3\varepsilon)^{\frac{p-1}{p}}}{(1-\varepsilon)^{\frac{p-1}{p}} - (1-3\varepsilon)^{\frac{p-1}{p}}} s^{\frac{N(p-2)}{\lambda}}.$$

If ε is sufficiently small, the right-hand side is a positive factor of $s^{N(p-2)/\lambda}$, and hence δ grows with s.

3.3 A Family of Counterexamples for $1 < p < 2$

When $1 < p < 2$, nonnegative solutions to the prototype equation (1.3) of Chapter 3 in some cylindrical domain E_T, might vanish identically in finite time. That is, there might exist a finite $T > 0$ such that

$$u(\cdot, t) = 0 \quad \text{in } E \quad \text{for all } t \ge T.$$

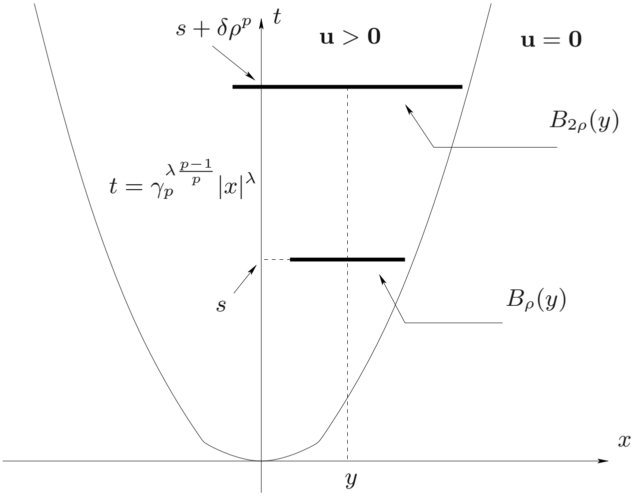

Fig. 3.2. The Barenblatt Solution

If E is a bounded domain with smooth boundary ∂E and u is the solution to the initial-boundary value problem, with bounded initial data and homogeneous Dirichlet data on ∂E, this extinction phenomenon occurs for all $1 < p < 2$ and the extinction time T can be estimated in terms of the initial datum ([41], Chapter VII § 2, and also [60]).

If $E = \mathbb{R}^N$ and u is the solution to the Cauchy problem with smooth and compactly supported initial datum, this phenomenon occurs for $1 < p < \frac{2N}{N+1}$ ([41], Chapter VII § 3, and also [60]).

It is apparent that for a cylinder $K_\rho(y) \times (s, s + \delta\rho^p)$ such that $u(\cdot, s) > 0$ on $K_\rho(y)$, the expansion of (3.1) does not hold true if $s + \delta\rho^p$ exceeds the extinction time T.

If $N = 1$, a family of such solutions can be constructed semi-explicitly, by separation of variables. Consider the boundary value problem

$$\begin{aligned} &u_t - (|u_x|^{p-2} u_x)_x = 0 \quad \text{in } [|x| < 1] \times [t > 0] \\ &u(-1, t) = u(1, t) = 0 \\ &u(\cdot, 0) = T^{\frac{1}{2-p}} X(\cdot; \mu) \end{aligned} \tag{3.5}$$

where $X(\cdot)$ is a nonnegative solution to

$$\begin{aligned} &-(|X'|^{p-2} X')' = \mu X \quad \text{in } (0, 1) \\ &X(-1) = X(1) = 0, \end{aligned} \tag{3.6}$$

for some $\mu > 0$. Whence such an $X(\cdot)$ is constructed, a solution to (3.5) is

$$u(x, t) = \big[T - (2 - p)\mu t\big]^{\frac{1}{2-p}} X(x; \mu).$$

A construction procedure for nonnegative solutions to (3.6) is in § 8.1.

3.4 The Expansion of Positivity in Some Intrinsic Geometry

These examples raise the natural question, whether a version of the expansion of positivity still holds, in some form, for supersolutions to equations (1.1)–(1.2), and (5.1)–(5.2) of Chapter 3 for $p \neq 2$, or for $m \neq 1$. Such a result would pave the way to a Harnack inequality when $p \neq 2$, or $m \neq 1$.

It turns out that the expansion of positivity continues to hold for these degenerate and singular equations, but in a time-intrinsic geometry.

In the next sections we make precise the notion of *intrinsic geometry* and state and prove the expansion of positivity in such a geometry, respectively for degenerate equations ($p > 2$ or $m > 1$) and singular equations ($1 < p < 2$ or $0 < m < 1$).

4 The Expansion of Positivity for Degenerate Quasilinear Parabolic Equations ($p > 2$)

Throughout this section let u be a nonnegative, local, weak supersolution to (1.1)–(1.2) of Chapter 3 in E_T, for $p > 2$. For $(y, s) \in E_T$, and some given positive number M, consider the cylinder

$$K_{8\rho}(y) \times (s, s + \frac{b^{p-2}}{(\eta M)^{p-2}}\delta\rho^p], \tag{4.1}$$

where b, η, δ are the constants given by Proposition 4.1, and $\rho > 0$ is so small that it is included in E_T.

Proposition 4.1 *Assume that for some $(y, s) \in E_T$ and some $\rho > 0$*

$$\big|[u(\cdot, s) \geq M] \cap K_\rho(y)\big| \geq \alpha\big|K_\rho(y)\big| \tag{4.2}$$

for some $M > 0$ and some $\alpha \in (0, 1)$. There exist constants η and δ in $(0, 1)$ and $\gamma, b > 1$ depending only on the data $\{p, N, C_o, C_1\}$, and α, such that either $\gamma C\rho > \min\{1\,, M\}$, or

$$u(\cdot, t) \geq \eta M \qquad a.e.\ in\ \ K_{2\rho}(y) \tag{4.3}$$

for all times

$$s + \frac{b^{p-2}}{(\eta M)^{p-2}}\tfrac{1}{2}\delta\rho^p \leq t \leq s + \frac{b^{p-2}}{(\eta M)^{p-2}}\delta\rho^p. \tag{4.4}$$

Remark 4.1 The cylinder in (4.1) is “intrinsic” to the supersolution itself, since its height depends on the lower bound M in (4.2). The conclusion (4.3)–(4.4) is analogous to the conclusion (2.2) of Proposition 2.1, except that the time is rescaled by a factor $(\eta M)^{2-p}$. In this sense Proposition 4.1 is an “intrinsic” expansion of positivity.

Remark 4.2 The constants η, δ, γ, and b are stable as $p \to 2$ and therefore the statement of Proposition 2.1, valid for the nondegenerate case $p = 2$, can be recovered from Proposition 4.1 by letting $p \to 2$. This stability of γ, η, and b will be established in § 6.

Remark 4.3 The proposition transforms the measure-theoretical information (4.2) into the pointwise expansion of positivity (4.3). The proof below shows that the functional dependence of η on the measure-theoretical parameter α is of the form

$$\eta = \eta_o \alpha B^{-\frac{1}{\alpha^d}} \tag{4.5}$$

for parameters η_o, B, d depending only on the data $\{p, N, C_o, C_1\}$. Such a dependence will be improved in Proposition 7.1 of Chapter 5.

4.1 Structure of the Proof

Assume $(y, s) = (0, 0)$ and let ϵ and δ be the numbers claimed by Lemma 1.1.

Following the proof for the nondegenerate case $p = 2$, one seeks to convert the information (1.1) originating from Lemma 1.1, into an estimate of the type of (2.4) of Lemma 2.1. The proof could then be concluded, as in the nondegenerate case, by an application of Lemma 3.1 of Chapter 3. The conclusion of this lemma holds, provided the number ν can be chosen so small as in (3.12) of Chapter 3 with ω replaced by $\epsilon_\nu M$. If $p = 2$, such a choice can be made independent of $(\epsilon_\nu M)$. If $p > 2$, the number ν can be determined in terms only of the data if θ is chosen to satisfy $\theta(\epsilon_\nu M)^{p-2} = 1$. Thus the smaller is ϵ_ν the longer is the cylinder $Q^+_{4\rho}(\theta)$. Therefore an information of the form of (1.1) would need to be derived over a large cylinder.

This is precisely the main difficulty of the proof. It is overcome by introducing a suitable change of the time variable, and the function u for which a version of (1.1) continues to hold over "large times."

4.2 Changing the Time Variable

We may assume $(y, s) = (0, 0)$. The assumption (4.2) implies

$$|[u(\cdot, 0) \ge \sigma M] \cap K_\rho| \ge \alpha |K_\rho| \quad \text{for all } \sigma \le 1. \tag*{(4.2)$'$}$$

The conclusion of Lemma 1.1 continues to hold, with the same parameters ϵ and δ, if one replaces M by σM, and yields

$$\Big|\Big[u\Big(\cdot, \frac{\delta \rho^p}{(\sigma M)^{p-2}}\Big) \ge \epsilon \sigma M\Big] \bigcap K_\rho\Big| \ge \tfrac{1}{2}\alpha |K_\rho| \quad \text{for all } \sigma \le 1.$$

For $\tau \ge 0$ set

$$\sigma_\tau = e^{-\frac{\tau}{p-2}} \tag{4.6}$$

and

$$w(x,\tau) \stackrel{\text{def}}{=} \frac{e^{\frac{\tau}{p-2}}}{M}(\delta\rho^p)^{\frac{1}{p-2}}\, u\Big(x, \frac{e^\tau}{M^{p-2}}\delta\rho^p\Big). \tag{4.7}$$

Then for all $\tau \ge 0$

$$\Big|\Big[u\Big(\cdot, \frac{e^\tau}{M^{p-2}}\,\delta\rho^p\Big) \ge \epsilon\frac{M}{e^{\frac{\tau}{p-2}}}\Big] \bigcap K_\rho\Big| \ge \tfrac{1}{2}\alpha|K_\rho|$$

which, in terms of $w(\cdot,\tau)$, means

$$|[w(\cdot,\tau) \ge k_o] \cap K_\rho| \ge \tfrac{1}{2}\alpha|K_\rho| \qquad \text{for all } \ \tau > 0,$$

where

$$k_o \stackrel{\text{def}}{=} \epsilon(\delta\rho^p)^{\frac{1}{p-2}}. \tag{4.8}$$

From this

$$|K_{4\rho} - [w(\cdot,\tau) < k_o]| \ge \tfrac{1}{2}\alpha 4^{-N}|K_{4\rho}| \qquad \text{for all } \ \tau > 0. \tag{4.9}$$

4.2.1 Relating w to the Evolution Equation

Since $u \ge 0$, by formal calculations

$$\begin{aligned} w_\tau &= \Big(\frac{e^{\frac{\tau}{p-2}}}{M}(\delta\rho^p)^{\frac{1}{p-2}}\Big)^{p-1} u_t + \frac{1}{p-2}\,\frac{e^{\frac{\tau}{p-2}}}{M}(\delta\rho^p)^{\frac{1}{p-2}}\, u \\ &\ge \Big(\frac{e^{\frac{\tau}{p-2}}}{M}(\delta\rho^p)^{\frac{1}{p-2}}\Big)^{p-1}\big[\operatorname{div}\mathbf{A}(x,t,u,Du) + B(x,t,u,Du)\big] \\ &= \operatorname{div}\tilde{\mathbf{A}}(x,\tau,w,Dw) + \tilde{B}(x,\tau,w,Dw) \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} &\tilde{\mathbf{A}} : (E \times \mathbb{R}^+) \times \mathbb{R}^{N+1} \to \mathbb{R}^N \\ &\tilde{B} : (E \times \mathbb{R}^+) \times \mathbb{R}^{N+1} \to \mathbb{R} \end{aligned}$$

satisfy the structure conditions

$$\begin{cases} \tilde{\mathbf{A}}(x,\tau,w,Dw)\cdot Dw \ge C_o|Dw|^p - \tilde{C}^p \\ |\tilde{\mathbf{A}}(x,\tau,w,Dw)| \le C_1|Dw|^{p-1} + \tilde{C}^{p-1} \\ |\tilde{B}(x,\tau,w,Dw)| \le C|Dw|^{p-1} + C\tilde{C}^{p-1} \end{cases} \qquad \text{a.e. in } \ E \times \mathbb{R}^+,$$

where C_o, C_1, and C are the constants appearing in the structure conditions (1.2) of Chapter 3, and

$$\tilde{C}(\tau) = C\,\frac{e^{\frac{\tau}{p-2}}}{M}(\delta\rho^p)^{\frac{1}{p-2}}. \tag{4.11}$$

The formal differential inequality (4.10) can be made rigorous by starting from the weak formulation (1.4)–(1.7) of Chapter 3, by operating the corresponding change of variables from t into τ, and by taking testing functions $\varphi \geq 0$. We will use (4.10) in space-time domains contained in $K_{8\rho} \times \mathbb{R}^+$.

Write the energy estimates for $(w-k)_-$, of the type of (2.3) of Chapter 3, over cylinders $Q^+_{8\rho}(\theta) \subset E \times \mathbb{R}^+$, as defined in (2.1)–(2.2) of Chapter 3, with bottom center at $(0,0)$, and in the new variables (x,τ). Precisely

$$\begin{aligned}
\operatorname*{ess\,sup}_{0<\tau<\theta(8\rho)^p} \int_{K_{8\rho}} (w-k)^2_-\zeta^p(x,\tau)dx + \iint_{Q^+_{8\rho}(\theta)} |D(w-k)_-\zeta|^p dx\, d\tau \\
\leq \gamma \iint_{Q^+_{8\rho}(\theta)} \big[(w-k)^p_-|D\zeta|^p + (w-k)^2_-|\zeta_\tau|\big]\, dx\, d\tau \\
+\gamma\{\tilde{C}[\theta(8\rho)^p]\}^p \iint_{Q^+_{8\rho}(\theta)} \chi_{[(w-k)_->0]}\zeta^p dx\, d\tau + \gamma C^p \iint_{Q^+_{8\rho}(\theta)} (w-k)^p_-\zeta^p dx\, d\tau
\end{aligned}$$

for a nonnegative, piecewise smooth cutoff function that vanishes on the parabolic boundary of $Q^+_{8\rho}(\theta)$. Choose ζ to be one on the cylinder

$$\mathcal{Q}_{4\rho}(\theta) = K_{4\rho} \times \big((4\rho)^p\theta, (8\rho)^p\theta\big]$$

and such that

$$|D\zeta| \leq \frac{1}{4\rho} \quad \text{and} \quad |\zeta_\tau| \leq \frac{1}{\theta(4\rho)^p}.$$

With these choices, the previous energy inequalities yield

$$\begin{aligned}
&\iint_{\mathcal{Q}_{4\rho}(\theta)} |D(w-k)_-|^p dx\, d\tau \\
&\leq \frac{\gamma k^p}{(4\rho)^p}|\mathcal{Q}_{4\rho}(\theta)|\Big(1 + \frac{1}{\theta k^{p-2}} + (C\rho)^p + \frac{\big\{\tilde{C}\big[\theta(8\rho)^p\big]\big\}^p(4\rho)^p}{k^p}\Big).
\end{aligned} \tag{4.12}$$

4.3 The Set Where w Is Small Can Be Made Small Within $\mathcal{Q}_{4\rho}(\theta)$ for Large θ

Lemma 4.1 *Let (4.2) hold and let k_o be defined by (4.8). For every $\nu > 0$, there exist $\epsilon_\nu \in (0,1)$ depending only on the data $\{p, N, C_o, C_1\}$ and α, and $\theta = \theta(k_o, \epsilon_\nu)$ depending only on k_o, ϵ_ν and the data, and $\gamma = \gamma(\theta)$ depending only on θ and the data, such that either*

$$\gamma(\theta)C\rho > \min\{1\,,\, M\}$$

or

$$|[w < \epsilon_\nu k_o] \cap \mathcal{Q}_{4\rho}(\theta)]| \leq \nu\, |\mathcal{Q}_{4\rho}(\theta)|.$$

Proof Write down the energy inequalities (4.12) for the level k_j and the parameter θ given by

$$k_j = \frac{1}{2^j}k_o \ \text{ for } \ j = 0, 1, \ldots, j_* \quad \text{and} \quad \theta = k_{j_*}^{2-p} = \Big(\frac{2^{j_*}}{k_o}\Big)^{p-2},$$

where $j_* \in \mathbb{N}$ is to be chosen depending only on the data $\{p, N, C_o, C_1\}$. The term involving $\tilde{C}$ is estimated by the definition (4.11) of $\tilde{C}(\tau)$ and the definition (4.8) of k_o. Thus

$$\frac{\{\tilde{C}[\theta(8\rho)^p]\}^p (4\rho)^p}{k_j^p} \le \bar{\gamma}(j_*, \text{data})^p \Big(\frac{\rho C}{M}\Big)^p.$$

Therefore, if

$$M > \bar{\gamma}(j_*, \text{data}) C\rho,$$

the last term is majorized by an absolute constant depending only on the data $\{p, N, C_o, C_1\}$ and the previous inequality yields

$$\iint_{\mathcal{Q}_{4\rho}(\theta)} |D(w-k_j)_-|^p dx\, d\tau \le \frac{\gamma k_j^p}{(4\rho)^p} |\mathcal{Q}_{4\rho}(\theta)| \tag{4.13}$$

for a constant γ depending only on the data $\{p, N, C_o, C_1\}$, and independent of j_*. Set

$$A_j(\tau) = [w(\cdot, \tau) < k_j] \cap K_{4\rho}, \qquad A_j = [w < k_j] \cap \mathcal{Q}_{4\rho}(\theta)$$

so that

$$|A_j| = \int_{\theta(4\rho)^p}^{\theta(8\rho)^p} |A_j(\tau)| d\tau.$$

By Lemma 2.2 of the Preliminaries

$$(k_j - k_{j+1}) |A_{j+1}(\tau)| \le \frac{\gamma \rho^{N+1}}{|K_{4\rho} - A_j(\tau)|} \int_{K_{4\rho} \cap [k_{j+1} < w(\cdot,\tau) < k_j]} |Dw| dx$$

for all $\tau \in (\theta(4\rho)^p, \theta(8\rho)^p]$. For all such τ, applying (4.9)

$$\tfrac{1}{2} k_j |A_{j+1}(\tau)| \le \frac{2\gamma 4^N \rho}{\alpha} \int_{K_{4\rho} \cap [k_{j+1} < w(\cdot,\tau) < k_j]} |Dw| dx.$$

Integrate this in $d\tau$ over $\big(\theta(4\rho)^p, \theta(8\rho)^p\big)$ and majorize the resulting integral on the right-hand side by Hölder's inequality, and by means of (4.13), to obtain

$$\begin{aligned} \tfrac{1}{2} k_j |A_{j+1}| &\le \gamma\rho \Big(\iint_{A_j - A_{j+1}} |Dw|^p dx\, d\tau \Big)^{\frac{1}{p}} |A_j - A_{j+1}|^{\frac{p-1}{p}} \\ &\le \gamma\rho \Big(\iint_{\mathcal{Q}_{4\rho}(\theta)} |D(w-k_j)_-|^p dx\, d\tau \Big)^{\frac{1}{p}} |A_j - A_{j+1}|^{\frac{p-1}{p}} \\ &\le \gamma k_j |\mathcal{Q}_{4\rho}(\theta)|^{\frac{1}{p}} |A_j - A_{j+1}|^{\frac{p-1}{p}}. \end{aligned}$$

From this, by taking the $\frac{p}{p-1}$ power of both sides, we arrive at the recursive inequalities

$$|A_{j+1}|^{\frac{p}{p-1}} \le \gamma |\mathcal{Q}_{4\rho}(\theta)|^{\frac{1}{p-1}} |A_j - A_{j+1}|$$

for a quantitative constant γ depending only on the data $\{p, N, C_o, C_1\}$ and α, and independent of j_*. Now add these for $j = 0, 1, \dots, j_* - 1$, and majorize the sum on the right-hand side by the corresponding telescopic series. This gives

$$(j_* - 1)|A_{j_*}|^{\frac{p}{p-1}} \le \gamma |\mathcal{Q}_{4\rho}(\theta)|^{\frac{p}{p-1}}.$$

Rewriting this as

$$|A_{j_*}| \le \left(\frac{\gamma}{j_*}\right)^{\frac{p-1}{p}} |\mathcal{Q}_{4\rho}(\theta)|,$$

proves the proposition for the choices

$$\epsilon_\nu = \frac{1}{2^{j_*}} \qquad \text{and} \qquad \nu = \left(\frac{\gamma}{j_*}\right)^{\frac{p-1}{p}}. \tag{4.14}$$

4.4 Expanding the Positivity of w

The measure-theoretical information in (4.9), valid for all $\tau > 0$, will be expanded in the space variables over the cube $K_{2\rho}$ for "times" τ sufficiently large.

Lemma 4.2 *Let (4.2) hold. There exist $\nu \in (0,1)$ and $\gamma(\nu) > 1$, that can be determined a priori only in terms of the data $\{p, N, C_o, C_1\}$ and α, such that either*

$$\gamma(\nu) C \rho > \min\{1\,,\, M\}$$

or

$$w(\cdot, \tau) \ge \tfrac{1}{2}\epsilon_\nu k_o \qquad \text{a.e. in } K_{2\rho} \times \left(\frac{(6\rho)^p}{(\epsilon_\nu k_o)^{p-2}}, \frac{(8\rho)^p}{(\epsilon_\nu k_o)^{p-2}}\right] \tag{4.15}$$

where ϵ_ν is the number claimed by Lemma 4.1 corresponding to ν.

Proof Apply (3.1)–(3.3) of Lemma 3.1 of Chapter 3 to w over the cylinder

$$\mathcal{Q}_{4\rho}(\theta) = (0, \tau_*) + Q^-_{4\rho}(\theta) \qquad \text{for } \tau_* = \theta(8\rho)^p.$$

The parameter $\xi\omega$ is replaced by $\epsilon_\nu k_o$ and $\mu_- \ge 0$ is neglected. Taking into account (3.12) of Chapter 3, and choosing $a = \frac{1}{2}$ gives

$$w(x, \tau) \ge \tfrac{1}{2}\epsilon_\nu k_o \qquad \text{for a.e. } (x, \tau) \in [(0, \tau_*) + Q^-_{2\rho}(\theta)]$$

provided $M > \gamma(\epsilon_\nu) C \rho$ and

$$\frac{\big|[w < \epsilon_\nu k_o] \cap \mathcal{Q}_{4\rho}(\theta)\big|}{\big|\mathcal{Q}_{4\rho}(\theta)\big|} \le \gamma^{-1} \left(\frac{1}{2}\right)^{N+2} \frac{[\theta(\epsilon_\nu k_o)^{p-2}]^{\frac{N}{p}}}{[1 + \theta(\epsilon_\nu k_o)^{p-2}]^{\frac{p+N}{p}}} = \nu.$$

Choosing now ν from (4.14) determines ϵ_ν and therefore θ quantitatively.

4.5 Expanding the Positivity of u

Return to the definitions (4.6)–(4.8) of τ, w, and k_o. As τ ranges over the interval in (4.15), $e^{\frac{\tau}{p-2}}$ ranges over

$$b_1 \overset{\text{def}}{=} \exp\left\{\frac{6^p}{(p-2)[\epsilon_\nu \epsilon \delta^{\frac{1}{p-2}}]^{p-2}}\right\} \le f(\tau) \le \exp\left\{\frac{8^p}{(p-2)[\epsilon_\nu \epsilon \delta^{\frac{1}{p-2}}]^{p-2}}\right\} \overset{\text{def}}{=} b_2$$

where b_1 and b_2 are constants that can be determined a priori only in terms of the data $\{p, N, C_o, C_1\}$, and are independent of ρ, M, and u. Translating Lemma 4.2 in terms of u and t gives

$$u(x,t) \ge \frac{\epsilon_\nu \epsilon M}{2b_2} \overset{\text{def}}{=} \eta M \qquad \text{for a.e. } x \in K_{2\rho}$$

for all times

$$\frac{b^{p-2}}{(\eta M)^{p-2}} \tfrac{1}{2} \delta \rho^p \le t \le \frac{b^{p-2}}{(\eta M)^{p-2}} \delta \rho^p$$

for a suitable b depending only on the data $\{p, N, C_o, C_1\}$. ■

5 The Expansion of Positivity for Singular Quasilinear Parabolic Equations $(1 < p < 2)$

Throughout this section we let u be a nonnegative, local, weak supersolution to (1.1)–(1.2) of Chapter 3 with $1 < p < 2$, and let the cylinder

$$(y,s) + Q_{16\rho}(\delta M^{2-p}) = K_{16\rho}(y) \times (s, s + \delta M^{2-p} \rho^p]$$

be contained in E_T.

Proposition 5.1 *Assume that for some $(y,s) \in E_T$ and some $\rho > 0$*

$$|[u(\cdot, s) \ge M] \cap K_\rho(y)| \ge \alpha |K_\rho(y)| \tag{5.1}$$

for some $M > 0$ and some $\alpha \in (0,1)$. There exist constants η, δ, and ε in $(0,1)$ and $\gamma > 1$ depending only on the data $\{p, N, C_o, C_1\}$, and α, such that either

$$\gamma C \rho > \min\{1\,,\, M\}$$

or

$$u(\cdot, t) \ge \eta M \qquad \textit{a.e. in } K_{2\rho}(y) \tag{5.2}$$

for all times

$$s + (1-\varepsilon)\delta M^{2-p} \rho^p \le t \le s + \delta M^{2-p} \rho^p. \tag{5.3}$$

Remark 5.1 The proposition transforms the measure-theoretical information (5.1) into the pointwise expansion of positivity (5.2). The proof below shows that the functional dependence of η on the measure-theoretical parameter α is of the form

$$\eta = \eta_o \alpha\, 2^{-\gamma_1/\alpha^{p+2}} \exp(-\gamma_2 \alpha^p 2^{\gamma_1/\alpha^{p+2}}), \tag{5.4}$$

for parameters $\eta_o, \gamma_1, \gamma_2$ depending only on the data $\{p, N, C_o, C_1\}$. It is not known whether the dependence can be improved to be power-like, as in the degenerate case $p > 2$, for the general *singular* equations (1.1)–(1.2) of Chapter 3.

Proof Assume $(y, s) = (0, 0)$, and let δ and ϵ in $(0, 1)$ be the numbers claimed by Lemma 1.1 depending only on the data $\{p, N, C_o, C_1\}$ and α. The conclusion of the lemma is that either $\gamma C\rho > \min\{1\,,\, M\}$, or

$$|[u(\cdot, t) > \epsilon M] \cap K_\rho| \ge \tfrac{1}{2}\alpha |K_\rho| \quad \text{for all } t \in \big(0, \delta M^{2-p}\rho^p\big]. \tag{5.5}$$

5.1 Transforming the Variables and the Equation

Let $\rho > 0$ be so that

$$Q_{16\rho}(\delta M^{2-p}) = K_{16\rho} \times (0, \delta M^{2-p}\rho^p] \subset E_T. \tag{5.6}$$

Introduce the change of variables and the new unknown function

$$z = \frac{x}{\rho}, \qquad -e^{-\tau} = \frac{t - \delta M^{2-p}\rho^p}{\delta M^{2-p}\rho^p}, \qquad v(z, \tau) = \frac{1}{M} u(x, t) e^{\frac{\tau}{2-p}}. \tag{5.7}$$

This maps the cylinder in (5.6) into $K_{16} \times (0, \infty)$ and transforms the equations (1.1)–(1.2) of Chapter 3 into

$$v_\tau - \operatorname{div}_z \bar{\mathbf{A}}(z, \tau, v, D_z v) = \bar{B}(z, \tau, v, D_z v) + \frac{1}{2-p} v \tag{5.8}$$

weakly in $K_{16} \times (0, \infty)$, where $\bar{\mathbf{A}}$, and $\bar{B}$ are measurable functions of their arguments, satisfying the structure conditions

$$\begin{cases} \bar{\mathbf{A}}(z, \tau, v, D_z v) \cdot D_z v \ge \delta C_o |D_z v|^p - \delta \bar{C}^p \\ |\bar{\mathbf{A}}(z, \tau, v, D_z v)| \le \delta C_1 |D_z v|^{p-1} + \delta \bar{C}^{p-1} \\ |\bar{B}(z, \tau, v, D_z v)| \le \delta \rho C |D_z v|^{p-1} + \delta \rho C \bar{C}^{p-1} \end{cases} \tag{5.9}$$

a.e. in $K_{16} \times (0, \infty)$, where C_o and C_1 are the constants in the structure conditions (1.2) of Chapter 3, δ is the number claimed by Lemma 1.1, and

$$\bar{C} = \bar{C}(\tau) = \rho \frac{C}{M}\, e^{\frac{\tau}{2-p}}.$$

In this setting, the information (5.5) becomes

$$\left|[v(\cdot,\tau) \geq \epsilon\, e^{\frac{\tau}{2-p}}] \cap K_1\right| \geq \tfrac{1}{2}\alpha|K_1| \quad \text{for all } \tau \in (0,+\infty). \tag{5.10}$$

Let $\tau_o > 0$ to be chosen and set

$$k_o = \epsilon\, e^{\frac{\tau_o}{2-p}}, \qquad \text{and} \qquad k_j = \frac{1}{2^j}k_o \quad \text{for } j = 0,1,\dots,j_*,$$

where j_* is to be chosen. With this symbolism (5.10) implies

$$|[v(\cdot,\tau) \geq k_j] \cap K_8| \geq \tfrac{1}{2}\alpha 8^{-N}|K_8| \quad \text{for all } \tau \in (\tau_o,+\infty) \tag{5.11}$$

and for all $j \in \mathbb{N}$. Introduce the cylinders

$$\begin{aligned} Q_{\tau_o} &= K_8 \times \left(\tau_o + k_o^{2-p}, \tau_o + 2k_o^{2-p}\right) \\ Q'_{\tau_o} &= K_{16} \times \left(\tau_o, \tau_o + 2k_o^{2-p}\right) \end{aligned}$$

and a nonnegative, piecewise smooth, cutoff function in Q'_{τ_o} of the form $\zeta(z,\tau) = \zeta_1(z)\zeta_2(\tau)$, where

$$\zeta_1 = \begin{cases} 1 & \text{in } K_8 \\ 0 & \text{in } \mathbb{R}^N - K_{16} \end{cases} \qquad |D\zeta_1| \leq \frac{1}{8},$$

$$\zeta_2 = \begin{cases} 0 & \text{for } \tau < \tau_o \\ 1 & \text{for } \tau \geq \tau_o + k_o^{2-p} \end{cases} \qquad 0 \leq \zeta_{2,\tau} \leq \frac{1}{k_o^{2-p}}.$$

Write down the energy estimates (2.3) of Chapter 3, for $(v-k_j)_-$ over Q'_{τ_o}, and for the indicated choice of cutoff function ζ. These are derived by taking $-(v-k_j)_-\zeta^p$ as a testing function in the weak formulation of (5.8). Discarding the nonpositive contribution of the right-hand side, coming from the nonnegative term $\frac{1}{2-p}v$, standard calculations give

$$\begin{aligned} &\iint_{Q'_{\tau_o}} |D(v-k_j)_-\zeta|^p dz\, d\tau \\ &\qquad \leq \gamma \iint_{Q'_{\tau_o}} \left[(v-k_j)_-^p |D\zeta|^p + (v-k_j)_-^2 \zeta_t\right] dz\, d\tau \\ &\qquad\quad + \gamma \bar{C}^p(\tau_o + 2k_o^{2-p}) \iint_{Q'_{\tau_o}} \chi_{[(v-k_j)_->0]} dz\, d\tau \\ &\qquad\quad + \gamma C^p \rho^p \iint_{Q'_{\tau_o}} (v-k_j)_-^p dz\, d\tau, \end{aligned}$$

where $\gamma = \tilde{\gamma}/\delta$, the constant $\tilde{\gamma}$ depends only on $\{p, N, C_o, C_1\}$, and δ is the parameter claimed by Lemma 1.1, and appearing in the transformed structure conditions (5.9). From this

$$\iint_{Q_{\tau_o}} |D(v-k_j)_-|^p dz\, d\tau \leq \gamma k_j^p |Q_{\tau_o}| \Big[2 + \frac{\bar{C}^p(\tau_o + 2k_o^{2-p})}{k_j^p} + C^p\rho^p\Big].$$

Taking into account the expressions of $\bar{C}$ and k_o, estimate

$$\frac{\bar{C}^p(\tau_o + 2k_o^{2-p})}{k_j^p} \le 2^{j_* p}\frac{C^p}{M^p}\rho^p e^{\frac{2p}{2-p}k_o^{2-p}}.$$

Suppose for the moment that j_* and k_o have been chosen, and set

$$\gamma(j_*, \tau_o) = 2^{j_*} e^{\frac{2}{2-p}k_o^{2-p}}. \tag{5.12}$$

Therefore either $M < \gamma(j_*, \tau_o)C\rho$, or the previous inequality yields

$$\iint_{Q_{\tau_o}} |D(v-k_j)_-|^p dz\, d\tau \le 4\gamma k_j^p |Q_{\tau_o}| \tag{5.13}$$

for a constant γ depending only on the data $\{p, N, C_o, C_1\}$, and δ.

5.2 Estimating the Measure of the Set $[v < k_j]$ Within Q_{τ_o}

Set

$$A_j(\tau) = [v(\cdot,\tau) < k_j] \cap K_8, \qquad A_j = [v < k_j] \cap Q_{\tau_o}.$$

By Lemma 2.2 of the Preliminaries, and (5.11)

$$\begin{aligned}(k_j - k_{j+1})|A_{j+1}(\tau)| &\le \frac{\gamma(N)}{|K_8 - A_j(\tau)|}\int_{K_8\cap[k_{j+1}<v(\cdot,\tau)<k_j]}|Dv|dz \\ &\le \frac{\gamma(N)}{\alpha}\int_{K_8\cap[k_{j+1}<v(\cdot,\tau)<k_j]}|Dv|dz\end{aligned}$$

for all $\tau \ge \tau_o$. Integrate this in $d\tau$ over $(\tau_o + k_o^{2-p}, \tau_o + 2k_o^{2-p})$, majorize the resulting integral on the right-hand side by the Hölder inequality, and use (5.13) to get

$$\begin{aligned}\frac{k_j}{2}|A_{j+1}| &\le \gamma(\text{data},\alpha)\iint_{A_j - A_{j+1}}|Dv|dz\, d\tau \\ &\le \gamma(\text{data},\alpha)\Big(\iint_{A_j-A_{j+1}}|Dv|^p dz\, d\tau\Big)^{\frac{1}{p}}|A_j - A_{j+1}|^{\frac{p-1}{p}} \\ &\le \gamma(\text{data},\alpha)\Big(\iint_{Q_{\tau_o}}|D(v-k_j)_-|^p dz\, d\tau\Big)^{\frac{1}{p}}|A_j - A_{j+1}|^{\frac{p-1}{p}} \\ &\le \gamma(\text{data},\alpha,\delta)k_j|Q_{\tau_o}|^{\frac{1}{p}}|A_j - A_{j+1}|^{\frac{p-1}{p}}.\end{aligned}$$

Taking the $\frac{p}{p-1}$ power yields the recursive inequalities

$$|A_{j+1}|^{\frac{p}{p-1}} \le \gamma(\text{data},\alpha,\delta)|Q_{\tau_o}|^{\frac{1}{p-1}}|A_j - A_{j+1}|.$$

Add these inequalities for $j = 0, 1, \dots, j_* - 1$, where j_* is an integer to be chosen, and majorize the sum on the right-hand side by the corresponding telescopic series. This gives

$$(j_* - 1)|A_{j_*}|^{\frac{p}{p-1}} \le \gamma(\text{data}, \alpha, \delta)|Q_{\tau_o}|^{\frac{p}{p-1}}.$$

Equivalently

$$|[v < k_{j_*}] \cap Q_{\tau_o}| \le \nu |Q_{\tau_o}| \quad \text{where} \quad \nu = \Big(\frac{\gamma(\text{data}, \alpha, \delta)}{j_*}\Big)^{\frac{p-1}{p}}. \tag{5.14}$$

Taking into account (1.3), the constant γ in (5.14) can be traced to be of the form $\gamma = \dfrac{\tilde{\gamma}(\text{data})}{\alpha^{p+2}}$.

5.3 Segmenting Q_{τ_o}

Assume momentarily that j_* and hence ν have been determined. By possibly increasing j_* to be not necessarily integer, without loss of generality we may assume that $(2^{j_*})^{2-p}$ is an integer. Then subdivide Q_{τ_o} into $(2^{j_*})^{2-p}$ cylinders, each of length $k_{j_*}^{2-p}$, by setting

$$\begin{aligned} &Q_n = K_8 \times \big(\tau_o + k_o^{2-p} + n k_{j_*}^{2-p}, \tau_o + k_o^{2-p} + (n+1) k_{j_*}^{2-p}\big) \\ &\text{for} \;\; n = 0, 1, \dots, (2^{j_*})^{2-p} - 1. \end{aligned}$$

For at least one of these, say Q_n, there must hold

$$|[v < k_{j_*}] \cap Q_n| \le \nu |Q_n|.$$

Apply Lemma 3.1 of Chapter 3 to v over Q_n with

$$\mu_- = 0, \quad \xi\omega = k_{j_*}, \quad a = \tfrac{1}{2}, \quad \theta = k_{j_*}^{2-p}.$$

It gives

$$v\big(z, \tau_o + k_o^{2-p} + (n+1)k_{j_*}^{2-p}\big) \ge \tfrac{1}{2} k_{j_*} \qquad \text{a.e. in } K_4$$

provided

$$\frac{|[v < k_{j_*}] \cap Q_n|}{|Q_n|} \le 2^{-\frac{N+p}{p}} \bar{\gamma}_o(\text{data}) = \nu.$$

Choose now j_*, and hence ν, from this and (5.14). Summarizing, for such a choice of j_*, and hence ν, there exists a time level τ_1 in the range

$$\tau_o + k_o^{2-p} < \tau_1 < \tau_o + 2k_o^{2-p} \tag{5.15}$$

such that

$$v(z, \tau_1) \ge \sigma_o e^{\frac{\tau_o}{2-p}} \qquad \text{where} \qquad \sigma_o = \epsilon\, 2^{-(j_*+1)}.$$

Remark 5.2 Notice that j_* and hence ν are determined only in terms of the data and are independent of the parameter τ_o, which is still to be chosen.

5.4 Returning to the Original Coordinates

In terms of the original coordinates and the original function $u(x,t)$ this implies

$$u(\cdot, t_1) \ge \sigma_o M e^{-\frac{\tau_1-\tau_o}{2-p}} \stackrel{\text{def}}{=} M_o \qquad \text{in } K_{4\rho},$$

where the time t_1 corresponding to τ_1 is computed from (5.7) and (5.15). Apply now Lemma 4.1 of Chapter 3 with M replaced by M_o and $\xi = 1$ over the cylinder

$$(t_1, 0) + Q_{4\rho}^+(\theta) = K_{4\rho} \times \big(t_1, t_1 + \theta(4\rho)^p\big].$$

By choosing

$$\theta = \nu_o M_o^{2-p} \qquad \text{where} \qquad \nu_o = \nu_o(\text{data})$$

the assumption (4.2) of Chapter 3 is satisfied, and the lemma yields

$$\left.\begin{aligned} u(\cdot,t) &\ge \tfrac{1}{2} M_o = \tfrac{1}{2}\sigma_o M e^{-\frac{\tau_1-\tau_o}{2-p}} \\ &\ge \tfrac{1}{2}\sigma_o e^{-\frac{2}{2-p}e^{\tau_o}} M \end{aligned}\right\} \quad \text{in} \quad K_{2\rho} \tag{5.16}$$

for all times

$$t_1 \le t \le t_1 + \nu_o M_o^{2-p}(4\rho)^p. \tag{5.17}$$

If the right-hand side equals $\delta M^{2-p}\rho^p$, then (5.16) and the conclusion (5.2) will hold for the time $t = \delta M^{2-p}\rho^p$. The transformed τ_o level is still undetermined, and it will be so chosen as to verify such a requirement. Precisely, taking into account the change of variables (5.7)

$$\delta M^{2-p}\rho^p e^{-\tau_1} = \delta M^{2-p}\rho^p - t_1 = \nu_o \sigma_o^{2-p} M^{2-p}(4\rho)^p e^{-(\tau_1-\tau_o)}$$

which implies

$$e^{\tau_o} = \frac{\delta}{4^p \nu_o \sigma_o^{2-p}}.$$

This determines quantitatively $\tau_o = \tau_o(\text{data})$. The proof of Proposition 5.1 is now completed by inserting such a τ_o on the right-hand side of (5.16) and in (5.17). In particular (5.16) holds for all times

$$t_1 = \delta M^{2-p}\rho^p - \nu_o M_o^{2-p}(4\rho)^p \le t \le \delta M^{2-p}\rho^p.$$

From the previous definitions and transformations one estimates

$$t_1 \le (1-\varepsilon)\delta M^{2-p}\rho^p, \qquad \text{where} \qquad \varepsilon = e^{-\tau_o - 2e^{\tau_o}}.$$

Notice that once j_* and τ_o are fixed, then the constant γ in (5.12) is also defined, only in terms of the data $\{p, N, C_o, C_1\}$ and α.

Remark 5.3 As it will be apparent in the next chapters, the Harnack inequality has different formulations, respectively when $\frac{2N}{N+1} < p < 2$ and $1 < p \le \frac{2N}{N+1}$. It is remarkable, however, that the expansion of positivity holds with the same statement in the full singular range $1 < p < 2$.

Remark 5.4 It might seem that two approaches for the degenerate case $p > 2$ and the singular case $1 < p < 2$ are similar, based as they are on an exponential-type change of variable, respectively (4.6)–(4.7) and (5.7). The two phenomena, however, are markedly different.

In the degenerate case, starting at time level s, the transformation itself chooses the final time level, as indicated in (4.4), in terms of the lower bound M. In the singular case, the final time level $\delta M^{2-p}\rho^p$ is fixed in terms of M, as indicated in (5.3). The structural constants only determine how the original time interval shrinks, about the upper limit, which remains fixed.

6 Stability of the Expansion of Positivity for $p \to 2$

The proof of Proposition 4.1 for the degenerate case $p > 2$ shows that the constants b and η in (4.3)–(4.4) depend on p as (see § 4.5)

$$b \approx \exp\Big(\gamma_b \frac{h^{p-2}}{p-2}\Big), \qquad \eta \approx \exp\Big(-\gamma_\eta \frac{k^{p-2}}{p-2}\Big)$$

for constants γ_b, γ_η, h, k all larger than 1, depending only on the data $\{N, C_o, C_1\}$, and independent of p. Thus the ratio $(b/\eta)^{p-2}$ that determines the "waiting time" needed to preserve and expand the positivity, deteriorates as $p \to \infty$. However, it is stable as $p \to 2$ and (4.4) remains meaningful for p near 2. On the other hand, $\eta(p) \to 0$ as $p \to 2$ and (4.3) becomes vacuous.

Likewise, in the proof of Proposition 5.1, for the singular case $1 < p < 2$, the change of variables (5.7) and the subsequent arguments, yield constants that deteriorate as $p \to 2$.

Nevertheless the conclusions of both Proposition 4.1, for $p > 2$, and Proposition 5.1 for $1 < p < 2$, continue to hold with constants that are stable as $p \to 2$, in the sense of (1.9) of Chapter 3. This is the content of the next proposition.

Proposition 6.1 *Let u be a nonnegative, local, weak solution to (1.1)–(1.2) of Chapter 3 for $p > 1$ in E_T. Let*

$$K_{8\rho}(y) \times (s, s + \frac{\delta\rho^p}{M^{p-2}}] \subset E_T$$

and assume that for some $(y, s) \in E_T$ and some $\rho > 0$

$$|[u(\cdot, s) \ge M] \cap K_\rho(y)| \ge \alpha |K_\rho(y)|$$

for some $M > 0$ and some $\alpha \in (0, 1)$. There exist constants $\gamma_* > 1$, δ, σ_*, η_* in $(0, 1)$, depending only on the data $\{N, C_o, C_1\}$ and α, and independent of (y, s), ρ, M, and p, such that if $|p - 2| < \sigma_*$, then either

$$\gamma_* C\rho > \min\{1\,,\, M\}$$

or

$$u(x,t) \geq \eta_* M \quad \textit{for all } x \in K_{2\rho}(y)$$

for all

$$s + \frac{\frac{1}{2}\delta\rho^p}{M^{p-2}} \leq t \leq s + \frac{\delta\rho^p}{M^{p-2}}.$$

Remark 6.1 The constants γ_*, δ, σ_*, and η_* are stable as $p \to 2$, in the sense of (1.9) of Chapter 3.

6.1 Proof of Proposition 6.1

Assume that $(y,s) = (0,0)$ and let $\epsilon(p)$ and $\delta(p)$ be the constants corresponding to α, claimed by Lemma 1.1. The lemma does not distinguish between $p > 2$ and $1 < p < 2$ and it implies

$$|[u(\cdot,t) < \epsilon M] \cap K_{4\rho}| > \tfrac{1}{2}\alpha 4^{-N}|K_{4\rho}|, \quad \text{for all } t \in (0, \delta M^{2-p}\rho^p). \tag{6.1}$$

By Remark 1.1 the constants $\epsilon(p)$ and $\delta(p)$ are stable as $p \to 2$. The proof now proceeds for p near 2 irrespective of the degeneracy ($p > 2$) or singularity ($1 < p < 2$) of the partial differential equation. For this reason we denote by $|p-2|$ the proximity of p to 2 from either side.

Lemma 6.1 *For every $\nu^* \in (0,1)$ there exist constants σ^*, $\epsilon_{\nu^*} \in (0,1)$ and $\gamma_* > 1$, depending only on the data $\{N, C_o, C_1\}$ and α and independent of u, M, p, and ρ, such that for all $|p-2| \leq \sigma_*$, either*

$$\gamma_* C\rho > \min\{1\,,\, M\}$$

or

$$|[u < \epsilon_{\nu^*} M] \cap \mathcal{Q}^+_{4\rho}(\delta M^{2-p})| \leq \nu_* |\mathcal{Q}^+_{4\rho}(\delta M^{2-p})|.$$

Proof Write down the energy inequalities in (2.3) of Chapter 3, for $(u-k_j)_-$, over the cylinder

$$\mathcal{Q}^+_{8\rho}(\delta M^{2-p})$$

for a nonnegative, piecewise smooth, cutoff function ζ that equals one on $\mathcal{Q}^+_{4\rho}(\delta M^{2-p})$, and such that

$$|D\zeta| \leq \frac{1}{4\rho} \quad \text{and} \quad |\zeta_t| \leq \frac{1}{\delta M^{2-p}\rho^p}.$$

The levels k_j are taken as

$$k_j = \frac{\epsilon M}{2^j} \quad \text{for } j = 0, 1, \dots, j_* \quad \text{where } j_* \in \mathbb{N} \text{ is to be chosen.}$$

The first term on the left-hand side is discarded and the integral involving $D(u-k_j)_-$ is minorized by extending it over $\mathcal{Q}^+_{4\rho}(\delta M^{2-p})$, which is the set where $\zeta = 1$. The right-hand side is majorized in a standard fashion and gives

$$\iint_{\mathcal{Q}^+_{4\rho}(\delta M^{2-p})} |D(u-k_j)_-|^p dx\,dt \le \gamma \frac{k_j^p}{\delta\rho^p} 2^{j_*|p-2|} |\mathcal{Q}^+_{4\rho}| \Big[1 + \frac{C^p\rho^p}{k_j^p} + C^p\rho^p\Big].$$

Assume momentarily that j_* has been chosen in terms only of the data and α. Then either $M < C2^{j_*}\rho$, or the previous inequality yields

$$\iint_{\mathcal{Q}^+_{4\rho}(\delta M^{2-p})} |D(u-k_j)_-|^p dx\,dt \le \gamma \frac{k_j^p}{\delta\rho^p} 2^{j_*|p-2|} |\mathcal{Q}^+_{4\rho}|.$$

The number j_* will be chosen shortly depending only on the data $\{N, C_o, C_1\}$ and α, and independent of u, M, ρ, and p. Assuming momentarily that such a choice has been made, choose $\sigma_* \in (0,1)$ so that $j_*|p-2| \le 1$ for all $|p-2| < \sigma_*$. This yields the energy estimates

$$\iint_{\mathcal{Q}^+_{4\rho}(\delta M^{2-p})} |D(u-k_j)_-|^p dx\,dt \le \frac{\gamma k_j^p}{\rho^p} |\mathcal{Q}^+_{4\rho}(\delta M^{2-p})| \tag{6.2}$$

for a constant γ depending only on the data $\{N, C_o, C_1\}$ and independent of u, M, ρ, and p, provided $M > C\gamma_*\rho$ for $\gamma_* = 2^{j_*}$.

Starting from these energy estimates, the proof can now be concluded as in the proof of Lemma 2.1 valid for nondegenerate equations. Precisely, set

$$A_j = [u < k_j] \cap \mathcal{Q}^+_{4\rho}(\delta M^{2-p})$$

and proceed as in that context by making use of (6.1) and (6.2), to arrive at the analog of (2.6)

$$|A_{j_*}| \le \Big(\frac{\gamma}{j_*}\Big)^{\frac{p-1}{p}} |\mathcal{Q}^+_{4\rho}(\delta M^{2-p})| \tag{6.3}$$

for a constant γ depending only on the data $\{N, C_o, C_1\}$ and independent of u, M, ρ, and p. Choosing

$$\epsilon_{\nu^*} = \frac{\epsilon}{2^{j_*}} \qquad \text{and} \qquad \nu^* = \Big(\frac{\gamma}{j_*}\Big)^{\frac{p-1}{p}} \tag{6.4}$$

proves the lemma. ■

To conclude the proof of Proposition 6.1, apply Lemma 3.1 of Chapter 3, with $\mu_- = 0$, $\xi = \epsilon_{\nu^*}$, $a = \frac{1}{2}$, $\omega = M$, $\theta = \delta M^{2-p}$ and ρ replaced by 2ρ. The lemma yields

$$u > \tfrac{1}{2}\epsilon_{\nu^*} M \quad \text{in } K_{2\rho} \times (\tfrac{1}{2}\delta\rho^p, \delta\rho^p),$$

provided

$$Y_o = \frac{|[u < \epsilon_{\nu^*}] \cap \mathcal{Q}^+_{4\rho}(\delta M^{2-p})|}{|\mathcal{Q}^+_{4\rho}(\delta M^{2-p})|} = \frac{|A_{j_*}|}{|\mathcal{Q}^+_{4\rho}(\delta M^{2-p})|} = \nu^*.$$

Here the number ν^* is chosen from (3.12) of Chapter 3 for $p > 1$. For $p > 2$ compute

$$
\begin{aligned}
Y_o &\le \frac{1}{\bar{\gamma}(\text{data})} \frac{[\delta M^{2-p}(\epsilon_{\nu^*} M)^{p-2}]^{\frac{N}{p}}}{[1+\delta M^{2-p}(\epsilon_{\nu^*} M)^{p-2}]^{\frac{N+p}{p}}} \\
&= \frac{1}{\bar{\gamma}(\text{data})} \frac{[\delta \epsilon^{p-2} 2^{j_*(2-p)}]^{\frac{N}{p}}}{[1+\delta \epsilon^{p-2} 2^{j_*(2-p)}]^{\frac{N+p}{p}}} = \nu^*.
\end{aligned}
$$

Stipulate to choose $|p-2| \le \sigma_*$ and then σ_* so small that $2^{j_*|p-2|} \in (1,2)$. Then, from (6.3)–(6.4) choose j_* so large as to satisfy this requirement. The calculations for $1 < p < 2$ are identical starting once more from (3.12) of Chapter 3. ■

The argument is a hybrid between the nondegenerate case of § 2 and the degenerate case of § 4 and the singular case of § 5. It mimics the degenerate or singular case in that the length of the cylinders is of the order of M^{2-p} thereby abiding to the notion of intrinsic geometry. If a lower bound of the type $\epsilon_{\nu^*} M = \epsilon 2^{-j_*} M$ is sought, then the intrinsic geometry required by Lemma 3.1 of Chapter 3 would require a cylinder of length $(\epsilon_{\nu^*} M)^{2-p}$, relative to ρ^p. However, because of the indicated choices $\epsilon_{\nu^*}^{p-2} \approx 1$ if $p \approx 2$. Roughly speaking the partial differential equation, while degenerate or singular, for $p \approx 2$ is "mildly degenerate or singular," and it transitions from its nondegenerate regime $p = 2$ to its degenerate regime $p > 2$ or singular regime $1 < p < 2$, in a stable manner.

7 The Expansion of Positivity for Porous Medium Type Equations

Throughout this section let u be a nonnegative, local, weak supersolution to (5.1)–(5.2) of Chapter 3 in E_T, for $m > 0$. For $(y, s) \in E_T$, and some given positive number M, consider the cylinders

$$
\begin{aligned}
&K_{8\rho}(y) \times (s, s + \frac{b^{m-1}}{(\eta M)^{m-1}} \delta\rho^2] \quad \text{for} \quad m > 1 \\
&K_{16\rho}(y) \times (s, s + \delta M^{1-m} \rho^2] \quad \text{for} \quad 0 < m < 1
\end{aligned}
$$

where b, δ, η are the constants given by Propositions 7.1 and 7.2, and $\rho > 0$ is so small that they are both included in E_T. The results of the previous sections are based solely on the following technical tools: (i) Lemmas 3.1 and 4.1 of Chapter 3, (ii) the discrete isoperimetric inequality of Lemma 2.2 and the embedding Proposition 4.1 of the Preliminaries, and (iii) the change of variables introduced respectively in (4.6)–(4.7) for the degenerate case $p > 2$ and in (5.7) for the singular case $1 < p < 2$.

For porous medium type equations Lemmas 3.1 and 4.1 of Chapter 3 have their exact counterpart respectively in Lemmas 7.1 and 8.1 for $m > 1$, and in Lemmas 10.1 and 11.1 for $0 < m < 1$ of Chapter 3.

The discrete isoperimetric inequality and the embeddings of the Preliminaries are facts of Classical Analysis, independent of partial differential equations. Therefore the expansion of positivity effect continues to hold for these equations, by essentially the same proof, whence one introduces changes of variables analogous to (4.6)–(4.7) for the degenerate case $m > 1$ and to (5.7) for the singular case $0 < m < 1$. Below we outline the main differences in the proofs by distinguishing the degenerate case $m > 1$ from the singular case $0 < m < 1$.

7.1 Expansion of Positivity When $m > 1$

The starting point is a time propagation of positivity similar to Lemma 1.1.

Lemma 7.1 *Assume that for some $(y, s) \in E_T$ and some $\rho > 0$*

$$|[u(\cdot, s) \geq M] \cap K_\rho(y)| \geq \alpha |K_\rho(y)|$$

for some $M > 0$ and some $\alpha \in (0, 1)$. There exist δ and ϵ in $(0, 1)$, depending only on the data $\{m, N, C_o, C_1\}$ and α, and independent of M, such that either $C\rho > 1$, or

$$|[u(\cdot, t) > \epsilon M] \cap K_\rho(y)| \geq \tfrac{1}{2}\alpha |K_\rho(y)| \quad \textit{for all } t \in \Big(s, s + \frac{\delta\rho^2}{M^{m-1}}\Big].$$

Proof Same as in Lemma 1.1 by minor changes. We may assume

$$\delta = \frac{\alpha^3}{\gamma 2^{10} N^2},$$

with ϵ as in Lemma 1.1. ■

Proposition 7.1 *Assume that for some $(y, s) \in E_T$ and some $\rho > 0$*

$$|[u(\cdot, s) \geq M] \cap K_\rho(y)| \geq \alpha |K_\rho(y)|$$

for some $M > 0$ and some $\alpha \in (0, 1)$. There exist constants $b > 1$, $\delta, \eta \in (0, 1)$, depending only on the data $\{m, N, C_o, C_1\}$ and α, and independent of (y, s), ρ, M, such that either $C\rho > 1$, or

$$u(\cdot, t) \geq \eta M \qquad \textit{in } K_{2\rho}(y)$$

for all times

$$s + \frac{b^{m-1}}{(\eta M)^{m-1}} \tfrac{1}{2} \delta \rho^2 \leq t \leq s + \frac{b^{m-1}}{(\eta M)^{m-1}} \delta \rho^2.$$

The constants b, δ, η deteriorate as $m \to \infty$, but they are stable as $m \to 1$.

Proof Assume $(y, s) = (0, 0)$ and let ϵ and δ be determined as in Lemma 7.1. The proof is almost identical to that of § 4 by means of the change of variables

$$w(x, \tau) \overset{\text{def}}{=} \frac{e^{\frac{\tau}{m-1}}}{M} (\delta\rho^2)^{\frac{1}{m-1}} u\Big(x, s + \frac{e^\tau}{M^{m-1}} \delta \rho^2\Big),$$

modulo the obvious changes in symbolism. The stability analysis of the constants for $m \approx 1$ is carried out as in § 6. ■

7.2 Expansion of Positivity When $0 < m < 1$

The starting point is a time propagation of positivity similar to Lemma 1.1.

Lemma 7.2 *Let* $0 < m < 1$ *and assume that for some* $(y, s) \in E_T$ *and some* $\rho > 0$

$$|[u(\cdot, s) \geq M] \cap K_\rho(y)| \geq \alpha |K_\rho(y)|$$

for some $M > 0$ *and some* $\alpha \in (0, 1)$. *There exist* δ *and* ϵ *in* $(0, 1)$, *depending only on the data* $\{m, N, C_o, C_1\}$ *and* α, *and independent of* M, *such that either* $C\rho > 1$, *or*

$$|[u(\cdot, t) > \epsilon M] \cap K_\rho(y)| \geq \tfrac{1}{2}\alpha |K_\rho(y)| \quad \textit{for all } \; t \in \Big(s, s + \frac{\delta\rho^2}{M^{m-1}}\Big].$$

Proof Assume $(y, s) = (0, 0)$, and consider the cylinder

$$Q_\rho^+(\delta M^{1-m}) = K_\rho \times (0, \delta M^{1-m}\rho^2]$$

where $\delta \in (0, 1)$ is to be chosen. In the weak formulation (5.5) of Chapter 3, take the test function

$$\varphi = -(u^m - M^m)_- \zeta^2$$

where $x \to \zeta(x)$ is a nonnegative, piecewise smooth cutoff function in K_ρ which equals one on $K_{(1-\sigma)\rho}$ and such that $|D\zeta| \leq (\sigma\rho)^{-1}$. Proceeding as in § 9 of Chapter 3 and enforcing the condition $C\rho \leq 1$ gives

$$\begin{aligned}\int_{K_\rho}\int_{u(x,t)}^{M} (M^m - s^m)_+ ds \zeta^2 dx \leq \int_{K_\rho}\int_{u(x,0)}^{M} (M^m - s^m)_+ ds\zeta^2 dx \\ + \gamma |K_\rho| \frac{\delta M^{m+1}}{\sigma^2}\end{aligned}$$

for all times $0 < t < \delta M^{1-m}\rho^2$. Enforcing the assumptions of the lemma, estimate

$$\begin{aligned}\int_{K_\rho}\int_{u(x,0)}^{M} (M^m - s^m)_+ ds\zeta^2 dx &\leq \frac{m}{m+1} M^{m+1}(1-\alpha)|K_\rho| \\ \int_{K_\rho}\int_{u(x,t)}^{M} (M^m - s^m)_+ ds\zeta^2 dx &\geq \int_{K_{(1-\sigma)\rho}\cap[u<\epsilon M]} \int_{u(x,t)}^{M} (M^m - s^m)_+ ds\, dx \\ &\geq \frac{m}{m+1}(1 - \frac{m+1}{m}\epsilon) M^{m+1} |A_{\epsilon M,(1-\sigma)\rho}(t)|.\end{aligned}$$

Therefore proceeding as in the proof of Lemma 1.1

$$|A_{\epsilon M,\rho}(t)| \leq \frac{1}{1 - \epsilon^{\frac{m+1}{m}}} \Big[(1-\alpha) + \gamma \frac{m+1}{m}\frac{\delta}{\sigma^2} + N\sigma\Big] |K_\rho|.$$

From here on, conclude as in the proof of Lemma 1.1. ■

Proposition 7.2 *Let* $0 < m < 1$ *and assume that for some* $(y,s) \in E_T$ *and some* $\rho > 0$

$$|[u(\cdot,s) \ge M] \cap K_\rho(y)| \ge \alpha |K_\rho(y)| \tag{7.1}$$

for some $M > 0$ *and some* $\alpha \in (0,1)$. *There exist constants* $\varepsilon, \delta, \eta \in (0,1)$, *depending only on the data* $\{m, N, C_o, C_1\}$ *and* α, *and independent of* (y,s), ρ, M, *such that either* $C\rho > 1$, *or*

$$u(\cdot,t) \ge \eta M \quad \text{in } K_{2\rho}(y) \tag{7.2}$$

for all times

$$s + (1-\varepsilon)\delta M^{1-m}\rho^2 \le t \le s + \delta M^{1-m}\rho^2.$$

The constants ε, δ, η *deteriorate as* $m \to 0$, *but they are stable as* $m \to 1$.

Proof The proof is similar to that of § 5. Nevertheless, since the particular structure of the energy estimates of § 9 of Chapter 3 brings about some differences, here we present the full proof. The arguments below show that the functional dependence of η on the measure-theoretical parameter α is of the form

$$\eta = \eta_o \alpha\, 2^{-\gamma_1/\alpha^4} \exp(-\gamma_2 \alpha^2 2^{\gamma_1/\alpha^4}), \tag{7.3}$$

for parameters $\eta_o, \gamma_1, \gamma_2$ depending only on the data $\{m, N, C_o, C_1\}$. It is not known whether the dependence can be improved to be power-like, for the general *singular* equations (5.1)–(5.2) of Chapter 3. ■

7.2.1 Transforming the Variables and the Equation

Assume $(y,s) = (0,0)$, let δ and ϵ be as determined in Lemma 7.2, and let $\rho > 0$ be so that

$$Q_{16\rho}(\delta M^{1-m}) = K_{16\rho} \times (0, \delta M^{1-m}\rho^2] \subset E_T.$$

Introduce the change of variables and the new unknown function

$$z = \frac{x}{\rho}, \quad -e^{-\tau} = \frac{t - \delta M^{1-m}\rho^2}{\delta M^{1-m}\rho^2}, \quad v(z,\tau) = \frac{1}{M} u(x,t) e^{\frac{\tau}{1-m}}. \tag{7.4}$$

This maps the cylinder $Q_{16\rho}(\delta M^{1-m})$ into $K_{16} \times (0,\infty)$ and transforms the equations (5.1)–(5.2) of Chapter 3 into

$$v_\tau - \operatorname{div}_z \bar{\mathbf{A}}(z,\tau,v,D_z v) = \bar{B}(z,\tau,v,D_z v) + \frac{1}{1-m} v \tag{7.5}$$

weakly in $K_{16} \times (0,\infty)$, where $\bar{\mathbf{A}}$, and $\bar{B}$ are measurable functions of their arguments, satisfying the structure conditions

$$\begin{cases} \bar{\mathbf{A}}(z,\tau,v,D_z v) \cdot D_z v \ge m\delta C_o v^{m-1} |D_z v|^2 - \delta \bar{C}^2 v^{m+1} \\ |\bar{\mathbf{A}}(z,\tau,v,D_z v)| \le m\delta C_1 v^{m-1} |D_z v| + \delta \bar{C} v^m \\ |\bar{B}(z,\tau,v,D_z v)| \le m\delta \bar{C} v^{m-1} |D_z v| + \delta \bar{C}^2 v^m \end{cases}$$

a.e. in $K_{16} \times (0, \infty)$. Here C_o and C_1 are the constants in the structure conditions (5.2) of Chapter 3, δ is the number claimed by Lemma 7.2, and $\bar{C} = \rho C$. In this setting, the information of Lemma 7.2 reads

$$\left|[v(\cdot,\tau) \ge \epsilon\, e^{\frac{\tau}{1-m}}] \cap K_1\right| \ge \tfrac{1}{2}\alpha |K_1| \quad \text{for all } \ \tau \in (0, +\infty).$$

Let $\tau_o > 0$ to be chosen and set

$$k_o = \epsilon\, e^{\frac{\tau_o}{1-m}}, \qquad \text{and} \qquad k_j = \frac{1}{2^j} k_o \quad \text{for } \ j = 0, 1, \dots, j_*,$$

where j_* is to be chosen. With this symbolism

$$|[v(\cdot,\tau) \ge k_j] \cap K_8| \ge \tfrac{1}{2}\alpha 8^{-N} |K_8| \quad \text{for all } \ \tau \in (\tau_o, +\infty) \tag{7.6}$$

and for all $j \in \mathbb{N}$. Introduce the cylinders

$$\begin{aligned} Q_{\tau_o} &= K_8 \times \left(\tau_o + k_o^{1-m}, \tau_o + 2k_o^{1-m}\right) \\ Q'_{\tau_o} &= K_{16} \times \left(\tau_o, \tau_o + 2k_o^{1-m}\right) \end{aligned}$$

and a nonnegative, piecewise smooth, cutoff function in Q'_{τ_o} of the form $\zeta(z,\tau) = \zeta_1(z)\zeta_2(\tau)$, where

$$\zeta_1 = \begin{cases} 1 & \text{in } \ K_8 \\ 0 & \text{in } \ \mathbb{R}^N - K_{16} \end{cases} \qquad\qquad |D\zeta_1| \le \tfrac{1}{8},$$

$$\zeta_2 = \begin{cases} 0 & \text{for } \ \tau < \tau_o \\ 1 & \text{for } \ \tau \ge \tau_o + k_o^{1-m} \end{cases} \qquad 0 \le \zeta_{2,\tau} \le \frac{1}{k_o^{1-m}}.$$

In the weak formulation of (7.5), analogous to (5.5) of Chapter 3, take as test function

$$-(v^m - k_j^m)_- \zeta^2 \quad \text{over} \quad Q'_{\tau_o},$$

for the indicated choice of cutoff function ζ. Performing calculations in all analogous to the ones of § 9 and 10 of Chapter 3, yields

$$\iint_{Q_{\tau_o}} |D(v - k_j)_-|^2 dz\, d\tau \le 2\gamma k_j^2 |Q_{\tau_o}| \tag{7.7}$$

for a constant γ depending only on the data $\{m, N, C_o, C_1\}$, and δ.

7.2.2 Estimating the Measure of the Set $[v < k_j]$ Within Q_{τ_o}

Set

$$A_j(\tau) = [v(\cdot,\tau) < k_j] \cap K_8 \qquad \text{and} \qquad A_j = [v < k_j] \cap Q_{\tau_o}.$$

By Lemma 2.2 of the Preliminaries, and (7.6)

$$
\begin{aligned}
(k_j - k_{j+1})|A_{j+1}(\tau)| &\le \frac{\gamma(N)}{|K_8 - A_j(\tau)|} \int_{K_8 \cap [k_{j+1} < v(\cdot,\tau) < k_j]} |Dv| dz \\
&\le \frac{\gamma(N)}{\alpha} \int_{K_8 \cap [k_{j+1} < v(\cdot,\tau) < k_j]} |Dv| dz
\end{aligned}
$$

for all $\tau \ge \tau_o$. Integrate this in $d\tau$ over $(\tau_o + k_o^{1-m}, \tau_o + 2k_o^{1-m})$, majorize the resulting integral on the right-hand side by the Hölder inequality, and use (7.7) to get

$$
\begin{aligned}
\frac{k_j}{2}|A_{j+1}| &\le \gamma(\text{data}, \alpha) \iint_{A_j - A_{j+1}} |Dv| dz\, d\tau \\
&\le \gamma(\text{data}, \alpha) \Big(\iint_{A_j - A_{j+1}} |Dv|^2 dz\, d\tau \Big)^{\frac{1}{2}} |A_j - A_{j+1}|^{\frac{1}{2}} \\
&\le \gamma(\text{data}, \alpha) \Big(\iint_{Q_{\tau_o}} |D(v - k_j)_-|^2 dz\, d\tau \Big)^{\frac{1}{2}} |A_j - A_{j+1}|^{\frac{1}{2}} \\
&\le \gamma(\text{data}, \alpha, \delta) k_j |Q_{\tau_o}|^{\frac{1}{2}} |A_j - A_{j+1}|^{\frac{1}{2}}.
\end{aligned}
$$

Taking the square yields the recursive inequalities

$$
|A_{j+1}|^2 \le \gamma(\text{data}, \alpha, \delta) |Q_{\tau_o}| |A_j - A_{j+1}|.
$$

Add these inequalities for $j = 0, 1, \dots, j_* - 1$, where j_* is an integer to be chosen, and majorize the sum on the right-hand side by the corresponding telescopic series. This gives

$$
(j_* - 1)|A_{j_*}|^2 \le \gamma(\text{data}, \alpha, \delta) |Q_{\tau_o}|^2.
$$

Equivalently

$$
|[v < k_{j_*}] \cap Q_{\tau_o}| \le \nu |Q_{\tau_o}| \quad \text{where} \quad \nu = \Big(\frac{\gamma(\text{data}, \alpha, \delta)}{j_*} \Big)^{\frac{1}{2}}. \tag{7.8}
$$

7.2.3 Segmenting Q_{τ_o}

Assume momentarily that j_* and hence ν have been determined. By possibly increasing j_* to be not necessarily integer, without loss of generality we may assume that $(2^{j_*})^{1-m}$ is an integer. Then subdivide Q_{τ_o} into $(2^{j_*})^{1-m}$ cylinders, each of length $k_{j_*}^{1-m}$, by setting

$$
\begin{aligned}
&Q_n = K_8 \times \big(\tau_o + k_o^{1-m} + n k_{j_*}^{1-m}, \ \tau_o + k_o^{1-m} + (n+1) k_{j_*}^{1-m} \big) \\
&\text{for } \ n = 0, 1, \dots, (2^{j_*})^{1-m} - 1.
\end{aligned}
$$

For at least one of these, say Q_n, there must hold

$$
|[v < k_{j_*}] \cap Q_n| \le \nu |Q_n|.
$$

Apply Lemma 10.1 of Chapter 3 to v over Q_n with $\xi\omega = k_{j_*}$, $a = \frac{1}{2}$, and $\theta = k_{j_*}^{1-m}$. This gives

$$v\big(z, \tau_o + k_o^{1-m} + (n+1)k_{j_*}^{1-m}\big) \geq \tfrac{1}{2} k_{j_*} \qquad \text{a.e. in } K_4,$$

provided $C\rho < 1$ and

$$\frac{|[v < k_{j_*}] \cap Q_n|}{|Q_n|} \leq 2^{-(N+2)^2} \bar{\gamma}_o(\text{data}) = \nu.$$

Choose now j_*, and hence ν, from this and (7.8). Summarizing, for such a choice of j_*, and hence ν, there exists a time level τ_1 in the range

$$\tau_o + k_o^{1-m} < \tau_1 < \tau_o + 2k_o^{1-m} \tag{7.9}$$

such that

$$v(z, \tau_1) \geq \sigma_o e^{\frac{\tau_o}{1-m}} \qquad \text{where} \qquad \sigma_o = \epsilon\, 2^{-(j_*+1)}.$$

Remark 7.1 Notice that j_* and hence ν are determined only in terms of the data and are independent of the parameter τ_o, which is still to be chosen.

7.2.4 Returning to the Original Coordinates

In terms of the original coordinates and the original function $u(x,t)$ this implies

$$u(\cdot, t_1) \geq \sigma_o M e^{-\frac{\tau_1 - \tau_o}{1-m}} \stackrel{\text{def}}{=} M_o \qquad \text{in } K_{4\rho}$$

where the time t_1 corresponding to τ_1 is computed from (7.4) and (7.9). Apply now Lemma 11.1 of Chapter 3 with M replaced by M_o and $\xi = 1$ over the cylinder

$$(0, t_1) + Q_{4\rho}^+(\theta) = K_{4\rho} \times \big(t_1, t_1 + \theta(4\rho)^2\big].$$

By choosing

$$\theta = \nu_o M_o^{1-m} \qquad \text{where} \qquad \nu_o = \nu_o(\text{data}),$$

the assumption (11.1) of Chapter 3 is satisfied, and the lemma yields

$$\begin{aligned} u(\cdot, t) &\geq \tfrac{1}{2} M_o = \tfrac{1}{2}\sigma_o M e^{-\frac{\tau_1 - \tau_o}{1-m}} \\ &\geq \tfrac{1}{2}\sigma_o e^{-\frac{2}{1-m}e^{\tau_o}} M \end{aligned} \qquad \text{in} \quad K_{2\rho} \tag{7.10}$$

for all times

$$t_1 \leq t \leq t_1 + \nu_o M_o^{1-m} (4\rho)^2. \tag{7.11}$$

If the right-hand side equals $\delta M^{1-m}\rho^2$, then (7.10) and the conclusion of Proposition 7.2 hold for the time $t = \delta M^{1-m}\rho^2$. The transformed τ_o level is still undetermined, and it will be so chosen as to verify such a requirement. Precisely, taking into account the change of variables (7.4)

$$\delta M^{1-m}\rho^2 e^{-\tau_1} = \delta M^{1-m}\rho^2 - t_1 = \nu_o\sigma_o^{1-m} M^{1-m}(4\rho)^2 e^{-(\tau_1-\tau_o)}$$

which implies

$$e^{\tau_o} = \frac{\delta}{16\nu_o\sigma_o^{1-m}}.$$

This determines quantitatively $\tau_o = \tau_o(\text{data})$. The proof of Proposition 7.2 with $0 < m < 1$ is now completed by inserting such a τ_o on the right-hand side of (7.10) and in (7.11). In particular (7.10) holds for all times

$$t_1 = \delta M^{1-m}\rho^2 - \nu_o M_o^{1-m}(4\rho)^2 \le t \le \delta M^{1-m}\rho^2.$$

From the previous definitions and transformations one estimates

$$t_1 \le (1-\varepsilon)\delta M^{1-m}\rho^2, \quad \text{where} \quad \varepsilon = e^{-\tau_o - 2e^{\tau_o}}.$$

8 Remarks and Bibliographical Notes

Proposition 2.1 was first established in [40]. The main idea is in realizing that the classical theorems of [36, 101] can be read in an "expanding fashion," instead of a "shrinking one," as originally conceived by DeGiorgi.

The notion of expansion of positivity is related to the so-called growth lemmas, introduced by Landis ([104]). Based on these lemmas, Landis gave alternative proofs of the results by DeGiorgi ([36]) and Moser ([120]), on the Hölder regularity and Harnack inequalities for solutions to second-order elliptic equations in divergence form. This approach is flexible enough as to adapt to equations in nondivergence form ([93, 94, 135], see also [3]).

For the homogeneous prototype degenerate equations (1.3) and (5.3) of Chapter 3, the expansion of positivity was realized in [39, 59] by means of comparison with suitable subsolutions.

In the full generality of Proposition 4.1, this expansion effect was established in [49], including the analysis of stability of the various parameters, either as $p \to 2$ or as $m \to 1$. The proof we present here is a simpler, more streamlined version of that in [49]. Some measure-theoretical lemmas are avoided, and the statements are shown to hold in the more general assumptions (2.1), (4.2), (5.2), with any $\alpha \in (0, 1]$ instead of $\alpha = 1$ as established in [49, 51, 54].

In the context of singular equations ($1 < p < 2$ or $0 < m < 1$) the proof of Proposition 5.1 was first given in [31] and reported in [41], Chapter IV, § 5. The proof is rather involved and not intuitive. The proof we present here follows an idea of [51] and [54]; it is more direct, being based on geometrical ideas. Both proofs require $p > 1$ and $m > 0$. The restriction is not only technical in view of the geometrical significance of the homogeneous, prototype equation (1.3) of Chapter 3 for $p = 1$ even in the elliptic case ([117]), and the homogeneous equation (5.3) of Chapter 3 for $m \to 0$ ([35, 44, 45]).

8.1 Solving (3.6)

Seek convex, symmetric about $x = 0$, smooth solutions. For these (3.6) is transformed into

$$X' = -\Big(\frac{p}{p-1}\Big)^{\frac{1}{p}}\big(C - \tfrac{1}{2}\mu X^2\big)^{\frac{1}{p}}, \quad \text{in } (0,1) \tag{8.1}$$
$$X'(0) = 0, \quad X(1) = 0$$

for positive parameters C and μ. In general this problem cannot be solved explicitly. However, one can show that solutions actually exist, by studying their qualitative behavior. By setting

$$K = \frac{2C}{\mu}$$
$$y = \alpha x, \quad \alpha = \left(\frac{K^{2-p}\mu p}{2(p-1)}\right)^{\frac{1}{p}} = \left(\frac{C^{2-p}\mu^{p-1}p}{2^{p-1}(p-1)}\right)^{\frac{1}{p}}$$
$$Y(y) = \frac{X(x)}{K},$$

problem (8.1) can be rewritten as

$$Y' = -\big(1 - Y^2\big)^{\frac{1}{p}}, \quad \text{in } (0,\alpha)$$
$$Y'(0) = 0, \quad Y(\alpha) = 0.$$

The parameter dependence is now transferred into α. Because of the two-point condition for a first-order differential equation, the problem may appear overdetermined. By standard ODE's theory, the family of solutions to the Cauchy problem

$$Y' = -\big(1 - Y^2\big)^{\frac{1}{p}}, \qquad Y'(0) = 0 \tag{8.2}$$

behave as in Figure 8.3 below. Such a Cauchy problem does not have a unique solution, and Y_{min}, Y_{max} represent the minimal and the maximal solutions, respectively. Now Y_{min} intersects the $Y = 0$ axis at y_{int}. By properly choosing the pair (C,μ), one may realize $\alpha = y_{int}$. There exist ∞^1 possible choices for (C,μ), that realize such a condition, and hence there exists an infinite number of functions X that solve the original boundary value problem. This is expected, as (3.6) is the 1–D case of the nonlinear eigenvalue problem

$$-\operatorname{div}(|Dw|^{p-2}Dw) = \mu w, \quad \text{in } E$$
$$w|_{\partial E} = 0.$$

By the results of [73], such a problem admits infinitely many solutions.

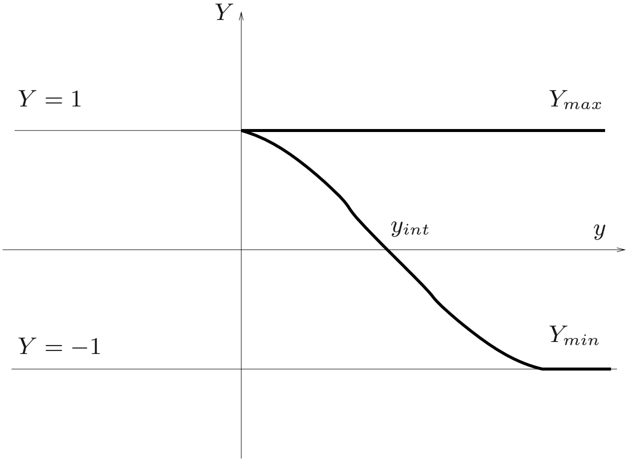

Fig. 8.3. Qualitative Behavior of the Solution to (8.2)

5

The Harnack Inequality for Degenerate Equations

1 The Intrinsic Harnack Inequality

1.1 The Mean Value Form of Moser's Harnack Inequality

Let u be a continuous, nonnegative, local, weak solution to the homogeneous, nondegenerate version of (1.1)–(1.2) of Chapter 3 in some domain E_T, where $p = 2$ and $C = 0$. Such solutions satisfy the Harnack inequality of Moser ([121, 122]), which we present in various forms. Fix $(x_o, t_o) \in E_T$ and $\rho > 0$ and construct the cylinders

$$\begin{aligned} Q_\rho^+(x_o, t_o) &= K_\rho(x_o) \times (t_o, t_o + \rho^2] \\ Q_\rho^-(x_o, t_o) &= K_\rho(x_o) \times (t_o - \rho^2, t_o]. \end{aligned}$$

The first has bottom "vertex" at (x_o, t_o) and the second has top "vertex" at (x_o, t_o). Assume that ρ is so small that $Q_{4\rho}^\pm(x_o, t_o) \subset E_T$. Fix $\sigma \in (0, 1)$ and inside $Q_\rho^\pm(x_o, t_o)$ construct the two subcylinders

$$\begin{aligned} Q_{\sigma\rho}^+ &= K_{\sigma\rho}(x_o) \times (t_o + (\sigma\rho)^2, t_o + \rho^2] \\ Q_{\sigma\rho}^- &= K_{\sigma\rho}(x_o) \times (t_o - \rho^2, t_o - (\sigma\rho)^2]. \end{aligned}$$

There exists a constant $\gamma(\sigma)$ depending only on the data $\{N, C_o, C_1\}$ and σ, and independent of (x_o, t_o) and ρ, such that

$$\sup_{Q_{\sigma\rho}^-} u \le \gamma(\sigma) \inf_{Q_{\sigma\rho}^+} u. \tag{1.1}$$

The two cylinders $Q_{\sigma\rho}^\pm$ are separated along the time axis by a distance of $2(\sigma\rho)^2$, and the constant $\gamma(\sigma) \to \infty$ as $\sigma \to 0$. However, $\gamma(\sigma)$ is stable as $\sigma \to 1$. Other than the indicated separation between $Q_{\sigma\rho}^+$ and $Q_{\sigma\rho}^-$, there is great freedom in choosing these cylinders. For example, one could take $\sigma \approx 1$, keep $Q_{\sigma\rho}^+$ fixed, and choose $Q_{\sigma\rho}^-$ with its top "vertex" at (x_o, t_o). This would

E. DiBenedetto et al., *Harnack's Inequality for Degenerate and Singular Parabolic Equations*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-1584-8_5,

keep it separated by a distance $(\sigma\rho)^2$ from $Q^+_{\sigma\rho}$. By possibly modifying the form of the constant γ, this would imply

$$u(x_o,t_o) \le \gamma \inf_{K_\rho(x_o)} u(\cdot, t_o+\rho^2). \tag{1.1$^+$}$$

Likewise one could take $\sigma \approx 1$, keep $Q^-_{\sigma\rho}$ fixed, and choose $Q^+_{\sigma\rho}$ with its bottom "vertex" at (x_o,t_o). This would keep it separated by a distance $(\sigma\rho)^2$ from $Q^-_{\sigma\rho}$. By possibly modifying the form of the constant γ, this would imply

$$\gamma^{-1} \sup_{K_\rho(x_o)} u(\cdot, t_o-\rho^2) \le u(x_o,t_o). \tag{1.1$^-$}$$

The inequalities $(1.1)^\pm$ are "forward" and "backward" in time. They could be combined to yield

$$\gamma^{-1} \sup_{K_\rho(x_o)} u(\cdot, t_o-\rho^2) \le u(x_o,t_o) \le \gamma \inf_{K_\rho(x_o)} u(\cdot, t_o+\rho^2) \tag{1.2}$$

for a constant γ depending only on the data $\{N, C_o, C_1\}$ and independent of $(x_o,t_o) \in E_T$ and $\rho > 0$, provided $Q^\pm_{4\rho}(x_o,t_o) \subset E_T$.

We call this the *mean value form* of the Harnack inequality for nonnegative solutions to nondegenerate parabolic equations. The terminology is suggested by the mean value property of harmonic functions. The latter implies that the value $u(x_o)$ at one point x_o of a nonnegative harmonic function u, controls its maximum and minimum in a ball centered at x_o ([43], Chapter II, § 5). The point (x_o,t_o) can be, roughly speaking, regarded as the "center" of the cylindrical domain $Q^-_\rho(x_o,t_o) \cup Q^+_\rho(x_o,t_o)$.

1.2 The Intrinsic Mean Value Harnack Inequality for Degenerate Equations

Let u be a continuous, nonnegative, local, weak solution to the degenerate equations (1.1)–(1.2) of Chapter 3 in some domain E_T, for $p > 2$. It is known that these solutions are locally bounded in E_T ([41], Chapter V), which henceforth we assume.

Fix $(x_o,t_o) \in E_T$ such that $u(x_o,t_o) > 0$ and construct the cylinders

$$(x_o,t_o) + Q^\pm_{4\rho}(\theta) \qquad \text{where} \qquad \theta = \left(\frac{c}{u(x_o,t_o)}\right)^{p-2} \tag{1.3}$$

and c is a given positive constant. These cylinders are "intrinsic" to the solution since their length is determined by the value of u at (x_o,t_o). Cylindrical domains of the form $K_\rho \times (0,\rho^p]$ reflect the natural, parabolic space-time dilations that leave the homogeneous, degenerate, prototype equation (1.3) of Chapter 3 invariant. The latter, however, is not homogeneous with respect to the solution u. The time dilation by a factor $u(x_o,t_o)^{2-p}$ is intended to restore the homogeneity, and the Harnack inequality holds in such an intrinsic geometry, as precisely stated in the following theorem.

Theorem 1.1 *Let u be a continuous, nonnegative, local, weak solution to the degenerate equations (1.1)–(1.2) of Chapter 3, in E_T. There exist positive constants c and γ depending only on the data $\{p, N, C_o, C_1\}$, such that for all intrinsic cylinders $(x_o, t_o) + Q^{\pm}_{4\rho}(\theta)$ as in (1.3), contained in E_T, either*

$$\gamma C\rho > \min\{1\,,\, u(x_o, t_o)\}$$

or

$$\gamma^{-1} \sup_{K_\rho(x_o)} u(\cdot, t_o - \theta\rho^p) \le u(x_o, t_o) \le \gamma \inf_{K_\rho(x_o)} u(\cdot, t_o + \theta\rho^p). \tag{1.4}$$

Thus the form (1.2) continues to hold for nonnegative solutions to the degenerate equations (1.1)–(1.2) of Chapter 3, although in their own intrinsic geometry, made precise by (1.3). In analogy with (1.2) and $(1.1)^{\pm}$, we call intrinsic, "forward" and "backward" Harnack inequalities, the right and left inequalities in (1.4).

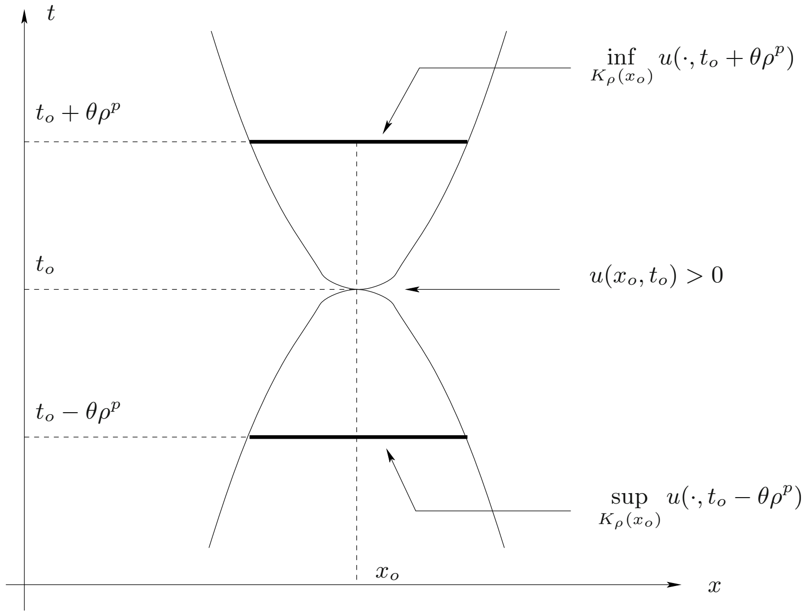

Fig. 1.1. Mean Value Harnack Inequality

Remark 1.1 The constants γ and c deteriorate as $p \to \infty$ in the sense that $\gamma(p), c(p) \to \infty$ as $p \to \infty$. However, they are stable as $p \to 2$ in the sense of (1.9) of Chapter 3. Thus by formally letting $p \to 2$ in (1.4), one recovers the classical Moser's Harnack inequality in the form (1.2).

Remark 1.2 Most of our arguments are based on the energy estimates and DeGiorgi-type lemmas of § 2–4 of Chapter 3 and the expansion of positivity of Chapter 4. According to the discussion in § 1.3 and Remarks 2.2, 3.1, and 4.3 of Chapter 3, a constant γ depends only on the data if it can be quantitatively determined a priori only in terms of $\{p, N, C_o, C_1\}$. The constant C appearing in the structure conditions (1.2) of Chapter 3, enters in the statement of Theorem 1.1 only through an alternative.

Remark 1.3 The constants γ in (1.4) and c in (1.3) are quantitatively connected. The first quantifies the upper bound in (1.4) and the constant c in (1.3) quantifies the "waiting time" for an intrinsic Harnack estimate to hold. The proof will determine these constants quantitatively only in terms of $\{p, N, C_o, C_1\}$. Whence they are determined, one may choose a *smaller* c, and hence a *shorter* "waiting time" provided the constant γ is chosen to be *larger* (§ 2.4).

Remark 1.4 The theorem has been stated for continuous solutions, to give meaning to $u(x_o, t_o)$. While it is known that local, weak solutions to (1.1)–(1.2) of Chapter 3, for $p > 2$, are locally Hölder continuous ([38, 41, 75]), the theorem continues to hold for almost all $(x_o, t_o) \in E_T$ and the corresponding cylinders $(x_o, t_o) + Q^{\pm}_{4\rho}(\theta) \subset E_T$. The intrinsic Harnack inequality, in turn, can be used to prove that these local solutions, irrespective of their signum, are indeed locally Hölder continuous within their domain of definition. This will be shown in § 4.

1.3 Significance of Theorem 1.1

The inequality (1.4) is "intrinsic" in that the waiting time from t_o to $t_o + \theta\rho^p$ depends on the solution at (x_o, t_o). Such an intrinsic dependence is a consequence of the intrinsic expansion of positivity of Chapter 4, and it cannot be removed. Indeed (1.4) is false in a geometry where θ is a constant independent of $u(x_o, t_o)$. This can be verified for the Barenblatt solution (§ 3.2 of Chapter 4), by the same arguments as in § 3.2 of Chapter 4. This effect is not due to the moving boundary of the Barenblatt solution, delimiting its support. Rather, it is structural to these degenerate equations as shown by the family of functions, parametrized by $T \in \mathbb{R}$,

$$u(x,t) = C(N,p)\Big(\frac{|x|^p}{T-t}\Big)^{\frac{1}{p-2}} \tag{1.5}$$

defined in $\mathbb{R}^N \times (-\infty, T)$, where

$$C(N,p) = \Big[\frac{1}{\lambda}\Big(\frac{p-2}{p}\Big)^{p-1}\Big]^{\frac{1}{p-2}} \qquad \text{and} \qquad \lambda = N(p-2) + p. \tag{1.6}$$

One verifies that this solves the prototype equation (1.3) of Chapter 3 for any $p > 2$, and it can be constructed by separating the variables. Fix $\rho > 0$ and

$t_o < T - (8\rho)^p$, so that the cylinders $(x_o, t_o) + Q^{\pm}_{4\rho}(1)$ are contained in the domain of definition of u, for all $x_o \in \mathbb{R}^N$. Then letting $x_o \to 0$ by keeping t_o and ρ fixed, shows that (1.4) fails if the geometry is not intrinsic, that is, if $\theta = 1$.

2 Proof of the Intrinsic, Forward Harnack Inequality in Theorem 1.1

Fix $(x_o, t_o) \subset E_T$, assume that $u(x_o, t_o) > 0$, and construct the cylinders $(x_o, t_o) + Q^{\pm}_{4\rho}(\theta) \subset E_T$ as in (1.3), where the constant $c \geq 1$ is to be determined. The change of variables

$$x \to \frac{x - x_o}{\rho} \qquad t \to u(x_o, t_o)^{p-2} \frac{t - t_o}{\rho^p}$$

maps these cylinders into $Q^{\pm}$, where

$$Q^+ = K_4 \times (0, 4^p c^{p-2}], \qquad Q^- = K_4 \times (-4^p c^{p-2}, 0].$$

Denoting again by (x, t) the transformed variables, the rescaled function

$$v(x, t) = \frac{1}{u(x_o, t_o)} u\Big(x_o + \rho x, t_o + \frac{t\rho^p}{u(x_o, t_o)^{p-2}}\Big)$$

is a bounded, nonnegative, weak solution to

$$v_t - \operatorname{div} \bar{\mathbf{A}}(x, t, v, Dv) = \bar{B}(x, t, v, Dv) \tag{2.1}$$

weakly in $Q = Q^+ \cup Q^-$, where $\bar{\mathbf{A}}$ and $\bar{B}$ satisfy the structure conditions

$$\begin{cases} \bar{\mathbf{A}}(x, t, v, Dv) \cdot Dv \geq C_o |Dv|^p - \bar{C}^p \\ |\bar{\mathbf{A}}(x, t, v, Dv)| \leq C_1 |Dv|^{p-1} + \bar{C}^{p-1} \\ |\bar{B}(x, t, v, Dv)| \leq C\rho |Dv|^{p-1} + C\rho\bar{C}^{p-1} \end{cases} \quad \text{a.e. in } Q \tag{2.2}$$

where

$$\bar{C} = \frac{C\rho}{u(x_o, t_o)} \tag{2.3}$$

and C_o, C_1, and C are as in (1.2) of Chapter 3. Moreover $v(0, 0) = 1$.

The theorem is a consequence of the following

Proposition 2.1 *There exist constants $\gamma_o \in (0, 1)$, $\gamma_1, \gamma_2 > 1$, that can be quantitatively determined a priori only in terms of the data $\{p, N, C_o, C_1\}$, and independent of $u(x_o, t_o)$, such that either*

$$\gamma_2 C\rho > \min\{1\,,\, u(x_o, t_o)\}$$

or

$$v(\cdot, \gamma_1) \geq \gamma_o \qquad \text{in } K_1.$$

2.1 Proof of Proposition 2.1

For $\tau \in [0,1)$, introduce the family of nested cylinders $\{Q_\tau^-\}$ with the same "vertex" at $(0,0)$, and the families of nonnegative numbers $\{M_\tau\}$ and $\{N_\tau\}$, defined by

$$Q_\tau^- = K_\tau \times (-\tau^p, 0], \qquad M_\tau = \sup_{Q_\tau^-} v, \qquad N_\tau = (1-\tau)^{-\beta},$$

where $\beta > 1$ is to be chosen. The two functions $[0,1) \ni \tau \to M_\tau, N_\tau$ are increasing, and $M_o = N_o = 1$ since $v(0,0) = 1$. Moreover $N_\tau \to \infty$ as $\tau \to 1$ whereas M_τ is bounded since v is locally bounded. Therefore the equation $M_\tau = N_\tau$ has roots and we let τ_* denote the largest one. By the continuity of v, there exists $(y,s) \in \bar{Q}_{\tau_*}$ such that

$$v(y,s) = M_{\tau_*} = N_{\tau_*} = (1-\tau_*)^{-\beta} \overset{\text{def}}{=} M. \tag{2.4}$$

Moreover

$$(y,s) + Q_r^- \subset Q^-_{\frac{1+\tau_*}{2}} \subset Q_1, \qquad \text{where } r \overset{\text{def}}{=} \tfrac{1}{2}(1-\tau_*). \tag{2.5}$$

Therefore by the definition of M_τ and N_τ

$$\sup_{(y,s)+Q_r^-} v \le \sup_{Q^-_{\frac{1+\tau_*}{2}}} v \le 2^\beta (1-\tau_*)^{-\beta} \overset{\text{def}}{=} M_*.$$

The parameter τ_*, and hence the upper bound M_*, is only known qualitatively, and β has to be chosen. The arguments below have the role of eliminating the qualitative knowledge of τ_* by a quantitative choice of β.

2.1.1 Local Largeness of v Near (y,s)

The largeness of v at (y,s) as expressed by (2.4), propagates to a full space-time neighborhood nearby (y,s). To render this quantitative, set

$$\xi = 1 - \frac{1}{2^{\beta+1}}, \qquad a = \frac{1 - \dfrac{3}{2}\dfrac{1}{2^{\beta+1}}}{1 - \dfrac{1}{2^{\beta+1}}}.$$

Lemma 2.1 *Either $\bar{C} \ge 1$, or*

$$|[v > \tfrac{1}{2}M] \cap [(y,s) + Q_r^-(M_*^{2-p})]| > \nu |Q_r^-(M_*^{2-p})| \tag{2.6}$$

where

$$\nu = \Big(\frac{1-a}{\gamma(\text{data})}\Big)^{N+p} \frac{\xi^{\frac{N(p-2)}{p}}}{(1+\xi^{p-2})^{\frac{p+N}{p}}}.$$

Proof Assume that $\bar{C} < 1$. If (2.6) is violated, apply Lemma 3.1 of Chapter 3, over the cylinder

$$(y,s) + Q_r^-(M_*^{2-p}) = K_r(y) \times (s - M_*^{2-p} r^p\,,\, s]$$

in the form (3.1)–(3.3), for the choices $\mu_+ = \omega = M_*$ and $\theta = M_*^{2-p}$, to conclude that

$$v(y,s) \le M_*(1 - a\xi) = \tfrac{3}{4}(1-\tau_*)^{-\beta},$$

contradicting (2.4). ∎

Remark 2.1 The indicated expressions of ξ, a, and ν imply that $\nu(\beta)$ depends on the data and β, but is independent of τ_*, and is stable as $p \to 2$. Such a constant will be made quantitative whence β is chosen, dependent only on the data. We continue to denote by ν such a constant, keeping in mind its dependence on β.

Corollary 2.1 *Either* $\bar{C} \ge 1$, *or there exists a time level*

$$s - M_*^{2-p} r^p \le \bar{s} \le s$$

such that

$$|[v(\cdot,\bar{s}) > \tfrac{1}{2}M] \cap K_r(y)| > \nu |K_r|. \tag{2.7}$$

2.2 Expanding the Positivity of v

Starting from (2.7) apply the expansion of positivity of Proposition 4.1 of Chapter 4 to the weak solution v to (2.1)–(2.2), with $\frac{1}{2}M$ and r given by (2.4)–(2.5) and $\alpha = \nu$. Then, taking into account the expression (2.3) of $\bar{C}$, either

$$u(x_o,t_o) \le \gamma_* C \rho \frac{r}{M} \tag*{$(2.8)_*$}$$

or

$$v(\cdot,t) \ge \eta_* M \qquad \text{in } K_{2r}(y) \tag*{$(2.9)_*$}$$

for all t in the range

$$\bar{s} + \frac{b_*^{p-2}}{(\eta_* M)^{p-2}} \tfrac{1}{2}\delta_* r^p \le t \le \bar{s} + \frac{b_*^{p-2}}{(\eta_* M)^{p-2}} \delta_* r^p \;=\; s_*. \tag*{$(2.10)_*$}$$

Remark 2.2 The constants $\{\gamma_*, b_*, \delta_*, \eta_*\}$ in $(2.8)_*$–$(2.10)_*$ depend on the data $\{N, p, C_o, C_1\}$ and β, through the constant $\nu(\beta)$ in (2.7). However, they are independent of the constant $\bar{C}$ in (2.3), as discussed in Remark 1.2. These constants are also independent of M and r. The parameter β is still to be chosen.

The expansion of positivity implies in particular

$$|[v(\cdot, s_*) > \eta_* M] \cap K_{2r}(y)| = |K_{2r}|. \tag{2.11}$$

Therefore the expansion of positivity of Proposition 4.1 of Chapter 4 can be applied again, starting at the time level s_*, with M replaced by $(\eta_* M)$, $\rho = 2r$, and $\alpha = 1$. It gives that either

$$u(x_o, t_o) \le \gamma C \rho \frac{2r}{\eta_* M} \qquad (2.8)_1$$

or

$$v(\cdot, t) \ge \eta(\eta_* M) \qquad \text{in } \; K_{4r}(y) \qquad (2.9)_1$$

for all t in the range

$$s_* + \frac{b^{p-2}}{[\eta(\eta_* M)]^{p-2}} \tfrac{1}{2} \delta (2r)^p \le t \le s_* + \frac{b^{p-2}}{[\eta(\eta_* M)]^{p-2}} \delta (2r)^p \; = \; s_1. \qquad (2.10)_1$$

Remark 2.3 The constants $\{\gamma, b, \delta, \eta\}$ in $(2.8)_1$–$(2.10)_1$ are different from the set of constants $\{\gamma_*, b_*, \delta_*, \eta_*\}$ in $(2.8)_*$–$(2.10)_*$. They depend on the data $\{N, p, C_o, C_1\}$ but they are no longer dependent on β. By the expansion of positivity of Proposition 4.1 of Chapter 4 these parameters depend only on $\{N, p, C_o, C_1\}$, and the measure-theoretical lower bound α in (4.2) of Chapter 4. Such a measure-theoretical lower bound in the current context is $\alpha = 1$, as provided by (2.11). The parameter β is still to be chosen.

Starting from $(2.9)_1$ the expansion of positivity can now be applied again with M replaced by $\eta(\eta_* M)$, and ρ replaced by $4r$, and $\alpha = 1$ to yield that either

$$u(x_o, t_o) \le \gamma C \rho \frac{4r}{\eta(\eta_* M)} \qquad (2.8)_2$$

or

$$v(\cdot, t) \ge \eta^2(\eta_* M) \qquad \text{in } \; K_{8r}(y) \qquad (2.9)_2$$

for all t in the range

$$s_1 + \frac{b^{p-2}}{[\eta^2(\eta_* M)]^{p-2}} \tfrac{1}{2} \delta (4r)^p \le t \le s_1 + \frac{b^{p-2}}{[\eta^2(\eta_* M)]^{p-2}} \delta (4r)^p \; = \; s_2 \qquad (2.10)_2$$

for the same set of parameters $\{\gamma, b, \delta, \eta\}$ as in $(2.8)_1$–$(2.10)_1$. These parameters depend on $\{p, N, C_o, C_1\}$ but they are independent of β.

The process can be iterated to yield that either

$$u(x_o, t_o) \le \gamma C \rho \frac{2^n r}{\eta^{n-1}(\eta_* M)} \qquad (2.8)_n$$

or

$$v(\cdot, t) \ge \eta^n(\eta_* M) \quad \text{in } \; K_{2^{n+1} r}(y) \qquad (2.9)_n$$

for all t in the range

$$\begin{aligned} s_{n-1} + \frac{b^{p-2}}{[\eta^n(\eta_* M)]^{p-2}} \tfrac{1}{2}\delta(2^n r)^p &\le t \\ &\le s_{n-1} + \frac{b^{p-2}}{[\eta^n(\eta_* M)]^{p-2}} \delta(2^n r)^p \;=\; s_n. \end{aligned} \tag{2.10$_n$}$$

2.3 Proof of Proposition 2.1 Concluded

Without loss of generality we may assume that $(1-\tau_*)$ is a negative, integral power of 2. Then choosing n so that $2^{n+1}r = 2$, the cube $K_2(y)$ covers the cube K_1 centered at $x = 0$, and

$$v(\cdot,t) \ge \eta^n(\eta_* M) \quad \text{in } K_1,$$

for all t in the interval $(2.10)_n$. For the indicated choice of n, and the values of M and r given by (2.4)–(2.5)

$$\begin{aligned} \eta^n(\eta_* M) = \eta^n \frac{\eta_*(\beta)}{(1-\tau_*)^\beta} &= \eta^n \frac{2^{-\beta}\eta_*(\beta)}{r^\beta} \\ &= (2^\beta \eta)^n \frac{\eta_*(\beta)}{(2^{n+1}r)^\beta} = (2^\beta\eta)^n \gamma_o, \end{aligned}$$

where

$$\gamma_o = 2^{-\beta}\eta_*(\beta). \tag{2.12}$$

To remove the qualitative knowledge of τ_* and hence n, choose β from $2^\beta\eta = 1$. Notice that such a choice is possible, since by Remark 2.3 the parameter η is independent of β. This makes γ_o quantitative. The time level s_n is computed from

$$s_n = s_* + \frac{b^{p-2}}{(\eta_* M)^{p-2}} \delta r^p \sum_{j=1}^{n} \Big(\frac{2^p}{\eta^{p-2}}\Big)^j.$$

Therefore the range of t for which $(2.9)_n$ holds, can be estimated as

$$s_* + \frac{b^{p-2}}{[\eta^n(\eta_* M)]^{p-2}} \tfrac{1}{2}\delta(2^n r)^p \le t \le s_* + \frac{b^{p-2}}{[\eta^n(\eta_* M)]^{p-2}} 2\delta(2^n r)^p.$$

From the previous choices one estimates

$$s_* + \bar\gamma_1 \le t \le s_* + 4\bar\gamma_1 \quad \text{where } \bar\gamma_1 = \Big(\frac{b}{\gamma_o}\Big)^{p-2}\frac{\delta}{2}.$$

By choosing η_* even smaller if necessary, we may insure that $\bar\gamma_1 \ge 1$ so that $s_* + \bar\gamma_1 \ge 0$ and hence

$$\gamma_1 = 3\bar\gamma_1 \tag{2.13}$$

is included in the times for which $(2.9)_n$ holds.

From Remark 2.3 it follows that b and η do not depend on η_*, and hence the assumption of possibly taking η_* smaller is justified.

Finally, from the indicated choices of n and β the alternatives $(2.8)_*$–$(2.8)_n$ can be rewritten as $u(x_o, t_o) \le \gamma_2 C\rho$ for $\gamma_2 = \frac{\gamma\gamma_*}{\gamma_o}$. ■

2.4 On the Connection Between γ and c

Examine now the connection between the constant γ that quantifies the bound above in (1.4) and the constant c in (1.3) that quantifies the "waiting time" for the forward, intrinsic Harnack estimate to hold.

The constant γ is generated by Proposition 2.1 and $\gamma = \gamma_o^{-1}$ with γ_o given by (2.12) in terms of β and $\eta_*(\beta)$. The "waiting time" is determined by (2.13).

Following Remark 2.2, the set of constants $\{\gamma, b, \delta, \eta\}$ are independent of $\{\gamma_*, b_*, \delta_*, \eta_*\}$. Whence the parameter β has been chosen the latter are quantitatively determined.

Having determined β and hence η_*, one may repeat the proof with the choice of a *smaller* η_*. In view of the indicated independence of $\{\gamma, b, \delta, \eta\}$ from the parameters $\{\gamma_*, b_*, \delta_*, \eta_*\}$, the process yields the same functional dependence as in (2.12)–(2.13).

Thus the "waiting time" can be made *larger* by choosing a smaller η_*, provided the constant γ is made larger as quantified by (2.12).

3 Proof of the Intrinsic, Backward Harnack Inequality in Theorem 1.1

Fix $(x_o, t_o) \in E_T$, assume $u(x_o, t_o) > 0$, and let $(x_o, t_o) + Q^{\pm}_{4\rho}(\theta)$ as in (1.3). Seek those values of $t < t_o$, if any, for which

$$u(x_o, t) = 2\gamma u(x_o, t_o) \tag{3.1}$$

where γ is the constant in the intrinsic, forward estimate (1.3)–(1.4), which by the results of the previous section, holds for all such intrinsic cylinders. If such a t does not exist

$$u(x_o, t) < 2\gamma u(x_o, t_o) \qquad \text{for all } \ t \in (t_o - \theta(4\rho)^p, t_o). \tag{3.2}$$

We establish by contradiction that this in turn implies

$$\sup_{K_\rho(x_o)} u(\cdot, t_o - \theta\rho^p) \le 2\gamma^2 u(x_o, t_o). \tag{3.3}$$

If not, by continuity there exists $x_* \in K_\rho(x_o)$ such that

$$u(x_*, t_o - \theta\rho^p) = 2\gamma^2 u(x_o, t_o).$$

Apply the intrinsic, forward Harnack inequality in (1.3)–(1.4) with (x_o, t_o) replaced by $(x_*, t_o - \theta\rho^p)$, to get

$$u(x_*, t_o - \theta\rho^p) \le \gamma \inf_{K_\rho(x_*)} u(\cdot, t_o - \theta\rho^p + \theta_*\rho^p) \tag{3.4}$$

where

$$\theta_* = \Big(\frac{c}{u(x_*, t_o - \theta\rho^p)}\Big)^{p-2}.$$

Now $x_o \in K_\rho(x_*)$ and, since $\gamma > 1$ and $p > 2$,

$$\begin{aligned} t_o - \theta\rho^p + \theta_*\rho^p &= t_o - \Big(\frac{c}{u(x_o, t_o)}\Big)^{p-2}\rho^p + \Big(\frac{c}{u(x_*, t_o - \theta\rho^p)}\Big)^{p-2}\rho^p \\ &= t_o - \Big(\frac{c}{u(x_o, t_o)}\Big)^{p-2}\rho^p + \frac{1}{(2\gamma^2)^{p-2}}\Big(\frac{c}{u(x_o, t_o)}\Big)^{p-2}\rho^p \\ &= t_o - \big[1 - (2\gamma^2)^{2-p}\big]\Big(\frac{c}{u(x_o, t_o)}\Big)^{p-2}\rho^p < t_o. \end{aligned}$$

Therefore from (3.2) and (3.4)

$$2\gamma^2 u(x_o, t_o) = u(x_*, t_o - \theta\rho^p) \le \gamma u(x_o, t_o - \theta\rho^p + \theta_*\rho^p) < 2\gamma^2 u(x_o, t_o).$$

The contradiction establishes (3.3).

3.1 There Exists $t < t_o$ Satisfying (3.1)

Let $\tau < t_o$ be the first time for which (3.1) holds. For such a time

$$t_o - \tau > \Big(\frac{c}{u(x_o, \tau)}\Big)^{p-2}\rho^p = \frac{1}{(2\gamma)^{p-2}}\Big(\frac{c}{u(x_o, t_o)}\Big)^{p-2}\rho^p. \tag{3.5}$$

Indeed, if such inequality were violated, by applying the intrinsic, forward Harnack inequality in (1.3)–(1.4) with (x_o, t_o) replaced by (x_o, τ) would give

$$2\gamma u(x_o, t_o) = u(x_o, \tau) \le \gamma u(x_o, t_o).$$

Set

$$s = t_o - \frac{1}{(2\gamma)^{p-2}}\Big(\frac{c}{u(x_o, t_o)}\Big)^{p-2}\rho^p.$$

From the definitions, the continuity of u and (3.5)

$$\tau < s < t_o \qquad \text{and} \qquad u(x_o, s) \le 2\gamma u(x_o, t_o).$$

We claim that

$$u(y, s) < 2\gamma u(x_o, t_o) \quad \text{for all } y \in K_\rho(x_o). \tag{3.6}$$

Proceeding by contradiction, let $y \in K_\rho(x_o)$ be such that

$$u(y,s) = 2\gamma u(x_o,t_o).$$

Apply the intrinsic, forward Harnack inequality in (1.3)–(1.4) with (x_o,t_o) replaced by (y,s) to obtain

$$u(y,s) \le \gamma \inf_{K_\rho(y)} u(\cdot, s+\theta_s\rho^p), \qquad \text{where } \theta_s = \Big(\frac{c}{u(y,s)}\Big)^{p-2}.$$

Using the definition of s and θ_s one computes

$$s + \theta_s \rho^p = t_o.$$

Therefore, since $y \in K_\rho(x_o)$

$$2\gamma u(x_o,t_o) = u(y,s) \le \gamma \inf_{K_\rho(y)} u(\cdot,t_o) \le \gamma u(x_o,t_o).$$

The contradiction implies that (3.6) holds true. Summarizing the results of these alternatives, either (3.3) holds or (3.6) is in force. The proof is now concluded by using the arbitrariness of ρ and by properly redefining γ. ■

4 The Intrinsic Harnack Inequality Implies the Hölder Continuity

The intrinsic Harnack inequality of Theorem 1.1 can be used to establish that local, weak solutions u to (1.1)–(1.2) of Chapter 3, in E_T, are locally Hölder continuous in E_T, irrespective of their signum, and permits one to exhibit a quantitative Hölder modulus of continuity. These solutions are locally bounded ([41]) in E_T. To streamline the presentation and symbolism, we assume that $u \in L^\infty(E_T)$. Let

$$\Gamma = \partial E_T - \bar{E} \times \{T\}$$

denote the parabolic boundary of E_T, and for a compact set $K \subset E_T$ introduce the *intrinsic*, parabolic p-distance from K to Γ by

$$p - \operatorname{dist}(K;\Gamma) \overset{\text{def}}{=} \inf_{\substack{(x,t)\in K\\(y,s)\in\Gamma}} \Big(|x-y| + \|u\|_{\infty,E_T}^{\frac{p-2}{p}} |t-s|^{\frac{1}{p}}\Big).$$

Theorem 4.1 *Let u be a bounded, local, weak solution to (1.1)–(1.2) of Chapter 3. Then u is locally Hölder continuous in E_T, and there exist constants $\gamma > 1$ and $\alpha \in (0,1)$ that can be determined a priori only in terms of the data $\{p, N, C_o, C_1\}$, such that for every compact set $K \subset E_T$,*

$$|u(x_1,t_1) - u(x_2,t_2)| \le \gamma \|u\|_{\infty,E_T} \Big(\frac{|x_1-x_2| + \|u\|_{\infty,E_T}^{\frac{p-2}{p}} |t_1-t_2|^{\frac{1}{p}}}{p - \operatorname{dist}(K;\Gamma)}\Big)^\alpha$$

for every pair of points (x_1,t_1), and $(x_2,t_2) \in K$.

Proof Fix a point in E_T, which up to a translation we take to be the origin of $\mathbb{R}^{N+1}$. For $\rho > 0$ consider the cylinder

$$Q_* = K_\rho \times (-\rho^2, 0]$$

with "vertex" at $(0, 0)$ and set

$$\mu_o^+ = \sup_{Q_*} u, \qquad \mu_o^- = \inf_{Q_*} u, \qquad \omega_o = \operatorname*{osc}_{Q_*} u = \mu_o^+ - \mu_o^- .$$

With ω_o at hand, construct now the cylinder of intrinsic geometry

$$Q_o = K_\rho \times (-\theta_o \rho^p, 0] \quad \text{where} \quad \theta_o = \left(\frac{c}{\omega_o}\right)^{p-2}$$

and c is a constant to be determined in terms only of the data $\{p, N, C_o, C_1, C\}$, and independent of u and ρ. If $\omega_o \ge c\rho$, then $Q_o \subset Q_*$. Theorem 4.1 is then a consequence of the following.

Proposition 4.1 *There exist constants $c, \gamma > 1$, and $\varepsilon, \delta \in (0, 1)$, that can be quantitatively determined only in terms of the data $\{p, N, C_o, C_1\}$, and independent of u and ρ, such that, if $\omega_o \ge c\rho$, setting $\rho_o = \rho$ and*

$$\omega_n = \delta\omega_{n-1}, \quad \theta_n = \left(\frac{c}{\omega_n}\right)^{p-2}, \quad \rho_n = \varepsilon\rho_{n-1}, \quad Q_n = Q^-_{\rho_n}(\theta_n)$$

for $n \in \mathbb{N}$, we have $Q_{n+1} \subset Q_n$, and either

$$\operatorname*{osc}_{Q_n} u \le 4\gamma \frac{C}{\varepsilon} \rho_n \qquad \text{or} \qquad \operatorname*{osc}_{Q_n} u \le \omega_n .$$

Proof We exhibit constants c, δ, ε depending only on the data $\{p, N, C_o, C_1\}$, such that if the statement holds for n, it continues to hold for $n + 1$. Thus assume Q_n has been constructed and that the statement holds up to n. Set

$$\mu_n^+ = \sup_{Q_n} u, \qquad \mu_n^- = \inf_{Q_n} u, \qquad \text{and} \qquad P_n = (0, -\tfrac{1}{2}\theta_n \rho_n^p).$$

The point P_n is roughly speaking the "mid-point" of Q_n. The two functions $(M_n - u)$ and $(u - m_n)$ are nonnegative weak solutions to (1.1) of Chapter 3 in Q_n. Either of these satisfies the intrinsic Harnack inequality with respect to P_o in Q_n, if its "intrinsic waiting time"

$$\left(\frac{c}{\mu_n^+ - u(P_n)}\right)^{p-2} \rho_n^p, \qquad \text{or} \qquad \left(\frac{c}{u(P_n) - \mu_n^-}\right)^{p-2} \rho_n^p$$

is of the order of $\theta_n \rho_n^p$. At least one of the two inequalities

$$\mu_n^+ - u(P_n) > \tfrac{1}{4}\omega_n, \qquad u(P_n) - \mu_n^- > \tfrac{1}{4}\omega_n$$

must hold. Assuming the first holds true, apply the intrinsic, forward Harnack inequality of Theorem 1.1. By possibly modifying the constant c appearing in (1.3) that determines the "waiting time," either

$$\gamma C\rho_n \geq \big(\mu_n^+ - u(P_n)\big) > \frac{1}{4}\omega_n \tag{4.1}$$

or

$$\inf_{Q^-_{\frac{1}{4}\rho_n}(\theta_n)} (\mu_n^+ - u) \geq \frac{1}{\gamma}\big(\mu_n^+ - u(P_n)\big) > \frac{1}{4\gamma}\omega_n. \tag{4.2}$$

Choosing

$$\delta = (1 - \frac{1}{4\gamma}) \qquad \text{and} \qquad \varepsilon = \frac{1}{4}\delta^{\frac{p-2}{p}}$$

one verifies that $Q_{n+1} \subset Q^-_{\frac{1}{4}\rho_n}(\theta_n) \subset Q_n$. Then if (4.1) occurs,

$$\operatorname*{osc}_{Q_{n+1}} u \leq C\tilde{\gamma}\rho_{n+1}, \qquad \text{for} \qquad \tilde{\gamma} = \frac{4\gamma}{\varepsilon}.$$

If (4.2) occurs, then

$$\mu_n^+ \geq \sup_{Q_{n+1}} u + \frac{1}{4\gamma}\omega_n.$$

From this, subtracting $\inf_{Q_{n+1}} u$ from both sides

$$\omega_n \geq \operatorname*{osc}_{Q_{n+1}} u + \frac{1}{4\gamma}\omega_n.$$

Thus

$$\operatorname*{osc}_{Q_{n+1}} u \leq \delta\omega_n = \omega_{n+1}.$$ ■

Proof (of Theorem 4.1 Concluded) From the construction of Proposition 4.1 it follows that

$$\operatorname*{osc}_{Q_n} u \leq C\frac{4\gamma}{\varepsilon}\rho_n + \omega_n.$$

By iteration

$$\operatorname*{osc}_{Q_n} u \leq \delta^n\omega_o + 4\gamma C\varepsilon^{n-1}\rho_o.$$

Let now $0 < r < \rho$ be fixed. There exists a nonnegative integer n such that

$$\varepsilon^{n+1}\rho \leq r \leq \varepsilon^n\rho.$$

This implies

$$(n+1) \geq \ln\Big(\frac{r}{\rho}\Big)^{\frac{1}{\ln\varepsilon}},$$

$$\delta^n \leq \frac{1}{\delta}\Big(\frac{r}{\rho}\Big)^{\alpha_1}, \qquad \alpha_1 = \frac{|\ln\delta|}{|\ln\varepsilon|},$$

$$\operatorname*{osc}_{Q_n} u \le \frac{1}{\delta}\omega_o\Big(\frac{r}{\rho}\Big)^{\alpha_1} + 4\gamma\frac{C}{\varepsilon}\Big(\frac{r}{\rho}\Big)\rho,$$

that is,

$$\operatorname*{osc}_{Q_n} u \le C_*(\omega_o + \rho)\Big(\frac{r}{\rho}\Big)^{\alpha}$$

where

$$C_* = \frac{1}{\delta} + 4\gamma\frac{C}{\varepsilon} \qquad \text{and} \qquad \alpha = \min\{\alpha_1\,,\,1\}.$$

To conclude the proof, we observe that since $\omega_n \le \omega_o$, the cylinder $Q_r(\theta_o)$ is included in Q_n, and therefore

$$\operatorname*{osc}_{Q_r(\theta_o)} u \le C_*(\omega_o + \rho)\Big(\frac{r}{\rho}\Big)^{\alpha}.$$

Statements of Hölder continuity over a compact set now follow by a standard covering argument. ∎

5 Liouville-Type Results

The Harnack inequality, while local in nature, has global implications. For example, harmonic functions defined in $\mathbb{R}^N$ and with one-sided bound are constant. This, known as the Liouville theorem, is solely a consequence of the Harnack inequality. As such it extends to solutions to homogeneous, elliptic partial differential equations in $\mathbb{R}^N$ with one-sided bound.

This property does not extend to nonnegative solutions to the heat equation in $\mathbb{R}^N \times \mathbb{R}$. A one-sided bound on these solutions does not imply that they are constant. The function

$$\mathbb{R} \times \mathbb{R} \ni (x,t) \to u(x,t) = e^{x+t}$$

is nonnegative, satisfies the heat equation in $\mathbb{R} \times \mathbb{R}$, and is not constant. The Liouville theorem continues to be false for nonnegative solutions to degenerate p-Laplacian type equations. The function constructed in (3.2) of Chapter 4 solves the homogeneous, prototype p-Laplacian equation (1.3) of Chapter 3 in $\mathbb{R} \times \mathbb{R}$, is nonnegative, and nonconstant. It is then natural to ask what kind of global properties are implied by the intrinsic Harnack inequality of Theorem 1.1.

Henceforth we let u be a solution to the degenerate ($p > 2$), homogeneous ($C = 0$) equations (1.1)–(1.2) of Chapter 3 in the semi-infinite strip

$$S_T = \mathbb{R}^N \times (-\infty, T) \qquad \text{for some fixed } T \in \mathbb{R}.$$

If u is bounded above (below) in S_T, set

$$M = \sup_{S_T} u, \qquad (m = \inf_{S_T} u)$$

and for points $(y,s) \in S_T$ for which $M > u(y,s)$, ($u(y,s) > m$, respectively) construct the intrinsic, backward p-paraboloid(s)

$$P_M(y,s) = \Big\{(x,t) \in S_T \mid t - s \le -\Big(\frac{c}{M - u(y,s)}\Big)^{p-2} |x-y|^p\Big\}$$

$$\left(P_m(y,s) = \Big\{(x,t) \in S_T \mid t - s \le -\Big(\frac{c}{u(y,s) - m}\Big)^{p-2} |x-y|^p\Big\}\right)$$

where c is the constant in the intrinsic Harnack inequality (1.3)–(1.4).

5.1 Two-Sided Bounds and Liouville-Type Results

Proposition 5.1 *If u is bounded above and below in S_T, then u is constant.*

The proof is an immediate consequence of the following lemma:

Lemma 5.1 *Let u be bounded below (above) in S_T. Then for all $x \in \mathbb{R}^N$*

$$\lim_{t\to-\infty} u(x,t) = \inf_{S_T} u, \qquad \Big(\lim_{t\to-\infty} u(x,t) = \sup_{S_T} u\Big),$$

and the limit is uniform for x ranging over a compact set $K \subset \mathbb{R}^N$ such that $K \times \{\tau\}$ is included in a p-paraboloid $P_m(y,s)$, ($P_M(y,s)$, respectively), for some $\tau < s$.

Proof Having fixed $\varepsilon > 0$, there exists $(y_\varepsilon, s_\varepsilon) \in S_T$, such that

$$u(x_\varepsilon, t_\varepsilon) - m = \frac{\varepsilon}{\gamma}$$

where γ is the constant in the intrinsic, backward Harnack inequality in (1.4). Applying such inequality to $(u - m)$ gives

$$m \le u(y,s) \le m + \varepsilon, \qquad \text{for all } (y,s) \in P_m(y_\varepsilon, s_\varepsilon).$$

Now, for any $x \in \mathbb{R}^N$, the half-line $[t < T] \times \{x\}$ enters the p-paraboloid $P_m(y_\varepsilon, s_\varepsilon)$ for some t. ■

Proposition 5.2 *Let u be bounded below in S_T and assume that*

$$\sup_{\mathbb{R}^N} u(\cdot, s) = M_s < +\infty \qquad \textit{for some } s < T.$$

Then u is constant in S_s.

Proof We may assume $m = 0$. The assumption implies

$$0 \le u(y,s) \le M_s < \infty \qquad \text{for all } y \in \mathbb{R}^N.$$

By the backward, intrinsic Harnack inequality in (1.3)–(1.4),

$$0 \le u \le \gamma M_s \qquad \text{in } P_m(y,s) \text{ for all } y \in \mathbb{R}^N.$$

Hence $0 \le u \le \gamma M_s$ in S_s, and by Proposition 5.1 u is constant in S_s. ■

Remark 5.1 In the statement of Theorem 1.1 it is required that $Q^{\pm}_{4\rho}(\theta)$ is contained in E_T. As a matter of fact, the proof of such a theorem shows that, once the constant c of (1.3) has been determined, there is no need to have further room above, and it is enough to assume that $t_o + \theta\rho^p < T$. On the other hand, the need of extra room *below* t_o is not a merely technical fact, and such an assumption cannot be removed.

5.2 Two-Sided Bound at One Point, as $t \to \infty$

It has been observed that a one-sided bound on u is not sufficient to infer that u is constant in S_T. Such a conclusion, however, holds if u has a two-sided bound as indicated by Proposition 5.1. The explicit solution (1.5)–(1.6) blows up as $t \to T$, and suggests that if u is defined in the whole $\mathbb{R}^N \times \mathbb{R}$, a condition weaker than a two-sided bound might imply that u is constant. The next proposition is in this direction; it asserts that it suffices to check the two-sided boundedness of u at a single point $y \in \mathbb{R}^N$, for large times, to conclude that u is constant.

Proposition 5.3 *Let u be defined and bounded below in $\mathbb{R}^N \times \mathbb{R}$. If*

$$\lim_{s\to+\infty} u(y,s) = \alpha \qquad \textit{for some } y \in \mathbb{R}^N \textit{ and some } \alpha \in \mathbb{R},$$

then u is constant.

Proof Assume $m = 0$, and $\alpha > 0$. There exists a sequence $\{s_n\} \to \infty$, such that for all arbitrary but fixed $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\alpha - \varepsilon < u(y, s_n) < \alpha + \varepsilon \qquad \text{for all } \ n \geq n_\varepsilon.$$

Fix $s > s_{n_\varepsilon}$, and define a sequence of radii $\{\rho_n\}$, such that

$$s_n - \Big(\frac{c}{\alpha+\varepsilon}\Big)^{p-2} \rho_n^p = s \quad \Longrightarrow \quad \rho_n = \Big[(s_n - s)\Big(\frac{\alpha+\varepsilon}{c}\Big)^{p-2}\Big]^{\frac{1}{p}}.$$

By the intrinsic, backward Harnack inequality in (1.3)–(1.4)

$$\sup_{K_{\rho n}} u\Big(\cdot, s_n - \Big(\frac{c}{u(y,s_n)}\Big)^{p-2} \rho_n^p\Big) \leq \gamma u(y, s_n) \leq \gamma(\alpha+\varepsilon)$$

which we rewrite as

$$\sup_{K_{\rho n}} u(\cdot, s) \leq \gamma(\alpha + \epsilon).$$

Now let $n \to \infty$ by keeping $s > s_{n_\varepsilon}$ fixed. Then $\rho_n \to \infty$ and the previous inequality implies

$$\sup_{\mathbb{R}^N} u(\cdot, s) = M_s \leq \gamma(\alpha+\varepsilon).$$

The conclusion now follows from Proposition 5.2, since $s > s_{n_\varepsilon}$ is arbitrary.■

Remark 5.2 Assuming $\alpha > 0$ for simplicity, the same argument continues to hold, if there exists a sequence $\{(y_n, s_n)\} \subset \mathbb{R}^N \times \mathbb{R}$ and $s \in \mathbb{R}$, such that $s_n \to +\infty$,

$$s_n - s = \left(\frac{c}{\alpha}\right)^{p-2} |y_n|^p,$$

and $\lim_{n\to+\infty} u(y_n, s_n) = \alpha$.

6 Subpotential Lower Bounds

The homogeneous, prototype equation (1.3) of Chapter 3 admits the family of Barenblatt-type subsolutions

$$\Gamma_{\nu,p}(x,t;y,s) = \frac{k\rho^\nu}{S^{\frac{\nu}{\lambda}}(t)} \left[1 - b(\nu,p)\left(\frac{|x-y|}{S^{\frac{1}{\lambda}}(t)}\right)^{\frac{p}{p-1}}\right]_+^{\frac{p-1}{p-2}}$$

where $\nu \geq N$, k and ρ are positive parameters and

$$S(t) = k^{p-2}\rho^{\nu(p-2)}(t-s) + \rho^\lambda \qquad t \geq s$$

$$b(\nu,p) = \frac{1}{\lambda^{\frac{1}{p-1}}} \frac{p-2}{p}, \qquad \lambda = \nu(p-2) + p.$$

One verifies that for $\nu = N$, the functions $\Gamma_{N,p}$ are exact solutions to the prototype, degenerate, homogeneous equation (1.3) of Chapter 3 for all $k, \rho > 0$ (see also § 3.2 of Chapter 4), whereas for $\nu > N$ they are subsolutions. Information on the behavior of nonnegative solutions to the prototype equation (1.3) of Chapter 3 can be derived by comparing them with $\Gamma_{\nu,p}$. For example, if $u(\cdot, s) \geq k$ in the ball $B_\rho(y)$, then the positivity of u expands over the balls

$$|x-y| < b(\nu,p)^{-\frac{p-1}{p}} S^{\frac{1}{\lambda}}(t)$$

at later times t. Moreover u does not decay faster than $t^{-\nu/\lambda}$ for $t \gg 1$ ([41], Chapter VI). One verifies that as $p \to 2$

$$\Gamma_{\nu,p}(x,t;y,s) \to \Gamma_\nu(x,t;y,s) = \frac{k\rho^\nu}{[(t-s)+\rho^2]^{\frac{\nu}{2}}} \exp\left[-\frac{|x-y|^2}{4[(t-s)+\rho^2]}\right]$$

pointwise in $\mathbb{R}^N \times [t > s]$. For $\nu = N$ the latter are exact solutions to the heat equation and one verifies that for $\nu > N$ they are subsolutions. In this sense the functions $\Gamma_{\nu,p}$ for $\nu \geq N$ are subpotentials.

Given the general quasilinear structure of (1.1)–(1.2) of Chapter 3, the functions $\Gamma_{\nu,p}$ are not subsolutions to these equations in any sense. Moreover no comparison principle holds. Nevertheless the "fundamental subsolutions" $\Gamma_{\nu,p}$ drive, in a sense made precise by Proposition 6.1 below, the structural behavior of nonnegative solutions to these degenerate, quasilinear equations.

The content of Propositions 6.1–6.2 is that these weak solutions are, locally, bounded below by one $\Gamma_{\nu,p}$ for some $\nu > N$, and thus they do not decay in space, faster than these "subpotentials."

Proposition 6.1 *Let u be a continuous, nonnegative, local, weak solution to the degenerate ($p > 2$), homogeneous ($C = 0$), quasilinear equations (1.1)–(1.2) of Chapter 3, in E_T. There exist positive constants $\gamma_o < 1$ and $\gamma_1 > 1$, depending only on the data $\{p, N, C_o, C_1\}$ and the constants c and γ appearing in the Harnack inequality (1.3)–(1.4) of Theorem 1.1, such that for all $(x_o, t_o) \in E_T$ such that $u(x_o, t_o) > 0$, and all $(x, t) \in E_T$ with*

$$K_{8|x-x_o|}(x_o) \subset E, \qquad \text{and} \qquad 0 < t - t_o < \tfrac{1}{4^p} t_o, \tag{6.1}$$

we have

$$u(x,t) \geq \gamma_o u(x_o,t_o)\Big\{1 - \gamma_1 \Big[\frac{|x-x_o|^p}{u(x_o,t_o)^{p-2}(t-t_o)}\Big]^{\frac{1}{p-1}}\Big\}_+^{\frac{p-1}{p-2}}. \tag{6.2}$$

Proof Fix (x, t) as in (6.1) and consider the line segment ℓ_o through (x_o, t_o) and (x, t)

$$\ell_o \qquad\qquad y - x_o = \frac{x - x_o}{t - t_o}(s - t_o), \quad t_o < s \leq t,$$

and the p-paraboloid with bottom vertex at (x_o, t_o)

$$\mathcal{P}_o \qquad\qquad s - t_o = \theta |y - x_o|^p, \qquad \text{where } \theta = \Big(\frac{c}{u(x_o,t_o)}\Big)^{p-2}.$$

By (6.1) $\ell_o \subset E_T$. If ℓ_o does not intersect $\mathcal{P}_o$ at points other than (x_o, t_o), then (6.2) follows from the intrinsic, forward Harnack inequality (1.3)–(1.4). Let then ℓ_o intersect $\mathcal{P}_o$ at (x_1, t_1) with $|x_1 - x_o| < |x - x_o|$, and

$$\begin{aligned} |x_1 - x_o|^{p-1} &= \Big(\frac{u(x_o,t_o)}{c}\Big)^{p-2} \frac{t - t_o}{|x - x_o|} \\ t_1 - t_o &= \Big(\frac{c}{u(x_o,t_o)}\Big)^{p-2} |x_1 - x_o|^p. \end{aligned}$$

Iterating this procedure gives a finite number of points (x_j, t_j), with $j = 1, \dots, n$, such that $t_o < t_1 < \cdots < t_n \leq t$ and

$$\begin{aligned} |x_{j+1} - x_j|^{p-1} &= \Big(\frac{u(x_j,t_j)}{c}\Big)^{p-2} \frac{t - t_o}{|x - x_o|} \\ t_{j+1} - t_j &= \Big(\frac{c}{u(x_j,t_j)}\Big)^{p-2} |x_{j+1} - x_j|^p, \end{aligned} \tag{6.3}$$

and where (x_n, t_n) is the first point not overcoming (x, t). Using the intrinsic, forward Harnack inequality of Theorem 1.1,

$$u(x_j, t_j) \leq \gamma\, u(x_{j+1}, t_{j+1}), \qquad j = 0, \dots, n-1,$$

provided the cylinder

$$(x_j,t_j)+Q_{4\rho_j}(\theta_j) \quad \text{with } \theta_j=\Big(\frac{c}{u(x_j,t_j)}\Big)^{p-2} \text{ and } \rho_j=|x_{j+1}-x_j|$$

is contained in E_T. This is the case if

$$t_j-(4\rho_j)^p\theta_j\geq 0$$

which, in view of (6.3), is verified if

$$t_j-4^p(t_{j+1}-t_j)\geq 0.$$

This last inequality holds true by the last of (6.1). A similar argument for the space variables guarantees the inclusion of

$$(x_j,t_j)+Q_{4\rho_j}(\theta_j)\subset E_T.$$

We infer that

$$u(x_j,t_j)\geq\gamma^{-j}u(x_o,t_o) \quad \text{for } j=1,\dots,n-1. \tag{6.4}$$

From this and (6.3) it follows that

$$\begin{aligned}|x-x_o|&\geq\sum_{j=0}^{n-1}|x_{j+1}-x_j|\\ &=\Big(\frac{1}{c^{p-2}}\frac{t-t_o}{|x-x_o|}\Big)^{\frac{1}{p-1}}\sum_{j=0}^{n-1}u(x_j,t_j)^{\frac{p-2}{p-1}}\\ &\geq\Big(\frac{t-t_o}{|x-x_o|}\Big)^{\frac{1}{p-1}}\Big(\frac{u(x_o,t_o)}{c}\Big)^{\frac{p-2}{p-1}}\sum_{j=0}^{n-1}\big(\gamma^{-\frac{p-2}{p-1}}\big)^j\\ &=\Big(\frac{t-t_o}{|x-x_o|}\Big)^{\frac{1}{p-1}}\Big(\frac{u(x_o,t_o)}{c}\Big)^{\frac{p-2}{p-1}}\frac{1-q^n}{1-q},\end{aligned} \tag{6.5}$$

where $q=\gamma^{-\frac{p-2}{p-1}}$. Then, combining (6.4) written for $j=n$ and (6.5), gives

$$\Big(\frac{u(x_n,t_n)}{u(x_o,t_o)}\Big)^{\frac{p-2}{p-1}}\geq q^n\geq 1-\gamma_1\Big(\frac{|x-x_o|^p}{u(x_o,t_o)^{p-2}(t-t_o)}\Big)^{\frac{1}{p-1}},$$

whenever the right-hand side is positive. Here

$$\gamma_1=\big(\gamma^{\frac{p-2}{p-1}}-1\big)\big(c\gamma^{-1}\big)^{\frac{p-2}{p-1}}. \tag{6.6}$$

The previous inequality is rewritten as

$$u(x_n,t_n)\geq u(x_o,t_o)\Big[1-\gamma_1\Big(\frac{|x-x_o|^p}{u(x_o,t_o)^{p-2}(t-t_o)}\Big)^{\frac{1}{p-1}}\Big]_+^{\frac{p-1}{p-2}}.$$

If $(x_n, t_n) = (x, t)$, this is (6.2) with $\gamma_o = 1$. Otherwise, a further application of the intrinsic, forward Harnack inequality (1.3)–(1.4) gives

$$u(x,t) \geq \gamma^{-1} u(x_n, t_n)$$

thereby establishing (6.2) with $\gamma_o = \min\{1\,,\,\gamma^{-1}\}$. ∎

The result of Proposition 6.1 can be improved to include an estimate of the decay in time of these solutions, provided u is a local solution in the whole half-space

$$S_\infty = \mathbb{R}^N \times (0, \infty)$$

which, for the remainder of this section, we assume.

Proposition 6.2 *Let $(x_o, t_o) \in S_\infty$ and assume that $u(x_o, t_o) > 0$. Then for all $x \in \mathbb{R}^N$ and all $t > 4t_o > 0$, we have*

$$\frac{u(x,t)}{u(x_o,t_o)} \geq \gamma_o \Big(\frac{t_o}{t}\Big)^a \Big[1 - \gamma_2 \Big(\frac{t}{t_o}\Big)^{a\frac{p-2}{p-1}} \Big(\frac{|x - x_o|^p}{u(x_o,t_o)^{p-2}(t - t_o)}\Big)^{\frac{1}{p-1}}\Big]_+^{\frac{p-1}{p-2}}, \tag{6.7}$$

where

$$a = \ln \gamma, \qquad \gamma_2 = \gamma_1 2^{\frac{1}{p-1}},$$

γ_o and γ_1 are the constants claimed in Proposition 6.1, and γ is the constant appearing in the Harnack inequality (1.3)–(1.4) of Theorem 1.1.

Proof Let k be that positive integer for which

$$(1+\sigma)^{k+1} t_o \leq t \leq (1+\sigma)^{k+2} t_o \qquad \text{for some fixed } \ 0 < \sigma \leq \tfrac{1}{3}\tfrac{1}{4^p},$$

and let $\tau = (1+\sigma)^k t_o$. Since the time τ satisfies (6.1), the estimate (6.2) holds for it, which we rewrite explicitly as

$$u(x,t) \geq \gamma_o u(x_o, \tau) \Big[1 - \gamma_1 \Big(\frac{|x - x_o|^p}{u(x_o,\tau)^{p-2}(t-\tau)}\Big)^{\frac{1}{p-1}}\Big]_+^{\frac{p-1}{p-2}}. \tag{6.8}$$

Next apply the intrinsic forward Harnack inequality (1.3)–(1.4) for the time levels $t_j = (1+\sigma)^j t_o$, for $j = 0, \dots, k-1$, and by keeping x_o fixed. The inequality is intrinsic and the "ending times" t_{j+1}, starting from t_j, are realized by a proper choice of ρ_j. This implies

$$u(x_o, t_o) \leq \gamma^k u(x_o, \tau),$$

that is,

$$u(x_o, \tau) \geq \Big(\frac{t_o}{\tau}\Big)^{\ln\gamma/\ln(1+\sigma)} u(x_o, t_o) \geq \Big(\frac{t_o}{t}\Big)^{\ln\gamma/\ln(1+\sigma)} u(x_o, t_o).$$

On the other hand,

$$t - \tau > (1+\sigma)^{-2} t \geq (1+\sigma)^{-2}(t - t_o).$$

Combining these estimates in (6.8) proves the proposition. ∎

Remark 6.1 The constants c and γ of Theorem 1.1 are stable as $p \to 2$. Thus letting $p \to 2$ in (6.7), and taking into account the structure (6.6) of the constant γ_1 and the expression of γ_2 gives

$$u(x,t) \ge \tilde{\gamma}_o u(x_o,t_o)\Big(\frac{t_o}{t}\Big)^a \exp\Big(-\tilde{\gamma}_1\frac{|x-x_o|^2}{t-t_o}\Big)$$

for constants $\tilde{\gamma}_o$ and $\tilde{\gamma}_1$ depending only on the data $\{N, C_o, C_1\}$ in the structure conditions (1.2) of Chapter 3 for $p = 2$. Thus for $p \to 2$ we recover the classical result of Moser ([121]), for nonnegative solutions to nondegenerate ($p = 2$), linear, parabolic equations with bounded and measurable coefficients.

7 A Weak Harnack Inequality for Positive Supersolutions

Local, weak supersolutions to the degenerate ($p > 2$) equations (1.1)–(1.2) of Chapter 3 in E_T, need not be continuous, and if nonnegative, they do not, in general, satisfy a Harnack estimate. Nevertheless they satisfy a weak, or integral, form of the Harnack inequality as expressed in the following theorem. To convey the main ideas, the theory is presented for homogeneous equations for which $C = 0$. The extension to nonhomogeneous equations $C \ne 0$ is achieved by the usual alternative that negating $C\rho > 1$, forces $C\rho \le 1$.

Throughout, for $y \in E$, let $\rho > 0$ be so small that $K_{16\rho}(y) \subset E$.

Theorem 7.1 *Let u be a nonnegative, local, weak supersolution to the degenerate ($p > 2$), homogeneous ($C = 0$) equations (1.1)–(1.2) of Chapter 3 in E_T. There exist positive constants c and γ, depending only on the data $\{p, N, C_o, C_1\}$, such that for a.e. $s \in (0,T)$*

$$\fint_{K_\rho(y)} u(x,s)dx \le c\Big(\frac{\rho^p}{T-s}\Big)^{\frac{1}{p-2}} + \gamma \inf_{K_{4\rho}(y)} u(\cdot,t) \tag{7.1}$$

for all times

$$s + \tfrac{1}{2}\theta\rho^p \le t \le s + \theta\rho^p$$

where

$$\theta = \min\Big\{c^{2-2}\frac{T-s}{\rho^p}\,,\ \Big(\fint_{K_\rho(y)} u(x,s)dx\Big)^{2-p}\Big\}. \tag{7.2}$$

The proof of Theorem 7.1 rests on the following improved version of the expansion of positivity for $p > 2$.

Proposition 7.1 *Let u be a nonnegative, local, weak supersolution to the degenerate ($p > 2$), homogeneous ($C = 0$) equations (1.1)–(1.2) of Chapter 3 in E_T. Assume that for some $(y,s) \in E_T$ and some $\rho > 0$*

$$\big|[u(\cdot,s) \ge M] \cap K_\rho(y)\big| \ge \alpha\big|K_\rho(y)\big| \tag{7.3}$$

for some $M > 0$ *and some* $\alpha \in (0,1)$. *There exist constants* η_o, δ *in* $(0,1)$ *and* $b, d > 1$ *depending only on the data* $\{p, N, C_o, C_1\}$, *and* α, *such that*

$$u(\cdot, t) \geq \eta_* M \quad a.e. \ in \ K_{2\rho}(y), \qquad where \ \eta_* = \eta_o \alpha^d, \tag{7.4}$$

for all times

$$s + \frac{b^{p-2}}{(\eta_* M)^{p-2}} \tfrac{1}{2}\delta\rho^p \leq t \leq s + \frac{b^{p-2}}{(\eta_* M)^{p-2}} \delta\rho^p \tag{7.5}$$

provided

$$K_{8\rho}(y) \times \Big(s\,, s + \frac{b^{p-2}}{(\eta_* M)^{p-2}}\delta\rho^p\Big] \subset E_T.$$

This improves Proposition 4.1 of Chapter 4 in that the functional dependence of the shrinking parameter η on the measure-theoretical parameter α is power-like as opposed to an exponential form, as indicated in (4.5) of Remark 4.3 of Chapter 4. Such a power-like dependence is made possible by the measure-theoretical Lemma 3.1 of the Preliminaries.

8 Proof of Proposition 7.1

Assume $(y, s) = (0, 0)$. By Lemma 1.1 of Chapter 4 there exist δ and ϵ in $(0,1)$, depending only on the data $\{p, N, C_o, C_1\}$, and α, and independent of M, such that

$$|[u(\cdot, t) > \epsilon M] \cap K_\rho| \geq \tfrac{1}{2}\alpha |K_\rho| \quad \text{ for all } t \in \Big(0, \frac{\delta\rho^p}{M^{p-2}}\Big]. \tag{8.1}$$

The dependence of δ and ϵ on α is traced in (1.3) of Chapter 4. Next, write down the energy estimates (2.3) of Chapter 3 for $(u - M)_-$, over the pair of cylinders

$$\mathcal{Q} = K_\rho \times \Big(\frac{\frac{1}{2}\delta}{M^{p-2}}\rho^p, \frac{\delta}{M^{p-2}}\rho^p\Big], \qquad \mathcal{Q}' = K_{2\rho} \times \Big(0, \frac{\delta}{M^{p-2}}\rho^p\Big].$$

The nonnegative, piecewise smooth, cutoff function ζ is taken to be equal to one on $\mathcal{Q}$, vanishing on the parabolic boundary of $\mathcal{Q}'$, and such that

$$0 \leq \zeta_t \leq \frac{4^p}{\delta M^{2-p}\rho^p}, \qquad |D\zeta| \leq \frac{4}{\rho}.$$

The resulting energy estimates are

$$C_o \iint_{\mathcal{Q} \cap [u \leq M]} |Du|^p dx\, dt \leq \gamma \frac{4^{p+1}}{\delta\rho^p} M^p |\mathcal{Q}|.$$

The change of variable

$$y = \frac{x}{\rho}, \qquad \tau = \frac{t}{\delta M^{2-p}\rho^p}, \qquad w = \frac{(M-u)_+}{M}$$

maps $\mathcal{Q}$ into $Q = K_1 \times (\frac{1}{2}, 1]$, and the previous estimate yields

$$\iint_Q |Dw|^p dy\, d\tau \le \frac{\gamma}{\alpha^{p+1}}|Q|, \tag{8.2}$$

where γ is a constant that depends only on $\{p, N, C_o, C_1\}$, and we have taken into account the dependence of δ on α given by (1.3) of Chapter 4. In terms of $z = (1-w)/\epsilon$, (8.1) reads

$$|[z(\cdot,\tau) > 1] \cap K_1| > \tfrac{1}{2}\alpha \quad \text{for all} \quad \tau \in (\tfrac{1}{2}, 1]. \tag{8.3}$$

From (8.2)–(8.3) it follows there exists $\tau_1 \in (\frac{1}{2}, 1]$, such that

$$\int_{K_1} |Dz(\cdot,\tau_1)| dy \le \frac{\gamma}{\epsilon\, \alpha^{\frac{p+1}{p}}}, \qquad \text{and} \qquad |[z(\cdot,\tau_1) > 1] \cap K_1| > \tfrac{1}{2}\alpha. \tag{8.4}$$

By Lemma 3.1 of the Preliminaries, applied with $\delta = \lambda = \frac{1}{2}$, there exist $y_o \in K_1$, and ε, that depend only on $\{p, N, C_o, C_1\}$ and α, such that

$$\Big|\big[w(\cdot,\tau_1) < 1 - \tfrac{1}{2}\epsilon\big] \cap K_\varepsilon(y_o)\Big| > \tfrac{1}{2}|K_\varepsilon(y_o)|.$$

The functional dependence of ε on the measure-theoretical parameter α, and on the constant appearing on the right-hand side of (8.4), is traced in (3.5) of the Preliminaries. Also the functional dependence of ϵ on the parameter α is traced in (1.3) of Chapter 4. Hence

$$\varepsilon = B_o^{-1}\alpha^{\frac{4p+1}{p}}, \qquad \epsilon = \tfrac{1}{8}\alpha \tag{8.5}$$

for an absolute constant $B_o > 1$ depending only on the data $\{p, N, C_o, C_1\}$ and independent of α. Returning to the original variables gives a time t_1 in the range

$$M^{2-p}\tfrac{1}{2}\delta\rho^p < t_1 \le M^{2-p}\delta\rho^p$$

where

$$\Big|\big[u(\cdot,t_1) > \tfrac{1}{2}\epsilon M\big] \cap K_{\varepsilon\rho}(x_o)\Big| > \tfrac{1}{2}|K_{\varepsilon\rho}(x_o)|.$$

This measure-theoretical information plays the same role as (4.2) in Proposition 4.1 of Chapter 4, with α being replaced by an absolute constant, and ρ being replaced by $\varepsilon\rho$. By the same proposition

$$u(x,t) \ge \tfrac{1}{2}\epsilon\bar{\eta}M \quad \text{for a.e. } x \in K_{2\varepsilon\rho}(x_o) \tag{8.6}$$

for all times

$$t_1 + \frac{b^{p-2}}{(\frac{1}{2}\epsilon\bar{\eta}M)^{p-2}}\tfrac{1}{2}\bar{\delta}(\varepsilon\rho)^p \le t \le t_1 + \frac{b^{p-2}}{(\frac{1}{2}\epsilon\bar{\eta}M)^{p-2}}\bar{\delta}(\varepsilon\rho)^p$$

where $\bar{\eta}$ and $\bar{\delta}$ are the constants η and δ claimed by Proposition 4.1 of Chapter 4, corresponding to the value $\frac{1}{2}$ of the parameter α. As such, these parameters are absolute numbers depending only on the data $\{p, N, C_o, C_1\}$ and independent of the value α as appearing in (7.3).

The expansion of positivity is now repeated starting from (8.6) with M replaced by $\frac{1}{2}\epsilon\bar{\eta}M$ and $\alpha = 1$, and continued, at each step with $\alpha = 1$. This implies that the constants $\bar{\eta}$ and $\bar{\delta}$ can be taken to be the same at each repetition of the expansion process, and yields

$$u(\cdot, t) > \tfrac{1}{2}\epsilon\bar{\eta}^n M \qquad \text{for a.e. } x \in K_{2^n\varepsilon\rho}(x_o)$$

for all times

$$t_{n-1} + \frac{b^{p-2}}{(\frac{1}{2}\epsilon\bar{\eta}^n M)^{p-2}}\tfrac{1}{2}\bar{\delta}(\varepsilon\rho)^p \le t \le t_{n-1} + \frac{b^{p-2}}{(\frac{1}{2}\epsilon\bar{\eta}^n M)^{p-2}}\bar{\delta}(\varepsilon\rho)^p.$$

To realize the expansion of positivity to the cube $K_{2\rho}$ take

$$2^n\varepsilon = 2 \quad \Longrightarrow \quad n = 1 + \ln \varepsilon^{-\frac{1}{\ln 2}} \quad \Longrightarrow \quad n = 1 + \log_{\frac{1}{\bar{\eta}}} \varepsilon^{\frac{\ln \bar{\eta}}{\ln 2}}.$$

Hence recalling the functional dependence of ε and ϵ on the parameter α as given in (8.5), yields the existence of $\eta_o \in (0, 1)$ and $d > 1$ depending only on the data and independent of α, such that (7.4)–(7.5) hold. ■

9 Proof of Theorem 7.1 by Alternatives

By a translation and dilation of the space and time variables, we may assume $y = 0$ and $\rho = 1$. Consider the cylinder $K_8 \times (0, T_1]$ where $T_1 > 1$ is to be chosen, and assume without loss of generality that it is contained in E_T.

The proof of Theorem 7.1 unfolds along two alternatives. Either in the cylinder $K_1 \times (0, 1]$ there exist a time level t_o and a $k > 1$, such that

$$\left|\left[u(\cdot, t_o) > k^{1+\frac{1}{d}}\right] \cap K_1\right| > \frac{1}{k^{\frac{1}{d}}}|K_1| \tag{9.1}$$

or such inequality is violated for all $k > 1$ and all $t_o \in (0, 1)$. Here $d > 1$ is the constant claimed in (7.4) of Proposition 7.1.

If some $k > 1$ and $t_o \in (0, 1)$ exist, satisfying (9.1), by Proposition 7.1

$$u(\cdot, t) \ge \eta_o k^{\frac{1}{d}} \qquad \text{a.e. in} \qquad K_2$$

for all times

$$t_o + \frac{b^{p-2}}{(\eta_o k^{\frac{1}{d}})^{p-2}}\tfrac{1}{2}\delta \le t \le t_o + \frac{b^{p-2}}{(\eta_o k^{\frac{1}{d}})^{p-2}}\delta.$$

By possibly reducing η_o, if needed, we may assume, without loss of generality, that

$$1 + \frac{b^{p-2}}{\eta_o^{p-2}} \tfrac{1}{2}\delta \le \frac{b^{p-2}}{\eta_o^{p-2}}\delta.$$

Hence we conclude that if (9.1) holds true for some $k > 1$ and some $t_o \in (0,1)$, then

$$u(\cdot, t) \ge \eta_o \qquad \text{a.e. in} \qquad K_2 \times \big(1 + \tfrac{1}{2}T_1\,,\, T_1\big] \tag{9.2}$$

where we have set

$$T_1 = \delta\Big(\frac{b}{\eta_o}\Big)^{p-2}. \tag{9.3}$$

If no $k > 1$ and $t_o \in (0,1)$ exist satisfying (9.1), then

$$\big|\big[u(\cdot,t) > k^{1+\frac{1}{d}}\big] \cap K_1\big| \le \frac{1}{k^{\frac{1}{d}}}|K_1| \tag{9.4}$$

for almost every time $t \in (0, 1]$ and all levels $k \ge 1$. Set

$$\sigma = \frac{1}{2}\frac{1}{d+1} \tag{9.5}$$

and compute

$$\begin{aligned}
\int_{K_1} u^\sigma(x,t)dx &= \sigma \int_0^\infty s^{\sigma-1}|[u > s] \cap K_1|ds \\
&= \sigma \int_0^1 s^{\sigma-1}|[u > s] \cap K_1|ds \\
&\quad + \sigma \int_1^\infty s^{\sigma-1}|[u > s] \cap K_1|ds \\
&\le \Big[\sigma \int_0^1 s^{\sigma-1}ds + \sigma \int_1^\infty s^{\sigma-1}s^{-\frac{1}{d+1}}ds\Big]|K_1| \\
&= \Big[1 + \sigma \int_1^\infty s^{-\sigma-1}\Big]|K_1| = 2|K_1|.
\end{aligned} \tag{9.6}$$

The next step in the proof is in transforming this absolute bound of the $L^\sigma(K_1)$ integral of $u(\cdot,t)$, uniform in $t \in (0,1)$, into an absolute bound of the L^q-norm of u over the cylinder $K_1 \times (0,1)$, for some $q > 1$.

10 A Reverse Hölder Inequality for Supersolutions

Raising the integrability of u is achieved by Moser's method ([120]). A standard application of this method proves the following lemma.

Lemma 10.1 *Let u be a nonnegative, local, weak supersolution to the degenerate ($p > 2$), homogeneous ($C = 0$) equations (1.1)–(1.2) of Chapter 3 in the cylindrical domain*

$$K \times (t_1, t_2)$$

where K is a cube in $\mathbb{R}^N$. Then for all $\varepsilon < 0$ with $\varepsilon \neq -1$,

$$\begin{aligned}
&\frac{p}{C_o|\varepsilon(1+\varepsilon)|}\sup_{t_1<t<t_2}\int_K u^{1+\varepsilon}\varphi^p dx + \int_{t_1}^{t_2}\int_K |Du|^p u^{-1+\varepsilon}\varphi^p dx\,dt \\
&\le \Big(\frac{C_1 p}{C_o \min\{1\,,\,|\varepsilon|\}}\Big)^p \int_{t_1}^{t_2}\int_K u^{p-1+\varepsilon}|D\varphi|^p dx\,dt \qquad (10.1)\\
&\quad + \frac{p}{C_o}\int_{t_1}^{t_2}\int_K u^{1+\varepsilon}\Big(\frac{1}{\min\{1\,,\,|\varepsilon|\}(1+\varepsilon)}\frac{\partial\varphi^p}{\partial t}\Big)_+ dx\,dt,
\end{aligned}$$

for every test-function

$$\varphi \in W^{1,2}\big(t_1,t_2;L^2(K)\big)\cap L^p\big(t_1,t_2;W_o^{1,p}(K)\big)$$

provided $u \ge \nu$ in $K\times(t_1,t_2)$ for some $\nu > 0$.

This, in turn, permits one to establish the following reverse Hölder inequality.

Lemma 10.2 *Let u be a nonnegative, local, weak supersolution to the degenerate ($p > 2$), homogeneous ($C = 0$) equations (1.1)–(1.2) of Chapter 3 in*

$$Q_1 = K_1 \times (0,1)$$

and assume that $u \ge 1$ in Q_1. For all q in the range

$$p-2 < q < p-1+\frac{p}{N}$$

and all s given by

$$s = p-2+\Big(1+\frac{p}{N}\Big)^{-(n+1)}(q-(p-2)), \quad n = 1,2,\dots$$

there is a positive constant γ, depending only on the data $\{p,N,C_o,C_1\}$, and q and s, such that

$$\Big(\int_0^{a^p}\int_{K_a} u^q dx\,dt\Big)^{\frac{1}{q-p+2}} \le \Big(\frac{\gamma}{(1-a)^{N+p}}\int_0^1\int_{K_1} u^s dx\,dt\Big)^{\frac{1}{s-p+2}} \qquad (10.2)$$

for all $a \in (\frac{1}{2},1)$.

Proof For $n \in \mathbb{N}$ set

$$\begin{aligned}
&\rho_o = 1, \qquad \rho_j = 1-(1-a)\frac{1-2^{-j}}{1-2^{-(n+1)}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad j = 0,1,\dots,n+1,\\
&K_j = K_{\rho_j}, \qquad Q_j = K_{\rho_j}\times(0,\rho_j^p] = Q_{\rho_j}(1).
\end{aligned}$$

For the pair of cylinders Q_j and Q_{j+1}, choose nonnegative, piecewise smooth, test functions φ_j, that equal 1 on Q_{j+1}, vanish on the conjugate parabolic boundary of Q_j

$$\partial^* Q_j = \partial Q_j - K_j \times \{0\},$$

and such that

$$|D\varphi_j| \le \gamma \frac{2^{j+1}}{(1-a)}, \qquad 0 \le \Big|\frac{\partial \varphi_j}{\partial t}\Big| \le \frac{2^{p(j+1)}}{(1-a)^p}, \qquad j = 0, 1, \ldots, n.$$

By the embedding Proposition 4.1 of the Preliminaries

$$\begin{aligned}
&\iint_{Q_{j+1}} u^{\kappa\alpha} dx\, dt \le \gamma \iint_{Q_j} (u^{\frac{\alpha}{p}} \varphi_j^{\frac{\beta}{p}})^{\kappa p} dx\, dt \\
&\le \gamma \iint_{Q_j} |D(u^{\frac{\alpha}{p}} \varphi_j^{\frac{\beta}{p}})|^p dx\, dt \Big(\sup_{0<t<\rho_j^p} \int_{K_j} (u^{\frac{\alpha}{p}} \varphi_j^{\frac{\beta}{p}})^{(\kappa-1)N} dx \Big)^{\frac{p}{N}},
\end{aligned} \tag{10.3}$$

for $\alpha \in \mathbb{R}$, $\beta \ge p$, and $\kappa > 1$. Choose

$$\alpha = p - 1 + \varepsilon, \qquad \kappa = 1 + \frac{p(1+\varepsilon)}{N(p-1+\varepsilon)}, \qquad \beta = \frac{p(p-1+\varepsilon)}{1+\varepsilon},$$

where $\varepsilon \in (-1, 0)$. By (10.1)

$$\begin{aligned}
&\sup_{0<t<\rho_j^p} \int_{K_j} (u^{\frac{\alpha}{p}} \varphi_j^{\frac{\beta}{p}})^{(\kappa-1)N} dx = \sup_{0<t<\rho_j^p} \int_{K_j} u^{1+\varepsilon} \varphi_j^p dx \\
&\qquad \le \frac{\gamma}{|\varepsilon|^p (1+\varepsilon)} \Big(\iint_{Q_j} u^{p-1+\varepsilon} |D\varphi_j|^p dx\, dt + \iint_{Q_j} u^{1+\varepsilon} \Big|\frac{\partial \varphi_j}{\partial t}\Big| dx\, dt \Big)
\end{aligned}$$

and

$$\begin{aligned}
&\iint_{Q_j} |D(u^{\frac{\alpha}{p}} \varphi_j^{\frac{\beta}{p}})|^p dx\, dt \\
&= \iint_{Q_j} \Big(\frac{\alpha}{p}\Big)^p u^{(\frac{\alpha}{p}-1)p} |Du|^p \varphi_j^{\beta} dx\, dt + \iint_{Q_j} \Big(\frac{\beta}{p}\Big)^p u^{\alpha} \varphi_j^{(\frac{\beta}{p}-1)p} |D\varphi_j|^p dx\, dt \\
&\le \gamma \Big(\iint_{Q_j} u^{\varepsilon-1} \varphi_j^p |Du|^p dx\, dt + \frac{1}{1+\varepsilon} \iint_{Q_j} u^{p-1+\varepsilon} |D\varphi_j|^p dx\, dt \Big) \\
&\le \frac{\gamma}{|\varepsilon|^p (1+\varepsilon)} \Big(\iint_{Q_j} u^{p-1+\varepsilon} |D\varphi_j|^p dx\, dt + \iint_{Q_j} u^{1+\varepsilon} \Big|\frac{\partial \varphi_j}{\partial t}\Big| dx\, dt \Big).
\end{aligned}$$

Combining these estimates gives

$$\begin{aligned}
&\iint_{Q_{j+1}} u^{p-1+\frac{p}{N}+\varepsilon(1+\frac{p}{N})} dx\, dt \\
&\le \Big[\frac{\gamma}{|\varepsilon|^p (1+\varepsilon)} \Big(\iint_{Q_j} u^{p-1+\varepsilon} |D\varphi_j|^p dx\, dt + \iint_{Q_j} u^{1+\varepsilon} \Big|\frac{\partial \varphi_j}{\partial t}\Big| dx\, dt \Big) \Big]^{1+\frac{p}{N}}.
\end{aligned}$$

Since $u \ge 1$, thanks to the assumptions on φ_j,

$$\iint_{Q_{j+1}} u^{p-1+\frac{p}{N}+\varepsilon(1+\frac{p}{N})} dx\,dt \le \Big(\frac{\gamma 2^{jp}}{|\varepsilon|^p(1+\varepsilon)(1-a)^p}\iint_{Q_j} u^{p-1+\varepsilon} dx\,dt\Big)^{1+\frac{p}{N}}.$$

Now set

$$h = 1+\frac{p}{N}, \qquad \varepsilon_j = h^j(\varepsilon_o+1)-1, \qquad \alpha_j = p-1+\varepsilon_j,$$

where

$$-1 < \varepsilon_o < -1+\gamma^{-n} \quad \text{and} \quad j=0,\dots,n.$$

Since

$$p-1+\frac{p}{N}+h\varepsilon_j = p-1+\varepsilon_{j+1}$$

these choices yield

$$\iint_{Q_{j+1}} u^{\alpha_{j+1}} dx\,dt \le \Big(\frac{\gamma 2^{jp}}{|\varepsilon_j|^p(1+\varepsilon_j)(1-a)^p}\iint_{Q_j} u^{\alpha_j} dx\,dt\Big)^h$$

for $j=0,1,\dots,n$. Choose

$$\varepsilon_o = -1+h^{-1-n}(q-(p-2)), \qquad \Rightarrow \qquad \alpha_o = s, \quad \alpha_{n+1}=q.$$

This choice implies the estimate

$$\frac{1}{|\varepsilon_j|^p(1+\varepsilon_j)} \le \frac{1}{|\varepsilon_n|^p(1+\varepsilon_o)} = \frac{h^p}{(p-1+\frac{p}{N}-q)^p(s-(p-2))}.$$

Then setting

$$\tilde{\gamma} = \frac{1}{(p-1+\frac{p}{N}-q)(s-(p-2))^{\frac{1}{p}}},$$

the previous inequality yields

$$\iint_{Q_{j+1}} u^{\alpha_{j+1}} dx\,dt \le \Big[\frac{\gamma\tilde{\gamma}^p 2^{jp}}{(1-a)^p}\iint_{Q_j} u^{\alpha_j} dx\,dt\Big]^h.$$

From this, by iteration

$$\iint_{Q_{n+1}} u^q dx\,dt \le \Big[\frac{\gamma\tilde{\gamma}^p}{(1-a)^p}\Big]^{\sum_{j=1}^{n+1} h^j} \prod_{j=0}^{n-1} 2^{p(n-j)h^{j+1}} \Big(\iint_{Q_o} u^s dx\,dt\Big)^{h^{n+1}}.$$

Now

$$p\sum_{j=1}^{n+1} h^j = \frac{ph}{h-1}(h^{n+1}-1) = (N+p)(h^{n+1}-1)$$

$$\prod_{j=0}^{n-1} 2^{p(n-j)h^{j+1}} = \Big(\prod_{j=1}^{n} 2^{pjh^{-j}}\Big)^{h^{n+1}} \le \Big(2^{\frac{ph}{(h-1)^2}}\Big)^{h^{n+1}}.$$

Thus

$$\Big(\iint_{Q_{n+1}} u^q dx\,dt\Big)^{\frac{1}{q-(p-2)}} \le \Big(\frac{\gamma\tilde{\gamma}^{N+p}}{(1-a)^{N+p}}\iint_{Q_o} u^s dx\,dt\Big)^{\frac{1}{s-(p-2)}}. \qquad \blacksquare$$

11 A Uniform Bound Above on the $L^q_{\rm loc}$-Integral of Supersolutions

Lemma 11.1 *Let u be a nonnegative, local, weak supersolution to the degenerate $(p>2)$, homogeneous $(C=0)$ equations (1.1)–(1.2) of Chapter 3 in $Q_1 = K_1 \times (0,1)$, satisfying (9.6). Then, for all q in the range*

$$p-2 < q < p-1+\tfrac{p}{N} \tag{11.1}$$

there exists a constant γ, depending only on the data $\{p,N,C_o,C_1\}$, and q, such that

$$\int_0^{(\frac{3}{4})^p}\int_{K_{\frac{3}{4}}} u^q dx\,dt \le \gamma.$$

Proof The supersolution $v=u+1$ satisfies the assumptions of Lemma 10.2. Moreover by (9.6)

$$\int_{K_1} v^\sigma(\cdot,t)dx \le \gamma \quad \text{for all } \ t\in(0,1) \tag{11.2}$$

for an absolute constant γ depending only on the data $\{p,N,C_o,C_1\}$, and with σ given by (9.5). Pick numbers

$$\tfrac{7}{8} < s < r < 1 \quad \text{ and set } \quad Q_s = K_s \times (0,1].$$

Let φ be a nonnegative, piecewise smooth, cutoff function on K_r such that

$$0\le\varphi\le 1, \qquad \varphi = 1 \ \text{ in } \ K_s \qquad |D\varphi| \le \frac{1}{r-s}.$$

By (11.2), and the embedding Proposition 4.1 of the Preliminaries, applied as in (10.3) with

$$\alpha = p-2+\sigma, \qquad \beta=\frac{p(p-2+\sigma)}{\sigma}, \qquad \kappa = 1+\frac{p\sigma}{N(p-2+\sigma)},$$

$$\begin{aligned}
&\iint_{Q_s} v^{p-2+\sigma(1+\frac{p}{N})} dx\,dt \\
&\qquad \le \gamma \iint_{Q_r} |D(v^{\frac{p-2+\sigma}{p}}\varphi)|^p dx\,dt \left(\sup_{0<t<1}\int_{K_1} v^\sigma(x,t)dx\right)^{\frac{p}{N}} \\
&\qquad \le \gamma \iint_{Q_r} |D(v^{\frac{p-2+\bar\sigma}{p}}\varphi)|^p dx\,dt.
\end{aligned}$$

By Lemma 10.1 with $\varepsilon = -1+\sigma$, and (11.2)

$$\begin{aligned}\iint_{Q_r} &|D(v^{\frac{p-2+\sigma}{p}}\varphi)|dx\,dt \\ &\le \gamma\Big(\sup_{0<t<1}\int_{K_r} v^\sigma(x,t)dx + \iint_{Q_r} v^{p-2+\sigma}|D\varphi|^p dx\,dt\Big) \\ &\le \gamma\Big(1+\iint_{Q_r} v^{p-2+\sigma}|D\varphi|^p dx\,dt\Big).\end{aligned}$$

By Young's inequality

$$\begin{aligned}\iint_{Q_r} v^{p-2+\sigma}|D\varphi|^p dx\,dt \le &\frac{1}{2\gamma}\iint_{Q_r} v^{p-2+\sigma(1+\frac{p}{N})}dx\,dt \\ &+\gamma\Big(\frac{1}{r-s}\Big)^{p+\frac{N(p-2+\sigma)}{\sigma}}.\end{aligned}$$

Combining the previous estimates yields

$$\begin{aligned}\iint_{Q_s} &v^{p-2+\sigma(1+\frac{p}{N})}dx\,dt \\ &\le \frac{1}{2}\iint_{Q_r} v^{p-2+\sigma(1+\frac{p}{N})}dx\,dt + \gamma\Big(\frac{1}{r-s}\Big)^{p+\frac{N(p-2+\sigma)}{\sigma}}.\end{aligned}$$

By the interpolation Lemma 5.2 of the Preliminaries

$$\int_0^1\int_{K_{\frac{7}{8}}} v^{p-2+\sigma(1+\frac{p}{N})}dx\,dt \le \gamma.$$

An application of (10.2) concludes the proof. ∎

12 An Integral Bound Below for Supersolutions

The previous argument provides a uniform bound above on the L^q-norm of supersolutions. Here we establish a lower bound on the L^1-norm of u.

Lemma 12.1 *Let u be a nonnegative, local, weak supersolution to the degenerate ($p>2$), homogeneous ($C=0$) equations (1.1)–(1.2) of Chapter 3 in $Q_1 = K_1\times(0,1)$ satisfying (9.6). There exists a constant γ, depending only on the data $\{p,N,C_o,C_1\}$, such that, for all $0<\tau<(\frac{5}{8})^p$,*

$$\int_0^\tau\int_{K_{\frac{5}{8}}} |Du|^{p-1}dx\,dt \le \gamma\tau^{\frac{1}{p}\frac{\varepsilon}{1+2\varepsilon}}$$

where

$$\varepsilon = \frac{p}{4N(p-1)} < 1.$$

Proof The supersolution $v = u + 1$ satisfies the assumptions of Lemma 10.2, and hence the conclusion of Lemma 11.1 holds for it. By Hölder's inequality

$$\int_0^\tau \int_{K_{\frac{5}{8}}} |Du|^{p-1} dx\, dt = \int_0^\tau \int_{K_{\frac{5}{8}}} |Dv|^{p-1} v^{-(1+\varepsilon)\frac{p-1}{p}} v^{(1+\varepsilon)\frac{p-1}{p}} dx\, dt$$
$$\le \Big(\int_0^{(\frac{5}{8})^p} \int_{K_{\frac{5}{8}}} |Dv|^p v^{-1-\varepsilon} dx\, dt \Big)^{\frac{p-1}{p}} \Big(\int_0^\tau \int_{K_{\frac{5}{8}}} v^{(p-1)(1+\varepsilon)} dx\, dt \Big)^{\frac{1}{p}}.$$

The second integral is estimated by Hölder's inequality and Lemma 11.1, as

$$\Big(\int_0^\tau \int_{K_{\frac{7}{8}}} v^{(p-1)(1+\varepsilon)} dx\, dt \Big)^{\frac{1}{p}}$$
$$\le \tau^{\frac{1}{p}\frac{\varepsilon}{1+2\varepsilon}} \Big(\int_0^{(\frac{7}{8})^p} \int_{K_{\frac{7}{8}}} v^{(p-1)(1+2\varepsilon)} dx\, dt \Big)^{\frac{1}{p}\frac{1+\varepsilon}{1+2\varepsilon}} \le \gamma \tau^{\frac{1}{p}\frac{\varepsilon}{1+2\varepsilon}}$$

for an absolute constant γ depending only on the data $\{p, N, C_o, C_1\}$. The first integral is estimated by means of Lemma 10.1, with a proper test function φ, and gives

$$\int_0^{(\frac{5}{8})^p} |Dv|^p v^{-1-\varepsilon} dx\, dt \le \gamma \int_0^{(\frac{7}{8})^p} \int_{K_{\frac{7}{8}}} v^{p-1-\varepsilon} |D\varphi|^p dx\, dt$$
$$+ \gamma \int_0^{(\frac{7}{8})^p} \int_{K_{\frac{7}{8}}} v^{1-\varepsilon} \Big| \frac{\partial \varphi}{\partial t} \Big| dx\, dt.$$

The proof is concluded by a further application of Lemma 11.1 and of Hölder's inequality. ∎

Lemma 12.2 *Let u be a nonnegative, local, weak supersolution to the degenerate ($p > 2$), homogeneous ($C = 0$) equations (1.1)–(1.2) of Chapter 3 in $Q_1 = K_1 \times (0, 1)$, satisfying (9.6). Assume that*

$$\int_{K_{\frac{1}{2}}} u(x, 0) dx \ge 2c_o$$

for some positive constant c_o depending only on the data $\{p, N, C_o, C_1\}$. There exists a time $\tau \in (0, (\frac{5}{8})^p]$ depending only on the data $\{p, N, C_o, C_1\}$ and c_o, such that

$$\inf_{0<t<\tau} \int_{K_{\frac{5}{8}}} u(x, t) dx \ge c_o.$$

Proof Take a piecewise smooth, cutoff function φ in $K_{\frac{5}{8}}$, such that

$$0 \le \varphi \le 1, \qquad \varphi = 1 \quad \text{in} \quad K_{\frac{1}{2}}, \qquad |D\varphi| \le 8.$$

Since u is a supersolution, using φ as test function in the weak formulation of (1.1)–(1.2) of Chapter 3, for any $t \in (0, (\frac{5}{8})^p]$

$$\int_{K_{\frac{5}{8}}} u(x,t)\varphi(x)dx \geq \int_{K_{\frac{5}{8}}} u(x,0)\varphi(x)dx - C_1 \int_0^{(\frac{5}{8})^p} \int_{K_{\frac{5}{8}}} |Du|^{p-1}|D\varphi| dx\, dt.$$

The proof is concluded by means of Lemma 12.1. ■

13 Proof of Theorem 7.1

Having fixed $(y, s) \in E_T$, set

$$M = \fint_{K_\rho(y)} u(x,s)dx,$$

and assume that $M > 0$. Introduce the change of variables

$$x \to \frac{x-y}{2\rho}, \qquad t \to M^{p-2}\frac{t-s}{\rho^p}, \qquad v \to \frac{u}{M}.$$

Then v is a supersolution in the cylinder

$$K_8 \times \left[0, M^{p-2}\frac{T-s}{\rho^p}\right]$$

and

$$\fint_{K_{\frac{1}{2}}} v(x,0)dx = 1. \tag{13.1}$$

Let T_1 be the number defined in (9.3). By taking η_o even smaller if necessary, we may assume that $T_1 > 1$. Let now $T_2 > 0$ to be chosen and assume that

$$M^{p-2}\frac{T-s}{\rho^p} > \max\{T_1; T_2\} > 1 \tag{13.2}$$

so that

$$K_1 \times (0,1) \subset K_8 \times \left(0, M^{p-2}\frac{T-s}{\rho^p}\right].$$

If there exist $k > 1$ and $t_o \in (0,1)$ satisfying (9.1), then (9.2) for v and the indicated rescaling proves the theorem. If (9.1) is violated for all $k > 1$ and all $t \in (0,1)$, then by Lemma 11.1

$$\int_0^t \int_{K_{\frac{5}{8}}} v(x,t)^q dx\, dt \leq \gamma \quad \text{for all} \quad 0 \leq t \leq \left(\tfrac{5}{8}\right)^p$$

for some $q > 1$ in the range (11.1). Moreover, by (13.1) and Lemma 12.2, there exists $c_o = 2^{-(N+1)}$, and

$$0 < \tau < \left(\tfrac{5}{8}\right)^p$$

depending only on the data $\{p, N, C_o, C_1\}$, such that

$$\int_{K_{\frac{5}{8}}} v(x,t)dx > c_o \quad \text{for all } \ t \in \big(0, \tau\big).$$

These two inequalities imply that there exists a time level $t_1 \in (0, \tau)$ at which simultaneously

$$\int_{K_{\frac{5}{8}}} v(x,t_1)dx > c_o \quad \text{and} \quad \int_{K_{\frac{5}{8}}} v(x,t_1)^q dx\, dt \le \gamma.$$

From this

$$\begin{aligned} c_o &\le \tfrac{1}{2}c_o \big|\big[v(\cdot,t_1) < \tfrac{1}{2}c_o\big] \cap K_{\frac{5}{8}}\big| \\ &\quad + \int_{K_{\frac{5}{8}}} \chi_{[v(\cdot,t_1)>\frac{1}{2}c_o]} v(x,t_1)dx \\ &\le \tfrac{1}{2}c_o + \big|\big[v(\cdot,t_1) \ge \tfrac{1}{2}c_o\big] \cap K_{\frac{5}{8}}\big|^{\frac{q-1}{q}} \Big(\int_{K_{\frac{5}{8}}} v(x,t_1)^q dx\, dt\Big)^{\frac{1}{q}} \\ &\le \tfrac{1}{2}c_o + \gamma^{\frac{1}{q}} \big|\big[v(\cdot,t_1) \ge \tfrac{1}{2}c_o\big] \cap K_{\frac{5}{8}}\big|^{\frac{q-1}{q}}. \end{aligned}$$

Hence,

$$\big|\big[v(\cdot,t_1) \ge \tfrac{1}{2}c_o\big] \cap K_{\frac{5}{8}}\big| \ge \alpha |K_{\frac{5}{8}}| \quad \text{for} \quad \alpha = \left(\tfrac{8}{5}\right)^N \left(\tfrac{1}{2}c_o\right)^{\frac{q}{q-1}} \gamma^{-\frac{1}{q-1}}$$

and one verifies that α depends only on the data $\{p, N, C_o, C_1\}$. These are precisely the assumptions of the expansion of positivity Proposition 7.1 and yield the existence of constants $\eta_*, \delta_* \in (0,1)$ and $b_* > 1$ depending only on the data and α, such that

$$v(\cdot,t) \ge \eta_* c_o \ \text{ a.e. in } \ K_{2\rho}(y), \qquad \text{where } \ \eta_* = \eta_o \alpha^d, \tag{13.3}$$

for all times

$$t_1 + \frac{b_*^{p-2}}{(\eta_* c_o)^{p-2}} \tfrac{1}{2}\delta_* \le t \le t_1 + \frac{b_*^{p-2}}{(\eta_* c_o)^{p-2}} \delta_*.$$

Set

$$T_2 = \frac{b_*^{p-2}}{(\eta_* c_o)^{p-2}} \delta_*. \tag{13.4}$$

By taking c_o or η_* smaller if needed, one may insure that $t_1 + \frac{1}{2}T_2 \ge 1$, and (13.3) holds for all times

$$1 + \frac{1}{2}T_2 < t < T_2.$$

If $T_2 < T_1$, then a further application of the expansion of positivity Proposition 7.1 implies the bound below (13.3), with new constants, still depending only on the data, for all times t as in (9.2). If $T_1 < T_2$, a similar argument holds. Thus we conclude that either the bound below (9.2) or the bound below (13.3), can be made to hold, by a suitable modification of the bounding constants, within a common interval of time

$$1 + \tfrac{1}{2}T_* < t < T_*, \qquad \text{where} \qquad T_* = \max\{T_1; T_2\}.$$

Returning to the original coordinates, proves Theorem 7.1 if (13.2) is in force. On the other hand, such a condition being violated, coincides precisely with the conclusion of Theorem 7.1. ∎

Remark 13.1 Combining the result of Theorem 7.1 with L^∞-bounds for nonnegative subsolutions (see [41], Chapter V), provides a different proof of Theorem 1.1. The proof of Theorem 1.1 as presented here, does not require distinct sup and inf estimates.

14 A Consequence of Theorem 7.1

If u is defined in the strip

$$S_T = \mathbb{R}^N \times (0, T],$$

then Theorem 7.1 can be sharpened to the following.

Proposition 14.1 *Let u be a nonnegative, weak supersolution to the degenerate ($p > 2$), homogeneous ($C = 0$) equations (1.1)–(1.2) of Chapter 3 in S_T. There exists a constant γ depending only on the data $\{p, N, C_o, C_1\}$, such that for all $(y, s) \in S_T$, and $\rho > 0$, and $\theta > 0$ such that $s + \theta < T$,*

$$\fint_{K_\rho(y)} u(x, s)dx \le \gamma\Big\{\Big(\frac{\rho^p}{\theta}\Big)^{\frac{1}{p-2}} + \Big(\frac{\theta}{\rho^p}\Big)^{\frac{N}{p}}\big(\inf_Q u\big)^{\frac{\lambda}{p}}\Big\}, \tag{14.1}$$

where λ is as in (1.6), and

$$Q = K_{4\rho}(y) \times \big(s + \tfrac{1}{2}\theta\,,\, s + \theta\big).$$

Proof Assume $(y, s) = (0, 0)$, and introduce the change of variables

$$v(x, t) = u(h\rho x, (h\rho)^p t),$$

where $h > 1$ is to be determined. By this rescaling

$$\fint_{K_\rho} u(x, 0)dx = \fint_{K_{1/h}} v(x, 0)dx \le h^N \fint_{K_1} v(x, 0)dx. \tag{14.2}$$

We may assume that

$$\fint_{K_1} v(x,0)dx > 2c\Big[\frac{(h\rho)^p}{T}\Big]^{\frac{1}{p-2}}, \tag{14.3}$$

where c is the constant given by Theorem 7.1, otherwise there is nothing to prove. From (7.1)–(7.2)

$$\fint_{K_1} v(x,0)dx \le c\Big(\frac{(h\rho)^p}{T}\Big)^{\frac{1}{p-2}} + \gamma \inf_{Q_*} u, \tag{14.4}$$

where

$$Q_* = K_4 \times (1 + \tfrac{1}{2}T_*, T_*)$$

and

$$T_* = c^{p-2}\Big(\frac{1}{2}\fint_{K_1} v(x,0)dx\Big)^{2-p} < \frac{T}{(h\rho)^p},$$

in view of (14.5). From this, (14.4) yields

$$\fint_{K_1} v(x,0)dx \le 2\gamma \inf_{Q_*} u. \tag{14.5}$$

The time θ in (14.1), being fixed, choose h from

$$\theta = T_*(h\rho)^p, \qquad \text{that is,} \qquad h^p = c^{2-p}\frac{\theta}{\rho^p}\Big(\frac{1}{2}\fint_{K_1} v(x,0)dx\Big)^{p-2}.$$

By (14.2), this choice of h yields

$$h^\lambda \ge c^{2-p}\frac{\theta}{\rho^p}\Big(\frac{1}{2}\fint_{K_\rho} u(x,0)dx\Big)^{p-2}.$$

We may assume

$$\fint_{K_\rho} u(x,0)dx > 2c\Big(\frac{\rho^p}{\theta}\Big)^{\frac{1}{p-2}},$$

otherwise there is nothing to prove, and this guarantees $h > 1$. In terms of u, from (14.2) and (14.5)

$$h^{-N}\fint_{K_\rho} u(x,0)dx \le 2\gamma \inf_{Q_{h,\theta}} u, \tag{14.6}$$

where

$$Q_{h,\theta} = K_{4h\rho} \times \big(\tfrac{1}{2}\theta, \theta\big).$$

Observe also that

$$\inf_{Q_*} v = \inf_{Q_{h,\theta}} u.$$

From this, the definition of h, and (14.5)

$$
\begin{aligned}
h^N &= \left(\tfrac{1}{2}c\right)^{\frac{N(p-2)}{p}} \left(\frac{\theta}{\rho^p}\right)^{\frac{N}{p}} \left(\fint_{K_1} v(x,0)dx\right)^{\frac{N(p-2)}{p}} \\
&\le (c\gamma)^{\frac{N(p-2)}{p}} \left(\frac{\theta}{\rho^p}\right)^{\frac{N}{p}} \left(\inf_{Q_2} u\right)^{\frac{N(p-2)}{p}} .
\end{aligned}
\tag{14.7}
$$

Combining (14.6) and (14.7), and recalling that $Q \subset Q_{h,\theta}$ since $h > 1$ proves the proposition. ■

15 Equations of the Porous Medium Type

The intrinsic Harnack inequality (1.3)–(1.4) and its consequences, including the Hölder continuity of solutions, Liouville-type results, and subpotential-type estimates, have counterparts for nonnegative, local, weak solutions to a large class of quasilinear degenerate parabolic equations that include equations of the porous medium type. We use the latter class to draw similarities in the statements and outline differences in the proofs.

Henceforth in this section we let u be a continuous, nonnegative, local, weak solution to the degenerate ($m > 1$) equations (5.1)–(5.2) of Chapter 3, in E_T. It is known that these solutions are locally bounded in E_T, which from now on we assume ([7]).

Continue to denote by $Q^{\pm}_{\rho}(\theta)$ and $(y,s) + Q^{\pm}_{\rho}(\theta)$ the cylinders introduced in (2.1)–(2.2) of Chapter 3, with $p = 2$.

15.1 The Intrinsic Harnack Inequality

Fix $(x_o, t_o) \in E_T$ such that $u(x_o, t_o) > 0$ and construct the cylinders

$$
(x_o, t_o) + Q^{\pm}_{\rho}(\theta) \qquad \text{where} \qquad \theta = \left(\frac{c}{u(x_o, t_o)}\right)^{m-1} \tag{15.1}
$$

and c is a given positive constant. These cylinders are "intrinsic" to the solution since their length is determined by the value of u at (x_o, t_o).

Theorem 15.1 *Let u be a continuous, nonnegative, local, weak solution to the degenerate equations (5.1)–(5.2) of Chapter 3. There exist positive constants c and γ depending only on the data $\{m, N, C_o, C_1\}$, such that for all intrinsic cylinders $(x_o, t_o) + Q^{\pm}_{4\rho}(\theta)$ as in (15.1), contained in E_T, either*

$$
\gamma C \rho > 1
$$

or

$$
\gamma^{-1} \sup_{K_\rho(x_o)} u(\cdot, t_o - \theta\rho^2) \le u(x_o, t_o) \le \gamma \inf_{K_\rho(x_o)} u(\cdot, t_o + \theta\rho^2). \tag{15.2}
$$

Thus the form (1.2) continues to hold for nonnegative solutions to the degenerate equations (5.1)–(5.2) of Chapter 3, although in their own intrinsic geometry, made precise by (15.1). In analogy with (1.2) and $(1.1)^{\pm}$, we call intrinsic, "forward" and "backward" Harnack inequalities, the right and left inequalities in (15.2).

Remark 15.1 The constants γ and c deteriorate as $m \to \infty$ in the sense that $\gamma(m), c(m) \to \infty$ as $m \to \infty$. However they are stable as $m \to 1$ in the sense of (5.9) of Chapter 3. Thus by formally letting $m \to 1$ in (15.2) one recovers the classical Moser's Harnack inequality in the form (1.2).

Remark 15.2 The proofs are based on the energy estimates and DeGiorgi-type lemmas of § 6–8 of Chapter 3 and the expansion of positivity of § 7 of Chapter 4. According to the discussion in § 5.3 and Remarks 6.2, 7.1, and 8.3 of Chapter 3, a constant γ depends only on the data if it can be quantitatively determined a priori only in terms of $\{m, N, C_o, C_1\}$. The constant C appearing in the structure conditions (5.2) of Chapter 3, enters in the statement of Theorem 15.1 only through an alternative.

Remark 15.3 The theorem has been stated for continuous solutions, to give meaning to $u(x_o, t_o)$. While it is known that local, weak solutions to (5.1)–(5.2) of Chapter 3, for $m > 1$, are locally Hölder continuous ([37, 47]), the theorem continues to hold for almost all $(x_o, t_o) \in E_T$ and the corresponding cylinders $(x_o, t_o) + Q^{\pm}_{4\rho}(\theta) \subset E_T$.

15.1.1 Significance of Theorem 15.1

The Harnack inequality (15.1)–(15.2) is "intrinsic" in that the waiting time from t_o to $t_o + \theta\rho^2$ depends on the solution at (x_o, t_o). Such an intrinsic dependence is a consequence of the intrinsic expansion of positivity of § 7 of Chapter 4, and it cannot be removed. Indeed (15.2) is false in a geometry where θ is a constant independent of $u(x_o, t_o)$. This can be verified for the Barenblatt solution, of the degenerate ($m > 1$) homogeneous, prototype equation (5.3) of Chapter 3 in $\mathbb{R}^N \times \mathbb{R}^+$

$$\begin{aligned} \Gamma_m(x,t) &= \frac{1}{t^{\frac{N}{\lambda}}}\left[1 - b(N,m)\left(\frac{|x|}{t^{\frac{1}{\lambda}}}\right)^2\right]_+^{\frac{1}{m-1}}, \quad t > 0 \\ b(N,m) &= \frac{N(m-1)}{2Nm\lambda}, \qquad \lambda = N(m-1) + 2. \end{aligned} \tag{15.3}$$

Arguing as in § 3.2 of Chapter 4, one verifies that Γ fails to satisfy the expansion of positivity of § 7 of Chapter 4, unless the geometry is "intrinsic." By the same calculations one also verifies that it fails to satisfy the Harnack inequality of Theorem 15.1, unless θ reflects the intrinsic geometry of the equation as indicated in (15.1). Further counterexamples include the one-parameter family

$$u(x,t) = \Big(\frac{m-1}{2m\lambda}\Big)^{\frac{1}{m-1}} \Big(\frac{|x|^2}{T-t}\Big)^{\frac{1}{m-1}} \qquad \text{in} \qquad \mathbb{R}^N \times (-\infty, T)$$

where λ is as in (15.3). This satisfies the homogeneous, prototype equation (5.3) of Chapter 3, and is the counterpart of (1.5)–(1.6). By similar arguments one verifies that it fails to satisfy the Harnack inequality of Theorem 15.1 if θ is independent of u. Finally the family of traveling wave solution,

$$u(x,t) = A(x_o + bt - x)_+^{\frac{1}{m-1}}, \qquad \text{where} \qquad A = b^{\frac{1}{m-1}},$$

is the counterpart of (3.2) of Chapter 4 and by similar arguments, is a counterexample to a Harnack inequality and expansion of positivity in geometries that are independent of the solution itself.

15.1.2 On the Proof of Theorem 15.1

The proof of Theorem 1.1 is based on the energy inequalities (2.3) of Chapter 3, the DeGiorgi-type Lemmas 3.1–4.1 of Chapter 3, and the expansion of positivity of Chapter 4. Now all these technical tools are available for the porous medium type equations and are provided in § 5–8 of Chapter 3 and § 7 of Chapter 4. With these at hand the proof parallels that of Theorem 1.1 almost verbatim.

Fix $(x_o, t_o) \subset E_T$, assume that $u(x_o, t_o) > 0$, and construct the cylinders $(x_o, t_o) + Q_{4\rho}^{\pm}(\theta) \subset E_T$ as in (15.1), where the constant $c \geq 1$ is to be determined. The change of variables

$$x \to \frac{x - x_o}{\rho} \qquad t \to u(x_o, t_o)^{m-1} \frac{t - t_o}{\rho^2}$$

maps these cylinders into $Q^{\pm}$, where

$$Q^+ = K_4 \times (0, 4^2 c^{m-1}], \qquad Q^- = K_4 \times (-4^2 c^{m-1}, 0].$$

Denoting again by (x,t) the transformed variables, the rescaled function

$$v(x,t) = \frac{1}{u(x_o, t_o)} u\Big(x_o + \rho x, t_o + \frac{t\rho^2}{u(x_o, t_o)^{m-1}}\Big)$$

is a bounded, nonnegative, weak solution to

$$v_t - \operatorname{div} \bar{\mathbf{A}}(x,t,v,Dv) = \bar{B}(x,t,v,Dv)$$

weakly in $Q = Q^+ \cup Q^-$, where $\bar{\mathbf{A}}$ and $\bar{B}$ satisfy the structure conditions

$$\begin{cases} \bar{\mathbf{A}}(x,t,v,Dv) \cdot Dv \geq mC_o v^{m-1} |Dv|^2 - \bar{C}^2 v^{m+1} \\ |\bar{\mathbf{A}}(x,t,v,Dv)| \leq mC_1 v^{m-1} |Dv| + \bar{C} v^m \\ |\bar{B}(x,t,v,Dv)| \leq m\bar{C} v^{m-1} |Dv| + \bar{C}^2 v^m \end{cases} \qquad \text{a.e. in } Q$$

where

$$\bar{C} = C\rho,$$

and C_o, C_1, and C are as in (5.2) of Chapter 3. Moreover $v(0,0) = 1$.

The theorem is then a consequence of a fact similar to Proposition 2.1, whose proof relies solely on the indicated technical tools. The proof can now be followed and completed with the obvious changes in language and symbolism.

16 Some Consequences of the Harnack Inequality

16.1 Hölder Continuity

The methods leading to the intrinsic Harnack inequality of Theorem 15.1 can be used to establish that local, weak solutions u to the degenerate $(m > 1)$ porous medium type equations (5.1)–(5.2) of Chapter 3, in E_T, are locally Hölder continuous in E_T, irrespective of their signum, and permit one to exhibit a quantitative Hölder modulus of continuity. These solutions are locally bounded ([7]) in E_T. To streamline the presentation and symbolism, we assume that $u \in L^\infty(E_T)$.

Let

$$\Gamma = \partial E_T - \bar{E} \times \{T\}$$

denote the parabolic boundary of E_T, and for a compact set $K \subset E_T$ introduce the *intrinsic*, parabolic m-distance from K to Γ by

$$m - \operatorname{dist}(K;\Gamma) \overset{\text{def}}{=} \inf_{\substack{(x,t)\in K\\(y,s)\in\Gamma}} \Big(|x-y| + \|u\|_{\infty,E_T}^{\frac{m-1}{2}} |t-s|^{\frac{1}{2}} \Big).$$

Theorem 16.1 *Let u be a bounded, local, weak solution to the degenerate $(m > 1)$ porous medium type equations (5.1)–(5.2) of Chapter 3. Then u is locally Hölder continuous in E_T, and there exist constants $\gamma > 1$ and $\alpha \in (0,1)$ that can be determined a priori only in terms of the data $\{m, N, C_o, C_1, C\}$, such that for every compact set $K \subset E_T$,*

$$|u(x_1,t_1) - u(x_2,t_2)| \le \gamma \|u\|_{\infty,E_T} \left(\frac{|x_1-x_2| + \|u\|_{\infty,E_T}^{\frac{m-1}{2}} |t_1-t_2|^{\frac{1}{2}}}{m - \operatorname{dist}(K;\Gamma)} \right)^{\alpha}$$

for every pair of points (x_1,t_1), and $(x_2,t_2) \in K$.

Proof Fix a point in E_T, which up to a translation we take to be the origin of $\mathbb{R}^{N+1}$. For $\rho > 0$ consider the cylinder

$$Q_\epsilon = K_\rho \times (-\rho^{2-(m-1)\epsilon}, 0]$$

where $\epsilon \in (0,1)$ is to be determined, and set

$$\mu_o^+ = \sup_{Q_\epsilon} u, \qquad \mu_o^- = \inf_{Q_\epsilon} u, \qquad \omega_o = \operatorname*{osc}_{Q_\epsilon} u = \mu_o^+ - \mu_o^-.$$

With ω_o at hand, construct now the cylinder of intrinsic geometry

$$Q_o = K_\rho \times (-\omega_o^{1-m}\rho^2, 0].$$

If $\omega_o \geq \rho^\epsilon$, then $Q_o \subset Q_\epsilon$. Theorem 16.1 is then a consequence of the following:

Proposition 16.1 *There exist constants $\gamma > 1$, and $\epsilon, \varepsilon, \delta \in (0,1)$, that can be quantitatively determined only in terms of the data $\{m, N, C_o, C_1\}$ and independent of u and ρ, such that, if $\omega_o \geq \rho^\epsilon$, setting $\rho_o = \rho$ and*

$$\omega_n = \delta\omega_{n-1}, \quad \rho_n = \varepsilon\rho_{n-1}, \quad Q_n = Q_{\rho_n}^-(\omega_n^{1-m}), \quad \textit{for } n = 0, 1, \dots$$

we have $Q_{n+1} \subset Q_n$, and either

$$\operatorname*{osc}_{Q_n} u \leq 4\gamma\frac{C}{\varepsilon}\rho_n \qquad \textit{or} \qquad \operatorname*{osc}_{Q_n} u \leq \omega_n.$$

The proposition is established by induction. Thus assume Q_n has been constructed and that the statement holds up to n. Set

$$\mu_n^+ = \sup_{Q_n} u, \quad \mu_n^- = \inf_{Q_n} u, \quad P_n = (0, -s_n), \quad s_n = \tfrac{1}{2}\omega_n^{1-m}\rho_n^2.$$

The main difference with the proof of Proposition 4.1 is that the two functions $(\mu_n^+ - u)$ and $(u - \mu_n^-)$, while nonnegative, need not be weak solutions to the porous medium type equations (5.1)–(5.2) of Chapter 3 in Q_n, and hence one cannot claim that they both satisfy the intrinsic Harnack inequality in subdomains of Q_n. However, it can be shown that at least one of them does satisfy the intrinsic Harnack inequality with respect to P_n, and this will suffice to establish the proposition. This is done by the same techniques of proving the Harnack inequality. We refrain from further elaborating on the modifications needed, since Theorem 16.1 is known from the literature ([47]). ■

16.2 Liouville-Type Results

The Liouville-type results of § 5 are possible since if u is a local weak solution to the homogeneous ($C = 0$), degenerate ($p > 2$), p-Laplacian type equations (1.1)–(1.2) of Chapter 3 with $C = 0$, then $u + \text{const}$ is a solution to a similar equation, with the same modulus of ellipticity. This is no longer the case for solutions to the porous medium type equations, and as a consequence Liouville-type statements hold in a rather restricted form.

Proposition 16.2 *Let u be a nonnegative, local, weak solution to the degenerate ($m > 1$), homogeneous ($C = 0$) equations (5.1)–(5.2) of Chapter 3 in*

$$S_T = \mathbb{R}^N \times (-\infty, T) \quad \textit{for some fixed } T \in \mathbb{R},$$

and let $\inf_{S_T} u = 0$. *Then for all* $x \in \mathbb{R}^N$

$$\lim_{t\to-\infty} u(x,t) = 0,$$

and the limit is uniform for x *ranging over a compact set* $K \subset \mathbb{R}^N$ *such that* $K \times \{\tau\}$ *is included in a* (y,s)*-paraboloid*

$$P(y,s) = \Big\{(x,t) \in S_T \;\Big|\; t - s \le -\Big(\frac{c}{u(y,s)}\Big)^{m-1} |x-y|^2\Big\}$$

for $(y,s) \in S_T$ *such that* $u(y,s) > 0$, *for some* $\tau < s$.

Proof Analogous to the first part of Lemma 5.1. ■

Proposition 5.1 still holds true for solutions to porous medium type equations.

Proposition 16.3 *If* u *is bounded above and below in* S_T, *then* u *is constant.*

Proof Set

$$m = \inf_{S_T} u, \quad M = \sup_{S_T} u, \quad \omega_o = M - m.$$

Fix $t_o < T$ and assume after a translation that $t_o = 0$. The proof uses the form of the Hölder continuity as expressed in Proposition 16.1, over a sequence of nested cylinders Q_n with "vertex" at $(0,0)$. Notice that, by the definition of ω_o, the requirement

$$\operatorname*{osc}_{Q_o} u \le \omega_o$$

is always satisfied, and hence the condition $\omega_o > \rho^\epsilon$ does not have to be enforced. Applying Proposition 16.1 recursively, starting from $\rho = \varepsilon^{-n} R$ for $R > 0$ fixed, gives

$$\operatorname*{osc}_{K_R \times (-\omega_n^{1-m} R^2, 0]} u \le \delta^n \omega_o$$

for all $n \in \mathbb{N}$. Let $R \to \infty$ for n fixed and then let $n \to \infty$. ■

16.3 Subpotential Lower Bounds

The prototype equation (5.3) of Chapter 3 admits the family of subsolutions

$$\Gamma_{\nu,m}(x,t;y,s) = \frac{k\rho^\nu}{S^{\frac{\nu}{\lambda}}(t)} \Big[1 - b(\nu,m)\Big(\frac{|x-\bar{x}|}{S^{\frac{1}{\lambda}}(t)}\Big)^2\Big]_+^{\frac{1}{m-1}}$$

where $\nu \ge N$, k and ρ are positive parameters and

$$S(t) = k^{m-1}\rho^{\nu(m-1)}(t-s) + \rho^\lambda \qquad t \ge s$$

$$b(\nu,m) = \frac{\nu(m-1)}{2Nm\lambda}, \qquad \lambda = \nu(m-1) + 2.$$

One verifies that for $\nu = N$, as in (15.3), the functions $\Gamma_{\nu,m}$ are exact solutions to the homogeneous prototype porous medium equation, for all $k, \rho > 0$, whereas for $\nu > N$ they are subsolutions. Decay estimates of the type of § 6 can be established, with no major change and we summarize them in the following proposition.

Proposition 16.4 *Let u be a nonnegative, local, weak solution to the degenerate $(m > 1)$, homogeneous $(C = 0)$ equations (5.1)–(5.2) of Chapter 3 in E_T. There exist positive constants $\gamma_o < 1$ and $\gamma_1 > 1$, depending only on the data $\{m, N, C_o, C_1\}$ and the constant c and γ appearing in the Harnack inequality (15.1)–(15.2) of Theorem 15.1, such that for all $(x_o, t_o) \in E_T$, such that $u(x_o, t_o) > 0$, and all $(x, t) \in E_T$ with*

$$K_{8|x-x_o|}(x_o) \subset E, \qquad \textit{and} \qquad 0 < t - t_o < \tfrac{1}{4^2} t_o,$$

we have

$$u(x,t) \geq \gamma_o u(x_o, t_o) \Big\{ 1 - \gamma_1 \Big[\frac{|x - x_o|^2}{u(x_o, t_o)^{m-1}(t - t_o)} \Big] \Big\}_+^{\frac{1}{m-1}}.$$

A decay estimate similar to that of Proposition 6.2 can be stated and proved analogously.

17 A Weak Harnack Inequality for Positive Supersolutions

Local, weak supersolutions to the degenerate $(m > 1)$ equations (5.1)–(5.2) of Chapter 3 in E_T, need not be continuous, and if nonnegative they do not, in general, satisfy a Harnack estimate. Nevertheless they satisfy a weak, or integral, form of the Harnack inequality as expressed in the following theorem. The theorem is stated for homogeneous equations for which $C = 0$. The extension to nonhomogeneous equations $C \neq 0$ is achieved by the usual alternative that negating $C\rho > 1$, forces $C\rho \leq 1$.

Throughout, for $y \in E$, let $\rho > 0$ be so small that $K_{16\rho}(y) \subset E$.

Theorem 17.1 *Let u be a nonnegative, local, weak supersolution to the degenerate $(m > 1)$, homogeneous $(C = 0)$ equations (5.1)–(5.2) of Chapter 3 in E_T. There exist positive constants c and γ, depending only on the data $\{m, N, C_o, C_1\}$, such that for a.e. $s \in (0, T)$*

$$\fint_{K_\rho(y)} u(x,s)dx \leq c \Big(\frac{\rho^2}{T - s} \Big)^{\frac{1}{m-1}} + \gamma \inf_{K_{4\rho}(y)} u(\cdot, t)$$

for all times

$$s + \tfrac{1}{2}\theta\rho^2 \leq t \leq s + \theta\rho^2$$

where

$$\theta = \min \Big\{ c^{1-m} \frac{T - s}{\rho^2}, \Big(\fint_{K_\rho(y)} u(x,s)dx \Big)^{1-m} \Big\}.$$

The proof closely follows the one for Theorem 7.1. Perhaps the most significant difference is in Lemmas 11.1–12.1, where $u+1$ remains a supersolution to the p-Laplacian, and not for the porous medium equation. This can be overcome, by assuming that supersolutions at hand can be approximated by supersolutions locally bounded away from zero.

If u is defined in the strip $S_T = \mathbb{R}^N \times (0,T]$, then Theorem 17.1 can be sharpened to the following.

Proposition 17.1 *Let u be a nonnegative, local, weak supersolution to the degenerate ($m > 1$), homogeneous ($C = 0$) equations (5.1)–(5.2) of Chapter 3 in S_T. There exists a constant γ depending only on the data $\{m, N, C_o, C_1\}$, such that for all $(y,s) \in S_T$, all $\rho > 0$, and $\theta > 0$ such that $s + \theta < T$,*

$$\fint_{K_\rho(y)} u(x,s)dx \le \gamma \Big\{ \Big(\frac{\rho^2}{\theta}\Big)^{\frac{1}{m-1}} + \Big(\frac{\theta}{\rho^2}\Big)^{\frac{N}{2}} \Big[\inf_{K_\rho(x_o)} u(\cdot, s+\theta) \Big]^{\frac{\lambda}{2}} \Big\},$$

where λ is as in (15.3).

18 Remarks and Bibliographical Notes

Moser's inequality is given in the form (1.1). The same form was given earlier by Hadamard [80] and Pini [127]. The form (1.1) for nonnegative solutions to elliptic equations appears in Landis [104] and for nonnegative solutions to parabolic equations in Krylov and Safonov [94]. The mean value form (1.2), while elementary, does not seem to be present in the literature.

The intrinsic, forward Harnack inequality of Theorem 1.1 was first established in [39] for the prototype equations (1.3) and (5.3) of Chapter 3, and later extended to equations of the form

$$u_t - \big(|Du|^{p-2} a_{ij}(x) u_{x_i}\big)_{x_j} = 0$$

in [30]. Another proof is in [32]. The proof of the intrinsic, forward Harnack inequality for nonnegative solutions to (1.1)–(1.2) of Chapter 3 with their full quasilinear structure has been established in [49], both for p-Laplacian and porous medium type equations. A different proof of the same result for *homogeneous* equations (1.1)–(1.2) is in [98], using a parabolic degenerate version of the Krylov–Safonov covering Lemma [94]. The proof in [49] makes no use of covering arguments. Recently there have been extensions to parabolic equations with coefficients in the Kato classes [109], or when the degeneracy depends also on Muckenhoupt-type weights [145]. The proof we present here is an unpublished simpler version of [49], where some measure-theoretical facts have been eliminated and the simpler, more streamlined version of the *expansion of positivity* of Chapter 4 has been used.

The intrinsic, backward Harnack inequality of Theorem 1.1 and its proof are taken from [52]. The lower bound in terms of the supremum of the solution at a *previous* time can be useful in studying the boundary Harnack inequality for the homogeneous equations (1.1)–(1.2) of Chapter 3. Results in this direction, for linear parabolic equations with bounded and measurable coefficients, are due to Salsa ([137]). See also [25], Chapter XIII, and the references therein.

The Hölder continuity of locally bounded solutions to (1.1)–(1.2) of Chapter 3, with $p > 2$, was first established in [38] and reported in [41]. Local solutions are locally bounded ([41]). The Hölder continuity of solutions to homogeneous systems of the form (1.3) was established in [47]. This contribution contains also a proof of the Hölder continuity of solutions to homogeneous, porous medium type equations. A new approach to Hölder continuity based on the expansion of positivity of Chapter 4 is in [75].

These continuity estimates were used in the proof of the intrinsic Harnack inequality in [39]. Here the approach is reversed: the Harnack inequality is established without any assumption on the modulus of continuity of solutions, and the Hölder regularity is then deduced from it. It should be pointed out, however, that this is an "interior" statement. A boundary modulus of continuity, under proper assumptions on ∂E and the boundary data, is in [41]. We do not know of a proof by which an interior Harnack estimate would yield a boundary modulus of continuity of solutions to these degenerate equations.

The use of the Harnack inequality to prove Liouville-type results for solutions to linear, nondegenerate, parabolic equations with principal part, in nondivergence form, is due to [78, 79]. Extensions have been given to a wide class of equations; see [91] for the case of hypoelliptic operators, still remaining within linear and nondegenerate classes.

Liouville-type results for solutions to *degenerate* equations as those in (1.1)–(1.2) of Chapter 3 appear to be new, and are taken from [56]. In particular, Propositions 5.2–5.3 seem to be new even in the context of the heat equation. The following is a further strengthening of Proposition 5.2.

Proposition 18.1 *Let u be bounded below in S_T and assume that*

$$\sup_{\mathbb{R}^N} u(\cdot, s) = M_s < +\infty \qquad \textit{for some } \; s < T.$$

Then u is constant in S_T.

Proof By Proposition 5.2 u is constant in S_s, and we may take such a constant to be zero. Now consider the Cauchy problem for the homogeneous, degenerate equations (1.1)–(1.2) of Chapter 3 in the strip $\mathbb{R}^N \times (s, T)$ with "initial datum" $u(\cdot, s) = 0$. Since the principal part $\eta \to \mathbf{A}(x, t, u, D\eta)$ is monotone at the origin (Lemma 1.1 of Chapter 3), such a problem admits only one nonnegative solution. Thus $u = 0$ in the whole S_T. ∎

Remark 18.1 It is not claimed here that, in the generality placed on the principal part $\mathbf{A}(x, t, u, \eta)$, the solution to the indicated Cauchy problem is unique. For this to occur, one would need a rather precise control on the growth of $x \to u(x,t)$ as $|x| \to \infty$, and the monotonicity of the map $(u,\eta) \to \mathbf{A}(x,t,u,\eta)$. This is stronger than the monotonicity of $\eta \to \mathbf{A}(x,t,u,\eta)$ stated in (12.1) of the *Remarks and Bibliographical Notes* of Chapter 3. The issue has been addressed in several contributions ([57, 32, 99, 6]).

The uniqueness statement used in the proof of Proposition 18.1 holds only because the initial datum is zero and $\eta \to \mathbf{A}(x,t,u,\eta)$ is monotone at the origin. The method relies on the techniques developed in [41], Chapter XI.

Corollary 18.1 *Let u be bounded below (above) in S_T. Then either u is constant, or for all $t < T$,*

$$\lim_{|x|\to\infty} u(x,t) = +\infty \, (-\infty).$$

Remark 18.2 The explicit solutions in (1.5)–(1.6) and in (3.2) of Chapter 4 exhibit the behavior indicated by Corollary 18.1.

The subpotential lower bounds of Propositions 6.1 and 6.2 were proved in [50] with a slightly different approach. They represent an extension of the analogous result proved by Moser for linear parabolic equations [121].

The results of Proposition 6.1 were proved in [74], using a Control Theory approach. Starting from a Hamilton–Jacobi differential inequality, and relying on Control Theory arguments, in [11] similar lower bounds are proved for a class of homogeneous degenerate and singular equations, satisfying the comparison principle, and which include the porous medium and the p-Laplacian equations. The proof depends in a fundamental way on the Aronson-Bénilan estimate [9], and hence on the comparison principle. No such principle holds for equations with the full quasilinear structure.

18.1 Weak Harnack Estimates

The idea of proving the weak Harnack inequality by first showing that the expansion of positivity holds with a power-like dependence on the quantity α introduced in (4.2), was originally used in [61] in the context of Q-minima, and functions in *elliptic* DeGiorgi classes. It has then been adapted to the parabolic p-Laplacian when $p > 2$ in [98]. In both cases the main technical tool is a proper version of the Krylov–Safonov covering lemma ([94, 93]). In the parabolic case, there is the extra difficulty that the covering lemma must have an intrinsic feature. Here we made no use of covering arguments, and used a different approach, namely, relying on the measure-theoretical Lemma 3.1 of the Preliminaries. The approach to the second alternative in the proof of Theorem 7.1 is analogous to a similar argument in [61], adapted by [98] to the parabolic case. Proposition 14.1 was first proved for *solutions* to the prototype equation (1.3) of Chapter 3 in [41], later extended to solutions to homogeneous equations (1.1)–(1.2) of Chapter 3 in [98].

Similar results for nonnegative supersolutions to the degenerate ($m > 1$), porous medium type equations (5.1)–(5.2) of Chapter 3, namely, Theorem 17.1 and Proposition 17.1, in the indicated generality, seem to be new.

6

The Harnack Inequality for Singular Equations

1 Supercritical, Singular Equations

Let u be a continuous, nonnegative, local, weak solution to the singular equations (1.1)–(1.2) of Chapter 3 in E_T, for p in the *supercritical* range

$$p_* = \tfrac{2N}{N+1} < p < 2. \tag{1.1}$$

Fix $(x_o, t_o) \in E_T$ such that $u(x_o, t_o) > 0$ and construct the cylinders

$$(x_o, t_o) + Q_\rho^\pm(\theta) \quad \text{where} \quad \theta = [u(x_o, t_o)]^{2-p}. \tag{1.2}$$

These cylinders are "intrinsic" to the solution since their length is determined by the value of u at (x_o, t_o). Cylindrical domains of the form $K_\rho \times (0, \rho^p]$ reflect the natural, parabolic space-time dilations that leave the homogeneous, singular, prototype equation (1.3) of Chapter 3 invariant. The latter, however, is not homogeneous with respect to the solution u. The time dilation by a factor $[u(x_o, t_o)]^{2-p}$ is intended to restore the homogeneity, and the Harnack inequality holds in such an intrinsic geometry, as made precise in Theorems 1.1–1.2 below. The first theorem establishes an intrinsic, mean value Harnack inequality in a form similar to Theorem 1.1 of Chapter 5, for degenerate equations ($p > 2$). This Harnack estimate is stable as $p \to 2$. The second theorem establishes a "time insensitive" mean value Harnack inequality, valid for all times t ranging in a neighborhood of t_o. This inequality is unstable as $p \to 2$. By counterexamples, it will be shown that for $1 < p \le \frac{2N}{N+1}$, neither of these theorems holds.

1.1 The Intrinsic, Mean Value, Harnack Inequality

Local weak solutions to (1.1)–(1.2) of Chapter 3, for p in the supercritical range (1.1), are locally bounded and locally Hölder continuous within their domain of definition ([41], Chapters IV–V). Having fixed $(x_o, t_o) \in E_T$, and cylinders $(x_o, t_o) + Q_\rho^\pm(\theta)$ as in (1.2), set

E. DiBenedetto et al., *Harnack's Inequality for Degenerate and Singular Parabolic Equations*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-1584-8_6,

$$\sup_{K_\rho(x_o)} u(x,t_o) = \mathcal{M} \tag{1.3}$$

and require that

$$(x_o,t_o) + Q^{\pm}_{8\rho}(\mathcal{M}^{2-p}) \subset E_T. \tag{1.4}$$

Specifically it is required that

$$\begin{aligned}\mathcal{Q}_{\mathcal{M}}(x_o,t_o) &= [(x_o,t_o) + Q^{-}_{8\rho}(\mathcal{M}^{2-p})] \cup [(x_o,t_o) + Q^{+}_{8\rho}(\mathcal{M}^{2-p})] \\ &= K_{8\rho}(x_o) \times (t_o - \mathcal{M}^{2-p}(8\rho)^p, t_o + \mathcal{M}^{2-p}(8\rho)^p] \subset E_T.\end{aligned} \tag{1.5}$$

The upper bound $\mathcal{M}$ is only known qualitatively, and accordingly it does not play any role in the proof, other than to insure that $(x_o,t_o) + Q^{\pm}_{8\rho}(\mathcal{M}^{2-p})$ are contained within the domain of definition of u.

Theorem 1.1 *Let u be a continuous, nonnegative, local, weak solution to the singular equations (1.1)–(1.2) of Chapter 3 in E_T, for p in the supercritical range (1.1). There exist constants $\epsilon \in (0,1)$ and $\gamma > 1$ depending only on the data $\{p, N, C_o, C_1\}$, such that for all intrinsic cylinders $(x_o,t_o) + Q^{\pm}_{8\rho}(\theta)$ as in (1.2), for which (1.4) holds, either*

$$C\rho > \min\{1\, ,\, u(x_o,t_o)\}$$

or

$$\begin{aligned}\gamma^{-1} \sup_{K_\rho(x_o)} u\big(\cdot, t_o - \epsilon\, u(x_o,t_o)^{2-p}\rho^p\big) &\le u(x_o,t_o) \\ &\le \gamma \inf_{K_\rho(x_o)} u\big(\cdot, t_o + \epsilon\, u(x_o,t_o)^{2-p}\rho^p\big).\end{aligned} \tag{1.6}$$

Thus the form (1.2) of Chapter 5, valid for nonnegative solutions to nondegenerate equations ($p = 2$), continues to hold for nonnegative solutions to supercritically singular equations, although in their own intrinsic geometry.

Remark 1.1 The intrinsic geometry enters here in two stages. First, it determines the cylinders $(x_o,t_o) + Q^{\pm}_{8\rho}(\theta)$, then the constant ϵ determines the relative "waiting time," within the cylinders $(x_o,t_o) + Q^{\pm}_{8\rho}(\theta)$ for the intrinsic Harnack estimate to hold. The proof will determine the constants γ and ϵ quantitatively, only in terms of the data $\{p, N, C_o, C_1\}$. Whence these constants are determined, the intrinsic Harnack inequality (1.2)–(1.6) continues to hold for a *smaller* ϵ provided we take a *larger* γ, and $\gamma(\epsilon) \to \infty$ as $\epsilon \to 0$. In all cases it is required that (1.4)–(1.5) be in force. The various constants, however, are dependent only on the data $\{p, N, C_o, C_1\}$ and are all independent of $\mathcal{M}$.

Remark 1.2 The constants γ and ϵ deteriorate as $p \to p_*$ in the sense that

$$\gamma(p), \epsilon(p)^{-1} \to \infty \qquad \text{as} \qquad p \to \tfrac{2N}{N+1}.$$

However, they are stable as $p \to 2$ in the sense of (1.9) of Chapter 3. Thus by formally letting $p \to 2$ in (1.6) one recovers the classical Moser's Harnack inequality in the form (1.2) of Chapter 5.

Remark 1.3 The proofs are based on the energy estimates and DeGiorgi-type lemmas of § 2–4 of Chapter 3 and the expansion of positivity of § 5–6 of Chapter 4. According to the discussion in § 1.3 and Remarks 2.2, 3.1, and 4.3 of Chapter 3, a constant γ depends only on the data if it can be quantitatively determined a priori only in terms of $\{p, N, C_o, C_1\}$. The constant C appearing in the structure conditions (1.2) of Chapter 3, enters in the statement of Theorem 1.1 only through an alternative.

Remark 1.4 The theorem has been stated for continuous solutions, to give meaning to $u(x_o, t_o)$. The intrinsic Harnack inequality, in turn, can be used to prove that these local solutions, irrespective of their signum, are indeed locally Hölder continuous within their domain of definition. This will be shown in § 10.

Remark 1.5 The intrinsic form of (1.6) is false in a time geometry independent of $u(x_o, t_o)$, as it can be verified for the family of counterexamples collected in § 3.3 of Chapter 4.

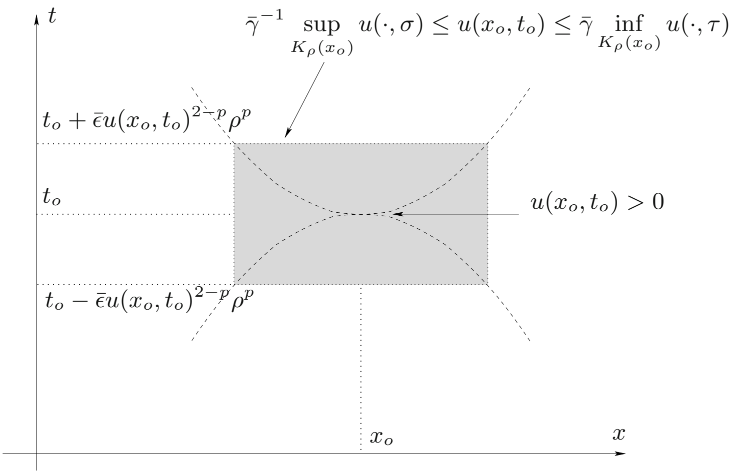

Fig. 1.1. Time-Insensitive Mean Value Harnack Inequality

1.2 Time-Insensitive, Intrinsic, Mean Value, Harnack Inequalities

Theorem 1.2 *Let u be a continuous, nonnegative, local, weak solution to the singular equations (1.1)–(1.2) of Chapter 3 in E_T, for p in the supercritical range (1.1). There exist constants $\bar\epsilon \in (0,1)$ and $\bar\gamma > 1$, depending only on the data $\{p, N, C_o, C_1\}$, such that for all intrinsic cylinders $(x_o, t_o) + Q^{\pm}_{8\rho}(\theta)$ as in (1.2), for which (1.4) holds, either*

$$C\rho > \min\{1\,,\, u(x_o, t_o)\}$$

or

$$\bar{\gamma}^{-1} \sup_{K_\rho(x_o)} u(\cdot, \sigma) \le u(x_o, t_o) \le \bar{\gamma} \inf_{K_\rho(x_o)} u(\cdot, \tau) \tag{1.7}$$

for any pair of time levels σ, τ *in the range*

$$t_o - \bar{\epsilon}\, u(x_o, t_o)^{2-p} \rho^p \le \sigma, \tau \le t_o + \bar{\epsilon}\, u(x_o, t_o)^{2-p} \rho^p. \tag{1.8}$$

The constants $\bar{\epsilon}$ *and* $\bar{\gamma}^{-1}$ *tend to zero either as* $p \to 2$ *or as* $p \to \frac{2N}{N+1}$.

Both right and left inequalities in (1.7) are insensitive to the times σ, τ, provided they range within the time-intrinsic geometry (1.8). For $\sigma = \tau = t_o$ the theorem yields

Corollary 1.1 (The Elliptic Harnack Inequality) *Let u be a continuous, nonnegative, local, weak solution to the singular equations (1.1)–(1.2) of Chapter 3 in E_T, for p in the supercritical range (1.1). For all intrinsic cylinders $(x_o, t_o) + Q^{\pm}_{8\rho}(\theta)$ as in (1.2) for which (1.4) holds, either*

$$C\rho > \min\{1\,,\, u(x_o, t_o)\}$$

or

$$\bar{\gamma}^{-1} \sup_{K_\rho(x_o)} u(\cdot, t_o) \le u(x_o, t_o) \le \bar{\gamma} \inf_{K_\rho(x_o)} u(\cdot, t_o) \tag{1.9}$$

for the same constant $\bar{\gamma}$ *as in Theorem 1.2.*

Thus, the right and left inequalities in (1.7) are simultaneously forward, backward, and elliptic Harnack estimates. Inequalities of this type, and in particular (1.9), are false for nonnegative solutions to the heat equation ([121]). This is reflected in (1.7)–(1.9), in that the constants $\bar{\epsilon}$ and $\bar{\gamma}^{-1}$ tend to zero as $p \to 2$. It turns out that these inequalities lose meaning also as p tends to the critical value $\frac{2N}{N+1}$ as discussed below.

1.3 On the Range (1.1) of *p*

The range of p in (1.1) is optimal for the intrinsic, forward in time Harnack estimate (1.2)–(1.6) to hold. Consider the Cauchy problem

$$u_t - \operatorname{div} |Du|^{p-2} Du = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+$$
$$u(\cdot, 0) = u_o \in L^s(\mathbb{R}^N), \qquad s = \frac{N(2-p)}{p}$$

for $1 < p < p_*$ and $u_o \ge 0$. Solutions exist and become extinct, abruptly, after a finite time T. Specifically, there exists a time T that can be determined a priori in terms of p, N, and $\|u_o\|_{s,\mathbb{R}^N}$, such that ([41], Chapter VII, § 3)

$$u(\cdot, t) > 0 \quad \text{for } t < T \quad \text{and} \quad u(\cdot, t) = 0 \quad \text{for } t > T.$$

Pick $(x_o, t_o) \in \mathbb{R}^N \times (0, T)$ where t_o is so close to T as to satisfy

$$T - t_o < 8^{-p} t_o$$

and choose $\rho > 0$ so large that

$$u(x_o, t_o)^{2-p} (8\rho)^p = T - t_o.$$

For such a choice

$$(x_o, t_o) + Q^{\pm}_{8\rho}(\theta) \subset \mathbb{R}^N \times \mathbb{R}^+ \quad \text{for } \theta \text{ as in (1.2).}$$

However, the intrinsic, forward Harnack estimate (1.2)–(1.6) fails.

When $1 < p \le \frac{2N}{N+1}$ also the elliptic version (1.9) fails, as shown by the following counterexample:

$$u(x,t) = \Big[|\lambda| \Big(\frac{p}{2-p}\Big)^{p-1}\Big]^{\frac{1}{2-p}} \frac{(T-t)_+^{\frac{1}{2-p}}}{|x|^{\frac{p}{2-p}}}$$
$$1 < p < \frac{2N}{N+2}, \qquad \lambda = N(p-2) + p. \tag{1.10}$$

This is a nonnegative, local, weak solution to the prototype p-Laplacian equation in $\mathbb{R}^N \times \mathbb{R}$. Such a solution is unbounded near $x = 0$ for all $t < T$ and finite otherwise. Therefore (1.9) fails to hold for cubes centered at the origin.

For $1 < p \le p_*$ the mere notion of weak solution is not sufficient to insure its local boundedness ([41], Chapter V, § 5). The weak solution (1.10) is indeed unbounded near $x = 0$. However, the lack of a Harnack estimate is not due to the possible unboundedness of the solutions. Consider the two-parameter family of functions

$$u(x,t) = (T-t)_+^{\frac{N+2}{4}} \big(a + b|x|^{\frac{2N}{N-2}}\big)^{-\frac{N}{2}}$$
$$N > 2, \qquad p = \tfrac{2N}{N+2} < p_* \tag{1.11}$$

where $a > 0$ and T are parameters, and

$$b = b(N,a) = \tfrac{N-2}{N^2} \Big(\tfrac{N+2}{4Na}\Big)^{\frac{N+2}{N-2}}.$$

They are nonnegative, locally bounded, weak solutions to the prototype p-Laplacian equation in $\mathbb{R}^N \times \mathbb{R}$ and they do not satisfy the Harnack estimates of Theorems 1.1–1.2. The same occurs for the critical value $p = p_*$ as shown by the following counterexample. The function

$$u(x,t) = \Big(|x|^{\frac{2N}{N-1}} + e^{bt}\Big)^{-\frac{N-1}{2}}$$
$$b = \tfrac{2N^{\frac{2N}{N+1}}}{N-1}, \quad N \ge 2, \quad p = \tfrac{2N}{N+1} = p_* \tag{1.12}$$

is a nonnegative solution to the prototype p-Laplacian equation in $\mathbb{R}^N \times \mathbb{R}$, and one verifies that it fails to satisfy the Harnack estimate in any one of the forward, backward, or elliptic form.

These remarks raise the question of what form the Harnack estimate might take for p in the subcritical range $1 < p \le p_*$. This issue will be addressed in § 11–15.

2 Main Components in the Proof of Theorems 1.1 and 1.2

The proofs of Theorems 1.1 and 1.2 are intertwined. In either case the key inequalities to establish are the right-hand side estimates in (1.6) and (1.7). The left estimates will follow from these by geometrical arguments (§ 5). In all all cases the proofs involve in an essential way the number

$$\lambda = N(p-2) + p.$$

The requirement that p be in the supercritical range (1.1) is equivalent to requiring that $\lambda > 0$. The main components of the proof are the expansion of positivity for quasilinear, singular equations of § 5–6 of Chapter 4, and an L^1_{loc}–L^∞_{loc} Harnack-type estimate valid for $\lambda > 0$, which we present next.

Theorem 2.1 *Let u be a nonnegative, local, weak solution to the singular equations (1.1)–(1.2) of Chapter 3 in E_T, for p in the supercritical range (1.1), so that $\lambda > 0$. There exists a positive constant γ depending only on the data $\{p, N, C_o, C_1\}$, such that for all cylinders*

$$K_{2\rho}(y) \times [s-(t-s), s+(t-s)] \subset E_T$$

either

$$C\rho > \min\Big\{1\,,\,\Big(\frac{t-s}{\rho^p}\Big)^{\frac{1}{2-p}}\Big\} \tag{2.1}$$

or

$$\sup_{K_\rho(y)\times[s,t]} u \le \frac{\gamma}{(t-s)^{\frac{N}{\lambda}}}\Big(\inf_{2s-t<\tau<t}\int_{K_{2\rho}(y)} u(x,\tau)dx\Big)^{\frac{p}{\lambda}} + \gamma\Big(\frac{t-s}{\rho^p}\Big)^{\frac{1}{2-p}}. \tag{2.2}$$

The constant $\gamma = \gamma(p) \to \infty$ as either $p \to 2$ or $p \to \frac{2N}{N+1}$.

Remark 2.1 Starting from K_ρ, the solution u is required to exist in a larger neighborhood $K_{2\rho}(y)$ and for times comparably larger and smaller than s.

The proof of Theorem 2.1 will be given in Appendix A. Assuming it for the moment, we proceed to prove the right-hand side Harnack estimate of Theorem 1.2. The analogous statement for Theorem 1.1 will be established in § 7.

3 The Right-Hand-Side Harnack Estimate of Theorem 1.2

Fix $(x_o, t_o) \in E_T$, determine the intrinsic cylinders $(x_o, t_o) + Q^{\pm}_{8\rho}(\theta)$ as in (1.2), and assume that (1.4)–(1.5) hold. Assume in addition that

$$C\rho \le \min\{1\,,\, u(x_o, t_o)\} \tag{3.1}$$

where C is the constant in the structure conditions (1.2) of Chapter 3.

Proposition 3.1 *Let u be a continuous, nonnegative, local, weak solution to (1.1)–(1.2) of Chapter 3 for p in the supercritical range (1.1). There exist positive constants $\bar{\epsilon}$ and $\bar{\gamma}$, that can be determined quantitatively, a priori only in terms of the data $\{p, N, C_o, C_1\}$, such that*

$$u(x_o, t_o) \le \bar{\gamma} \inf_{K_\rho(x_o)} u(\cdot, t) \tag{3.2}$$

for all times

$$t_o - \bar{\epsilon} u(x_o, t_o)^{2-p} \rho^p \le t \le t_o + \bar{\epsilon} u(x_o, t_o)^{2-p} \rho^p. \tag{3.3}$$

The constants $\bar{\epsilon}$ and $\bar{\gamma}^{-1}$ tend to zero either as $p \to 2$ or as $p \to \frac{2N}{N+1}$.

Introduce the change of variables and unknown function

$$x \to \frac{x - x_o}{\rho}, \qquad t \to \frac{t - t_o}{u(x_o, t_o)^{2-p} \rho^p}, \qquad v = \frac{u}{u(x_o, t_o)}. \tag{3.4}$$

This maps the cylinder $\mathcal{Q}_{\mathcal{M}}(x_o, t_o)$ in (1.5) into

$$\mathcal{Q}_{\mathcal{M}} = K_8 \times \Big(- \Big(\frac{\mathcal{M}}{u(x_o, t_o)} \Big)^{2-p} 8^p, \Big(\frac{\mathcal{M}}{u(x_o, t_o)} \Big)^{2-p} 8^p \Big]. \tag{3.5}$$

Relabeling by x, t the new coordinates, v is a weak solution to

$$v_t - \operatorname{div} \bar{\mathbf{A}}(x, t, v, Dv) = \bar{B}(x, t, v, Dv) \quad \text{in} \quad \mathcal{Q}_{\mathcal{M}}. \tag{3.6}$$

Taking into account (3.1), the transformed functions $\bar{\mathbf{A}}$ and $\bar{B}$ satisfy the structure conditions

$$\begin{cases} \bar{\mathbf{A}}(x, t, v, Dv) \cdot Dv \ge C_o |Dv|^p - 1 \\ |\bar{\mathbf{A}}(x, t, v, Dv)| \le C_1 |Dv|^{p-1} + 1 \\ |\bar{B}(x, t, v, Dv)| \le |Dv|^{p-1} + 1 \end{cases}, \tag{3.7}$$

where C_o and C_1 are the constants appearing in (1.2) of Chapter 3. Establishing Proposition 3.1 consists in finding positive constants $\bar{\epsilon}$ and $\bar{\gamma}$, depending only on the data, such that

$$v(\cdot, t) \ge \bar{\gamma}^{-1} \quad \text{in } K_1 \quad \text{for all } t \in [-\bar{\epsilon}, \bar{\epsilon}]. \tag{3.8}$$

4 Locating the Supremum of v in K_1

For $\tau \in (0,1)$ introduce the family of nested expanding cubes $\{K_\tau\}$ centered at the origin, and the increasing families of positive numbers

$$M_\tau = \sup_{K_\tau} v, \qquad N_\tau = (1-\tau)^{-\beta}, \tag{4.1}$$

where β is a positive parameter to be fixed later. By the definition, $M_o = N_o$ and as $\tau \to 1$, $N_\tau \to \infty$, whereas M_τ remains finite. Therefore the equation $M_\tau = N_\tau$ has roots. Denoting by τ_* the largest root

$$M_{\tau_*} = (1-\tau_*)^{-\beta} \quad \text{and} \quad M_\tau \le N_\tau \quad \text{for all } \tau \ge \tau_*.$$

Since v is continuous, the supremum M_{τ_*} is achieved at some $\bar{x} \in K_{\tau_*}$. Choose $\bar{\tau} \in (0,1)$ from

$$(1-\bar{\tau})^{-\beta} = 4(1-\tau_*)^{-\beta} \quad \text{i.e.,} \quad \bar{\tau} = 1 - 4^{-\frac{1}{\beta}}(1-\tau_*).$$

Set also

$$2r \stackrel{\text{def}}{=} \bar{\tau} - \tau_* = (1 - 4^{-\frac{1}{\beta}})(1-\tau_*). \tag{4.2}$$

For these choices, $K_{2r}(\bar{x}) \subset K_{\bar{\tau}}$, $M_{\bar{\tau}} \le N_{\bar{\tau}}$, and

$$\begin{aligned}\sup_{K_{\tau_*}} v(\cdot,0) = (1-\tau_*)^{-\beta} = v(\bar{x},0) \le \sup_{K_{2r}(\bar{x})} v(\cdot,0) \\ \le \sup_{K_{\bar{\tau}}} v(\cdot,0) \le 4(1-\tau_*)^{-\beta}.\end{aligned}$$

The information on τ_* is only qualitative. By using the parameter β we will eliminate such a qualitative dependence from our arguments. The qualitative information on $\mathcal{M}$ plays no role in this process. It only insures that the cylinder $\mathcal{Q}_{\mathcal{M}}(x_o,t_o)$ is within the domain of definition of u. Because of this interplay between qualitative and quantitative information, our quantitative arguments below are devised not to depend on β, $\mathcal{M}$, and ρ.

5 Estimating the Sup of v in Some Intrinsic Neighborhood About $(\bar{x},0)$

Consider the cylinder centered at $(\bar{x},0)$

$$\begin{aligned}Q_{2r} &= [(\bar{x},0) + Q_{2r}^-(\theta_*)] \cup [(\bar{x},0) + Q_{2r}^+(\theta_*)] \\ &= K_{2r}(\bar{x}) \times (-\theta_*(2r)^p, \theta_*(2r)^p]\end{aligned}$$

where

$$\theta_* = (1-\tau_*)^{-\beta(2-p)}. \tag{5.1}$$

Such a cylinder is included in the box $\mathcal{Q}_{\mathcal{M}}$ introduced in (3.5) since

$$\begin{aligned}\theta_*(2r)^p &= (1-\tau_*)^{-\beta(2-p)}(1-4^{-\frac{1}{\beta}})^p(1-\tau_*)^p \\ &\le (1-\tau_*)^{-\beta(2-p)} = \Big(\frac{u(\bar{x},0)}{u(x_o,t_o)}\Big)^{2-p} \le \Big(\frac{\mathcal{M}}{u(x_o,t_o)}\Big)^{2-p}.\end{aligned}$$

Lemma 5.1 *There exists a positive constant γ_1, depending only on the data $\{p, N, C_o, C_1\}$, and independent of β, $\mathcal{M}$, and ρ, such that*

$$\sup_{Q_r} v \le \gamma_1(1-\tau_*)^{-\beta}.$$

The constant $\gamma_1 \to \infty$ as $p \to 2$ and as $p \to p_*$.

Proof Apply (2.2) of Theorem 2.1 to the function v, solution to (3.6)–(3.7), over the pair of cylinders $Q_r \subset Q_{2r}$. Apply it first for the choice

$$s = 0 \qquad \text{and} \qquad t = \theta_*(2r)^p$$

and apply it again, for the choice

$$s = -\theta_*(2r)^p \qquad \text{and} \qquad t = 0.$$

Taking into account the structure conditions (3.7), and the definition (5.1) of θ_*, the condition (2.1) is always violated. Therefore Theorem 2.1 with these stipulations gives

$$\begin{aligned}\sup_{Q_r} v &\le \gamma(1-\tau_*)^{-\beta\frac{N(p-2)}{\lambda}}\Big(\fint_{K_{2r}(\bar{x})} v(x,0)dx\Big)^{\frac{p}{\lambda}} + \gamma 2^{\frac{p}{2-p}}\theta_*^{\frac{1}{2-p}} \\ &\le \gamma\big(4^{\frac{p}{\lambda}} + 2^{\frac{p}{2-p}}\big)(1-\tau_*)^{-\beta} = \gamma_1(1-\tau_*)^{-\beta}.\end{aligned}$$ ■

Introduce next the cylinder

$$Q_r(\bar{\delta}\theta_*) = K_r(\bar{x}) \times (-\bar{\delta}\theta_* r^p, \bar{\delta}\theta_* r^p] \subset Q_{2r}$$

where $\bar{\delta} \in (0,1)$ is to be chosen.

Lemma 5.2 *There exist numbers $\bar{\delta}$, $\bar{c}$, and α in $(0,1)$, depending only on the data $\{p, N, C_o, C_1\}$, and independent of β, $\mathcal{M}$, and ρ, such that*

$$|[v(\cdot,t) \ge \bar{c}(1-\tau_*)^{-\beta}]| > \alpha|K_r| \quad \textit{for all} \quad t \in [-\bar{\delta}\theta_* r^p\,,\,\bar{\delta}\theta_* r^p] \tag{5.2}$$

where θ_* is defined in (5.1). The constants $\bar{\delta}$, $\bar{c}$, and α tend to zero either as $p \to 2$ or as $\lambda \to 0$, that is, as p tends to the critical value $\frac{2N}{N+1}$.

Proof Apply (2.2) of Theorem 2.1 to the function v, solution to (3.6)–(3.7), over the pair of cylinders $Q_{\frac{1}{2}r}(\bar{\delta}\theta_*) \subset Q_r(\bar{\delta}\theta_*)$, for the choices $s = 0$ and

$t = \bar{\delta}\theta_* r^p$. Taking into account the structure conditions (3.7), and the definition (5.1) of θ_*, the condition (2.1) is always violated. Therefore for all $t \in [-\bar{\delta}\theta_* r^p, \bar{\delta}\theta_* r^p]$

$$
\begin{aligned}
(1-\tau_*)^{-\beta} = v(\bar{x},0) \le \sup_{K_{\frac{1}{2}r}(\bar{x})} v(\cdot,0) &\le \frac{\gamma(1-\tau_*)^{-\beta\frac{N(p-2)}{\lambda}}}{\bar{\delta}^{\frac{Np}{\lambda}}} \left(\fint_{K_r} v(x,t)dx \right)^{\frac{p}{\lambda}} \\
&\quad + \gamma(2\bar{\delta})^{\frac{1}{2-p}} (1-\tau_*)^{-\beta}.
\end{aligned}
$$

Choose $\bar{\delta}$ from

$$
\gamma(2\bar{\delta})^{\frac{1}{2-p}} \le \frac{1}{2} \qquad \text{and set} \qquad \gamma_2 = 2\gamma, \quad \gamma_3 = \frac{2^{\frac{N(2-p)}{\lambda}}\gamma_2}{\bar{\delta}^{\frac{Np}{\lambda}}}.
$$

For such choices, the constants $\bar{\delta}$, γ_2, and γ_3 depend only on the data $\{p, N, C_o, C_1\}$, and are independent of β. Then for all $t \in [-\bar{\delta}\theta_* r^p, \bar{\delta}\theta_* r^p]$

$$
\frac{1}{\gamma_2}(1-\tau_*)^{-\beta} \le \frac{(1-\tau_*)^{-\beta\frac{N(p-2)}{\lambda}}}{\bar{\delta}^{\frac{Np}{\lambda}}} \left(\fint_{K_r} v(x,t)dx \right)^{\frac{p}{\lambda}}.
$$

From this for $\bar{c} \in (0,1)$

$$
\begin{aligned}
\frac{1}{\gamma_3}(1-\tau_*)^{-\beta} &\le \frac{(1-\tau_*)^{-\beta\frac{N(p-2)}{\lambda}}}{2^{\frac{N(2-p)}{\lambda}}} \left(\fint_{K_r} v(x,t)dx \right)^{\frac{p}{\lambda}} \\
&\le (1-\tau_*)^{-\beta\frac{N(p-2)}{\lambda}} \left(\fint_{K_r} v(x,t)\chi_{[v(\cdot,t)<\bar{c}(1-\tau_*)^{-\beta}]}dx \right)^{\frac{p}{\lambda}} \\
&\quad + (1-\tau_*)^{-\beta\frac{N(p-2)}{\lambda}} \left(\fint_{K_r} v(x,t)\chi_{[v(\cdot,t)\ge\bar{c}(1-\tau_*)^{-\beta}]}dx \right)^{\frac{p}{\lambda}} \\
&\le \bar{c}^{\frac{p}{\lambda}}(1-\tau_*)^{-\beta} \\
&\quad + \gamma_1^{\frac{p}{\lambda}}(1-\tau_*)^{-\beta} \left(\fint_{K_r} \chi_{[v(\cdot,t)\ge\bar{c}(1-\tau_*)^{-\beta}]}dx \right)^{\frac{p}{\lambda}}.
\end{aligned}
$$

To prove (5.2) choose

$$
\bar{c}^{\frac{p}{\lambda}} = \frac{1}{2\gamma_3} \qquad \text{and set} \qquad \alpha = \frac{1}{\gamma_1}\left(\frac{1}{2\gamma_3}\right)^{\frac{\lambda}{p}}.
$$

6 Expanding the Positivity of v

The information provided by Lemma 5.2 is the assumption required by the expansion of positivity Proposition 5.1 of Chapter 4 for all

$$-\bar{\delta}\theta_* r^p \le s \le \bar{\delta}\theta_* r^p.$$

Apply then this expansion of positivity to v with

$$\rho = r, \qquad M = \bar{c}(1-\tau_*)^{-\beta}$$

and for s ranging in the indicated interval. It gives

$$v(\cdot,t) > \eta\bar{c}(1-\tau_*)^{-\beta} \qquad \text{in } K_{2r}(\bar{x}) \tag{6.1$_1$}$$

and for all times

$$-\bar{\delta}\theta_* r^p + (1-\varepsilon)\delta M^{2-p} r^p < t < \bar{\delta}\theta_* r^p \tag{6.2$_1$}$$

for constants $\delta, \varepsilon \in (0,1)$ depending only on the data $\{p, N, C_o, C_1\}$ and the constant α in (5.2), which itself is determined only in terms of the data $\{p, N, C_o, C_1\}$.

Apply again Proposition 5.1 of Chapter 4 with $\rho = 2r$, and M replaced by ηM, and for all s ranging in the interval $(6.2)_1$. It gives

$$v(\cdot,t) > \eta^2\bar{c}(1-\tau_*)^{-\beta} \qquad \text{in } K_{4r}(\bar{x}) \tag{6.1$_2$}$$

for all times

$$-\bar{\delta}\theta_* r^p + (1-\varepsilon)\delta M^{2-p} r^p + (1-\varepsilon)\delta(\eta M)^{2-p}(2r)^p < t < \bar{\delta}\theta_* r^p. \tag{6.2$_2$}$$

Notice that the constants ε, δ, η claimed by Proposition 5.1 of Chapter 4 depend only on the data $\{p, N, C_o, C_1\}$ and the number α in (5.1) of Chapter 4. In view of $(6.1)_1$, such a constant α is one. Therefore the numbers ε, δ, η determined by this second application of Proposition 5.1 of Chapter 4, starting from $(6.1)_1$ and leading to $(6.1)_2$ and $(6.2)_2$, can be taken as equal to those in $(6.1)_1$–$(6.2)_1$, originating from (5.2). After n iterations, this process gives

$$v(\cdot,t) > \eta^n\bar{c}(1-\tau_*)^{-\beta} \qquad \text{in } K_{2^{n+1}r} \tag{6.1$_n$}$$

for all times

$$-\bar{\delta}\theta_* r^p + (1-\varepsilon)\delta \sum_{j=0}^{n-1} (\eta^j M)^{2-p}(2^j r)^p < t < \bar{\delta}\theta_* r^p. \tag{6.2$_n$}$$

Recall the definition (4.2) of r and choose n so large that

$$1 \le 2^n r \le 2 \quad \text{ which implies } \quad (1-\tau_*)^{-1} > 2^{n-2}(1-4^{-\frac{1}{\beta}}).$$

Since τ_* is only known qualitatively also n is qualitative. We remove such a qualitative dependence for a suitable choice of β as follows. Taking into account the lower bound in $(6.1)_n$ and the previous choice of n

$$\eta^n\bar{c}(1-\tau_*)^{-\beta} > \bar{c}2^{-2\beta}(1-4^{-\frac{1}{\beta}})^\beta(\eta 2^\beta)^n.$$

Choose β so large that

$$\eta 2^{\beta} = 1 \qquad \text{and set} \qquad \bar{\gamma}^{-1} = \bar{c} 2^{-2\beta}(1 - 4^{-\frac{1}{\beta}})^{\beta}.$$

Finally, by choosing $\bar{c}$ even smaller if necessary, we may insure that

$$\sum_{j=0}^{\infty} (1-\varepsilon)\delta(\eta^j M)^{2-p}(2^j r)^p \le \tfrac{1}{2}\bar{\delta}\theta_* r^p.$$

Thus

$$v(\cdot, t) \ge \bar{\gamma}^{-1} \qquad \text{in} \quad K_1 \tag{6.3}$$

for all times

$$-\tfrac{1}{2}\bar{\delta}\theta_* r^p < t < \bar{\delta}\theta_* r^p. \tag{6.4}$$

As indicated earlier the information on τ_* is only qualitative and as such, the range of times in (6.4) is qualitative. However, from the definition (4.2) of r and (5.1) of θ_*

$$\begin{aligned} \tfrac{1}{2}\bar{\delta}\theta_* r^p &= \tfrac{1}{2^{p+1}}\bar{\delta}(1 - 4^{-\frac{1}{\beta}})^p (1-\tau_*)^{-\beta(2-p)}(1-\tau_*)^p \\ &\ge \tfrac{1}{2^{p+1}}\bar{\delta}(1 - 4^{-\frac{1}{\beta}})^p = \bar{\epsilon} \end{aligned}$$

provided $\beta \ge p/(2-p)$, which we may assume by possibly taking η smaller if necessary. Thus (6.3) holds for all times $t \in (-\bar{\epsilon}, \bar{\epsilon})$ and establishes (3.8), and hence the right-hand-side estimate (3.2)–(3.3) of Proposition 3.1.

7 Proof of the Right-Hand-Side Harnack Inequality of Theorem 1.1

The estimates in the proof of Theorem 1.2 deteriorate as $p \to 2$ and as $p \to \frac{2N}{N+1}$. Stable estimates for $p \to 2$ required in the proof of the right-hand-side inequality of Theorem 1.1 can be derived as in § 6 of Chapter 4 by almost identical arguments. As remarked in that context, there exists $\sigma_* \in (0,1)$, that can be determined a priori only in terms of $\{N, C_o, C_1\}$, and independent of p, such that for $|p-2| < \sigma_*$, the expansion of positivity for nonnegative solutions to the class of equations (1.1)–(1.2) of Chapter 3, behaves as if these equations were neither degenerate nor singular.

Henceforth we let σ_* be the number claimed by Proposition 6.1 of Chapter 4 and let $|p-2| < \sigma_*$. With such a restriction at hand, a "forward," intrinsic Harnack inequality can be derived for nonnegative, local, solutions to these equations, by the same arguments as in § 2 of Chapter 5, both for the degenerate case $p > 2$ and for the singular case $p < 2$.

Having fixed $(x_o, t_o) \in E_T$ such that $u(x_o, t_o) > 0$ construct the cylinder

$$(x_o, t_o) + Q^{\pm}_{8\rho}(\theta) \subset E_T$$

as in (1.2). Introduce the change of variables (3.4) which maps $(x_o,t_o)+Q^{\pm}_{8\rho}(\theta)$ into $Q^{\pm}_8$, and define a function v which solves (3.6)–(3.7) in $Q^-_8\cup Q^+_8$. Notice that no assumption of the type (1.3)–(1.5) is made in this context.

For $\tau\in[0,1)$, introduce the family of nested cylinders $\{Q^-_\tau\}$ with the same "vertex" at $(0,0)$, and the families of nonnegative numbers $\{M_\tau\}$ and $\{N_\tau\}$, defined by

$$Q^-_\tau = K_\tau\times(-\tau,0], \qquad M_\tau=\sup_{Q^-_\tau} v, \qquad N_\tau=(1-\tau)^{-\beta}$$

where $\beta>1$ is to be chosen. Notice that unlike in (4.1), where M_τ was defined as the supremum of v in the cube K_τ at time level $t=0$, here M_τ is defined as the supremum of v in the full cylinder Q^-_τ. Thus the proof departs from the arguments of § 3–6 and follows instead the proof for the degenerate case $p>2$ in § 2 of Chapter 5. In particular no use is made of the $L^1_{\rm loc}-L^\infty_{\rm loc}$ estimates of Theorem 2.1, with the goal of generating constants that are stable as $p\to 2$.

Let τ_* be the largest root of the equation $M_\tau=N_\tau$, and let $(\bar x,\bar t)\in\bar Q_{\tau_*}$ be a point where v achieves its maximum M_{τ_*}. Consider the cylinder

$$Q_o=\big[|x-\bar x|<\tfrac12(1-\tau_*)\big]\times\big(\bar t-\tfrac12(1-\tau_*),\bar t\big]\subset Q^-_{\frac12(1+\tau_*)}.$$

From the definitions

$$\begin{aligned} v(\bar x,\bar t)=M_{\tau_*}&=(1-\tau_*)^{-\beta}\le\sup_{Q_o} v\\ &\le \sup_{Q^-_{\frac12(1+\tau_*)}} v\le N_{\frac12(1+\tau_*)}=2^\beta(1-\tau_*)^{-\beta}. \end{aligned}$$

Set

$$r=\tfrac12(1-\tau_*), \qquad\text{and}\qquad M_\beta=2^\beta(1-\tau_*)^{-\beta}$$

and consider the cylinder with "vertex" at $(\bar x,\bar t)$

$$(\bar x,\bar t)+Q^-_r(M_\beta^{2-p})=K_r(\bar x)\times\big(\bar t-M_\beta^{2-p}r^p,\bar t\big]. \tag{7.1}$$

This can be taken as the starting cylinder in the proof of the "forward," intrinsic Harnack inequality (1.3)–(1.4) of Theorem 1.1 of Chapter 5, provided its geometry is "intrinsic," that is, if

$$\sup_{(\bar x,\bar t)+Q^-_r(M_\beta^{2-p})} v\le M_\beta.$$

This occurs if $(\bar x,\bar t)+Q^-_r(M_\beta^{2-p})\subset Q_o$, or equivalently if

$$2^{\beta(2-p)}(1-\tau_*)^{-\beta(2-p)}(1-\tau_*)^{p-1}=2^{p-1}. \tag{7.2}$$

Assuming this inclusion for the moment, proceed as in the proof of the "forward," intrinsic Harnack inequality (1.3)–(1.4) of Theorem 1.1 of Chapter 5.

The proof will determine quantitatively the constants $\epsilon \in (0,1)$ and $\gamma > 1$ by a quantitative determination of the parameter β depending only on the data $\{p, N, C_o, C_1\}$ and stable as $p \to 2$.

Condition (7.2) does not enter in the determination of β. It is needed only to insure that $(\bar{x}, \bar{t}) + Q_r^-(M_\beta^{2-p})$ possesses the correct intrinsic geometry, and is contained within the domain of definition of v. Having determined β, the condition (7.2) is satisfied by choosing p so close to 2 as to insure that $\beta(2-p) = p-1$. The right-hand-side Harnack inequality (1.2)–(1.6) then holds with constants ϵ and γ stable for

$$|p-2| < \sigma_{**} = \min\left\{\sigma_* \,,\; (1-\sigma_*)\beta^{-1}\right\}. \tag{7.3}$$

To establish the right-hand-side inequality of Theorem 1.1 assume first that

$$\frac{2N}{N+1} < p \leq 2 - \sigma_{**} \tag{7.4}$$

and proceed as in the proof of Theorem 1.2. This will produce constants $\bar{\gamma}(p)$ and $\bar{\epsilon}(p)$ that deteriorate as $p \to 2$. For p in the range

$$2 - \sigma_{**} < p < 2 \tag{7.5}$$

proceed as above, to establish the inequality with constants that are stable as $p \to 2$.

Remark 7.1 We stress that for p in the range (7.5) no use has been made of Theorem 2.1, whose constant $\gamma(p) \to \infty$ as $p \to 2$. Also the qualitative information of (1.3)–(1.5) is not needed. Indeed, whence β has been determined, the number σ_{**} can be quantitatively chosen so small to insure (7.2) and hence that the cylinder in (7.1) is contained within the domain of definition of v.

7.1 On the Functional Relation $\gamma = \gamma(\epsilon)$

For p in the range (7.4) the proof of the right-hand-side, intrinsic Harnack inequality (1.2)–(1.6) is a particular case of the right-hand-side inequality (1.7)–(1.8) of Theorem 1.2. Having fixed p in such a range and having determined $\bar{\epsilon}(p)$ and $\bar{\gamma}(p)$, the inequality continues to hold for any smaller $\bar{\epsilon}$ for the same constant $\bar{\gamma}$.

For p in the range (7.5), the proof of the Harnack inequality departs from the arguments of § 3–6 and follows instead the proof for the degenerate case $p > 2$ in § 2 of Chapter 5. As pointed out in Remark 1.3 and § 2.4 of that chapter, the constants ϵ and γ have a functional dependence, made quantitative by (2.12)–(2.13) of Chapter 5. Following the same arguments of § 2.4 of Chapter 5, and taking into account that $p < 2$, we conclude that, having determined ϵ and γ, the parameter ϵ can be taken to be smaller, provided γ is taken larger, following their functional dependence.

8 Proof of the Left-Hand-Side Harnack Inequality in Theorem 1.2

Having fixed $(x_o, t_o) \in E_T$ construct the intrinsic cylinders $(x_o, t_o) + Q^{\pm}_{2\rho}(\theta)$ as in (1.2), and assume that (1.3)–(1.5) are in force, with ρ replaced by 2ρ. Let now $x_* \in K_\rho$ be a point where $u(\cdot, t_o)$ attains its maximum, and construct cylinders

$$(x_*, t_o) + Q^{\pm}_\rho(\theta_*) \quad \text{where} \quad \theta_* = [u(x_*, t_o)]^{2-p}.$$

These cylinders are intrinsic and the analogues of (1.3)–(1.5) are satisfied. Hence by the right-hand-side Harnack inequality (3.2) of Proposition 3.1, and assuming that (1.3) holds, one has

$$\sup_{K_\rho(x_o)} u(\cdot, t_o) \le \bar{\gamma}\, u(x_o, t_o). \tag{8.1}$$

Apply Theorem 2.1 over the cubes $K_{\frac{1}{2}\rho}(x_o) \subset K_\rho(x_o)$ for the time levels

$$t_o - \bar{\epsilon}\, u(x_o, t_o)^{2-p} \rho^p < s < t_o - \tfrac{1}{2}\bar{\epsilon}\, u(x_o, t_o)^{2-p}\rho^p$$
$$< t_o < t < t_o + \bar{\epsilon}\, u(x_o, t_o)^{2-p}\rho^p$$

so that

$$\tfrac{1}{2}\bar{\epsilon}\, u(x_o, t_o)^{2-p}\rho^p \le t - s \le 2\bar{\epsilon}\, u(x_o, t_o)^{2-p}\rho^p.$$

With these choices, either

$$C\rho > \min\{1\,,\, \bar{\epsilon}^{\frac{1}{2-p}}\, u(x_o, t_o)\}$$

or

$$\begin{aligned}\sup_{K_{\frac{1}{2}\rho}(x_o)} u(\cdot, \sigma) &\le \frac{\gamma}{\bar{\epsilon}^{\frac{N}{\lambda}}\, u(x_o, t_o)^{\frac{N(2-p)}{\lambda}}} \Big(\fint_{K_\rho(x_o)} u(x, t_o) dx \Big)^{\frac{p}{\lambda}} \\ &\quad + \gamma (2\bar{\epsilon})^{\frac{1}{2-p}}\, u(x_o, t_o) \\ &\le \big(\gamma \bar{\gamma}^{\frac{p}{\lambda}} \bar{\epsilon}^{-\frac{N}{\lambda}} + \gamma (2\bar{\epsilon})^{\frac{1}{2-p}} \big) u(x_o, t_o) \\ &= \bar{\bar{\gamma}} u(x_o, t_o)\end{aligned}$$

for all σ in the range (1.8). The proof is concluded by suitably redefining the various constants and the the radius ρ. ■

9 Proof of the Left-Hand-Side Harnack Inequality in Theorem 1.1

Assume $u(x_o, t_o) \ge C\rho$, and let ϵ and γ be the constants appearing on the right-hand-side Harnack inequality (1.2)–(1.6) of Theorem 1.1. By the arguments of § 7 these constants are stable as $p \to 2$. For p in the range (7.3) the left-hand-side inequality (1.6) of Theorem 1.1 follows from the left-hand-side inequality (1.7) of Theorem 1.2 as established in the previous section. The proof, however, uses Theorem 2.1 whose constant $\gamma(p) \to \infty$ as $p \to 2$. Below we will give a proof for p in the range (7.5) which is independent of Theorem 2.1, and with constants that are stable as $p \to 2$. The proof is based on applying the right-hand-side Harnack inequality of (1.6) at points in a neighborhood of (x_o, t_o). For each such point one would have to verify the analogues of the qualitative requirements (1.3)–(1.5). However, as discussed in Remark 7.1, for p in the range (7.5) such requirements are not needed. Set

$$\bar{t} = t_o - \epsilon\, u(x_o, t_o)^{2-p} \rho^p.$$

Let $\alpha \in (0, 1)$ to be chosen, consider the cube $K_{\alpha\rho}(x_o)$, and introduce the set

$$\mathcal{U}_\alpha = K_{\alpha\rho}(x_o) \cap [u(\cdot, \bar{t}) \le \gamma u(x_o, t_o)].$$

Since u is continuous, $\mathcal{U}_\alpha$ is closed. The parameter α will be chosen, depending only on γ, such that $\mathcal{U}_\alpha$ is also open. Then if $\mathcal{U}_\alpha$ is not empty, it coincides with $K_{\alpha\rho}$, thereby establishing the left-hand-side Harnack inequality in (1.2)–(1.6), modulo a suitable redefinition of ρ and ϵ.

Assume momentarily that $\mathcal{U}_\alpha$ is not empty, and fix $z \in \mathcal{U}_\alpha$. Since u is continuous, there exists a cube $K_\varepsilon(z) \subset K_{\alpha\rho}(x_o)$ such that

$$u(y, \bar{t}) \le 2\gamma u(x_o, t_o) \qquad \text{for all } \; y \in K_\varepsilon(z). \tag{9.1}$$

For each $y \in K_\varepsilon(z)$ construct the intrinsic p-paraboloid

$$\mathcal{P}(y, \bar{t}) = [|t - \bar{t}| \ge \epsilon\, u(y, \bar{t})^{2-p} |x - y|^p].$$

If $(x_o, t_o) \in \mathcal{P}(y, \bar{t})$, by the right-hand-side Harnack inequality in (1.2)–(1.6)

$$u(y, \bar{t}) \le \gamma u(x_o, t_o)$$

and hence $y \in \mathcal{U}_\alpha$. This occurs if

$$\epsilon\, u(y, \bar{t})^{2-p} |y - x_o|^p \le \epsilon\, (2\gamma)^{2-p} u(x_o, t_o)^{2-p} |y - x_o|^p \le \epsilon\, u(x_o, t_o)^{2-p} \rho^p,$$

that is, if

$$|y - x_o| < \alpha\rho, \qquad \text{where} \qquad \alpha = (2\gamma)^{\frac{p-2}{p}}.$$

The right-hand-side Harnack inequality can be applied since, in view of (9.1), the cylinder

$$(y, \bar{t}) + Q^{\pm}_{8\rho}(\bar{\theta}) \qquad \text{with} \qquad \bar{\theta} = u(y, \bar{t})^{2-p}$$

can be assumed to be contained in E_T.

It remains to show that $\mathcal{U}_\alpha \ne \emptyset$. Having determined α, consider the cylinder

$$K_{\alpha\rho}(x_o) \times (\bar{t}, \bar{t} + \nu_o(\gamma u(x_o, t_o))^{2-p} (\alpha\rho)^p],$$

where $\nu_o \in (0,1)$ is to be chosen, depending only on the data $\{p, N, C_o, C_1\}$. Such a cylinder crosses the time level t_o if

$$t_o - \epsilon\, u(x_o,t_o)^{2-p}\rho^p + \nu_o(\gamma u(x_o,t_o))^{2-p}(\alpha\rho)^p > t_o.$$

Recalling the value of α, this occurs if

$$\nu_o\gamma^{2-p}\alpha^p > \epsilon \quad\Longrightarrow\quad \epsilon < \nu_o 2^{p-2}$$

which, by reducing ϵ if necessary, we assume. Note that such a reduction of ϵ is possible by increasing γ accordingly, as discussed in § 7.1. If $\mathcal{U}_\alpha = \emptyset$, then

$$u(\cdot,\bar{t}) > \gamma u(x_o,t_o) \quad \text{in} \quad K_{\alpha\rho}(x_o).$$

Apply Lemma 4.1 of Chapter 3, with 2ρ replaced by $\alpha\rho$, and with

$$a = \tfrac{1}{2}, \quad \xi = 1, \quad M = \gamma u(x_o,t_o), \quad \theta = \nu_o(\gamma u(x_o,t_o))^{2-p}$$

where ν_o is the number in (4.2) of Chapter 3. For such a choice of θ, (4.2) is satisfied and the lemma yields

$$u(x,t_o) > \tfrac{1}{2}\gamma u(x_o,t_o) \quad \text{for all } x \in K_{\frac{1}{2}\alpha\rho}(x_o).$$

Computing this for $x = x_o$ gives a contradiction if $\gamma > 2$, which without loss of generality we may assume. The proof is concluded by suitably redefining the various constants and the radius ρ. ∎

10 Some Consequences of the Harnack Inequality

10.1 Local Hölder Continuity

The forward, intrinsic Harnack inequality (1.2)–(1.6) of Theorem 1.1 can be used to establish the local Hölder continuity of local, weak solutions to (1.1)–(1.2) of Chapter 3, irrespective of their signum, provided p is in the singular, supercritical range (1.1). In particular a theorem in all similar to Theorem 4.1 of Chapter 5 holds true, with essentially the same proof.

However, it is known that locally bounded solutions to these equations are locally Hölder continuous in E_T for all $p > 1$ ([41]). A more precise connection between Harnack estimates and Hölder continuity will be discussed in § 14, where the Hölder continuity will be derived from a weaker form of a Harnack inequality, valid for all $1 < p < 2$.

10.2 A Liouville-Type Result

Let u be a continuous, local, weak solution to the homogeneous ($C = 0$) singular equations (1.1)–(1.2) of Chapter 3, in $\mathbb{R}^N \times \mathbb{R}$, for p in the singular, supercritical range (1.1). Assume further that u has a one-sided bound, say for example either $u \le k$ or $u \ge h$ for constants k or h.

It was observed in § 5 of Chapter 5 that in the degenerate ($p > 2$) and nondegenerate ($p = 2$) case, such solutions do not, in general, satisfy a Liouville property analogous to that for harmonic functions with one-sided bound. It was also proved that a two-sided bound was needed for a Liouville property to hold. In the singular, supercritical range (1.1), the situation is different, due to the elliptic form (1.9) of the Harnack inequality.

Proposition 10.1 *Let u be a continuous, local, weak solution to the homogeneous ($C = 0$) equations (1.1)–(1.2) of Chapter 3, in $\mathbb{R}^N \times \mathbb{R}$, for p in the singular, supercritical range (1.1). If u has a one-sided bound, then it is constant.*

Proof Let u be bounded below and let m denote its infimum in $\mathbb{R}^N \times \mathbb{R}$. Then $v - m$ is a nonnegative solution to the homogeneous ($C = 0$) equations (1.1)–(1.2) of Chapter 3, whose infimum in $\mathbb{R}^N \times \mathbb{R}$ is zero. By the Harnack inequality (1.9), for any $\rho > 0$

$$v(x_o, t_o) \le \bar{\gamma} \inf_{K_\rho(x_o)} v(\cdot, t_o).$$

Now let $\rho \to +\infty$ and deduce that $v(x, t_o) = 0$ for all $x \in \mathbb{R}^N$. The left-hand-side, intrinsic Harnack inequality (1.2)–(1.6) now implies that $v \equiv 0$. ■

Remark 10.1 The proposition is false for $1 < p \le \frac{2N}{N+1}$, as evidenced by the counterexamples in § 1.3. The functions in (1.11) for p subcritical, and in (1.12) for p critical, are nonnegative, not identically zero, weak solutions to the prototype p-Laplacian equation, in $\mathbb{R}^N \times \mathbb{R}$. However, these explicit solutions are all unbounded in every half-space

$$S_T = \mathbb{R}^N \times (-\infty, T) \quad \text{for all fixed } T \in \mathbb{R}.$$

This raises the issue as to whether solutions to the singular ($1 < p < 2$), homogeneous ($C = 0$) equations (1.1)–(1.2) of Chapter 3, with two-sided bound in some half-space S_T, are constant in S_T, for p in the whole singular range $1 < p < 2$.

Proposition 10.2 *Let u be a solution to the singular* ($1 < p < 2$) *homogeneous* ($C = 0$) *equations (1.1)–(1.2) of Chapter 3. If u is bounded above and below in some half-space S_T, then u is constant in S_T.*

Proof The proof is almost identical to that of Proposition 16.3 of Chapter 5. ■

11 Critical and Subcritical Singular Equations

Let u be a nonnegative, local, weak solution to the singular equations (1.1)–(1.2) of Chapter 3 in E_T, for p in the *critical* and *subcritical* range

$$1 < p \le p_* = \tfrac{2N}{N+1}. \tag{11.1}$$

By the examples and counterexamples of § 1.3 a Harnack estimate in any of the forms (1.6)–(1.9), fails to hold when p is in the range (11.1). Nevertheless, for p in such a range, a different form of a Harnack estimate holds with constants depending on the ratio of some integral norms of the solution u. Fix $(x_o, t_o) \in E_T$ and ρ such that $K_{4\rho}(x_o) \subset E$, and introduce the quantity

$$\theta = \Big[\varepsilon\Big(\fint_{K_\rho(x_o)} u^q(\cdot, t_o)dx\Big)^{\frac{1}{q}}\Big]^{2-p} \tag{11.2}$$

where $\varepsilon \in (0,1)$ is to be chosen, and $q \ge 1$ is arbitrary. If $\theta > 0$, assume that

$$(x_o, t_o) + Q^-_{8\rho}(\theta) = K_{8\rho}(x_o) \times (t_o - \theta(8\rho)^p, t_o] \subset E_T$$

and set

$$\sigma = \left[\frac{\Big(\fint_{K_\rho(x_o)} u^q(\cdot, t_o)dx\Big)^{\frac{1}{q}}}{\Big(\fint_{K_{4\rho}(x_o)} u^r(\cdot, t_o - \theta\rho^p)dx\Big)^{\frac{1}{r}}}\right]^{\frac{rp}{\lambda_r}},$$

$$M_q = \Big(\sup_{t_o - \theta\rho^p < s \le t_o} \fint_{K_{2\rho}} u^q(\cdot, s)dx\Big)^{\frac{1}{q}}, \tag{11.3}$$

where $r \ge 1$ is any number such that

$$\lambda_r = N(p-2) + rp > 0. \tag{11.4}$$

Theorem 11.1 *Let u be a nonnegative, locally bounded, local, weak solution to the singular equations (1.1)–(1.2) of Chapter 3, for $1 < p < 2$ in E_T. Introduce θ as in (11.2) and assume that $\theta > 0$. There exist constants $\varepsilon \in (0,1)$, and $\gamma, \beta > 1$, depending only on the data $\{p, N, C_o, C_1\}$ and the parameters q, r, such that either*

$$C\rho > \min\{1\,,\, M_q\,,\, M_r\} \tag{11.5}$$

or

$$\inf_{(x_o,t_o)+Q^-_\rho(\frac{1}{2}\theta)} u \ge \gamma^{-1}\sigma^\beta \sup_{(x_o,t_o)+Q^-_\rho(\theta)} u \tag{11.6}$$

where σ is defined in (11.3), $q \ge 1$ and $r \ge 1$ satisfies (11.4). The constants $\varepsilon \to 0$, and $\gamma, \beta \to \infty$ as either $\lambda_r \to 0$ or $\lambda_r \to \infty$.

Remark 11.1 The estimate is vacuous if $\theta = 0$. This does occur for certain solutions to (1.1) of Chapter 3 for t_o larger than the extinction time ([60]). An explicit example is in (1.11).

Remark 11.2 Inequality (11.6) is not a Harnack inequality per se, since σ depends on the solution itself. It would reduce to a Harnack inequality if $\sigma \geq \sigma_o > 0$ for some absolute constant σ_o depending only on the data. This, however, in general is not the case, in view of the counterexamples of § 1.3. Further comments in this direction are in Remark 14.1.

Remark 11.3 Inequality (11.6) actually holds for nonnegative solutions to (1.1)–(1.2) of Chapter 3 for all $1 < p < 2$, provided $r \geq 1$ satisfies (11.4). For supercritical p one has $\lambda = \lambda_1 > 0$, and (11.4) can be realized for $r = 1$. However, for $\lambda > 0$ the strong form of a Harnack estimate holds (Theorems 1.1 and 1.2). Therefore (11.6), while true for all $1 < p < 2$, is of significance only for the critical and subcritical values in (11.1). In this sense (11.6) can be regarded as a "weak" form of a Harnack estimate valid for all $1 < p < 2$. Nevertheless, (11.6) is sufficient to establish the local Hölder continuity of locally bounded, weak solutions to (1.1)–(1.2) of Chapter 3, irrespective of their signum, as we show in § 14.

12 Components of the Proof of Theorem 11.1

The first is the expansion of positivity presented in Proposition 5.1 of Chapter 4, which holds for nonnegative, local solutions to the singular, quasilinear parabolic equations (1.1)–(1.2) of Chapter 3, for all $1 < p < 2$.

12.1 L^r_{loc}–L^∞_{loc} Harnack-Type Estimates for $r \geq 1$ Such That $\lambda_r > 0$

Theorem 12.1 *Let u be a nonnegative, locally bounded, local, weak solution to the singular equations (1.1)–(1.2) of Chapter 3, in E_T, for $1 < p < 2$, and let $r \geq 1$ satisfy (11.4). There exists a positive constant γ_r depending only on the data $\{p, N, C_o, C_1\}$, and r, such that either*

$$C\rho > \min\Big\{1\,,\,M_r\,,\,\Big(\frac{t-s}{\rho^p}\Big)^{\frac{1}{2-p}}\Big\} \tag{12.1}$$

or

$$\begin{aligned}\sup_{K_\rho(y)\times[s,t]} u \leq \frac{\gamma_r}{(t-s)^{\frac{N}{\lambda_r}}}\Big(\int_{K_{2\rho}(y)} u^r(x, 2s-t)dx\Big)^{\frac{p}{\lambda_r}}\\ + \gamma_r\Big(\frac{t-s}{\rho^p}\Big)^{\frac{1}{2-p}},\end{aligned} \tag{12.2}$$

for all cylinders

$$K_{2\rho}(y) \times [s-(t-s), s+(t-s)] \subset E_T. \tag{12.3}$$

The constant $\gamma_r \to \infty$ if either $\lambda_r \to 0$ or $\lambda_r \to \infty$.

Remark 12.1 The values of u in the upper part of the cylinder (12.3) are estimated by the integral of u on the lower base of the cylinder.

Remark 12.2 Theorem 12.1 assumes that u is locally bounded, and turns such a *qualitative* information into the *quantitative* estimate (12.2) in terms of the $L^r_{\rm loc}$ integrability of $u(\cdot,t)$. A discussion on the local boundedness of solution to these singular equations is in § 21.3.

12.2 $L^r_{\rm loc}$ Estimates Backward in Time

Proposition 12.1 *Let u be a nonnegative, local, weak solution to the singular equations (1.1)–(1.2) of Chapter 3, in E_T, for $1<p<2$, and assume that $u\in L^r_{\rm loc}(E_T)$ for some $r>1$. There exists a positive constant γ depending only on the data $\{p,N,C_o,C_1\}$ and r, such that either*

$$C\rho > \min\{1\,,\,M_r\},$$

where

$$M_r=\Big(\sup_{\tau\le s\le t}\fint_{K_{2\rho}} u^r(x,s)dx\Big)^{\frac{1}{r}},$$

or

$$\sup_{\tau\le s\le t}\int_{K_\rho(y)} u^r(x,s)dx \le \gamma\int_{K_{2\rho}(y)} u^r(x,\tau)dx + \gamma\Big[\frac{(t-\tau)^r}{\rho^{\lambda_r}}\Big]^{\frac{1}{2-p}}, \tag{12.4}$$

for all cylinders

$$K_{2\rho}(y)\times[\tau,t]\subset E_T.$$

The proof of Theorem 12.1 and Proposition 12.1 will be given in Appendix A. Here we assume them and proceed to establish Theorem 11.1.

13 Estimating the Positivity Set of the Solutions

Having fixed $(x_o,t_o)\in E_T$, assume it coincides with the origin, write $K_\rho(0)=K_\rho$, and introduce the quantity θ as in (11.2), which is assumed to be positive. Assume moreover that (11.5) is always violated, that is,

$$C\rho\le\min\{1\,,\,M_q\,,\,M_r\}.$$

Apply (12.4) for $r=q$, $y=0$, and $s\in(-\theta\rho^p,0]$. Using the definition (11.2) of θ gives

$$\int_{K_\rho} u^q(\cdot,0)dx\le\bar\gamma_q\int_{K_{2\rho}} u^q(\cdot,\tau)dx+\bar\gamma_q\varepsilon^q\int_{K_\rho} u^q(\cdot,0)dx,$$

for all $q\ge1$ and all $\tau\in(-\theta\rho^p,0]$, for a constant $\bar\gamma_q$ depending only on the data $\{p,N,C_o,C_1\}$ and q. Choosing ε from

$$\bar{\gamma}_q \varepsilon^q \le \tfrac{1}{2}$$

yields

$$\int_{K_{2\rho}} u^q(\cdot,\tau)dx \ge \frac{1}{2\bar{\gamma}_q} \int_{K_\rho} u^q(\cdot,0)dx \tag{13.1}$$

for all $\tau \in (-\theta\rho^p, 0]$. Next apply the sup-estimate (12.2) over the cylinder

$$K_{2\rho} \times \big(-\tfrac{1}{2}\theta\rho^p, 0\big]$$

with $r \ge 1$ such that $\lambda_r > 0$, to get

$$\begin{aligned}
\sup_{K_{2\rho}\times(-\frac{1}{2}\theta\rho^p,0]} u &\le \frac{\gamma_r[\omega_N(4\rho)^N]^{\frac{p}{\lambda_r}}}{(\theta\rho^p)^{\frac{N}{\lambda_r}}} \Big(\fint_{K_{4\rho}} u^r(\cdot,-\theta\rho^p)dx\Big)^{\frac{1}{r}\frac{rp}{\lambda_r}} + \gamma_r \theta^{\frac{1}{2-p}} \\
&\le \frac{\gamma_r'}{\varepsilon^{\frac{N(2-p)}{\lambda_r}}} \frac{1}{\sigma} \Big(\fint_{K_\rho} u^q(\cdot,0)dx\Big)^{\frac{1}{q}} + \gamma_r'\varepsilon \Big(\fint_{K_\rho} u^q(\cdot,0)dx\Big)^{\frac{1}{q}} \\
&= \gamma_r'\varepsilon \Big(1 + \frac{1}{\sigma\,\varepsilon^{\frac{rp}{\lambda_r}}}\Big) \Big(\fint_{K_\rho} u^q(\cdot,0)dx\Big)^{\frac{1}{q}}
\end{aligned}$$

for a constant γ_r' depending only on the data $\{p, N, C_o, C_1\}$ and r. One verifies that $\gamma_r' \to \infty$, as either $\lambda_r \to 0$ or $\lambda_r \to \infty$.

Assume momentarily that $0 < \sigma < 1$ so that in the round brackets containing σ, the second term dominates the first. In such a case

$$\sup_{K_{2\rho}\times(-\frac{1}{2}\theta\rho^p,0]} u \le \frac{1}{\varepsilon'\sigma} \Big(\fint_{K_\rho} u^q(\cdot,0)dx\Big)^{\frac{1}{q}} \overset{\text{def}}{=} M, \tag{13.2}$$

where

$$\varepsilon' = \frac{\varepsilon^{\frac{N(2-p)}{\lambda_r}}}{2\gamma_r'}.$$

From this

$$\varepsilon'\sigma M = \Big(\fint_{K_\rho} u^q(\cdot,0)dx\Big)^{\frac{1}{q}}. \tag{13.3}$$

Let $\nu \in (0,1)$ to be chosen. Using (13.3) and (13.1) estimate

$$\begin{aligned}
(\varepsilon'\sigma M)^q &\le 2^{N+1}\bar{\gamma}_q \fint_{K_{2\rho}} u^q(\cdot,\tau)dx \\
&\le 2^{N+1}\bar{\gamma}_q \bigg(\fint_{K_{2\rho}\cap[u<\nu\sigma M]} u^q(\cdot,\tau)dx + \fint_{K_{2\rho}\cap[u\ge\nu\sigma M]} u^q(\cdot,\tau)dx \bigg) \\
&\le 2^{N+1}\bar{\gamma}_q \nu^q (\sigma M)^q + 2^{N+1}\bar{\gamma}_q M^q \frac{|[u(\cdot,\tau) > \nu\sigma M] \cap K_{2\rho}|}{|K_{2\rho}|}
\end{aligned}$$

for all $\tau \in (-\frac{1}{2}\theta\rho^p, 0]$. From this

$$|[u(\cdot,\tau) > \nu\sigma M] \cap K_{2\rho}| \geq \alpha\sigma^q |K_{2\rho}|, \tag{13.4}$$

where

$$\alpha = \frac{\varepsilon'^q - \nu^q \, 2^{N+1}\bar{\gamma}_q}{2^{N+1}\bar{\gamma}_q}$$

for all $\tau \in (-\frac{1}{2}\theta\rho^p, 0]$. By choosing $\nu \in (0,1)$ sufficiently small, only dependent on the data $\{p, N, C_o, C_1\}$ and $\bar{\gamma}_q$, we can insure that $\alpha \in (0,1)$ depends only on the data $\{p, N, C_o, C_1\}$ and q, and is independent of σ. We summarize:

Proposition 13.1 *Let u be a nonnegative, locally bounded, local, weak solution to the singular equations (1.1)–(1.2) of Chapter 3, for $1 < p < 2$. Fix $(x_o, t_o) \in E_T$, let $K_{4\rho}(x_o) \subset E$, and let θ and σ be defined by (11.2)–(11.3) for some $\varepsilon \in (0,1)$. For every $r \geq 1$ satisfying (11.4) and every $q \geq 1$, there exist constants $\varepsilon, \nu, \alpha \in (0,1)$, depending only on the data $\{p, N, C_o, C_1\}$ and q and r, such that*

$$|[u(\cdot,t) > \nu\sigma M] \cap K_{2\rho}(x_o)| \geq \alpha\sigma^q |K_{2\rho}|$$

for all $t \in (t_o - \frac{1}{2}\theta\rho^p, t_o]$.

13.1 A First Form of the Harnack Inequality

The definitions (11.2) of θ and the parameters ε' and α imply that

$$\tfrac{1}{2}\theta = \epsilon(\nu\sigma M)^{2-p} \qquad \text{where } \epsilon = \tfrac{1}{2}\Big(\frac{\varepsilon\varepsilon'}{\nu}\Big)^{2-p}.$$

By Proposition 5.1 of Chapter 4 with M replaced by $\nu\sigma M$ and α replaced by $\alpha\sigma^q$, there exist constants η and δ in $(0,1)$, depending on the data $\{p, N, C_o, C_1\}$, and α, σ, and ϵ, such that

$$u(\cdot,t) > \eta(\alpha\sigma^q, \epsilon)\nu\sigma M \quad \text{in } K_{4\rho}(x_o),$$

for all times

$$t \in (t_o - \tfrac{1}{2}\theta\rho^p + \delta(\nu\sigma M)^{2-p}(2\rho)^p \,,\, t_o]$$

where δ includes the quantity $1 - \varepsilon$ of Proposition 5.1 of Chapter 4. Without loss of generality, we can assume that this time interval contains $(t_o - \frac{1}{4}\theta\rho^p, t_o]$.

Proposition 13.2 *Let u be a nonnegative, locally bounded, local, weak solution to the singular equations (1.1)–(1.2) of Chapter 3, for $1 < p < 2$. Fix $(x_o, t_o) \in E_T$, let $K_{4\rho}(x_o) \subset E$, and let θ and σ be defined by (11.2)–(11.3) for some $\varepsilon \in (0,1)$. For every $r \geq 1$ satisfying (11.4) and every $q \geq 1$, there exist constants $\varepsilon, \delta \in (0,1)$, and a continuous, increasing function $\sigma \to f(\sigma)$ defined in $\mathbb{R}^+$ and vanishing at $\sigma = 0$, that can be quantitatively determined a priori only in terms of the data $\{p, N, C_o, C_1\}$ and q and r, such that*

$$\inf_{K_{4\rho}(x_o)} u(\cdot,t) \geq f(\sigma) \sup_{(x_o,t_o)+Q_{2\rho}(\frac{1}{4}\theta)} u, \tag{13.5}$$

for all

$$t \in (t_o - \tfrac{1}{4}\theta\rho^p\,,\, t_o]$$

provided $(x_o, t_o) + Q_{8\rho}(\theta) \subset E_T$.

Remark 13.1 The form of the function $f(\cdot)$ is given in (5.4) of Chapter 4, for constants depending only on the data, q and r.

Remark 13.2 The function $f(\cdot)$ depends on θ only through the parameter ε in the definition (11.2) of θ.

Remark 13.3 The inequality (13.5) is a Harnack-type estimate of the same form as that stated in § 1, where, however, the constant $f(\sigma)$ depends on the solution itself, through σ defined in (11.3), as a proper quotient of the $L^q_{\rm loc}$ and $L^r_{\rm loc}$ averages of u, respectively, at time $t = t_o$ on the cube $K_\rho(x_o)$, and at time $t = t_o - \theta\rho^p$ on the cube $K_{4\rho}(x_o)$.

Remark 13.4 The inequality (13.5) has been derived by assuming that $0 < \sigma < 1$. If $\sigma \geq 1$, the same proof gives (13.5) where $f(\sigma) \geq f(1)$, thereby establishing a strong form of the Harnack estimate for these solutions. As shown in § 1.3, such a strong form is false for p in the critical, and subcritical range $1 < p \leq \frac{2N}{N+1}$.

It turns out that (13.5) is actually sufficient to establish that any locally bounded, possibly of variable sign, local, weak solution to the singular equations (1.1)–(1.2) of Chapter 3 for $1 < p < 2$, is locally Hölder continuous in E_T. In turn, such a Hölder continuity permits one to improve the lower bound in (13.5) by estimating $f(\cdot)$ to a power of its argument, as indicated in (11.6).

14 The First Form of the Harnack Inequality Implies the Hölder Continuity of u

Let u be a locally bounded, local, weak solution to the singular equations (1.1)–(1.2) of Chapter 3, for $1 < p < 2$, in E_T, possibly of variable sign. Let

$$\delta \in (0,1), \quad C, A > 1, \quad R, \omega > 0$$

and for a fixed $(x_o, t_o) \in E_T$ construct the sequences

$$R_o = R, \quad R_n = \frac{R}{C^n}; \qquad \omega_o = \omega, \quad \omega_{n+1} = \delta\omega_n \quad \text{for } n = 0, 1, 2, \dots$$

and the cylinders

$$Q_n = K_{R_n}(x_o) \times \left(t_o - \left(\frac{\omega_n}{A}\right)^{2-p} R_n^p, t_o \right] \quad \text{for } n = 1, 2, \dots .$$

Following Theorem 4.1 and Proposition 4.1 of Chapter 5, the solution u is Hölder continuous at $(x_o, t_o) \in E_T$ if the constants $\delta \in (0,1)$ and $A, C > 1$ can be determined a priori depending only on the data $\{p, N, C_o, C_1\}$, and independent of u and (x_o, t_o), such that

$$Q_{n+1} \subset Q_n \subset Q_o \subset E_T \quad \text{and} \quad \operatorname*{ess\,osc}_{Q_n} u \le \omega_n$$

for all $n = 0, 1, \ldots$. We will show that (13.5) permits one to construct such sequences for an arbitrary $(x_o, t_o) \in E_T$.

Having fixed $(x_o, t_o) \in E_T$, assume it coincides with the origin of $\mathbb{R}^{N+1}$ and for $\rho > 0$ set

$$R_o = 4\rho \quad \text{and} \quad Q = K_{4\rho} \times (-(4\rho)^p, 0], \tag{14.1}$$

where ρ is so small that $Q \subset E_T$. Set also

$$\mu_o^+ = \operatorname*{ess\,sup}_{Q} u, \quad \mu_o^- = \operatorname*{ess\,inf}_{Q} u, \quad \omega_o = \mu_o^+ - \mu_o^- = \operatorname*{ess\,osc}_{Q} u.$$

Since u is locally bounded in E_T, without loss of generality we may assume that $\omega_o \le 1$ so that

$$Q_o \overset{\text{def}}{=} K_{4\rho} \times \Big(-\Big(\frac{\omega_o}{A}\Big)^{2-p} (4\rho)^p, 0\Big] \subset Q \subset E_T \quad \text{and} \quad \operatorname*{ess\,osc}_{Q_o} u \le \omega_o$$

for a number $A \ge 1$ to be chosen. Now set

$$\mu^+ = \operatorname*{ess\,sup}_{Q_o} u, \qquad \mu^- = \operatorname*{ess\,inf}_{Q_o} u, \qquad \bar{\omega} = \operatorname*{ess\,osc}_{Q_o} u,$$

and introduce the two functions defined in Q_o

$$v_+ = \mu^+ - u, \qquad v_- = u - \mu^-.$$

Without loss of generality we may assume that

$$\mu^+ - \tfrac{1}{4}\omega_o \ge \mu^- + \tfrac{1}{4}\omega_o. \tag{14.2}$$

Indeed otherwise $\bar{\omega} \le \frac{1}{2}\omega_o$ and thus passing from Q to any smaller cylinder the essential oscillation of u is reduced by a factor $\frac{1}{2}$, and there is nothing to prove. Then either

$$\begin{aligned} &|[v_-(\cdot, 0) \ge \tfrac{1}{4}\omega_o] \cap K_\rho| \ge \tfrac{1}{2}|K_\rho| \quad \text{or} \\ &|[v_+(\cdot, 0) \ge \tfrac{1}{4}\omega_o] \cap K_\rho| > \tfrac{1}{2}|K_\rho|. \end{aligned} \tag{14.3}$$

Indeed by virtue of (14.2)

$$[u \le \mu^+ - \tfrac{1}{4}\omega_o] \cap K_\rho \supset [u \le \mu^- + \tfrac{1}{4}\omega_o] \cap K_\rho.$$

Therefore, if the first of (14.3) is violated, then

$$|[u \le \mu^+ - \tfrac{1}{4}\omega_o] \cap K_\rho| > \tfrac{1}{2}|K_\rho|.$$

Compute and estimate the values $\theta_\pm$, as defined by (11.2), relative to the functions $v_\pm$, over K_ρ at the time level $t = 0$. Assuming the first of (14.3) holds,

$$\begin{aligned} \omega_o^q &\ge \frac{1}{|K_\rho|}\int_{K_\rho} \big(u(\cdot,0) - \mu^-\big)^q dx \\ &\ge \frac{1}{|K_\rho|}\int_{K_\rho \cap [v_- > \frac{1}{4}\omega_o]} [u(\cdot,0) - \mu^-]^q dx \ge \frac{1}{2}\Big(\frac{\omega_o}{4}\Big)^q. \end{aligned}$$

Therefore, if the first of (14.3) holds,

$$\frac{1}{2^{\frac{2-p}{q}}}\Big(\frac{\omega_o}{4A_o}\Big)^{2-p}\rho^p \le \theta_-\rho^p \le \Big(\frac{\omega_o}{A_o}\Big)^{2-p}\rho^p \qquad \text{for } A_o^{-1} = \varepsilon, \tag{14.4}$$

and we have the inclusion

$$K_{4\rho} \times (-\theta_-\rho^p, 0] \subset K_{4\rho} \times \Big(-\Big(\frac{\omega_o}{A_o}\Big)^{2-p}\rho^p, 0\Big].$$

Similar estimates hold for θ_+ if the second of (14.3) is in force. By the structure conditions (1.2) of Chapter 3 both $v_\pm$ are solutions to (1.1)–(1.2) of Chapter 3 for the same constants C_o and C_1 and hence the Harnack-type inequality (13.5) holds for either v_- or v_+, i.e.,

$$\inf_{Q_{4\rho}(\frac{1}{4}\theta_\pm)} v_\pm \ge f(\sigma_\pm) \sup_{Q_{2\rho}(\frac{1}{4}\theta_\pm)} v_\pm, \tag{14.5}$$

where $\sigma_\pm$ are defined as in (11.3) for $v_\pm$. By virtue of (14.4), which holds for either θ_- or θ_+, and Remark 13.1, the function $f(\cdot)$ can be taken to be the same. Assume now that the first of (14.3) holds true. Then as shown before,

$$\fint_{K_\rho} v_-^q(\cdot,0)dx \ge \frac{1}{|K_\rho|}\int_{K_\rho \cap [v \ge \frac{1}{4}\omega_o]} v_-^q(x,0)dx \ge \frac{1}{2}\big(\frac{\omega_o}{4}\big)^q.$$

On the other hand,

$$\fint_{K_{4\rho}} v_-^r(x, -\theta_-\rho^p)dx \le \omega_o^r,$$

and therefore recalling the definition (11.3) of σ_-

$$f(\sigma_-) \ge f\Big[\Big(\frac{1}{4\cdot 2^{1/q}}\Big)^{\frac{pr}{\lambda_r}}\Big] \overset{\text{def}}{=} 1 - \delta$$

for $\delta \in (0,1)$ depending only on the data $\{p, N, C_o, C_1\}$, q and r. This and (14.5) imply

$$\inf_{K_{4\rho}\times(-\frac{1}{4}\theta\rho^p,0]} v_- \ge (1-\delta) \sup_{K_{2\rho}\times(-\frac{1}{2}\theta\rho^p,0]} v_-$$

from which

$$\operatorname*{ess\,osc}_{Q_1} u \le \omega_1 = \delta\omega_o,$$

where

$$Q_1 = K_\rho \times \Big(-\Big(\frac{\omega_o}{A}\Big)^{2-p}\rho^p, 0\Big] \quad \text{and} \;\; A = 2^{1/q}\, 4^{1+\frac{1}{2-p}} A_o. \tag{14.6}$$

This and (14.4) determine A depending only on the data $\{p, N, C_o, C_1\}$, q and r. Taking into account (14.1), the cylinder Q_1 is determined from Q_o by the indicated choice of A and for $C = 4$. A similar argument holds if the second of (14.3) is in force. This process can now be iterated and continued to yield:

Proposition 14.1 *Let u be a locally bounded, local, weak solution to the singular equations (1.1)–(1.2) of Chapter 3 for $1 < p < 2$, in E_T. There exist constants $\bar\gamma > 1$ and $\epsilon_o \in (0,1)$, depending only on the data $\{p, N, C_o, C_1\}$, q and r, such that for all $(x_o, t_o) \in E_T$, setting*

$$M = \operatorname*{ess\,sup}_{(x_o,t_o)+Q_R^-(1)} u \qquad \textit{for} \qquad (x_o,t_o)+Q_R^-(1) \subset E_T,$$

we have

$$\operatorname*{ess\,osc}_{(x_o,t_o)+Q_\rho^-(\theta_M)} u \le \bar\gamma M \Big(\frac{\rho}{R}\Big)^{\epsilon_o}, \qquad \textit{where}\;\; \theta_M = \Big(\frac{M}{A}\Big)^{2-p}$$

for all $0 < \rho \le R$, and all cylinders

$$(x_o,t_o) + Q_\rho^-(\theta_M) \subset (x_o,t_o) + Q_R^-(1) \subset E_T.$$

Remark 14.1 Returning to Remark 11.2, the previous arguments show that either σ_+ or σ_- are bounded below by an absolute, positive constant σ_o. Thus (14.5) implies that either $\mu^+ - u$ or $u - \mu^-$ satisfy a strong form of the Harnack inequality. By the remarks of § 1.3, a strong form of the Harnack estimate does not hold simultaneously for $\mu^+ - u$ and $u - \mu^-$.

15 Proof of Theorem 11.1 Concluded

Assume (x_o, t_o) coincides with the origin of $\mathbb{R}^{N+1}$ and determine ν and α as in § 13. We may assume that

$$|[u(\cdot,0) \le \nu\sigma M] \cap K_\rho| > 0.$$

Indeed, otherwise (13.4) would hold with $\alpha\sigma^q = 1$ and the proof could be repeated leading to (13.5) with f depending only on the data $\{p, N, C_o, C_1\}$ and independent of σ. Moreover by (13.2)

$$\sup_{K_{2\rho}\times(-\frac{1}{2}\theta\rho^p,0]} u \le M$$

with θ given by (11.2). Since u is locally Hölder continuous, there exists $x_1 \in K_\rho$ such that

$$u(x_1,0) = \nu\sigma M.$$

Using the parameter A claimed by Proposition 14.1, construct the cylinder with "vertex" at $(x_1,0)$

$$(x_1,0) + Q_{2r}^-\Big[\Big(\frac{\nu\sigma M}{A}\Big)^{2-p} r^p\Big] \subset K_{2\rho}\times\big(-\tfrac{1}{4}\theta\rho^p,0\big].$$

In the definition (11.2) of θ and the choice (14.4) and (14.6) of the parameter A, such an inclusion can be realized by possibly increasing A by a fixed quantitative factor depending only on the data, and by choosing r sufficiently small. Assuming the choice of r has been made, by Proposition 14.1

$$|u(x,t) - u(x_1,0)| \le \bar\gamma M\Big(\frac{r}{\rho}\Big)^{\epsilon_o}$$

for all

$$(x,t) \in \tilde Q_1 \stackrel{\text{def}}{=} (x_1,0) + Q_r^-\Big[\Big(\frac{\nu\sigma M}{A}\Big)^{2-p} r^p\Big].$$

From this

$$u(x,t) \ge \tfrac{1}{2}\nu\sigma M \qquad \text{for all } (x,t)\in\tilde Q_1,$$

provided r is chosen to be so small that

$$\frac{\bar\gamma}{\nu\sigma}\Big(\frac{r}{\rho}\Big)^{\epsilon_o} = \frac{1}{2}, \qquad \text{that is,} \qquad r = \varepsilon_1\sigma^{\frac{1}{\epsilon_o}}\rho, \qquad \text{where } \varepsilon_1 = \Big(\frac{\nu}{2\bar\gamma}\Big)^{\frac{1}{\epsilon_o}}.$$

Therefore by Proposition 5.1 of Chapter 4

$$u \ge \eta(\nu\sigma M) \quad \text{in} \quad (x_1,0) + Q_{2r}^-\Big[\Big(\frac{\eta(\nu\sigma M)}{A}\Big)^{2-p}(2r)^p\Big]$$

for an absolute constant $\eta\in(0,1)$. This process can now be iterated to give

$$u \ge \eta^n(\nu\sigma M) \quad \text{in} \quad (x_1,0) + Q_{2^n r}^-\Big[\Big(\frac{\eta^n(\nu\sigma M)}{A}\Big)^{2-p}(2^n r)^p\Big]$$

for all $n\in\mathbb{N}$. Choose n as the smallest integer for which

$$2^n r \ge 4\rho, \qquad \text{that is,} \qquad n \ge \log_2\Big(\frac{4}{\varepsilon_1\sigma^{\frac{1}{\epsilon_o}}}\Big).$$

For such a choice

$$u \ge \gamma\sigma^\beta M \quad \text{in } Q_{2\rho}^-\Big[\Big(\frac{\gamma\sigma^\beta M}{A}\Big)^{2-p}\rho^p\Big]$$

for some $\beta = \beta(\text{data})$. ∎

16 Supercritical, Singular Equations of the Porous Medium Type

Let u be a continuous, nonnegative, local, weak solution to the singular equations (5.1)–(5.2) of Chapter 3 in E_T, for m in the *supercritical* range

$$m_* = \tfrac{(N-2)_+}{N} < m < 1. \tag{16.1}$$

Fix $(x_o, t_o) \in E_T$ such that $u(x_o, t_o) > 0$ and construct the cylinders

$$(x_o, t_o) + Q_\rho^\pm(\theta), \quad \text{where} \quad \theta = [u(x_o, t_o)]^{1-m}. \tag{16.2}$$

These cylinders are "intrinsic" to the solution since their length is determined by the value of u at (x_o, t_o). Cylindrical domains of the form $K_\rho \times (0, \rho^2]$ reflect the natural, parabolic space-time dilations that leave the homogeneous, singular, prototype equation (5.3) of Chapter 3 invariant. The latter, however, is not homogeneous with respect to the solution u. The time dilation by a factor $[u(x_o, t_o)]^{1-m}$ is intended to restore the homogeneity, and the Harnack inequality holds in such an intrinsic geometry, as made precise in Theorems 16.1 and 16.2 below. The first theorem establishes an intrinsic, mean value Harnack inequality in a form similar to Theorem 15.1 of Chapter 5, for degenerate equations ($m > 1$). This Harnack estimate is stable as $m \to 1$. The second theorem establishes a "time-insensitive" mean value Harnack inequality, valid for all times t ranging in a neighborhood of t_o. This inequality is unstable as $m \to 1$. By counterexamples, it will be shown that for $0 < m \le \frac{(N-2)_+}{N}$, neither of these theorems holds.

16.1 The Intrinsic, Mean Value, Harnack Inequality

Local weak solutions to (5.1)–(5.2) of Chapter 3, for m in the supercritical range (16.1), are locally bounded and locally Hölder continuous within their domain of definition (Appendix B, § B.5 and § B.8). Having fixed $(x_o, t_o) \in E_T$, and cylinders $(x_o, t_o) + Q_\rho^\pm(\theta)$ as in (16.2), set

$$\sup_{K_\rho(x_o)} u(x, t_o) = \mathcal{M} \tag{16.3}$$

and require that

$$(x_o, t_o) + Q_{8\rho}^\pm(\mathcal{M}^{1-m}) \subset E_T. \tag{16.4}$$

Specifically it is required that

$$\begin{aligned} \mathcal{Q}_{\mathcal{M}}(x_o, t_o) &= [(x_o, t_o) + Q_{8\rho}^-(\mathcal{M}^{1-m})] \cup [(x_o, t_o) + Q_{8\rho}^+(\mathcal{M}^{1-m})] \\ &= K_{8\rho}(x_o) \times (t_o - \mathcal{M}^{1-m}(8\rho)^2, t_o + \mathcal{M}^{1-m}(8\rho)^2] \subset E_T. \end{aligned} \tag{16.5}$$

The upper bound $\mathcal{M}$ is only known qualitatively, and accordingly it does not play any role in the proof other than to insure that $(x_o, t_o) + Q_{8\rho}^\pm(\mathcal{M}^{1-m})$ are contained within the domain of definition of u.

Theorem 16.1 *Let u be a continuous, nonnegative, local, weak solution to the singular equations (5.1)–(5.2) of Chapter 3 in E_T, for m in the supercritical range (16.1). There exist constants $\epsilon \in (0,1)$ and $\gamma > 1$ depending only on the data $\{m, N, C_o, C_1\}$, such that for all intrinsic cylinders $(x_o, t_o) + Q^{\pm}_{8\rho}(\theta)$, for which (16.4) holds, either*

$$C\rho > 1$$

or

$$\begin{aligned}\gamma^{-1} \sup_{K_\rho(x_o)} u\big(\cdot, t_o - \epsilon\, u(x_o, t_o)^{1-m}\rho^2\big) &\le u(x_o, t_o)\\ &\le \gamma \inf_{K_\rho(x_o)} u\big(\cdot, t_o + \epsilon\, u(x_o, t_o)^{1-m}\rho^2\big).\end{aligned} \tag{16.6}$$

Thus the form (1.2) of Chapter 5, valid for nonnegative solutions to nondegenerate equations ($m = 1$), continues to hold for nonnegative solutions to supercritically singular equations, although in their own intrinsic geometry.

Remark 16.1 The intrinsic geometry enters here in two stages. First, it determines the cylinders $(x_o, t_o) + Q^{\pm}_{8\rho}(\theta)$, then the constant ϵ determines the relative "waiting time," within the cylinders $(x_o, t_o) + Q^{\pm}_{8\rho}(\theta)$ for the intrinsic Harnack estimate to hold. The proof will determine the constants γ and ϵ quantitatively, only in terms of the data $\{m, N, C_o, C_1\}$. Whence these constants are determined, the intrinsic Harnack inequality (16.2)–(16.6) continues to hold for a *smaller* ϵ, provided we take a *larger* γ, and $\gamma(\epsilon) \to \infty$ as $\epsilon \to 0$. In all cases it is required that (16.3)–(16.5) be in force. The various constants, however, are dependent only on the data $\{m, N, C_o, C_1\}$ and are all independent of $\mathcal{M}$.

Remark 16.2 The constants γ and ϵ deteriorate as $m \to m_*$ in the sense that

$$\gamma(m), \epsilon(m)^{-1} \to \infty \qquad \text{as} \qquad m \to \tfrac{(N-2)_+}{N}.$$

However, they are stable as $m \to 1$ in the sense of (5.9) of Chapter 3. Thus by formally letting $m \to 1$ in (16.6) one recovers the classical Moser's Harnack inequality in the form (1.2) of Chapter 5.

Remark 16.3 The proofs are based on the energy estimates and DeGiorgi-type lemmas of § 9–11 of Chapter 3 and the expansion of positivity of § 7 of Chapter 4. According to the discussion in § 5.3 and Remarks 9.2, 10.1, and 11.3 of Chapter 3, a constant γ depends only on the data if it can be quantitatively determined a priori only in terms of $\{m, N, C_o, C_1\}$. The constant C appearing in the structure conditions (5.2) of Chapter 3, enters in the statement of Theorems 16.1 and 16.2, only through an alternative.

Remark 16.4 The theorem has been stated for continuous solutions, to give meaning to $u(x_o, t_o)$. The Hölder continuity of *nonnegative* solutions will be proved in § B.6–B.13 of Appendix B.

Remark 16.5 The intrinsic form (16.6) depends on the intrinsic expansion of positivity of Chapter 4, and it cannot be removed. Indeed (16.6) is false in a time geometry independent of $u(x_o, t_o)$. This can be verified for the family of examples and counterexamples collected in § 3.3 of Chapter 4, properly adapted for porous medium type equations. Explicit examples of solutions to porous medium type equations are collected in [150], and are discussed in § 16.3.

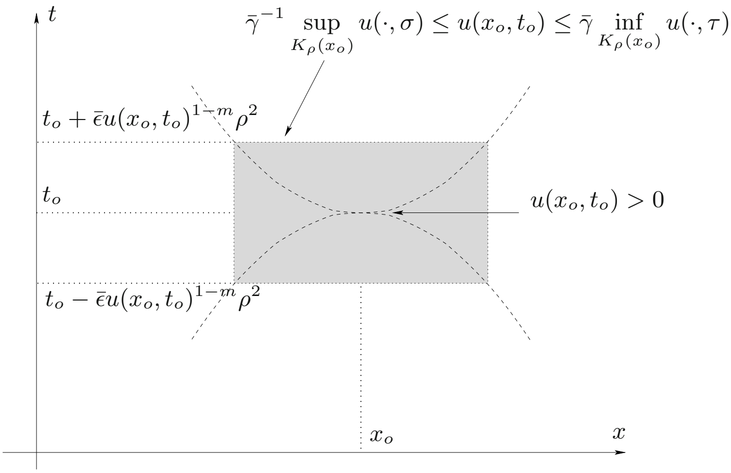

Fig. 16.2. Time-Insensitive Mean Value Harnack Inequality

16.2 Time-Insensitive, Intrinsic, Mean Value, Harnack Inequalities

Theorem 16.2 *Let u be a continuous, nonnegative, local, weak solution to the singular equations (5.1)–(5.2) of Chapter 3 in E_T, for m in the supercritical range (16.1). There exist constants $\bar\epsilon \in (0,1)$ and $\bar\gamma > 1$, depending only on the data $\{m, N, C_o, C_1\}$, such that for all intrinsic cylinders $(x_o, t_o) + Q^{\pm}_{8\rho}(\theta)$ as in (16.2), for which (16.4) holds, either*

$$C\rho > 1$$

or

$$\bar\gamma^{-1} \sup_{K_\rho(x_o)} u(\cdot,\sigma) \le u(x_o,t_o) \le \bar\gamma \inf_{K_\rho(x_o)} u(\cdot,\tau) \tag{16.7}$$

for any pair of time levels σ, τ in the range

$$t_o - \bar\epsilon\, u(x_o,t_o)^{1-m}\rho^2 \le \sigma, \tau \le t_o + \bar\epsilon\, u(x_o,t_o)^{1-m}\rho^2. \tag{16.8}$$

The constants $\bar\epsilon$ and $\bar\gamma^{-1}$ tend to zero as either $m \to 1$ or $m \to \frac{(N-2)_+}{N}$.

Both right and left inequalities in (16.7) are insensitive to the times σ, τ, provided they range within the time-intrinsic geometry (16.8). For $\sigma = \tau = t_o$ the theorem yields

Corollary 16.1 (The Elliptic Harnack Inequality) *Let u be a continuous, nonnegative, local, weak solution to the singular equations (5.1)–(5.2) of Chapter 3 in E_T, for m in the supercritical range (16.1). For all intrinsic cylinders $(x_o, t_o) + Q^{\pm}_{8\rho}(\theta)$ as in (16.2), for which (16.4) holds, either*

$$C\rho > 1$$

or

$$\bar{\gamma}^{-1} \sup_{K_\rho(x_o)} u(\cdot, t_o) \le u(x_o, t_o) \le \bar{\gamma} \inf_{K_\rho(x_o)} u(\cdot, t_o) \tag{16.9}$$

for the same constant $\bar{\gamma}$ as in Theorem 16.2.

Thus, the right and left inequalities in (16.7) are simultaneously forward, backward, and elliptic Harnack estimates. Inequalities of this type, and in particular (16.9), are false for nonnegative solutions to the heat equation ([121]). This is reflected in (16.7)–(16.9), in that the constants $\bar{\epsilon}$ and $\bar{\gamma}^{-1}$ tend to zero as $m \to 1$. It turns out that these inequalities lose meaning also as m tends to the critical value $\frac{(N-2)_+}{N}$ as discussed below.

16.3 On the Range (16.1) of m

The range of m in (16.1) is optimal for the intrinsic, forward in time Harnack estimate (16.2)–(16.6) to hold. Consider nonnegative weak solutions to the Cauchy problem

$$\begin{aligned} u_t - \Delta u^m = 0 \quad &\text{in } \mathbb{R}^N \times \mathbb{R}^+ \\ u(\cdot, 0) = u_o \in L^s(\mathbb{R}^N), \qquad &s = \frac{N(1-m)}{2} \end{aligned}$$

for $0 < m < m_*$ and $u_o \ge 0$. Solutions exist and become extinct, abruptly, after a finite time T. Specifically, there exists a time T, that can be determined a priori in terms of m, N, and $\|u_o\|_{s,\mathbb{R}^N}$, such that ([41], Chapter VII, § 11, and [59])

$$u(\cdot, t) > 0 \quad \text{for } t < T \quad \text{and} \quad u(\cdot, t) = 0 \quad \text{for } t > T.$$

Pick $(x_o, t_o) \in \mathbb{R}^N \times (0, T)$ where t_o is so close to T as to satisfy

$$T - t_o < 8^{-2} t_o$$

and choose $\rho > 0$ so large that

$$u(x_o, t_o)^{1-m} (8\rho)^2 = T - t_o.$$

For such a choice

$$(x_o, t_o) + Q^{\pm}_{8\rho}(\theta) \subset \mathbb{R}^N \times \mathbb{R}^+ \quad \text{for } \theta \text{ as in (16.2).}$$

However, the intrinsic, forward Harnack estimate (16.2)–(16.6) fails.

When $0 < m \le \frac{(N-2)_+}{N}$ also the elliptic version (16.9) fails, as shown by the following counterexample.

$$\begin{gathered} u(x,t) = \left(\frac{2|\lambda| m}{1-m}\right)^{\frac{1}{1-m}} \frac{(T-t)_+^{\frac{1}{1-m}}}{|x|^{\frac{2}{1-m}}} \\ 0 < m < \frac{(N-2)_+}{N+2}, \qquad \lambda = N(m-1)+2. \end{gathered} \tag{16.10}$$

This is a nonnegative, local, weak solution to the prototype porous medium equation in $\mathbb{R}^N \times \mathbb{R}$. Such a solution is unbounded near $x = 0$ for all $t < T$ and finite otherwise. Therefore (16.9) fails to hold for cubes centered at the origin.

For $0 < m \le m_*$ the mere notion of weak solution is not sufficient to insure its local boundedness ([41], Chapter V, § 11 and [59]). The weak solution (16.10) is indeed unbounded near $x = 0$. However, the lack of a Harnack estimate is not due to the possible unboundedness of the solutions. Consider the two-parameter family of functions

$$\begin{gathered} u(x,t) = (T-t)_+^{\frac{N+2}{4}} \left(a + \frac{|x|^2}{4aN}\right)^{-\frac{N+2}{2}} \\ N > 2, \qquad m = \tfrac{(N-2)_+}{N+2} < m_*, \end{gathered} \tag{16.11}$$

where $a > 0$ and T are parameters. These functions are nonnegative, locally bounded, weak solutions to the prototype porous medium equation in $\mathbb{R}^N \times \mathbb{R}$ and they do not satisfy the Harnack estimates of Theorems 16.1 and 16.2. The same occurs for the critical value $m = m_*$ as shown by the following two-parameter family of counterexamples. The functions

$$u(x,t) = \left(a|x|^2 + ke^{2Nat}\right)^{-\frac{N+1}{2}} \quad N \ge 3, \quad m = \frac{(N-2)_+}{N} \tag{16.12}$$

for positive parameters a and k, are nonnegative solutions to the prototype porous medium equation in $\mathbb{R}^N \times \mathbb{R}$, and one verifies that they fail to satisfy the Harnack estimate in any one of the forward, backward, or elliptic form.

These remarks raise the question of what form the Harnack estimate might take for m in the critical and subcritical range $0 < m \le m_*$. This issue will be addressed in § 19–20.

17 On the Proof of Theorems 16.1 and 16.2

The proof of these theorems follows the arguments of § 3–9. These arguments in turn depend on the number

$$\lambda = N(m-1)+2. \tag{17.1}$$

The requirements that m be in the supercritical range (16.1) is equivalent to requiring that $\lambda > 0$. The main components of the proof are the expansion of positivity for quasilinear, singular equations of the porous medium type, as presented in § 7 of Chapter 4, and an $L^1_{\rm loc}$–$L^\infty_{\rm loc}$ Harnack-type estimate, valid for $\lambda > 0$, analogous to Theorem 2.1, which we state next.

17.1 An $L^1_{\rm loc}$–$L^\infty_{\rm loc}$ Harnack-Type Estimate

Theorem 17.1 *Let u be a nonnegative, local, weak solution to (5.1)–(5.2) of Chapter 3 for m in the supercritical range (16.1). There exists a positive constant γ depending only on the data $\{m, N, C_o, C_1\}$, such that for all cylinders*

$$K_{2\rho}(y)\times[s-(t-s), s+(t-s)]\subset E_T$$

either

$$C\rho > 1$$

or

$$\sup_{K_\rho(y)\times[s,t]} u \le \frac{\gamma}{(t-s)^{\frac{N}{\lambda}}}\Big(\inf_{2s-t<\tau<t}\int_{K_{2\rho}(y)} u(x,\tau)dx\Big)^{\frac{2}{\lambda}} + \gamma\Big(\frac{t-s}{\rho^2}\Big)^{\frac{1}{1-m}}$$

where λ is defined in (17.1).

Theorem 17.1 will be established in Appendix B. Assuming it for the moment, we proceed to indicate the minor changes in the proof, along § 3–9.

17.2 The Right-Hand-Side Harnack Estimate of Theorem 16.2

Fix $(x_o, t_o) \in E_T$, determine the intrinsic cylinders $(x_o, t_o) + Q^{\pm}_{8\rho}(\theta)$ as in (16.2), and assume that (16.3)–(16.5) hold. Assume in addition that

$$C\rho \le 1 \tag{17.2}$$

where C is the constant in the structure conditions (5.2) of Chapter 3.

Proposition 17.1 *Let u be a continuous, nonnegative, local, weak solution to (5.1)–(5.2) of Chapter 3 in $\mathcal{Q}_{\mathcal{M}}$, for m in the supercritical range (16.1). There exist positive constants $\bar\epsilon$ and $\bar\gamma$, that can be determined quantitatively, a priori only in terms of the data $\{m, N, C_o, C_1\}$, such that*

$$u(x_o,t_o) \le \bar\gamma \inf_{K_\rho(x_o)} u(\cdot,t)$$

for all times

$$t_o - \bar\epsilon u(x_o,t_o)^{1-m}\rho^2 \le t \le t_o + \bar\epsilon u(x_o,t_o)^{1-m}\rho^2.$$

The constants $\bar\epsilon$ *and* $\bar\gamma^{-1}$ *tend to zero, as either* $m \to 1$ *or* $m \to \frac{(N-2)_+}{N}$.

Introduce the change of variables and unknown function

$$x \to \frac{x-x_o}{\rho}, \qquad t \to \frac{t-t_o}{u(x_o,t_o)^{1-m}\rho^2}, \qquad v = \frac{u}{u(x_o,t_o)}.$$

This maps the cylinder $\mathcal{Q}_{\mathcal{M}}(x_o,t_o)$ in (16.4) into

$$\mathcal{Q}_{\mathcal{M}} = K_8 \times \Big(- \Big(\frac{\mathcal{M}}{u(x_o,t_o)}\Big)^{1-m} 8^2, \Big(\frac{\mathcal{M}}{u(x_o,t_o)}\Big)^{1-m} 8^2 \Big].$$

Relabeling by x,t the new coordinates, v is a weak solution to

$$v_t - \operatorname{div} \bar{\mathbf{A}}(x,t,v,Dv) = \bar B(x,t,v,Dv) \quad \text{in} \quad \mathcal{Q}_{\mathcal{M}}. \tag{17.3}$$

Taking into account (17.2), the transformed functions $\bar{\mathbf{A}}$ and $\bar B$ satisfy the structure conditions

$$\begin{cases} \bar{\mathbf{A}}(x,t,v,Dv)\cdot Dv \ge mC_o v^{m-1}|Dv|^2 - v^{m+1} \\ |\bar{\mathbf{A}}(x,t,v,Dv)| \le mC_1 v^{m-1}|Dv| + v^m \\ |\bar B(x,t,v,Dv)| \le m v^{m-1}|Dv| + v^m \end{cases} \tag{17.4}$$

where C_o and C_1 are the constants appearing in (5.2) of Chapter 3. Establishing Proposition 17.1 consists in finding positive constants $\bar\epsilon$ and $\bar\gamma$, depending only on the data, such that

$$v(\cdot,t) \ge \bar\gamma^{-1} \quad \text{in } K_1 \quad \text{for all } t \in [-\bar\epsilon, \bar\epsilon].$$

The proof of this inequality involves the partial differential equations (17.3)–(17.4) only through Theorem 17.1 applied to the local solution v, and the expansion of positivity of § 7 of Chapter 4, applied to the same local solution v. The remaining arguments are based on measure-theoretical considerations, and harmonic analysis, independent of the particular partial differential equation at hand. They can be repeated almost verbatim as in § 3–9.

18 Some Consequences of the Harnack Inequality

18.1 Analyticity of Nonnegative Solutions

If the equation is homogeneous ($C = 0$), and $\mathbf{A}$ is locally analytic in all its arguments, within its domain of definition, then nonnegative weak solutions are locally analytic in the space variables and at least Lipschitz continuous in time, as made precise by the following proposition.

Proposition 18.1 *Let u be a nonnegative, local, weak solution to the singular, homogeneous ($C = 0$) equations (5.1)–(5.2) of Chapter 3 for m in the supercritical range (16.1). Assume moreover that* $\mathbf{A}$*, whenever well defined, is a locally analytic function of its arguments. Fix $(x_o, t_o) \in E_T$, such that $u(x_o, t_o) > 0$, and construct cylinders $(x_o, t_o) + Q^{\pm}_{8\rho}(\theta)$ as in (16.2), which in addition satisfy (16.3)–(16.5).*

Then there exists a positive constant γ, depending only on the data $\{m, N, C_o, C_1\}$ and independent of u, such that for every multi-index α,

$$|D^\alpha u(x_o, t_o)| \le \frac{\gamma^{|\alpha|+1}|\alpha|!}{\rho^{|\alpha|}} u(x_o, t_o).$$

Moreover for every nonnegative integer k,

$$\left|\frac{\partial^k}{\partial t^k} u(x_o, t_o)\right| \le \frac{\gamma^{2k+1}(k!)^2}{\rho^{2k}} [u(x_o, t_o)]^{1-(1-m)k}.$$

Proof By Theorem 16.2

$$0 < \bar{\gamma}^{-1} u(x_o, t_o) \le u(x, t) \le \bar{\gamma} u(x_o, t_o)$$

for all (x, t) in

$$K_\rho(x_o) \times (t_o - \bar{\epsilon} u(x_o, t_o)^{1-m} \rho^2 \, , \, t_o + \bar{\epsilon} u(x_o, t_o)^{1-m} \rho^2).$$

Therefore in such a cylinder the equation is no longer singular or degenerate. The proposition now follows from the classical results of Friedman [70] (see also [90]). The structural assumptions (5.2) of Chapter 3 insure that the matrix (A_{i,p_j}) is positive definite. ■

The conclusion of Proposition 18.1, for all multi-indices α and for $k = 1$, continues to hold for points $(x_o, t_o) \in E_T$ such that $u(x_o, t_o) = 0$. For this it suffices to apply Proposition 18.1 to a sequence of local approximations $\{u_n\}_{n \in \mathbb{N}}$ of u, each satisfying the equation, and bounded below in a neighborhood of (x_o, t_o).

18.2 Hölder Continuity

As already noticed in Remark 16.4, locally bounded, weak solutions to the porous medium type equations (5.1)–(5.2) of Chapter 3, in E_T, irrespective of their signum, and for all $m > 0$, are locally Hölder continuous in E_T.

However, such a regularity cannot be directly derived from the intrinsic Harnack inequality of either Theorem 16.1 or Theorem 16.2. Known proofs of Hölder continuity of a solution u from the Harnack inequality, are based, one way or another, on applying the Harnack estimate to the functions $u \pm k$ for a suitable constant k ([49, 51, 119, 120, 121], and also § 4 of Chapter 5). On the other hand, if u is a solution to the prototype porous medium equation (5.3) of Chapter 3, then $u - k$, for a nonzero constant k, is not a solution to the same equation. Therefore there is no guarantee that either $u-k$ or $k-u$ would satisfy the Harnack estimate. Even if this approach were viable, it would be limited to singular equations for m in the supercritical range (16.1), since, as discussed in § 16.3, the Harnack inequality is no longer valid for m in the critical and subcritical range $0 < m \le \frac{(N-2)_+}{N}$.

These remarks suggest that for the singular porous medium type equations (5.1)–(5.2) of Chapter 3 the issues of the local Hölder continuity of its local, weak solutions, and that of the Harnack inequality for its local, nonnegative, weak solutions are separate and neither implies the other.

However, the same methods leading to the Harnack estimates of Theorems 16.1 and 16.2 can be used to establish the local Hölder continuity of locally bounded, weak solutions to these singular equations, irrespective of their signum, and for all $m > 0$. They also permit one to exhibit a quantitative Hölder modulus of continuity. The results being local, assume without loss of generality that $u \in L^\infty(E_T)$. Let

$$\Gamma = \partial E_T - \bar{E} \times \{T\}$$

denote the parabolic boundary of E_T, and for a compact set $K \subset E_T$ introduce the *intrinsic*, parabolic m-distance from K to Γ by

$$m - \operatorname{dist}(K;\Gamma) \overset{\text{def}}{=} \inf_{\substack{(x,t)\in K\\(y,s)\in\Gamma}} \left(|x-y| + \|u\|_{\infty,E_T}^{\frac{m-1}{2}} |t-s|^{\frac{1}{2}} \right).$$

Theorem 18.1 *Let u be a bounded, local, weak solution to the singular porous medium type equations (5.1)–(5.2) of Chapter 3, for $0 < m < 1$. Then u is locally Hölder continuous in E_T, and there exist constants $\gamma > 1$ and $\alpha \in (0,1)$ that can be determined a priori only in terms of the data $\{m, N, C_o, C_1\}$ and C, such that for every compact set $K \subset E_T$,*

$$|u(x_1,t_1) - u(x_2,t_2)| \le \gamma \|u\|_{\infty,E_T} \left(\frac{|x_1 - x_2| + \|u\|_{\infty,E_T}^{\frac{m-1}{2}} |t_1 - t_2|^{\frac{1}{2}}}{m - \operatorname{dist}(K;\Gamma)} \right)^{\alpha}$$

for every pair of points (x_1,t_1), and $(x_2,t_2) \in K$.

For a fixed $(x_o,t_o) \in E_T$ and fixed numbers

$$\delta \in (0,1), \quad b > 1, \quad R, \omega > 0$$

construct the numbers

$$R_o = R, \quad R_n = \frac{R}{b^n}; \qquad \omega_o = \omega, \quad \omega_{n+1} = \delta\omega_n \quad \text{for } n = 0,1,2,\dots$$

and the cylinders

$$Q_n = K_{R_n}(x_o) \times \left(t_o - {\omega_n}^{1-m} R_n^2, t_o \right] \quad \text{for } n = 1,2,\dots.$$

The function u is Hölder continuous at $(x_o, t_o) \in E_T$ if there exist constants $\delta \in (0,1)$ and $b > 1$, independent of u and (x_o, t_o), such that

$$Q_{n+1} \subset Q_n \subset Q_o \subset E_T \quad \text{and} \quad \operatorname*{ess\,osc}_{Q_n} u \le \omega_n \tag{18.1}$$

for all $n = 0, 1, \dots$. Having fixed $(x_o, t_o) \in E_T$, assume it coincides with the origin of $\mathbb{R}^{N+1}$ and for $\rho > 0$ set

$$R_o = \rho \quad \text{and} \quad Q = K_\rho \times (-\rho^2, 0],$$

where ρ is so small that $Q \subset E_T$. Set also

$$\mu_o^+ = \operatorname*{ess\,sup}_Q u, \quad \mu_o^- = \operatorname*{ess\,inf}_Q u, \quad \omega_o = \mu_o^+ - \mu_o^- = \operatorname*{ess\,osc}_Q u.$$

Since u is locally bounded in E_T, without loss of generality we may assume that $\omega_o \le 1$ so that

$$Q_o = K_\rho \times \Big(- {\omega_o}^{1-m} \rho^2, 0 \Big] \subset Q \subset E_T, \quad \text{and} \quad \operatorname*{ess\,osc}_{Q_o} u \le \omega_o.$$

Thus (18.1) holds for $n = 0$. Methods analogous to those leading to the Harnack inequality, and indeed more general as to include the whole singular range $0 < m < 1$, permit one to determine numbers $b > 1$ and $\delta \in (0,1)$ depending only on the set of data $\{m, N, C_o, C_1\}$ and C, and independent of u and (x_o, t_o) for which (18.1) holds for all n. A complete proof of the Hölder continuity of *nonnegative* solutions to (5.1)–(5.2) of Chapter 3 in the whole range $0 < m < 1$ will be given in § B.6–B.13 of Appendix B. A consequence of the previous arguments is:

Proposition 18.2 *Let u be a locally bounded, local, weak solution to the singular equations (5.1)–(5.2) of Chapter 3 for $0 < m < 1$, in E_T. There exist constants $\bar\gamma > 1$ and $\epsilon_o \in (0,1)$, depending only on the data $\{m, N, C_o, C_1\}$ and C, such that for all $(x_o, t_o) \in E_T$, setting*

$$M = \operatorname*{ess\,sup}_{(x_o,t_o)+Q_R^-(1)} |u| \qquad \textit{for} \qquad (x_o, t_o) + Q_R^-(1) \subset E_T,$$

we have

$$\operatorname*{ess\,osc}_{(x_o,t_o)+Q_\rho^-(\theta_M)} u \le \bar\gamma M \Big(\frac{\rho}{R}\Big)^{\epsilon_o} \qquad \textit{where} \qquad \theta_M = M^{1-m}$$

for all $0 < \rho \le R$ and all cylinders

$$(x_o, t_o) + Q_\rho^-(\theta_M) \subset (x_o, t_o) + Q_R^-(1) \subset E_T.$$

18.3 A Liouville-Type Result

Proposition 18.3 *Let u be a nonnegative, continuous, local, weak solution to the homogeneous ($C = 0$), singular equations (5.1)–(5.2) of Chapter 3, in $\mathbb{R}^N \times \mathbb{R}$, for m in the singular, supercritical range (16.1). Then u is constant.*

Proof For $T \in \mathbb{R}$ denote by S_T the half-space

$$S_T = \mathbb{R}^N \times (-\infty, T] \quad \text{for some } \ T \in \mathbb{R}$$

By (16.6)

$$0 \le \inf_{S_T} u \le u \le \sup_{S_T} u < \infty.$$

Therefore u is quantitatively bounded above and below in S_T. The proof is now concluded as in Proposition 16.3 of Chapter 5 by making use of the form of the Hölder continuity as expressed by Proposition 18.2. ■

Remark 18.1 The proposition is false for $m = 1$ as discussed in § 5 of Chapter 5.

Remark 18.2 If u is a solution to the prototype porous medium equation (5.3) of Chapter 3, then $u - k$, for a nonzero constant k is not a solution to the same equation. For this reason the proof of Proposition 18.3 does not permit to replace the assumption of u being nonnegative, with u having a one-sided bound.

Remark 18.3 The proposition is false for $0 < m \le \frac{(N-2)_+}{N}$, as evidenced by the counterexamples in § 16.3. The functions in (16.11) for m subcritical, and in (16.12) for m critical, are nonnegative, not identically zero, weak solutions to the prototype porous medium equation, in $\mathbb{R}^N \times \mathbb{R}$. However, these solutions are all unbounded in every S_T. This raises the issue as to whether solutions to the singular ($0 < m < 1$), homogeneous ($C = 0$) equations (5.1)–(5.2) of Chapter 3, with two-sided bound in some half-space S_T are constant in S_T, for m in the whole singular range $0 < m < 1$.

Proposition 18.4 *Let u be a solution to the singular ($0 < m < 1$) homogeneous ($C = 0$) equations (5.1)–(5.2) of Chapter 3. If u is bounded above and below in some half-space S_T, then u is constant in S_T.*

Proof The proof is almost identical to that of Proposition 16.3 of Chapter 5. ■

19 Critical and Subcritical Singular Equations of the Porous Medium Type

Let u be a nonnegative, locally bounded, local, weak solution to (5.1)–(5.2) of Chapter 3 for m in the critical and subcritical range

$$0 < m \le m_* = \tfrac{(N-2)_+}{N} < 1. \tag{19.1}$$

By the examples and counterexamples of § 16.3 a Harnack estimate in any of the forms (16.6)–(16.9) fails to hold when m is in the critical and subcritical range (19.1). Nevertheless a different form of a Harnack estimate holds for m in such a range, with constants depending on the ratio of some integral averages of the solution u. Fix $(x_o, t_o) \in E_T$ and ρ such that $K_{4\rho}(x_o) \subset E$, and introduce the quantity

$$\theta = \Big[\varepsilon\Big(⨍_{K_\rho(x_o)} u^q(\cdot, t_o)dx\Big)^{\frac{1}{q}}\Big]^{1-m} \tag{19.2}$$

where $\varepsilon \in (0,1)$ is to be chosen, and $q \ge 1$ is arbitrary. If $\theta > 0$, assume that

$$(x_o, t_o) + Q^-_{8\rho}(\theta) = K_{8\rho}(x_o) \times (t_o - \theta(8\rho)^2, t_o] \subset E_T$$

and set

$$\sigma = \left[\frac{\Big(⨍_{K_\rho(x_o)} u^q(\cdot, t_o)dx\Big)^{\frac{1}{q}}}{\Big(⨍_{K_{4\rho}(x_o)} u^r(\cdot, t_o - \theta\rho^2)dx\Big)^{\frac{1}{r}}}\right]^{\frac{2r}{\lambda_r}} \tag{19.3}$$

where $r \ge 1$ is any number such that

$$\lambda_r = N(m-1) + 2r > 0. \tag{19.4}$$

Theorem 19.1 *Let u be a nonnegative, locally bounded, local, weak solution to the singular equations (5.1)–(5.2) of Chapter 3, for $0 < m < 1$ in E_T. Introduce θ as in (19.2) and assume that $\theta > 0$. There exist constants $\varepsilon \in (0,1)$, and $\gamma, \beta > 1$, depending only on the data $\{m, N, C_o, C_1\}$ and the parameters q, r, such that either*

$$C\rho > 1$$

or

$$\inf_{(x_o,t_o)+Q^-_\rho(\frac{1}{2}\theta)} u \ge \gamma^{-1}\sigma^\beta \sup_{(x_o,t_o)+Q^-_\rho(\theta)} u, \tag{19.5}$$

where σ is defined in (19.3), $q \ge 1$ and $r \ge 1$ satisfies (19.4). The constants $\varepsilon \to 0$, and $\gamma, \beta \to \infty$ as either $\lambda_r \to 0$ or $\lambda_r \to \infty$.

Remark 19.1 The estimate is vacuous if $\theta = 0$. This does occur for certain solutions to (5.1) of Chapter 3 for t_o larger than the extinction time ([59, 60]). An explicit example is in (16.11).

Remark 19.2 Inequality (19.5) is not a Harnack inequality per se, since σ depends on the solution itself. It would reduce to a Harnack inequality if $\sigma \ge \sigma_o > 0$ for some absolute constant σ_o depending only on the data. This, however, cannot occur since a Harnack inequality for solutions to (5.1)–(5.2) of Chapter 3 does not hold, as shown by the counterexamples of § 16.3.

Remark 19.3 Inequality (19.5) actually holds for nonnegative solutions to (5.1)–(5.2) of Chapter 3 for all $m > 0$, provided $r \geq 1$ satisfies (19.4). For supercritical m one has $\lambda = \lambda_1 > 0$, and (19.4) can be realized for $r = 1$. However, for $\lambda > 0$ the strong form of a Harnack estimate holds (Theorems 16.1 and 16.2). Therefore (19.5), while true for all $0 < m < 1$, is of significance only for the critical and subcritical values in (19.1). In this sense (19.5) can be regarded as a "weak" form of a Harnack estimate valid for all $0 < m < 1$.

20 On the Proof of Theorem 19.1

The first component of the proof is the expansion of positivity presented in Proposition 7.2 of Chapter 4, and valid for all $0 < m < 1$.

20.1 $L^r_{\rm loc}$–$L^\infty_{\rm loc}$ Harnack-Type Estimates for $r \geq 1$ Such That $\lambda_r > 0$

Theorem 20.1 *Let u be a nonnegative, locally bounded, local, weak solution to the singular, porous medium type equations (5.1)–(5.2) of Chapter 3, in E_T, for $0 < m < 1$, and let $r \geq 1$ satisfy (19.4). There exists a positive constant γ_r depending only on the data $\{m, N, C_o, C_1\}$, and r, such that either*

$$C\rho > 1$$

or

$$\sup_{K_\rho(y)\times[s,t]} u \leq \frac{\gamma_r}{(t-s)^{\frac{N}{\lambda_r}}}\Big(\int_{K_{2\rho}(y)} u^r(x, 2s-t)dx\Big)^{\frac{2}{\lambda_r}} + \gamma_r\Big(\frac{t-s}{\rho^2}\Big)^{\frac{1}{1-m}} \tag{20.1}$$

for all cylinders

$$K_{2\rho}(y) \times [s-(t-s), s+(t-s)] \subset E_T.$$

The constant $\gamma_r \to \infty$ if either $\lambda_r \to 0$ or $\lambda_r \to \infty$.

The theorem, which will be established in Appendix B, assumes that u is locally bounded, and turns such a *qualitative* information, into the *quantitative* estimate (20.1) in terms of the $L^r_{\rm loc}$ integrability of $u(\cdot, t)$. A discussion on the local boundedness of solutions to these singular equations is in § 21.5.

20.2 $L^r_{\rm loc}$ Estimates Backward in Time

Proposition 20.1 *Let u be a nonnegative, local, weak solution to the singular equations of the porous medium type (5.1)–(5.2) of Chapter 3, in E_T, for $0 < m < 1$, and assume that $u \in L^r_{\rm loc}(E_T)$ for some $r > 1$. There exists* *a positive constant* γ *depending only on the data* $\{m, N, C_o, C_1\}$ *and* r, *such that either*

$$C\rho > 1$$

or

$$\sup_{\tau\le s\le t}\int_{K_\rho(y)} u^r(x,s)dx \le \gamma\int_{K_{2\rho}(y)} u^r(x,\tau)dx + \gamma\Big[\frac{(t-\tau)^r}{\rho^{\lambda_r}}\Big]^{\frac{1}{1-m}}$$

for all cylinders

$$K_{2\rho}(y)\times[\tau,t]\subset E_T.$$

The proof of Theorem 20.1 and Proposition 20.1 will be given in Appendix B. Here we assume them and indicate how to establish Theorem 19.1.

20.3 Main Points of the Proof

With these two facts at hand, the proof now follows the same steps as the proof of Theorem 11.1. The first is in establishing a weaker form of (19.5).

Proposition 20.2 *Let* u *be a nonnegative, locally bounded, local, weak solution to the singular porous medium type equations (5.1)–(5.2) of Chapter 3, for* $0 < m < 1$. *Fix* $(x_o, t_o) \in E_T$, *let* $K_{4\rho}(x_o) \subset E$, *and let* θ *and* σ *be defined by (19.2)–(19.3) for some* $\varepsilon \in (0,1)$. *For every* $r \ge 1$ *satisfying (19.4) and every* $q \ge 1$, *there exist constants* $\varepsilon, \delta \in (0,1)$, *and a continuous, increasing function* $\sigma \to f(\sigma)$ *defined in* $\mathbb{R}^+$ *and vanishing at* $\sigma = 0$, *that can be quantitatively determined a priori only in terms of the data* $\{m, N, C_o, C_1\}$, q *and* r, *such that*

$$\inf_{K_{4\rho}(x_o)} u(\cdot,t) \ge f(\sigma) \sup_{(x_o,t_o)+Q_{2\rho}(\frac14\theta)} u,$$

for all

$$t\in(t_o-\tfrac14\theta\rho^2\,,\,t_o],$$

provided $(x_o,t_o)+Q_{8\rho}(\theta)\subset E_T$.

The proof of this proposition follows the same arguments as those in § 13, making use of Theorem 20.1 and Proposition 20.1. The arguments are measure-theoretical and not linked to a specific partial differential equation. The equation only enters in establishing Theorem 20.1 and Proposition 20.1, and in the expansion of positivity of Proposition 7.2 of Chapter 4.

Application of such an expansion of positivity shows that the function $f(\cdot)$ can be taken of the same form as given in (7.3) of Chapter 4, for constants depending only on the data, q and r.

The next step in the proof is in improving the dependence on σ so that $f(\sigma)$ can be replaced by σ^β for some $\beta > 0$ depending only on the data $\{m, N, C_o, C_1\}$. This is realized by making use of the Hölder continuity of u in the form stated in Proposition 18.2 (see also the proof in Appendix B). The technical arguments to this end are in § 15, which, being independent of any partial differential equation, apply to any function with the indicated properties.

21 Remarks and Bibliographical Notes

For nonnegative solutions to the prototype, homogeneous equation (1.3) of Chapter 3, intrinsic Harnack inequalities in the forward form (1.2)–(1.6), and the elliptic form (1.9), were established in a series of contributions ([39, 59, 60]), collected and reorganized in [41]. The parameter p was in the supercritical range (1.1) and only the right-hand sides of (1.6) and (1.7) were established. These proofs, one way or another, are based on the comparison principle, by comparing locally the solutions to the homogeneous prototype p-Laplacian equation with either the explicit Barenblatt solutions with pole at (x_o, t_o)

$$\Gamma_p(x,t;x_o,t_o) = \frac{1}{(t-t_o)^{\frac{N}{\lambda}}}\Big[1 + C_p\Big(\frac{|x-x_o|}{(t-t_o)^{\frac{1}{\lambda}}}\Big)^{\frac{p}{p-1}}\Big]^{\frac{p-1}{p-2}} \tag{21.1}$$

with

$$C_p = \Big(\frac{1}{\lambda}\Big)^{\frac{1}{p-1}}\frac{2-p}{p}, \qquad \lambda = N(p-2)+p. \tag{21.2}$$

as in [41], or with some suitably constructed subsolution as in [59].

The right-hand-side estimates of Theorems 1.1 and 1.2, for the full quasi-linear structure of the singular equations (1.1)–(1.2) of Chapter 3, and p in the supercritical range (1.1), were established in [51]. The left-hand-side estimates and thus the complete mean value form of these intrinsic Harnack inequalities were proved in [52].

The functions in § 1.3 to serve as counterexample to a Harnack estimate for critical and subcritical p were constructed by adapting similar procedures of [126, 77], for the porous medium equation.

The first proof of Theorem 2.1 for the homogeneous, singular, prototype p-Laplacian equation (1.3) of Chapter 3 is in [41]. The proof adapts to singular ($1 < p < 2$) equations with the full quasilinear structure of (1.1)–(1.2) of Chapter 3, and as such is established in [51]. Here it is reported in Appendix A for completeness and in view of its importance. The proof actually holds for all $1 < p < 2$, provided the local weak solutions u at hand are locally bounded. This is the case for p in the supercritical range (1.1) as established in [41] Chapter V. If p is subcritical, weak solutions might become unbounded and the boundedness of u must be postulated. Further remarks on boundedness and unboundedness of weak solutions are in § 21.1.

The technique in § 4 of locating the supremum of v through the numbers M_τ and N_τ follows an idea of Landis ([104]). Our approach, however, departs from [104], in that the sup is located at a single time level, as opposed to within an entire parabolic cylinder. This permits one to generate, in conjunction with Theorem 2.1, a Harnack estimate insensitive to time. The remainder of the proof in § 5–7 is taken from [51]. The techniques of § 8–9 leading to a mean value form of the Harnack inequality are taken from [52].

The Liouville-type Propositions 10.1 and 10.2 seem to be new in the literature, and are taken from [56].

21.1 About Theorem 11.1

For the prototype p-Laplacian equation (1.3) of Chapter 3, a result similar to Theorem 11.1 is in [22]. The arguments make extensive use of the comparison principle, by comparing the solution with either the solution to a suitable Cauchy problem or with the solutions to a boundary value problem in bounded cylindrical domains, with homogeneous data on its lateral boundary. For the latter a fine analysis of the solutions is in [60], and applied in [22].

Harnack estimates are unrelated to maximum and comparison principles and are more linked to the structural properties of elliptic and parabolic equations, as made precise by the pioneering work of Moser [119, 120, 121, 122] and Nash [123], and refined in [10, 141, 147, 148, 66].

However, independently of the maximum principle, the significance of the Harnack-type estimates (11.6) and (13.5) is not entirely understood. While valid for all $1 < p < 2$, it does not transition seamlessly from the subcritical range $1 < p < \frac{2N}{N+1}$, across the critical value $p = \frac{2N}{N+1}$, to the supercritical range $\frac{2N}{N+1} < p < 2$, where a strong form of the Harnack inequality holds (Theorem 1.1). Perhaps the most significant consequence is the Hölder continuity of the solutions established in § 14 and taken from [53]. Harnack-type estimates with coefficients depending on the solution itself do not, in general, imply the Hölder continuity of the solutions. It is the specific form of σ as introduced in (11.2)–(11.3), and its measure-theoretical interpretation, that permits one to establish that local solutions are indeed locally Hölder continuous (§ 14 and [53]).

The form of σ is not uniquely defined. For example, one could take $r > 1$ and $q \geq 1$ arbitrarily large, provided $\lambda_r > 0$. Also by taking $q = 1$ and using the Harnack-type inequality (2.2) in the $L^1_{\rm loc}$ topology (see also (A.1.2) of Appendix A), one could estimate

$$\fint_{K_\rho(x_o)} u(\cdot, t_o)dx \geq \gamma_o \fint_{K_{\frac{1}{2}\rho}(x_o)} u(\cdot, t_o - \theta\rho^p)dx$$

for a constant γ_o depending only on the data $\{p, N, C_o, C_1\}$, for the same quantity θ introduced in (11.2). Then σ could be given the form

$$\sigma = \left[\frac{\displaystyle\fint_{K_{\frac{1}{2}\rho}(x_o)} u(\cdot, t_o - \theta\rho^p)dx}{\displaystyle\Big(\fint_{K_{4\rho}(x_o)} u^r(\cdot, t_o - \theta\rho^p)dx\Big)^{\frac{1}{r}}} \right]^{\frac{rp}{\lambda_r}}.$$

Thus the ratio of the two integral averages making up σ is effected at the same time level $t_o - \theta\rho^p$. Recently Fornaro and Vespri [68] have found that for the singular equations (1.1)–(1.2) of Chapter 3, which in addition satisfy the comparison principle, the two integral averages making up σ can be taken over cubes of equal radius. This class of equations includes for example

$$u_t - \big(a_{ij}(x,t)|Du|^{p-2}u_{x_i}\big)_{x_j} = 0 \quad \text{weakly in } E_T,$$

where the coefficients a_{ij} are only locally bounded and measurable, and the matrix (a_{ij}) is locally elliptic in E_T. For this class of equations σ could be taken of the form

$$\sigma = \left[\frac{\displaystyle\fint_{K_\rho(x_o)} u(\cdot, t_o - \theta\rho^p)dx}{\Big(\displaystyle\fint_{K_\rho(x_o)} u^r(\cdot, t_o - \theta\rho^p)dx\Big)^{\frac{1}{r}}} \right]^{\frac{rp}{\lambda_r}}. \tag{21.3}$$

The interest in this form is that if $p > \frac{2N}{N+1}$, then r can be taken to be $r = 1$ and hence $\sigma = 1$. This in (11.6) of Theorem 11.1 would recover the strong form of the Harnack estimate of Theorem 1.1 for supercritical $p > \frac{2N}{N+1}$. The transition, however, is not seamless, since for $r = 1$ and $\lambda_r = 0$ the quantity σ in (21.3) is not defined, and in addition the constants in Theorem 11.1 deteriorate as $\lambda_r \to 0$.

21.2 On the Weak Harnack Inequality

The weak Harnack inequalities of Chapter 5, § 7–14 and 17 are not known to hold for singular equations, neither in the supercritical, nor in the critical and subcritical range of the parameters p and m.

It is not even clear what form such inequalities should have, and whether they transition seamlessly from the subcritical to the supercritical range, or not. For the prototype equations (1.3) and (5.3) of Chapter 3 partial results are in [22] and [21], respectively.

21.3 On the Boundedness of Weak Solutions

Theorem 12.1 has been stated in a unified way for all $p \in (1, 2]$. The theorem merely turns the *qualitative* information that u is locally bounded, into the *quantitative* estimate (12.2). Its proof, however, reveals two critical values of p:

$$1 \le p_{**} = \tfrac{2N}{N+2} < p \le p_* = \tfrac{2N}{N+1} < 2.$$

When p is larger than its *least* critical value p_{**}, that is, when

$$p > \max\big\{1\,,\, \tfrac{2N}{N+2}\big\}, \tag{21.4}$$

the integrability condition $u \in L^r_{\mathrm{loc}}$ with $\lambda_r > 0$ is a consequence of the notion of weak solution. Indeed by the parabolic embedding of Proposition 4.1 of the Preliminaries,

$$u \in L^m_{loc}(E_T) \quad \text{with} \quad m = \frac{N+2}{N}p \quad \text{and} \quad \lambda_m > 0. \tag{21.5}$$

For p in the range (21.4), the proof of Theorem 12.1 only uses the energy estimates of Proposition 2.1 of Chapter 3, and the indicated local integrability condition (see Appendix A). Thus for $p > p_{**}$, local, weak solutions to (1.1)–(1.2) of Chapter 3, in E_T, with no further requirement, are locally bounded in E_T, with a quantitative estimate provided by (12.1)–(12.2). In particular no local boundedness is required. This is the content of Theorem 3.1 of [41], Chapter V, § 3. When p is in the sub-sub-critical range

$$1 < p \le p_{**} = \tfrac{2N}{N+2}, \qquad \text{for} \qquad N \ge 3, \tag{21.6}$$

the local boundedness of a local, weak solution to these singular equations is not guaranteed by the mere notion of weak solution. The function in (1.10) is an explicit example of a weak, unbounded solution to the homogeneous, prototype p-Laplacian equation (1.3) of Chapter 3. This raises the issue of formulating sufficient conditions on solutions to these singular equations to be locally bounded.

The proof of Theorem 12.1 uses the energy estimates of Proposition 2.1 of Chapter 3, and the local integrability $u \in L^r_{\mathrm{loc}}(E_T)$, with $\lambda_r > 0$. When p is in the range (21.6) the latter is not a consequence of the notion of weak solution as in (21.5) and it must be postulated.

For p in such a sub-sub-critical range, the proof of the theorem also involves an interpolation procedure that requires that u be at least *qualitatively* locally bounded (see Appendix A). Therefore Theorem 12.1 could be regarded as a sufficient condition to boundedness, provided u can be locally approximated by locally bounded, weak solutions to equations similar to (1.1) of Chapter 3, with possibly smooth principal part $\mathbf{A}$ and lower order terms B, satisfying uniformly the structure conditions (1.2). If these approximating solutions are uniformly in L^r_{loc}, by Theorem 12.1 they are uniformly locally bounded. Then, by the results of [41] Chapter IV, they are uniformly Hölder continuous, and hence (12.2) is preserved in the limit. Thus, modulo the indicated local approximation, the local boundedness of u is insured by the integrability $u \in L^r_{\mathrm{loc}}(E_T)$ for some $r \ge 1$ satisfying (11.4). This is the content of Theorem 5.1 of [41], Chapter V, § 5-(i). The possibility of approximating these weak solutions by locally bounded ones, depends, in general, on the form of $\mathbf{A}$ and B, and is related to their local uniqueness or "uniqueness in the small" ([102], Chapter 1).

21.4 On the Two Critical Values $p_{**} = \frac{2N}{N+2} < \frac{2N}{N+1} = p_*$

The interplay between the numbers λ_r and p_{**} and p_* is not completely understood. The largest of them (p_*) discriminates between Harnack estimates and lack of them and existence and nonexistence of the Barenblatt p-potentials (21.1). The smallest of them (p_{**}) discriminates between boundedness and unboundedness of solutions.

$$\underbrace{\overbrace{1<p\le p_{**}}^{\text{no Harnack}}}_{\text{unbounded}} = \underbrace{\overbrace{\tfrac{2N}{N+2}<p\le \tfrac{2N}{N+1}}^{\text{no Harnack}} = \overbrace{p_*<p\le 2}^{\text{Harnack}}}_{\text{boundedness}}$$

A better understanding of these connections might come from a more refined notion of weak solution. Attempts in this direction are in [58].

21.5 Singular Equations of the Porous Medium Type

For nonnegative solutions to the prototype singular, porous medium equation (5.3), and m supercritical, the right-hand-side, intrinsic, Harnack inequality (16.6) was first proved in [59] by means of comparison principles. For general quasilinear singular equation Theorems 16.1 and 16.2 were established in [51].

The $L^1_{\rm loc}$–$L^\infty_{\rm loc}$ estimates of Theorem 17.1, while paralleling similar facts for the p-Laplacian type equations, in the full generality of quasilinear, singular parabolic equations, do not seem to appear in the literature. We have collected them in Appendix B.

The analyticity of solutions for supercritical m, and in the context of the prototype porous medium type equation was observed in [60], and in the context of fully quasilinear equations in [51].

The Liouville-type Propositions 18.3, while restricted to nonnegative solutions, and 18.4 seem to be new in the literature.

21.5.1 On the Hölder Continuity

Locally bounded, weak solutions to the porous medium type equations (5.1)–(5.2) of Chapter 3, in E_T, irrespective of their signum, and for all $m>0$, are locally Hölder continuous in E_T. For $m>1$ a formal proof is in [47]. For the singular case $0<m<1$ this fact follows from analogous proofs for singular p-Laplacian type equations ([41], Chapter IV, § 15, and [31, 59]). However, the literature does not have a formal, independent proof of this fact, dedicated to local weak solutions to singular, porous medium type equations, with full quasilinear structure. For completeness, in § B.6–B.13 of Appendix B, we have included a self-contained formal proof for nonnegative solutions to such singular equations.

21.5.2 On Theorem 19.1

An estimate similar to (19.5) is in [21] for nonnegative solutions to the prototype equation (5.3) of Chapter 3, by means of maximum and comparison principles. The arguments in [21] and [22] for the prototype p-Laplacian and porous medium equations are conceptually and technically similar. In § 21.1 we have already observed that maximum and comparison principles are unrelated to Harnack-type estimates.

The $L^r_{\rm loc}$–$L^\infty_{\rm loc}$ estimate of Theorem 20.1, in the context of the porous medium type equation, and with full quasilinear structure, appears to be new. Local weak solutions to these equations for m in the range (19.1) need not be locally bounded. An explicit counterexample is in (16.10). A sufficient condition for a local solution u to be locally bounded is that $u \in L^r_{\rm loc}(E_T)$, where $r \ge 1$ satisfies $\lambda_r > 0$. The number λ_r, defined in (19.4) is analogous to the number λ in (17.1). If $\lambda > 0$, that is, if $m > m_*$, then local solutions are locally bounded. If $\lambda \le 0$, that is, if $m \le m_*$, the mere notion of local weak solution u does not guarantee its local boundedness. The latter is restored if, in addition to the notion of solution, one requires that u be sufficiently integrable, as to guarantee $\lambda_r > 0$.

21.5.3 Remarks on the Local Analyticity of These Solutions

Let u be a nonnegative, locally bounded, local, weak solution to the homogeneous ($C = 0$), porous medium type equations (5.1)–(5.2) of Chapter 3, for m in the critical and/or subcritical range (19.1). Fix $(x_o, t_o) \in E_T$ and assume that $u(x_o, t_o) > 0$. If $\mathbf{A}$ is analytic in all its arguments, within its domain of definition, then u is analytic at (x_o, t_o) in the space variables, and at least Lipschitz continuous in time. The radius of convergence $r_{x_o}(\sigma)$ of the space-expansion of $u(\cdot, t_o)$ at x_o depends on σ and $r_{x_o}(\sigma) \to 0$ as $\sigma \to 0$. This can be made quantitative, exactly as in Proposition 18.1 with the constant γ depending on σ and such that $\gamma(\sigma) \to \infty$ as $\sigma \to 0$. Proposition 18.1 holds also for solutions to the following boundary value problem:

$$\begin{cases} u_t - \Delta u^m = 0 & \text{in } E_T, \\ u|_{\partial E} = 0, \\ u(\cdot, 0) = u_o \in L^{1+m}(E), \end{cases}$$

where E is a bounded domain in $\mathbb{R}^N$ with boundary of class C^2, and $\frac{(N-2)_+}{N+2} < m < 1$. Thus there seem to be at least two critical values of m, that discriminate between different behaviors of solutions,

$$0 \le m_{**} = \tfrac{(N-2)_+}{N+2} \le m_* = \tfrac{(N-2)_+}{N} < 1.$$

21.6 On the Two Critical Values $0 \le m_{**} \le m_* < 1$

The interplay between the numbers λ_r and m_{**} and m_* is not completely understood. The largest of them (m_*) discriminates between Harnack estimates and lack of them and existence and nonexistence of the Barenblatt m-potentials with pole at (x_o, t_o)

$$\Gamma_m(x, t; x_o, t_o) = \frac{1}{(t - t_o)^{\frac{N}{\lambda}}} \Big(1 + b(N, m) \frac{|x - x_o|^2}{(t - t_o)^{\frac{1}{\lambda}}}\Big)^{\frac{1}{m-1}} \tag{21.7}$$

with

$$b(N,m) = \frac{N(1-m)}{2Nm\lambda}, \qquad \lambda = N(m-1)+2. \tag{21.8}$$

Considerations analogous to those of § 21.3 imply that if $m > \frac{(N-2)_+}{N+2}$, then weak solutions are locally bounded, even though, if nonnegative, they might not satisfy the Harnack inequality. Indeed, due to the definition of solution we assumed in § 5 of Chapter 3, when $m_{**} < m < m_*$ the integrability condition $u \in L^r$ with $\lambda_r > 0$ is automatically satisfied by the parabolic embedding of Proposition 4.1 of the Preliminaries. For porous medium type equations, as for p-Laplacian equations, one can directly say that the smallest of them (m_{**}) discriminates between boundedness and unboundedness of solutions.

$$\underbrace{\overbrace{0 < m \le m_{**}}^{\text{no Harnack}}}_{\text{unbounded}} = \underbrace{\overbrace{\tfrac{(N-2)_+}{N+2} < m \le \tfrac{(N-2)_+}{N}}^{\text{no Harnack}} = \overbrace{m_* < m \le 1}^{\text{Harnack}}}_{\text{boundedness}}$$

7

Homogeneous Monotone Singular Equations

1 Monotone Structure Conditions

We restrict our attention to parabolic equations of the type of (1.1)–(1.5) of Chapter 3 which are singular ($1 < p < 2$), homogeneous ($C = 0$), and with structure conditions that insure existence and uniqueness of basic Dirichlet boundary value problems in cylindrical domains E_T for a bounded open set $E \subset \mathbb{R}^N$ with smooth boundary ∂E. Specifically, let E be a domain in $\mathbb{R}^N$ and consider quasilinear, singular parabolic partial differential equations of the form

$$u_t - \operatorname{div} \mathbf{A}(x,t,u,Du) = 0 \quad \text{weakly in } E_T \tag{1.1}$$

where the function

$$E_T \times \mathbb{R} \times \mathbb{R}^N \ni (x,t,z,\eta) \to \mathbf{A}(x,t,z,\eta)$$

is only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x,t,z,\eta)\cdot\eta \geq C_o|\eta|^p \\ |\mathbf{A}(x,t,z,\eta)| \leq C_1|\eta|^{p-1} \end{cases} \quad \text{a.e. in } E_T \times \mathbb{R} \times \mathbb{R}^N, \tag{1.2}$$

where $1 < p < 2$, and C_o and C_1 are given positive constants. The principal part $\mathbf{A}$ is required to be monotone in the variable η in the sense

$$(\mathbf{A}(x,t,z,\eta_1) - \mathbf{A}(x,t,z,\eta_2))\cdot(\eta_1 - \eta_2) \geq 0 \tag{1.3}$$

for all variables in the indicated domains and Lipschitz continuous in the variable z, that is,

$$|\mathbf{A}(x,t,z_1,\eta) - \mathbf{A}(x,t,z_2,\eta)| \leq \Lambda|z_1 - z_2|(1 + |\eta|^{p-1}) \tag{1.4}$$

for some given $\Lambda > 0$, and for the variables in the indicated domains. A prototype example is

$$u_t - \left(a_{ij}(x,t)|Du|^{p-2}u_{x_i}\right)_{x_j} = 0 \quad \text{weakly in } E_T \tag{1.5}$$

E. DiBenedetto et al., *Harnack's Inequality for Degenerate and Singular Parabolic Equations*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-1584-8_7,

where the matrix (a_{ij}) is only measurable and positive-definite in E_T.

The notion of local solution to (1.1)–(1.4) is the same as that to (1.1)–(1.5) of Chapter 3.

Let now E be a bounded open set in $\mathbb{R}^N$ with smooth boundary ∂E, fix $T > 0$, and consider the boundary value problem

$$\begin{aligned} & u \in C(0,T;L^2(E)) \cap L^p(0,T;W^{1,p}(E)) \\ & u_t - \operatorname{div} \mathbf{A}(x,t,u,Du) = 0 \quad \text{ weakly in } E_T \\ & u(\cdot,t)\,\big|_{\partial E} = g(\cdot,t) \in L^p\big(0,T;W^{1-\frac{1}{p}}(\partial E)\big) \\ & u(\cdot,0) = u_o \in L^2(E). \end{aligned} \tag{1.6}$$

With these specifications, the Dirichlet data $g(\cdot,t)$ on ∂E for a.e. $t \in (0,T)$ are taken in the sense of the traces of functions in $W^{1,p}(E)$ and the initial datum u_o is taken in the sense of continuous functions in t with values in $L^2(E)$.

The existence of solutions to (1.6) has been discussed under a number of different assumptions (see, for example, [108, 86]). For *continuous* boundary value problems, under suitable approximation conditions, the existence of solutions can be deduced from their regularity properties as in [131].

Proposition 1.1 *Let* $\mathbf{A}$ *satisfy the structure conditions (1.2)–(1.4). Then the boundary value problem (1.6) has at most one solution.*

Proof For $\varepsilon > 0$ let $H_\varepsilon(\cdot)$ be the approximation to the Heaviside function

$$H_\varepsilon(s) = \begin{cases} 1 & \text{for } s \ge \varepsilon; \\ \dfrac{s}{\varepsilon} & \text{for } 0 \le s < \varepsilon; \\ 0 & \text{for } s < 0. \end{cases} \tag{1.7}$$

If u and v are two weak solutions to (1.6), in their respective weak formulation take the test function $H_\varepsilon(u-v)$ and subtract the expressions so obtained to get

$$\begin{aligned} & \int_E \Big(\int_0^{(u-v)_+(x,t)} H_\varepsilon(s)ds \Big) dx \\ & \quad + \int_0^t \int_E H_\varepsilon'(u-v)\big[\mathbf{A}(x,\tau,u,Du) - \mathbf{A}(x,\tau,u,Dv)\big] \cdot (Du - Dv) dx\, d\tau \\ & = \int_0^t \int_E H_\varepsilon'(u-v)\big(\mathbf{A}(x,\tau,v,Dv) - \mathbf{A}(x,\tau,u,Dv)\big) \cdot (Du - Dv) dx\, d\tau \end{aligned}$$

for all $t \in (0,T)$. The second term on the left-hand side is discarded by the monotonicity (1.3) of $\mathbf{A}$. As $\varepsilon \to 0$, the first term tends to

$$\int_E \Big(\int_0^{(u-v)_+(x,t)} H_\varepsilon(s) ds \Big) dx \to \int_E (u-v)_+(x,t) dx,$$

for all $t \in (0,T)$. The term on the right-hand side is estimated by making use of the Lipschitz continuity (1.4), to yield

$$\begin{aligned}
&\Big| \int_0^t \int_E H_\varepsilon'(u-v)\big(\mathbf{A}(x,t,v,Dv) - \mathbf{A}(x,t,u,Dv)\big) \cdot (Du - Dv) dx\, d\tau \Big| \\
&\le \frac{\Lambda}{\varepsilon} \int_0^t \int_{E\cap[0<u-v<\varepsilon]} (u-v)(1+|Dv|^{p-1})|Du - Dv| dx\, d\tau \\
&\le \Lambda \int_0^t \int_{E\cap[0<u-v<\varepsilon]} (1+|Dv|^{p-1})|Du - Dv| dx\, d\tau \to 0 \ \text{ as } \ \varepsilon \to 0. \qquad \blacksquare
\end{aligned}$$

Corollary 1.1 (Weak Comparison Principle) *Let* $\mathbf{A}$ *satisfy the structure conditions (1.2)–(1.4). Let* u_i *for* $i = 1, 2$ *be weak solutions to (1.6) corresponding to initial and boundary data* $u_{o,i}$ *and* g_i *in the indicated functional classes. If*

$$u_{o,1} \le u_{o,2} \ \ \text{a.e. in } E \qquad \text{and} \qquad g_1 \le g_2 \ \ \text{a.e. in } \partial E \times (0,T),$$

then $u_1 \le u_2$ *a.e. in* E_T*.*

In what follows we take p in the supercritical range

$$\frac{2N}{N+1} < p < 2. \tag{1.8}$$

The more stringent structure conditions (1.2)–(1.4) on $\mathbf{A}$ afford a wider spectrum of techniques, including the comparison principle, yielding some improvements to the theory. Below we discuss one such improvement.

1.1 A Less Constrained Harnack Inequality

The intrinsic Harnack inequalities of Theorems 1.1 and 1.2 of Chapter 6 require that the intrinsic cylinders $(x_o, t_o) + Q_\rho^\pm(\theta)$ defined in (1.2) of Chapter 6 are well within the domain of definition of the solution. This is quantified by the requirements (1.3)–(1.5) of the same chapter. As indicated in that context, the various constants of the Harnack inequalities and the structure of the proof are independent of $\mathcal{M}$ and the indicated requirements have only the role of stipulating that the arguments are carried within the domain of definition of u. Now in some applications, the Harnack estimate might need to be applied repeatedly at a point (x_o, t_o), and in a sequence of neighboring points. Applications of this kind include subpotential lower bounds similar to those of § 6 of Chapter 5. For this kind of results to hold, the “interior” requirements (1.3)–(1.5) of Chapter 6 would have to be verified at a given point (x_o, t_o) and at a sequence of points (x_j, t_j) other than (x_o, t_o). This makes the procedure cumbersome and unclear to implement. For this reason it would be desirable to have a Harnack estimate for intrinsic cylinders

$$u(x_o,t_o)>0, \quad (x_o,t_o)+Q^{\pm}_{8\rho}(\theta)\subset E_T, \quad \theta=[u(x_o,t_o)]^{2-p} \tag{1.9}$$

with no further reference, albeit qualitative, to the quantity $\mathcal{M}$ in (1.3) of Chapter 6. The homogeneous structures of (1.1)–(1.4) permit one to establish such a form of a Harnack estimate.

2 The Intrinsic Harnack Inequality

In all statements below let u be a continuous, nonnegative, local, weak solution to the singular equations (1.1)–(1.4) in E_T, for p in the supercritical range (1.8). Moreover for a fixed $(x_o,t_o)\in E_T$ and $\rho>0$ construct intrinsic cylinders of the form of (1.9).

Theorem 2.1 (The Intrinsic, Mean Value, Harnack Inequality) *There exist constants $\epsilon\in(0,1)$ and $\gamma>1$ depending only on the data $\{p,N,C_o,C_1\}$, such that*

$$\begin{aligned}\gamma^{-1}\sup_{K_\rho(x_o)} u(\cdot,t_o-\epsilon\, u(x_o,t_o)^{2-p}\rho^p) &\le u(x_o,t_o)\\ &\le \gamma \inf_{K_\rho(x_o)} u(\cdot,t_o+\epsilon\, u(x_o,t_o)^{2-p}\rho^p).\end{aligned} \tag{2.1}$$

The constants $\epsilon,\gamma^{-1}\to 0$ as $p\to\frac{2N}{N+1}$, but they are stable as $p\to 2$.

Theorem 2.2 (Time-Insensitive, Intrinsic, Mean Value, Harnack Inequalities) *There exist constants $\bar\epsilon\in(0,1)$ and $\bar\gamma>1$, depending only on the data $\{p,N,C_o,C_1\}$, such that*

$$\bar\gamma^{-1}\sup_{K_\rho(x_o)} u(\cdot,\sigma)\le u(x_o,t_o)\le\bar\gamma\inf_{K_\rho(x_o)} u(\cdot,\tau) \tag{2.2}$$

for any pair of time levels σ,τ in the range

$$t_o-\bar\epsilon\, u(x_o,t_o)^{2-p}\rho^p\le\sigma,\tau\le t_o+\bar\epsilon\, u(x_o,t_o)^{2-p}\rho^p. \tag{2.3}$$

The constants $\bar\epsilon,\bar\gamma^{-1}\to 0$ either as $p\to\frac{2N}{N+1}$ or as $p\to 2$.

Comments on these theorems can be formulated as in § 1.1–1.3 of Chapter 6. In all cases the key inequality to establish is the right-hand-side estimate in (2.3), as the remaining ones follow from this by stability estimates and geometrical arguments. We state and prove independently such a right-hand-side estimate, to stress its independence of the requirements (1.3)–(1.5) of Chapter 6.

2.1 The Right-Hand-Side Harnack Estimate of Theorem 2.2

Proposition 2.1 *There exist constants* $\bar{\epsilon} \in (0,1)$ *and* $\bar{\gamma} > 1$*, depending only on the data* $\{p, N, C_o, C_1\}$*, such that*

$$u(x_o, t_o) \le \bar{\gamma} \inf_{K_\rho(x_o)} u(\cdot, t) \tag{2.4}$$

for all times t in the range (2.3). The constants $\bar{\epsilon}, \bar{\gamma}^{-1} \to 0$ *either as* $p \to \frac{2N}{N+1}$ *or as* $p \to 2$*.*

The new element of the proof is a novel form of the expansion of positivity based on the comparison principle, afforded by the new structures (1.2)–(1.4). The proof continues to use the $L^1_{\rm loc}$–$L^\infty_{\rm loc}$ Harnack-type estimate given in Theorem 2.1 of Chapter 6, valid for

$$\lambda = N(p-2) + p > 0, \qquad \text{i.e., for} \qquad \tfrac{2N}{N+1} < p < 2. \tag{2.5}$$

3 Proof of Proposition 2.1

Introduce the change of variables and unknown function

$$z \to \frac{x - x_o}{\rho}, \qquad \tau \to \frac{t - t_o}{u(x_o, t_o)^{2-p} \rho^p}, \qquad v = \frac{u}{u(x_o, t_o)} \tag{3.1}$$

which maps

$$[(x_o, t_o) + Q^+_{8\rho}(\theta)] \cup [(x_o, t_o) + Q^-_{8\rho}(\theta)] \ \text{ into } \ Q_8 = K_8 \times (-8^p, 8^p].$$

Relabeling by x, t the new coordinates, v is a weak solution to

$$v_t - \operatorname{div} \bar{\mathbf{A}}(x, t, v, Dv) = 0 \quad \text{ in } \ Q_8, \tag{3.2}$$

where the transformed function $\bar{\mathbf{A}}(x, t, v, Dv)$ satisfies the same structure conditions (1.2)–(1.4) with the same constants C_o and C_1. Establishing the proposition consists in finding positive constants $\bar{\epsilon}$ and $\bar{\gamma}$, depending only on the data $\{p, N, C_o, C_1\}$, such that

$$v(\cdot, t) \ge \bar{\gamma}^{-1} \quad \text{ in } \ K_1 \ \text{ for all } \ t \in [-\bar{\epsilon}, \bar{\epsilon}]. \tag{3.3}$$

The change of variables (3.1) is identical to the one in (3.4) of Chapter 6. The latter, however, was effected within the cylinder $\mathcal{Q}_{\mathcal{M}}(x_o, t_o)$ introduced in (1.3)–(1.5) of the same chapter.

3.1 Largeness of v in Q_1

For $\tau \in (0,1)$ introduce the family of nested expanding cubes $\{K_\tau\}$ centered at the origin, and the increasing family of positive numbers

$$M_\tau = \sup_{K_\tau} v \qquad \text{and} \qquad N_\tau = (1-\tau)^{-\beta}$$

exactly as in (4.1) of Chapter 6, where $\beta > 0$ is a parameter to be chosen. Proceeding exactly as in § 4 of that chapter, and using the same notation, locate a point $\bar{x} \in K_{\tau_*}$ such that

$$\begin{aligned} &v(\bar{x},0) = (1-\tau_*)^{-\beta}, \qquad \sup_{K_{2r}(\bar{x})} v(\cdot,0) \le 4(1-\tau_*)^{-\beta} \\ &2r = \big(1-4^{-\frac{1}{\beta}}\big)(1-\tau_*), \quad \theta_* = (1-\tau_*)^{-\beta}. \end{aligned} \tag{3.4}$$

We stress that τ_* is only known qualitatively, and the parameter β is to be chosen. Proceeding as in § 5 of Chapter 6, the next step in the proof is to identify an intrinsic cylinder Q_{2r} centered at $(\bar{x},0)$ where the supremum of v is of the same order as $(1-\tau_*)^{-\beta}$. The height of such an intrinsic cylinder is $\theta_*(2r)^p$, with r and θ_* given by (3.4). Hence Q_{2r} would be included within the domain of definition of v if

$$\theta_*(2r)^p = (1-\tau_*)^{-\beta(2-p)}\big(1-4^{-\frac{1}{\beta}}\big)(1-\tau_*)^p \le 1.$$

This is realized by choosing

$$\beta = \frac{p}{2-p}. \tag{3.5}$$

In the context of the proof in § 4–6 of Chapter 6, for singular equations with general quasilinear structure, the parameter β was left free. Moreover, by construction

$$\theta_*(2r)^p = (1-\tau_*)^{-\beta(2-p)}\big(1-4^{-\frac{1}{\beta}}\big)(1-\tau_*)^p \le \Big(\frac{\mathcal{M}}{u(x_o,t_o)}\Big)^{2-p}$$

where $\mathcal{M}$ was introduced in (1.3) of Chapter 6. Hence, by virtue of the requirements (1.3)–(1.5) of that chapter, the inclusion of Q_{2r} within the domain of definition of v was guaranteed irrespective of the choice of β.

In what follows the parameter β is fixed as in (3.5). However, we will continue to denote it by β to trace its role, as opposed to the role of β in the context of the proof in § 4–6 of Chapter 6.

Lemma 3.1 *There exists a positive constant γ_1, depending only on the data $\{p, N, C_o, C_1\}$, and independent of β and ρ, such that*

$$\sup_{Q_r} v \le \gamma_1(1-\tau_*)^{-\beta}.$$

The constant $\gamma_1 \to \infty$ as $p \to 2$ and as $p \to \frac{2N}{N+1}$.

Proof Identical to Lemma 5.1 of Chapter 6. ∎

Lemma 3.2 *There exist numbers $\bar{\delta}$, $\bar{c}$, and α in $(0,1)$, depending only on the data $\{p, N, C_o, C_1\}$, and independent of β and ρ, such that*

$$|[v(\cdot,t) \geq \bar{c}(1-\tau_*)^{-\beta}]| > \alpha |K_r| \tag{3.6}$$

for all times

$$t \in [-\bar{\delta}\theta_* r^p \, , \, \bar{\delta}\theta_* r^p] \qquad \textit{where} \;\; \theta_* = (1-\tau_*)^{-\beta(2-p)} \tag{3.7}$$

with β fixed by (3.5) and r defined by (3.4). The constants $\bar{\delta}$, $\bar{c}$, and α tend to zero either as $p \to 2$ or as $p \to \frac{2N}{N+1}$.

Proof Identical to Lemma 5.2 of Chapter 6. ∎

3.2 Expanding the Positivity of v

The information provided by Lemma 3.2 is precisely the assumption required by the expansion of positivity of Proposition 5.1 of Chapter 4 for all times in (3.7). Apply then this expansion of positivity to v with

$$\rho = r \qquad \text{and} \qquad M = \bar{c}(1-\tau_*)^{-\beta}, \qquad \text{with } \; \beta = \frac{p}{2-p}$$

and for s ranging in the indicated interval. It gives

$$v(\cdot,t) > \eta\bar{c}(1-\tau_*)^{-\frac{p}{2-p}} \qquad \text{in } K_{2r}(\bar{x}) \tag{3.8}$$

and for all times

$$-\bar{\delta}\theta_* r^p + (1-\varepsilon)\delta M^{2-p} r^p < t < \bar{\delta}\theta_* r^p \tag{3.9}$$

for constants δ, $\bar{\delta}$, and ε in $(0,1)$ depending only on the data $\{p, N, C_o, C_1\}$, and the constant α, which itself is determined only in terms of the data.

The parameter $\bar{\delta}$ being fixed by Lemma 3.2, the number δ can be chosen even smaller, if needed to insure that the range of t in (3.9) includes negative values of time. For example, by choosing $\delta = \bar{\delta}$ and taking into account the definitions of M, θ_*, and r in (3.4), one computes

$$-\bar{\delta}\theta_* r^p + (1-\varepsilon)\delta M^{2-p} r^p = -\tfrac{1}{2^p}\Big(1 - \tfrac{1}{4^{\frac{2-p}{p}}}\Big)^p \bar{\delta}\big[1-(1-\varepsilon)\bar{c}^p\big] \overset{\text{def}}{=} -\delta_*.$$

A smaller δ in (3.9) would generate a smaller η in (3.8), whose choice we assume has been made. By these choices, and due to the definitions of θ_*, r, M and the value (3.5) of β, the size of the time interval in (3.9) does not depend on τ_*, and can be taken to be

$$-\delta_* \leq t \leq \delta_* \tag{3.10}$$

for a positive constant δ_* depending only on the data $\{p, N, C_o, C_1\}$. Notice that having determined δ_* as indicated, the lower bound (3.8) continues to hold with the same constants, for t in the range (3.10) for a further reduced δ_*.

In § 6 of Chapter 6, with β kept as a free parameter to be chosen, the lower bound (3.8) was iterated n times by repeated application of the expansion of positivity of Proposition 5.1 of Chapter 4. This would give

$$v \geq \bar{c}\eta^n(1-\tau_*)^{-\beta} \quad \text{over intrinsic cylinders of radius } \ 2^n(1-\tau_*).$$

Then n and β were chosen so that

$$2^n(1-\tau_*) = 4 \quad \text{ and } \quad \eta 2^\beta = 1$$

to yield the lower bound in (6.3)–(6.4) of Chapter 6.

Since the parameter β has already been chosen in (3.5) this procedure cannot be applied here. We will use instead a novel form of the expansion of positivity based on the comparison principle, which expands the positivity of u over an intrinsic cylinder of radius of the order of 1, at once in a single step.

3.2.1 Expanding the Positivity of v to Full Cube K_1 by the Comparison Principle

Consider the boundary value problem

$$\begin{aligned}
&w \in L^\infty(-\delta_*, 1; L^2(K_4(\bar{x}))) \cap L^p(-\delta_*, 1; W_o^{1,p}(K_4(\bar{x}))),\\
&w_t - \operatorname{div} \bar{\mathbf{A}}(x,t,w,Dw) = 0, \qquad \text{in } K_4(\bar{x}) \times (-\delta_*, 1],\\
&w \,\Big|_{\partial K_4(\bar{x})} = 0,\\
&w(x,-\delta_*) = \begin{cases} \eta\bar{c}(1-\tau_*)^{-N}, & x \in K_{2r}(\bar{x}),\\ 0, & x \in K_4(\bar{x}) - K_{2r}(\bar{x}).\end{cases}
\end{aligned} \tag{3.11}$$

Conditions (1.2)–(1.4) allow us to apply Theorems 1.1 and 1.2 of [108], Chapter 2: this insures that a solution to (3.11) indeed exists. Such a solution w is unique by Proposition 1.1.

The function v is larger than w on the parabolic boundary of $K_4 \times (-\delta_*, 1]$. Indeed

$$w(\cdot,t) \leq v(\cdot,t) \quad \text{ on } \ \partial K_4(\bar{x}) \ \text{ for all } \ t \in (-\delta_*, 1]$$

and for $t = -\delta_*$ on $K_4(\bar{x})$,

$$\begin{aligned}
v(\cdot,-\delta_*) - w(\cdot,-\delta_*) &\geq \eta\bar{c}(1-\tau_*)^{-\frac{p}{2-p}} - \eta\bar{c}(1-\tau_*)^{-N}\\
&\geq \eta\bar{c}(1-\tau_*)^{-N}\big[(1-\tau_*)^{-\frac{\lambda}{2-p}} - 1\big] > 0
\end{aligned}$$

since $\lambda > 0$. Therefore, by the comparison principle

$$v \geq w \quad \text{ in } \ K_4(\bar{x}) \times [-\delta_*, 1].$$

To prove Proposition 2.1, it suffices to determine constants $\bar{\gamma}$ and $\bar{\epsilon}$, depending only on the data, such that

$$w(x,t) \geq \bar{\gamma}^{-1} \quad \text{in } K_4 \text{ for all } t \in [-\bar{\epsilon}, \bar{\epsilon}].$$

Up to a translation assume $\bar{x} = 0$. By Proposition A.1.1 of Appendix A, for all $t \in [-\delta_*, \delta_*]$

$$\int_{K_1} w(x, -\delta_*)dx \leq \gamma \int_{K_2} w(x,t)dx + \gamma \delta_*^{\frac{1}{2-p}}$$

and by the definition of $w(\cdot, -\delta_*)$

$$\int_{K_1} w(x, -\delta_*)dx = \eta \bar{c} \nu_o \quad \text{where } \nu_o = \Big(1 - \frac{1}{4^{\frac{2-p}{p}}}\Big)^N.$$

The choice of δ_* can be made as to satisfy

$$\delta_*^{\frac{1}{2-p}} = \tfrac{1}{2}\gamma^{-1}\eta\bar{c}\nu_o. \tag{3.12}$$

This is obvious if the constant δ_* in (3.10) exceeds the right-hand side of (3.12), by taking a smaller δ_*. If the constant δ_* in (3.10) is less than the right-hand side of (3.12), one can take a smaller η as to satisfy (3.12), and for such an η (3.8) would continue to be in force. For such a choice of δ_*,

$$\tfrac{1}{2^{N+1}} \frac{\eta\bar{c}\nu_o}{\gamma} |K_2| \leq \int_{K_2} w(x,t)dx \quad \text{for all } t \in (-\tfrac{1}{2}\delta_*, \tfrac{1}{2}\delta_*).$$

Next, from Theorem 2.1 of Chapter 6, applied with $y = 0$, $s = 0$, $t = \frac{1}{2}\delta_*$, and $\rho = 2$, we estimate

$$\sup_{K_2 \times [-\frac{1}{2}\delta_*, \frac{1}{2}\delta_*]} w \leq \gamma \delta_*^{-\frac{N}{\lambda}} (\eta\bar{c}\nu_o)^{\frac{p}{\lambda}} + \gamma \delta_*^{\frac{1}{2-p}} = \gamma_* \eta \bar{c} \nu_o$$

where we have taken into account the choice (3.12) of δ_*, for γ_* depending only on the data. For all $t \in (-\frac{1}{2}\delta_*, \frac{1}{2}\delta_*)$ estimate

$$\begin{aligned}
\int_{K_2} w(x,t)dx \leq & \int_{K_2 \cap [w(\cdot,t) < \frac{1}{2^{N+2}} \frac{\eta\bar{c}\nu_o}{\gamma}]} w(x,t)dx \\
& + \int_{K_2 \cap [w(\cdot,t) \geq \frac{1}{2^{N+2}} \frac{\eta\bar{c}\nu_o}{\gamma}]} w(x,t)dx \\
\leq & \tfrac{1}{2^{N+2}} \frac{\eta\bar{c}\nu_o}{\gamma} |K_2| + \gamma_* \eta \bar{c}\nu_o \big|[w(\cdot,t) \geq \tfrac{1}{2^{N+2}} \frac{\eta\bar{c}\nu_o}{\gamma}] \cap K_2\big|.
\end{aligned}$$

Combining these inequalities yields

$$\big|[w(\cdot,t) \geq \tfrac{1}{2^{N+2}} \frac{\eta\bar{c}\nu_o}{\gamma}] \cap K_2\big| \geq \alpha |K_2|, \quad \text{where } \alpha = \frac{1}{4\gamma\gamma_*|K_2|}$$

for all $t \in [-\frac{1}{2}\delta_*, \frac{1}{2}\delta_*]$. By the expansion of positivity of Proposition 5.1 of Chapter 4

$$w(x,t) \geq \tfrac{1}{2^{N+2}} \frac{\eta^2 \bar{c} \nu_o}{\gamma} \quad \text{in } K_4(\bar{x}) \quad \text{for all } t \in [-\bar{\epsilon}, \bar{\epsilon}]$$

for a sufficiently small $\bar{\epsilon}$ depending only on the data $\{p, N, C_o, C_1\}$. ■

The proof of the left-hand-side inequalities in Theorems 2.1 and 2.2, as well as the stability of these estimates as $p \to 2$, is identical to that of Theorems 1.1 and 1.2 of Chapter 6, as presented in § 8–9 of that chapter.

4 Subpotential Lower Bounds

Return to the Barenblatt solutions to the prototype p-Laplacian equation, introduced in (3.3)–(3.4) of Chapter 4. While introduced in the context of degenerate equations ($p > 2$), they are well defined also for $p < 2$ provided

$$\lambda > 0 \quad \Longleftrightarrow \quad \tfrac{2N}{N+1} < p < 2, \tag{4.1}$$

that is, if p is in the singular supercritical range (4.1). For p in such a range we rewrite the Barenblatt solution with "pole" at (x_o, t_o) as

$$\Gamma_p(x,t;x_o,t_o) = \frac{1}{(t-t_o)^{\frac{N}{\lambda}}} \Big[1 + C_p \Big(\frac{|x-x_o|}{(t-t_o)^{\frac{1}{\lambda}}} \Big)^{\frac{p}{p-1}} \Big]^{\frac{p-1}{p-2}} \tag{4.2}$$

with

$$C_p = \Big(\frac{1}{\lambda}\Big)^{\frac{1}{p-1}} \frac{2-p}{p}, \qquad \lambda = N(p-2) + p. \tag{4.3}$$

As $p \to 2$ this converges pointwise to the heat potential with pole at (x_o, t_o):

$$\Gamma(x,t;x_o,t_o) = \frac{1}{(t-t_o)^{N/2}} e^{-\frac{|x-x_o|^2}{4(t-t_o)}}.$$

In this sense the Barenblatt solutions (4.2)–(4.3) are the p-potentials of the prototype p-Laplacian equation (1.3) of Chapter 3. In view of (4.1), these p-potentials cease to exist for p in the critical and subcritical range $1 < p \leq \frac{2N}{N+1}$. Thus the intrinsic Harnack inequalities of Theorems 2.1 and 2.2 cease to hold precisely when these p-potentials cease to exist.

These p-potentials drive, in the sense made precise by Proposition 4.1 below, the structural behavior of nonnegative solutions to the singular, homogeneous, quasilinear equations (1.1)–(1.4).

Proposition 4.1 *Let u be a nonnegative, local, weak solution to the singular equations (1.1)–(1.4) in E_T, with p in the supercritical range (4.1), and let ϵ and γ be the constants in the intrinsic Harnack inequality (2.1) of Theorem 2.1. For every $(x_o, t_o) \in E_T$ such that $u(x_o, t_o) > 0$, and all (x,t) in E_T with*

$$K_{16|x-x_o|}(x_o) \subset E \quad \textit{and} \quad 0 < t - t_o < \epsilon 8^{-p} t_o, \tag{4.4}$$

we have

$$\frac{u(x,t)}{u(x_o,t_o)} \ge \gamma_o \Big[1 + \gamma_1 u(x_o,t_o)^{\frac{2-p}{p-1}} \Big(\frac{|x-x_o|}{(t-t_o)^{\frac{1}{p}}} \Big)^{\frac{p}{p-1}} \Big]^{\frac{p-1}{p-2}}, \tag{4.5}$$

where

$$\gamma_o = \min\{1\,,\gamma^{-1}\} \quad \textit{and} \quad \gamma_1 = (\gamma^{\frac{2-p}{p-1}} - 1)\epsilon^{\frac{1}{p-1}}.$$

Proof Fix (x,t) as in (4.4) and consider the line segment ℓ_o through (x_o,t_o) and (x,t)

$$\ell_o \qquad y - x_o = \frac{x-x_o}{t-t_o}(s-t_o), \quad t_o < s \le t$$

and the p-paraboloid with bottom vertex at (x_o,t_o)

$$\mathcal{P}_o \qquad s - t_o = \epsilon\, u(x_o,t_o)^{2-p}|y-x_o|^p.$$

By (4.4) $\ell_o \subset E_T$. If ℓ_o does not intersect $\mathcal{P}_o$ at points other than (x_o,t_o), then (4.5) follows from the intrinsic, forward Harnack inequality (2.1). Let then ℓ_o intersect $\mathcal{P}_o$ at (x_1,t_1) with $|x_1-x_o| < |x-x_o|$, and

$$|x_1-x_o|^{p-1} = \frac{1}{\epsilon\, u(x_o,t_o)^{2-p}} \frac{t-t_o}{|x-x_o|}$$

$$t_1 - t_o = \epsilon\, u(x_o,t_o)^{p-2}|x_1-x_o|^p.$$

Iteration of this process gives a finite sequence of points (x_j,t_j), with $j = 1,\dots,n$, such that $t_o < t_1 < \cdots < t_n \le t$, and

$$\begin{aligned} |x_{j+1}-x_j|^{p-1} &= \frac{1}{\epsilon\, u(x_j,t_j)^{2-p}} \frac{t-t_o}{|x-x_o|} \\ t_{j+1}-t_j &= \epsilon\, u(x_j,t_j)^{2-p}|x_{j+1}-x_j|^p. \end{aligned} \tag{4.6}$$

and where (x_n,t_n) is the first point not overcoming (x,t). Using the intrinsic, forward Harnack inequality of Theorem 2.1,

$$u(x_j,t_j) \le \gamma u(x_{j+1},t_{j+1}), \qquad j = 0,\dots,n-1,$$

provided the cylinder

$$(x_j,t_j) + Q_{8\rho_j}(\theta_j) \quad \text{with } \theta_j = u(x_j,t_j)^{2-p} \text{ and } \rho_j = |x_{j+1}-x_j|$$

is contained in E_T. This is the case if

$$t_j - (8\rho_j)^p\theta_j \ge 0$$

which, in view of (4.6), is verified if

$$t_j - \frac{8^p}{\epsilon}(t_{j+1} - t_j) \geq 0.$$

This in turn holds true by virtue of the last of (4.4). A similar argument for the space variables guarantees the inclusion of

$$(x_j, t_j) + Q_{8\rho_j}(\theta_j) \subset E_T.$$

We infer that

$$u(x_j, t_j) \geq \gamma^{-j}\, u(x_o, t_o). \tag{4.7}$$

This and (4.6) imply

$$\begin{aligned}
|x - x_o| &\geq \sum_{j=0}^{n-1} |x_{j+1} - x_j| \\
&= \frac{1}{\epsilon^{\frac{1}{p-1}}}\Big(\frac{t - t_o}{|x - x_o|}\Big)^{\frac{1}{p-1}} \sum_{j=0}^{n-1}\Big(\frac{1}{u(x_j, t_j)}\Big)^{\frac{2-p}{p-1}} \\
&\geq \Big(\frac{t - t_o}{|x - x_o|}\Big)^{\frac{1}{p-1}}\Big(\frac{1}{\epsilon\, u(x_n, t_n)^{2-p}}\Big)^{\frac{1}{p-1}} \sum_{j=0}^{n-1}\big(\gamma^{\frac{p-2}{p-1}}\big)^{n-j} \\
&= \Big(\frac{t - t_o}{|x - x_o|}\Big)^{\frac{1}{p-1}}\Big(\frac{1}{\epsilon\, u(x_n, t_n)^{2-p}}\Big)^{\frac{1}{p-1}} \frac{q(1 - q^n)}{1 - q}
\end{aligned}$$

where $q = \gamma^{\frac{p-2}{p-1}}$. From this

$$q^n + \frac{1 - q}{q}\big(\epsilon\, u(x_n, t_n)^{2-p}\big)^{\frac{1}{p-1}}\Big(\frac{|x - x_o|}{(t - t_o)^{\frac{1}{p}}}\Big)^{\frac{p}{p-1}} \geq 1.$$

On the other hand, (4.7) written for $j = n$ gives

$$\Big(\frac{u(x_n, t_n)}{u(x_o, t_o)}\Big)^{\frac{2-p}{p-1}} \geq q^n.$$

Combining these estimates proves the proposition if $(x_n, t_n) = (x, t)$. This, however, can be assumed without loss of generality, by a possible further application of the Harnack inequality, and by possibly slightly modifying the constant γ if needed. ■

Remark 4.1 Notice that (4.5) gives the same decay in the space variables as the Barenblatt "fundamental solution" of (4.2).

Remark 4.2 An estimate of the decay of u in time can be derived by considering the sequence

$$t_o = s > 0, \quad t_1 = (1 + \sigma)s, \quad \dots, \quad t_k = (1 + \sigma)^k s = \tau,$$

with $0 < \sigma \leq \frac{1}{3}\frac{\epsilon}{8^p}$. A repeated application of the forward Harnack inequality (2.1) for sufficiently large s yields

$$u(0, \tau) \geq \Big(\frac{s}{\tau}\Big)^{\ln\gamma/\ln(1+\sigma)} u(0, s).$$

Remark 4.3 The proof of (4.5) hinges on applying the Harnack inequality (2.1) of Theorem 2.1 in intrinsic cylinders $(x_j, t_i) + Q_{8\rho_j}(\theta_j)$. The only requirement for the Harnack estimate to hold is that these cylinders be contained within the domain of definition of the solution u. Part of the proof is to verify such inclusion at each jth step.

The Harnack inequality (1.6) of Theorem 1.1 of Chapter 6 holds for nonnegative solutions to singular equations ($\frac{2N}{N+1} < p < 2$) with full quasilinear structure such as (1.1)–(1.2) of Chapter 3. However, for such general structures, Theorem 1.1 of Chapter 6 requires the further inclusion (1.3)–(1.5). As remarked in that context, this requirement is only qualitative as all constants and all arguments are independent of the number $\mathcal{M}$. If such qualitative assumptions could be removed from Theorem 1.1 of Chapter 6, then the sub-potential lower bounds of Proposition 4.1 could be extended to nonnegative weak solutions to supercritical singular equations with full quasilinear structure. Such an extension would be of significance as these general quasilinear equations do not satisfy any form of the comparison principle and thus their solutions are not comparable to the Barenblatt potentials (4.2)–(4.3). This is the case for the degenerate case $p > 2$ as presented in § 6 of Chapter 5.

5 Monotone Structures for Singular Porous Medium Type Equations

Consider parabolic equations of the type of (5.1)–(5.6) of Chapter 3 which are singular ($0 < m < 1$), homogeneous ($C = 0$), and with structure conditions that insure existence and uniqueness of basic Dirichlet boundary value problems in cylindrical domains E_T for a bounded open set $E \subset \mathbb{R}^N$ with smooth boundary ∂E. Specifically, let E be a domain in $\mathbb{R}^N$ and consider quasilinear, singular parabolic partial differential equations of the form

$$u_t - \operatorname{div} \mathbf{A}(x, t, u, D|u|^{m-1}u) = 0 \quad \text{weakly in } E_T \tag{5.1}$$

where the vector-valued function

$$E_T \times \mathbb{R} \times \mathbb{R}^N \ni (x, t, z, \eta) \to \mathbf{A}(x, t, z, \eta)$$

is only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x, t, z, \eta) \cdot \eta \geq C_o|\eta|^2 \\ |\mathbf{A}(x, t, z, \eta)| \leq C_1|\eta| \end{cases} \quad \text{a.e. in } E_T \times \mathbb{R} \times \mathbb{R}^N, \tag{5.2}$$

where C_o and C_1 are given positive constants. The principal part $\mathbf{A}$ is required to be monotone in the variable η in the sense

$$(\mathbf{A}(x, t, z, \eta_1) - \mathbf{A}(x, t, z, \eta_2)) \cdot (\eta_1 - \eta_2) \geq 0 \tag{5.3}$$

for all variables in the indicated domains and Lipschitz continuous in the variable $|z|^{m-1}z$, that is,

$$|\mathbf{A}(x,t,z_1,\eta)-\mathbf{A}(x,t,z_2,\eta)|\le \Lambda\Big||z_1|^{m-1}z_1-|z_2|^{m-1}z_2\Big|(1+|\eta|) \tag{5.4}$$

for some given $\Lambda>0$, and for the variables in the indicated domains. A prototype example is

$$u_t-\big(m\,a_{ij}(x,t)|u|^{m-1}u_{x_i}\big)_{x_j}=0 \quad \text{weakly in } E_T \tag{5.5}$$

where the matrix (a_{ij}) is only measurable and positive-definite in E_T. The notion of local solution to (5.1)–(5.4) is the same as that to (5.1)–(5.6) of Chapter 3.

Here we require a stronger notion of solution; precisely, weak sub(super)-solutions are also required to be in the class

$$u\in W^{1,1}_{\rm loc}\big(0,T;L^1_{\rm loc}(E)\big). \tag{5.6}$$

Let now E be a bounded open set in $\mathbb{R}^N$ with smooth boundary ∂E, fix $T>0$, and consider the boundary value problem

$$\begin{aligned}
&u\in C(0,T;L^{m+1}(E))\cap W^{1,1}(0,T;L^1(E))\\
&|u|^m\in L^2(0,T;W^{1,2}(E))\\
&u_t-\operatorname{div}\mathbf{A}(x,t,u,D|u|^{m-1}u)=0 \quad \text{weakly in } E_T\\
&|u|^{m-1}u(\cdot,t)\,\big|_{\partial E}=g(\cdot,t)\in L^2\big(0,T;W^{\frac12}(\partial E)\big)\\
&u(\cdot,0)=u_o\in L^{m+1}(E).
\end{aligned} \tag{5.7}$$

With these specifications, the Dirichlet data $g(\cdot,t)$ on ∂E for a.e. $t\in(0,T)$, are taken in the sense of the traces of functions in $W^{1,2}(E)$ and the initial datum u_o is taken in the sense of continuous functions in t with values in $L^{m+1}(E)$.

The existence of solutions to (5.6)–(5.7) has been discussed under a number of different assumptions (see, for example, [108, 86, 133]). For *continuous* boundary value problems, under suitable approximation conditions, the existence of solutions can be deduced from their regularity properties as in [131].

Proposition 5.1 *Let* $\mathbf{A}$ *satisfy the structure conditions (5.2)–(5.4). Then the boundary value problem (5.7) has at most one solution.*

Proof If u and v are two weak solutions to (5.7), in their respective weak formulation take the test function

$$H_\varepsilon(\xi) \quad \text{with} \quad \xi=|u|^{m-1}u-|v|^{m-1}v$$

where $H_\varepsilon(\cdot)$ is the approximation to the Heaviside function introduced in (1.7), and subtract the expressions so obtained to get

$$
\begin{aligned}
&\int_0^t \int_E (u-v)_\tau H_\varepsilon(\xi) dx\, d\tau \\
&+ \int_0^t \int_E H'_\varepsilon(\xi)\big[\mathbf{A}(x,\tau,u,D|u|^{m-1}u) - \mathbf{A}(x,\tau,u,D|v|^{m-1}v)\big] \cdot D\xi\, dx\, d\tau \\
&= \int_0^t \int_E H'_\varepsilon(\xi)\big(\mathbf{A}(x,\tau,v,D|v|^{m-1}v) - \mathbf{A}(x,\tau,u,D|v|^{m-1}v)\big) \cdot D\xi\, dx\, d\tau
\end{aligned}
$$

for all $t \in (0,T)$. The second term on the left-hand side is discarded by the monotonicity (5.3) of $\mathbf{A}$. As $\varepsilon \to 0$, the first term tends to

$$
\int_0^t \int_E (u-v)_{+,\tau} dx\, d\tau = \int_E (u-v)_+(x,t) dx,
$$

for all $t \in (0,T)$. The term on the right-hand side is estimated by making use of the Lipschitz continuity (5.4), and is majorized by

$$
\begin{aligned}
&\frac{\Lambda}{\varepsilon} \int_0^t \int_{E\cap[0<\xi<\varepsilon]} \xi(1+|D|v|^{m-1}v|)|D\xi| dx\, d\tau \\
&\le \Lambda \int_0^t \int_{E\cap[0<\xi<\varepsilon]} (1+|D|v|^{m-1}v|)|D\xi| dx\, d\tau \to 0 \;\text{ as }\; \varepsilon \to 0. \qquad \blacksquare
\end{aligned}
$$

Corollary 5.1 (Weak Comparison Principle) *Let* $\mathbf{A}$ *satisfy the structure conditions (5.2)–(5.4). Let* u_i *for* $i = 1, 2$ *be weak solutions to (5.6)–(5.7) corresponding to initial and boundary data* $u_{o,i}$ *and* g_i *in the indicated functional classes. If*

$$
u_{o,1} \le u_{o,2} \;\text{ a.e. in }\; E \quad \text{and} \quad g_1 \le g_2 \;\text{ a.e. in }\; \partial E \times (0,T),
$$

then $u_1 \le u_2$ *a.e. in* E_T*.*

In what follows we take m in the supercritical range

$$
\frac{(N-2)_+}{N} < m < 1. \tag{5.8}
$$

The more stringent structure conditions (5.2)–(5.4) on $\mathbf{A}$ afford a wider spectrum of techniques, and permit one to improve the theory in several directions. Here we mention one such improvement.

5.1 A Less Constrained Harnack Inequality

The intrinsic Harnack inequalities of Theorems 16.1 and 16.2 require that the intrinsic cylinders $(x_o, t_o) + Q_\rho^\pm(\theta)$ defined in (16.2) of Chapter 6 are well within the domain of definition of the solution. This is quantified by the requirements (16.3)–(16.5) of the same chapter. As indicated in that context, the various constants of the Harnack inequalities and the structure of the proof are independent of $\mathcal{M}$. However, this would make cumbersome applying the Harnack inequality to a sequence of points (x_j, t_j) and radii ρ_j, for each of which the requirements (16.3)–(16.5) of Chapter 6 would have to be verified. Applications of this kind include subpotential lower bounds similar to those of § 16.3 of Chapter 5. The structure conditions (5.2)–(5.4) permit one to establish a Harnack estimate for intrinsic cylinders

$$u(x_o, t_o) > 0, \quad (x_o, t_o) + Q^{\pm}_{8\rho}(\theta) \subset E_T, \quad \theta = [u(x_o, t_o)]^{1-m} \tag{5.9}$$

with no further reference, albeit qualitative, to the quantity $\mathcal{M}$ in (16.3) of Chapter 6.

6 The Intrinsic Harnack Inequality

In all statements below let u be a continuous, nonnegative, local, weak solution to the singular equations (5.1)–(5.4) in E_T, for m in the supercritical range (5.8). Moreover for a fixed $(x_o, t_o) \in E_T$ and $\rho > 0$ construct intrinsic cylinders of the form of (5.9).

Theorem 6.1 (The Intrinsic, Mean Value, Harnack Inequality) *There exist constants $\epsilon \in (0, 1)$ and $\gamma > 1$ depending only on the data $\{m, N, C_o, C_1\}$, such that*

$$\begin{aligned}\gamma^{-1} \sup_{K_\rho(x_o)} u\big(\cdot, t_o - \epsilon\, u(x_o, t_o)^{1-m}\rho^2\big) &\le u(x_o, t_o) \\ &\le \gamma \inf_{K_\rho(x_o)} u\big(\cdot, t_o + \epsilon\, u(x_o, t_o)^{1-m}\rho^2\big).\end{aligned} \tag{6.1}$$

The constants $\epsilon, \gamma^{-1} \to 0$ as $m \to \frac{(N-2)_+}{N}$, but they are stable as $m \to 1$.

Theorem 6.2 (Time-Insensitive, Intrinsic, Mean Value, Harnack Inequalities) *There exist constants $\bar{\epsilon} \in (0, 1)$ and $\bar{\gamma} > 1$, depending only on the data $\{m, N, C_o, C_1\}$, such that*

$$\bar{\gamma}^{-1} \sup_{K_\rho(x_o)} u(\cdot, \sigma) \le u(x_o, t_o) \le \bar{\gamma} \inf_{K_\rho(x_o)} u(\cdot, \tau) \tag{6.2}$$

for any pair of time levels σ, τ in the range

$$t_o - \bar{\epsilon}\, u(x_o, t_o)^{1-m}\rho^2 \le \sigma, \tau \le t_o + \bar{\epsilon}\, u(x_o, t_o)^{1-m}\rho^2. \tag{6.3}$$

The constants $\bar{\epsilon}$ and $\bar{\gamma}^{-1}$ tend to zero as either $m \to 1$ or $m \to \frac{(N-2)_+}{N}$.

Comments on these theorems can be formulated as in § 16.1–16.3 of Chapter 6. In all cases the key inequality to establish is the right-hand-side estimate in (6.3), as the remaining ones follow from this by stability estimates and geometrical arguments. We state independently such a right-hand-side estimate, to stress its independence of the requirements (16.3)–(16.5) of Chapter 6.

6.1 The Right-Hand-Side Harnack Estimate of Theorem 2.2

Proposition 6.1 *There exist constants $\bar{\epsilon} \in (0,1)$ and $\bar{\gamma} > 1$, depending only on the data $\{m, N, C_o, C_1\}$, such that*

$$u(x_o, t_o) \le \bar{\gamma} \inf_{K_\rho(x_o)} u(\cdot, t) \tag{6.4}$$

for all times t in the range (6.3). The constants $\bar{\epsilon}, \bar{\gamma}^{-1} \to 0$ either as $m \to \frac{(N-2)_+}{N}$ or as $m \to 1$.

The new element of the proof is a novel form of the expansion of positivity based on the comparison principle, afforded by the new structure (5.2)–(5.4). The proof continues to use the $L^1_{\rm loc}$–$L^\infty_{\rm loc}$ Harnack-type estimate given in Theorem 17.1 of Chapter 6, valid for

$$\lambda = N(m-1) + 2 > 0, \qquad \text{i.e., for} \qquad \tfrac{(N-2)_+}{N} < m < 1. \tag{6.5}$$

The remaining arguments are essentially identical to those of § 3 with the obvious modifications. In particular, conditions (5.2)–(5.4) insure that the analog of (3.11) in this context has indeed a solution (see, for example, [133, 86]). Such a solution w is unique by Proposition 5.1.

6.2 Subpotential Lower Bounds

Return to the Barenblatt solution to the prototype porous medium equation, introduced in (15.3) of Chapter 5. While introduced in the context of degenerate equations ($m > 1$), such a solution is well defined also for $m < 1$, provided (6.5) holds. For m in such a range we rewrite the Barenblatt solution with "pole" at (x_o, t_o) as

$$\Gamma_m(x, t; x_o, t_o) = \frac{1}{(t - t_o)^{\frac{N}{\lambda}}} \Big(1 + b(N, m) \frac{|x - x_o|^2}{(t - t_o)^{\frac{2}{\lambda}}} \Big)^{\frac{1}{m-1}} \tag{6.6}$$

with

$$b(N, m) = \frac{N(1-m)}{2Nm\lambda}, \qquad \lambda = N(m-1) + 2. \tag{6.7}$$

As $m \to 1$ this converges pointwise to the heat potential with pole at (x_o, t_o):

$$\Gamma(x, t; x_o, t_o) = \frac{1}{(t - t_o)^{N/2}} e^{-\frac{|x - x_o|^2}{4(t - t_o)}}.$$

In this sense the Barenblatt solutions (6.6)–(6.7) are the m-potentials of the prototype porous medium equation (5.3) of Chapter 3. In view of (6.5), these m-potentials cease to exist for m in the critical and subcritical range $0 < m \le \frac{(N-2)_+}{N}$. Thus the intrinsic Harnack inequalities of Theorems 6.1 and 6.2 cease to hold precisely when these m-potentials cease to exist. These m-potentials drive, in the sense made precise by Proposition 6.2 below, the structural behavior of nonnegative solutions to the singular, homogeneous, quasilinear equations (5.1)–(5.4).

Proposition 6.2 *Let u be a nonnegative, local, weak solution to (5.1)–(5.4) with m in the supercritical range (6.5), and let ϵ and γ be the constants in the intrinsic Harnack inequality (6.1) of Theorem 6.1. For every $(x_o, t_o) \in E_T$ such that $u(x_o, t_o) > 0$ and all (x, t) in E_T with*

$$K_{16|x-x_o|}(x_o) \subset E \qquad \textit{and} \qquad 0 < t - t_o < \frac{\epsilon}{8^2} t_o$$

we have

$$\frac{u(x,t)}{u(x_o,t_o)} \geq \gamma_o \Big(1 + \gamma_1 \frac{u(x_o,t_o)^{1-m}|x-x_o|^2}{t-t_o}\Big)^{\frac{1}{m-1}},$$

where

$$\gamma_o = \min\{1\,,\,\gamma^{-1}\} \qquad \textit{and} \qquad \gamma_1 = (\gamma^{1-m} - 1)\epsilon.$$

The proof is almost identical to that of Proposition 4.1. Remarks analogous to Remarks 4.1–4.3 apply to these porous medium type equations. ■

7 Remarks and Bibliographical Notes

The idea of using the comparison principle to establish Harnack estimates for solutions to degenerate and/or singular equations appears first in [30] in the context of equations of the porous medium type. The proofs of Propositions 1.1 and 2.1 are adapted from [30]. The subpotential lower estimates of § 4 are taken from [51]. Further discussion on sub(super)-solutions to homogeneous, monotone, parabolic p-Laplacian equations is in [92].

8

Appendix A

A.1 An $L^1_{\rm loc}$-Form of the Harnack Inequality for All $p \in (1,2)$

Proposition A.1.1 *Let u be a nonnegative, local, weak solution to the singular equations (1.1)–(1.2) of Chapter 3, for $1 < p < 2$, in E_T. There exists a positive constant γ depending only on the data $\{p, N, C_o, C_1\}$, such that for all cylinders $K_{2\rho}(y) \times [s,t] \subset E_T$, either*

$$C\rho > \min\{1\,,\,\epsilon\} \qquad where \qquad \epsilon = \Big(\frac{t-s}{\rho^p}\Big)^{\frac{1}{2-p}} \tag{A.1.1}$$

or

$$\sup_{s<\tau<t} \int_{K_\rho(y)} u(x,\tau)dx \le \gamma \inf_{s<\tau<t} \int_{K_{2\rho}(y)} u(x,\tau)dx + \gamma\Big(\frac{t-s}{\rho^\lambda}\Big)^{\frac{1}{2-p}} \tag{A.1.2}$$

where

$$\lambda = N(p-2) + p.$$

The constant $\gamma = \gamma(p) \to \infty$ either as $p \to 2$ or as $p \to 1$.

For $\lambda > 0$, the parameter p is in the singular, supercritical range (1.1) of Chapter 6, and if $\lambda \le 0$, p is in the singular, critical and subcritical range (11.1) of Chapter 6. However, the Harnack-type estimate (A.1.2), in the topology of $L^1_{\rm loc}$, holds true for all $1 < p < 2$ and accordingly, λ could be of either sign.

A.1.1 Auxiliary Lemmas

Lemma A.1.1 *Let u be a nonnegative, local, weak supersolution to the singular equations (1.1)–(1.2) of Chapter 3, for $1 < p < 2$, in E_T. There exists a positive constant γ depending only on the data $\{p, N, C_o, C_1\}$, such that for all cylinders $K_\rho(y) \times [s,t] \subset E_T$, and all $\sigma \in (0,1)$, either (A.1.1) holds, or*

E. DiBenedetto et al., *Harnack's Inequality for Degenerate and Singular Parabolic Equations*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-1584-8_8,

$$\int_s^t \int_{K_{\sigma\rho}(y)} \tau^{\frac{1}{p}} (u+\epsilon)^{-\frac{2}{p}} |Du|^p \zeta^p dx\, d\tau \le \frac{\gamma\rho}{(1-\sigma)^p} \left(\frac{t-s}{\rho^\lambda}\right)^{\frac{1}{p}} \left(\mathcal{S}+\epsilon\rho^N\right)^{2\frac{p-1}{p}},$$

where ϵ is defined by (A.1.1), and

$$\mathcal{S} = \sup_{s<\tau<t} \int_{K_\rho(y)} u(x,\tau)dx.$$

The constant $\gamma(p) \to \infty$ as either $p \to 1, 2$.

Proof Assume $(y,s) = (0,0)$, fix $\sigma \in (0,1)$, and let $x \to \zeta(x)$ be a nonnegative, piecewise smooth cutoff function in K_ρ that vanishes outside K_ρ, equals one on $K_{\sigma\rho}$, and such that

$$|D\zeta| \le \frac{1}{(1-\sigma)\rho}.$$

In the weak formulation (1.5) of Chapter 3 take the test function

$$\varphi = -t^{\frac{1}{p}} (u+\epsilon)^{1-\frac{2}{p}} \zeta^p \qquad \text{for some } \epsilon > 0,$$

modulo a Steklov averaging process. This gives

$$\begin{aligned}
\frac{2-p}{p} C_o \int_0^t \int_{K_\rho} & \tau^{\frac{1}{p}} (u+\epsilon)^{-\frac{2}{p}} |Du|^p \zeta^p dx\, d\tau \\
&\le \frac{p}{2(p-1)} t^{\frac{1}{p}} \int_{K_\rho} (u+\epsilon)^{2\frac{p-1}{p}} (x,t) \zeta^p dx \\
&+ pC_1 \int_0^t \int_{K_\rho} \tau^{\frac{1}{p}} (u+\epsilon)^{1-\frac{2}{p}} |Du|^{p-1} \zeta^{p-1} |D\zeta| dx\, d\tau \\
&+ \frac{2-p}{p} C^p \int_0^t \int_{K_\rho} \tau^{\frac{1}{p}} (u+\epsilon)^{-\frac{2}{p}} \zeta^p dx\, d\tau \\
&+ C^p \int_0^t \int_{K_\rho} \tau^{\frac{1}{p}} (u+\epsilon)^{1-\frac{2}{p}} \zeta^p dx\, d\tau \\
&+ pC^{p-1} \int_0^t \int_{K_\rho} \tau^{\frac{1}{p}} (u+\epsilon)^{1-\frac{2}{p}} \zeta^{p-1} |D\zeta| dx\, d\tau \\
&+ C \int_0^t \int_{K_\rho} \tau^{\frac{1}{p}} (u+\epsilon)^{1-\frac{2}{p}} |Du|^{p-1} \zeta^p dx\, d\tau.
\end{aligned}$$

From this, by repeated application of Young's inequality

$$
\begin{aligned}
\int_0^t \int_{K_\rho} \tau^{\frac{1}{p}} (u+\epsilon)^{-\frac{2}{p}} |Du|^p \zeta^p dx\, d\tau \le \gamma t^{\frac{1}{p}} \int_{K_\rho} (u+\epsilon)^{2\frac{p-1}{p}} (x,t) \zeta^p dx \\
+ \gamma \int_0^t \int_{K_\rho} \tau^{\frac{1}{p}} (u+\epsilon)^{p-\frac{2}{p}} \left(|D\zeta|^p + C^p \zeta^p\right) dx\, d\tau \\
+ \gamma C^p \int_0^t \int_{K_\rho} \tau^{\frac{1}{p}} (u+\epsilon)^{-\frac{2}{p}} \zeta^p dx\, d\tau
\end{aligned}
$$

where $\gamma = \gamma(\text{data})$ tends to ∞ either as $p \to 2$ or as $p \to 1$. By Hölder's inequality

$$
\begin{aligned}
&\gamma t^{\frac{1}{p}} \int_{K_\rho} (u+\epsilon)^{2\frac{p-1}{p}} (x,t) \zeta^p dx \\
&\qquad \le \gamma t^{\frac{1}{p}} \rho^{\frac{N(2-p)}{p}} \left(\sup_{0\le\tau\le t} \int_{K_\rho} u(x,\tau) dx + \epsilon (2\rho)^N \right)^{2\frac{p-1}{p}} \\
&\qquad \le \gamma \rho \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{p}} \left(\mathcal{S} + \epsilon \rho^N \right)^{2\frac{p-1}{p}}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\gamma \int_0^t \int_{K_\rho} \tau^{\frac{1}{p}} (u+\epsilon)^{p-\frac{2}{p}} (|D\zeta|^p + C^p \zeta^p) dx\, d\tau \\
&\qquad \le \gamma \frac{1 + C^p \rho^p}{(1-\sigma)^p \rho^p} \int_0^t \int_{K_\rho} \tau^{\frac{1}{p}} (u+\epsilon)^{p-2} (u+\epsilon)^{2\frac{p-1}{p}} dx\, d\tau \\
&\qquad \le \gamma \frac{1 + C^p \rho^p}{(1-\sigma)^p} \left(\frac{t}{\rho^p} \right) \epsilon^{p-2} t^{\frac{1}{p}} \sup_{0\le\tau\le t} \int_{K_\rho} (u+\epsilon)^{2\frac{p-1}{p}} dx \\
&\qquad \le \gamma \rho \frac{1 + C^p \rho^p}{(1-\sigma)^p} \left(\frac{t}{\rho^p} \right) \epsilon^{p-2} \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{p}} \left(\mathcal{S} + \epsilon \rho^N \right)^{2\frac{p-1}{p}}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\gamma C^p \int_0^t \int_{K_\rho} \tau^{\frac{1}{p}} (u+\epsilon)^{-\frac{2}{p}} \zeta^p dx\, d\tau \\
&\qquad \le \gamma \rho \left(\frac{C\rho}{\epsilon} \right)^p \left(\frac{t}{\rho^p} \right) \epsilon^{p-2} \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{p}} \left(\mathcal{S} + \epsilon \rho^N \right)^{2\frac{p-1}{p}}.
\end{aligned}
$$

Combining these estimates,

$$
\begin{aligned}
&\int_0^t \int_{K_\rho} \tau^{\frac{1}{p}} (u+\epsilon)^{-\frac{2}{p}} |Du|^p \zeta^p dx\, d\tau \\
&\le \frac{\gamma\rho}{(1-\sigma)^p} \left\{ 1 + \left[1 + (C\rho)^p + \frac{(C\rho)^p}{\epsilon^p} \right] \left(\frac{t}{\rho^p} \right) \epsilon^{p-2} \right\} \left(\frac{t}{\rho^\lambda} \right)^{\frac{1}{p}} \left(\mathcal{S} + \epsilon \rho^N \right)^{2\frac{p-1}{p}}.
\end{aligned}
$$

To prove Lemma A.1.1, choose ϵ as in (A.1.1) and stipulate that C violates the first of (A.1.1). ■

Lemma A.1.2 *Let u be a nonnegative, local, weak supersolution to the singular equations (1.1)–(1.2) of Chapter 3, for $1 < p < 2$, in E_T. There exists a positive constant γ depending only on the data, such that for all cylinders $K_\rho(y) \times [s,t] \subset E_T$, and all $\sigma \in (0,1)$, either (A.1.1) holds, or*

$$\frac{1}{\rho}\int_s^t \int_{K_{\sigma\rho}(y)} |Du|^{p-1} dx\, d\tau \le \delta\mathcal{S} + \frac{\gamma(p)}{[\delta^2(1-\sigma)^p]^{\frac{p-1}{2-p}}}\left(\frac{t-s}{\rho^\lambda}\right)^{\frac{1}{2-p}}$$

for all $\delta \in (0,1)$. The constant $\gamma(p) \to \infty$ as either $p \to 1, 2$.

Proof Continue to assume that $(y,s) = (0,0)$ and that C violates (A.1.1). By Hölder's and Young's inequalities,

$$\begin{aligned}
\int_0^t \int_{K_{\sigma\rho}} |Du|^{p-1} dx\, d\tau &= \int_0^t \int_{K_{\sigma\rho}} \left[\tau^{\frac{1}{p}\frac{p-1}{p}} (u+\epsilon)^{-\frac{2}{p}\frac{p-1}{p}} |Du|^{p-1}\right] \\
&\quad \times \left[\tau^{-\frac{1}{p}\frac{p-1}{p}} (u+\epsilon)^{\frac{2}{p}\frac{p-1}{p}}\right] dx\, d\tau \\
&\le \left(\int_0^t \int_{K_{\sigma\rho}} \tau^{\frac{1}{p}} (u+\epsilon)^{-\frac{2}{p}} |Du|^p dx\, d\tau\right)^{\frac{p-1}{p}} \\
&\quad \times \left(\int_0^t \int_{K_{\sigma\rho}} \tau^{\frac{1}{p}-1} (u+\epsilon)^{2\frac{p-1}{p}} dx\, d\tau\right)^{\frac{1}{p}} \\
&\le \frac{\gamma\rho}{(1-\sigma)^{p-1}} \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{p}} \left(\mathcal{S} + \epsilon\rho^N\right)^{2\frac{p-1}{p}} \\
&\le \delta\rho\mathcal{S} + \frac{\gamma\rho}{\delta^{\frac{2(p-1)}{2-p}} (1-\sigma)^{\frac{p(p-1)}{2-p}}} \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}}.
\end{aligned}$$

■

A.1.2 Proof of Proposition A.1.1

Assume $(y,s) = (0,0)$ and for $n = 0,1,2\ldots$ set

$$\rho_n = \sum_{j=1}^n \frac{1}{2^j}\rho, \quad K_n = K_{\rho_n}; \qquad \tilde\rho_n = \frac{\rho_n + \rho_{n+1}}{2}, \quad \tilde K_n = K_{\tilde\rho_n}$$

and let $x \to \zeta_n(x)$ be a nonnegative, piecewise smooth cutoff function in $\tilde K_n$ that equals one on K_n, and such that $|D\zeta_n| \le 2^{n+2}/\rho$. In the weak formulation of (1.1)–(1.2) of Chapter 3 take ζ_n as a test function, to obtain

$$\begin{aligned}
\int_{\tilde K_n} u(x,\tau_1)\zeta_n dx &\le \int_{\tilde K_n} u(x,\tau_2)\zeta_n dx \\
&\quad + \frac{2^{n+2}}{\rho}[C_1 + (C\rho)] \left|\int_{\tau_1}^{\tau_2} \int_{\tilde K_n} |Du|^{p-1} dx\, d\tau\right| \\
&\quad + 2^{n+2+N}\left(\frac{C\rho}{\epsilon}\right)^{p-1} [1 + (C\rho)] \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}}
\end{aligned}$$

for any two time levels τ_1 and τ_2 in $[0,t]$, where ϵ is defined in (A.1.1). Therefore, if C violates the first of (A.1.1),

$$\int_{K_n} u(x,\tau_1)dx \le \int_{K_{2\rho}} u(x,\tau_2)dx + \frac{\gamma 2^n}{\rho}\int_0^t \int_{\tilde{K}_n} |Du|^{p-1}dx\,d\tau + 2^{n+2}\left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}}.$$

As time level τ_2 take one for which

$$\int_{K_{2\rho}} u(x,\tau_2)dx = \inf_{0\le\tau\le t}\int_{K_{2\rho}} u(x,\tau)dx \overset{\text{def}}{=} \mathcal{I}.$$

Also set

$$\mathcal{S}_n = \sup_{0\le\tau\le t}\int_{K_n} u(x,\tau)dx.$$

Since $\tau_1 \in [0,t]$ is arbitrary, the previous inequality yields

$$\mathcal{S}_n \le \mathcal{I} + \frac{\gamma 2^n}{\rho}\int_0^t \int_{\tilde{K}_n} |Du|^{p-1}dx\,d\tau + \gamma 2^n\left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}}.$$

The term involving $|Du|$ is estimated above by applying Lemma A.1.2 over the pair of cubes

$$\tilde{K}_n \subset K_{n+1} \quad \text{for which} \quad (1-\sigma) = 2^{-(n+2)},$$

and for

$$\delta = \gamma^{-1}2^{-n-2}\varepsilon_o,$$

where $\varepsilon_o \in (0,1)$ is to be chosen. For these choices

$$\frac{2^{n+2}}{\rho}\int_0^t \int_{\tilde{K}_n} |Du|^{p-1}dx\,d\tau \le \varepsilon_o\mathcal{S}_{n+1} + \gamma(p,\varepsilon_o)b^n\left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}},$$

where $b = 2^{p^2}$. Combining these remarks gives the recursive inequalities

$$\mathcal{S}_n \le \varepsilon_o\mathcal{S}_{n+1} + \gamma(\text{data},\varepsilon_o)b^n\left[\mathcal{I} + \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}}\right].$$

From this, by iteration

$$\mathcal{S}_o \le \epsilon_o^n\mathcal{S}_n + \gamma(\text{data},\varepsilon_o)\left[\mathcal{I} + \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}}\right]\sum_{i=1}^{n-1}(\varepsilon_o b)^i.$$

Choose ϵ_o so that the last term is majorized by a convergent series, and let $n \to \infty$. ∎

A.1.3 An Estimate for Supersolutions

Both the supporting Lemmas A.1.1 and A.1.2 are valid for supersolutions, as stated. However, the proof of Proposition A.1.1 requires the full notion of solution, since the pair (τ_1, τ_2) is arbitrary, and in particular nonordered.

A parallel but weaker statement holds for supersolutions.

Proposition A.1.2 *Let u be a nonnegative, local, weak supersolution to the singular equations (1.1)–(1.2) of Chapter 3, for $1 < p < 2$, in E_T. There exists a positive constant γ depending only on the data $\{p, N, C_o, C_1\}$, such that for all cylinders $K_{2\rho}(y) \times [s,t] \subset E_T$, either (A.1.1) holds, or*

$$\sup_{s<\tau<t} \int_{K_\rho(y)} u(x,\tau)dx \le \gamma \int_{K_{2\rho}(y)} u(x,t)dx + \gamma \Big(\frac{t-s}{\rho^\lambda}\Big)^{\frac{1}{2-p}} \tag{A.1.3}$$

where

$$\lambda = N(p-2) + p.$$

The constant $\gamma = \gamma(p) \to \infty$ either as $p \to 2$ or as $p \to 1$.

Proof A standard adaptation of the previous argument. ■

A.2 L^r_{loc}–L^∞_{loc} Estimates

Proposition A.2.1 *Let u be a locally bounded, local, weak sub(super)-solution to the singular equations (1.1)–(1.2) of Chapter 3 for $1 < p < 2$, in E_T, and let $r \ge 1$ such that*

$$\lambda_r = N(p-2) + rp > 0.$$

There exists a positive constant γ depending only on the data $\{p, N, C_o, C_1\}$, such that for all cylinders $K_\rho(y) \times [2s-t, t] \subset E_T$, either (A.1.1) holds, or

$$\sup_{K_{\frac{1}{2}\rho}(y)\times[s,t]} u_\pm \le \gamma \Big(\frac{\rho^p}{t-s}\Big)^{\frac{N}{\lambda_r}} \Big(\frac{1}{\rho^N(t-s)} \int_{2s-t}^t \int_{K_\rho(y)} u_\pm^r dx\, d\tau\Big)^{\frac{p}{\lambda_r}} + \Big(\frac{t-s}{\rho^p}\Big)^{\frac{1}{2-p}}.$$

Proof The proof will be given for nonnegative weak subsolutions, the proof for the remaining case being identical. Assume $(y,s) = (0,0)$ and for fixed $\sigma \in (0,1)$ and $n = 0, 1, 2, \dots$ set

$$\rho_n = \sigma\rho + \frac{1-\sigma}{2^n}\rho, \qquad t_n = -\sigma t - \frac{1-\sigma}{2^n} t,$$
$$K_n = K_{\rho_n}, \qquad Q_n = K_n \times (t_n, t).$$

This is a family of nested and shrinking cylinders with common "vertex" at $(0,t)$, and by construction

$$Q_o = K_\rho \times (-t, t) \quad \text{and} \quad Q_\infty = K_{\sigma\rho} \times (-\sigma t, t).$$

Having assumed that u is locally bounded in E_T, set

$$M = \operatorname*{ess\,sup}_{Q_o} \max\{u, 0\}, \qquad M_\sigma = \operatorname*{ess\,sup}_{Q_\infty} \max\{u, 0\}.$$

We first find a relationship between M and M_σ. Denote by ζ a nonnegative, piecewise smooth cutoff function in Q_n that equals one on Q_{n+1}, and has the form $\zeta(x,t) = \zeta_1(x)\zeta_2(t)$, where

$$\zeta_1 = \begin{cases} 1 & \text{in } K_{n+1} \\ 0 & \text{in } \mathbb{R}^N - K_n \end{cases} \qquad |D\zeta_1| \le \frac{2^{n+1}}{(1-\sigma)\rho}$$

$$\zeta_2 = \begin{cases} 0 & \text{for } t \le t_n \\ 1 & \text{for } t \ge t_{n+1} \end{cases} \quad 0 \le \zeta_{2,t} \le \frac{2^{n+1}}{(1-\sigma)t};$$

introduce the increasing sequence of levels $k_n = k - 2^{-n}k$, where $k > 0$ is to be chosen, and in the weak formulation (5.1) of Chapter 3, take the test functions $(u - k_{n+1})_+\zeta^p$. The energy estimates (2.3) of Chapter 3 yield

$$\begin{aligned} \sup_{t_n \le \tau \le t} \int_{K_n} [(u-k_{n+1})_+\zeta]^2(x,\tau)dx + \iint_{Q_n} |D[(u-k_{n+1})_+\zeta]|^p dx\, d\tau \\ \le \frac{\gamma 2^{np}}{(1-\sigma)^p\rho^p}\left[1 + (C\rho)^p\right] \iint_{Q_n} (u-k_{n+1})_+^p dx\, d\tau \qquad \text{(A.2.1)} \\ + \frac{\gamma 2^n}{(1-\sigma)t} \iint_{Q_n} (u-k_{n+1})_+^2 dx\, d\tau + \gamma C^p \iint_{Q_n} \chi_{[(u-k_{n+1})_+>0]} dx\, d\tau. \end{aligned}$$

A.2.1 Proof of Proposition A.2.1 for p in the Range $\max\{1, \frac{2N}{N+2}\} < p < 2$

This amounts to taking $\lambda_r > 0$ with $r \in [1,2]$. Estimate

$$\iint_{Q_n} (u-k_{n+1})_+^p \, dx\, d\tau \le \gamma \frac{2^{(2-p)n}}{k^{2-p}} \iint_{Q_n} (u-k_n)_+^2 \, dx\, d\tau$$

$$\iint_{Q_n} \chi_{[u>k_{n+1}]} dx\, d\tau \le \gamma \frac{2^{2n}}{k^2} \iint_{Q_n} (u-k_n)_+^2 \, dx\, d\tau.$$

Then the energy estimates (A.2.1) yield

$$\begin{aligned} \sup_{t_n < \tau \le t} \int_{K_n} [(u-k_{n+1})_+\zeta]^2(x,\tau)dx + \iint_{Q_n} |D[(u-k_{n+1})_+\zeta]|^p dx\, d\tau \\ \le \frac{\gamma 2^{2n}}{(1-\sigma)^p}\left(\frac{1+(C\rho)^p}{\rho^p k^{2-p}} + \frac{1}{t} + \frac{C^p}{k^2}\right) \iint_{Q_n} (u-k_n)_+^2 dx\, d\tau. \end{aligned}$$

If condition (A.1.1) is violated, this implies further

$$\sup_{t_n<\tau\le t}\int_{K_n}[(u-k_{n+1})_+\zeta]^2(x,\tau)dx+\iint_{Q_n}|D[(u-k_{n+1})_+\zeta]|^p dx\,d\tau$$
$$\le\frac{\gamma 2^{2n}}{(1-\sigma)^p t}\Big[\Big(\frac{t}{\rho^p}\Big)k^{p-2}+1+\Big(\frac{t}{\rho^p}\Big)^{\frac{2}{2-p}}\frac{1}{k^2}\Big]\iint_{Q_n}(u-k_n)_+^2 dx\,d\tau.$$

The last term in [...] is estimated by stipulating to take

$$k\ge\Big(\frac{t}{\rho^p}\Big)^{\frac{1}{2-p}}. \tag{A.2.2}$$

This gives the inequalities

$$\sup_{t_n<\tau\le t}\int_{K_n}[(u-k_{n+1})_+\zeta]^2(x,\tau)dx+\iint_{Q_n}|D[(u-k_{n+1})_+\zeta]|^p dx\,d\tau$$
$$\le\frac{\gamma 2^{2n}}{(1-\sigma)^p t}\iint_{Q_n}(u-k_n)_+^2 dx\,d\tau.$$

By Hölder's inequality and the embedding Proposition 4.1 of the Preliminaries

$$\begin{aligned}\iint_{Q_{n+1}}(u-k_{n+1})_+^2 dx\,d\tau &\le\Big(\iint_{Q_n}[(u-k_{n+1})_+\zeta]^{p\frac{N+2}{N}}dx\,d\tau\Big)^{\frac{2N}{p(N+2)}}\\&\quad\times\Big(\iint_{Q_n}\chi_{[(u-k_{n+1})_+>0]}dx\,d\tau\Big)^{1-\frac{2N}{p(N+2)}}\\&\le\gamma\Big(\iint_{Q_n}|D[(u-k_{n+1})_+\zeta]|^p dx\,d\tau\Big)^{\frac{2N}{p(N+2)}}\\&\quad\times\Big(\sup_{t_n\le\tau\le t}\int_{K_n}[(u-k_{n+1})_+\zeta]^2(x,\tau)dx\Big)^{\frac{2}{N+2}}\\&\quad\times\Big(\iint_{Q_n}\chi_{[(u-k_{n+1})_+>0]}dx\,d\tau\Big)^{1-\frac{2N}{p(N+2)}}\\&\le\gamma\Big(\frac{2^{2n}}{(1-\sigma)^p t}\Big)^{\frac{2}{p}\frac{N+p}{N+2}}\Big(\iint_{Q_n}(u-k_n)_+^2 dx\,d\tau\Big)^{\frac{2N}{p(N+2)}+\frac{2}{N+2}}\\&\quad\times\Big(\iint_{Q_n}\chi_{[(u-k_{n+1})_+>0]}dx\,d\tau\Big)^{1-\frac{2N}{p(N+2)}}.\end{aligned}$$

Estimate

$$\iint_{Q_n}\chi_{[(u-k_{n+1})_+>0]}dx\,d\tau\le\frac{2^{2n+1}}{k^2}\iint_{Q_n}(u-k_n)_+^2 dx\,d\tau$$

and set

$$Y_n=\frac{1}{|Q_n|}\iint_{Q_n}(u-k_n)_+^2 dx\,d\tau=⨍⨍_{Q_n}(u-k_n)_+^2 dx\,d\tau.$$

Then the previous recursive inequalities can be written more concisely as

$$Y_{n+1} \le \frac{\gamma b^n}{(1-\sigma)^{p\frac{2}{q}\frac{N+p}{N}} k^{\frac{2}{q}(q-2)}} \Big(\frac{\rho^p}{t}\Big)^{\frac{2}{q}} Y_n^{1+\frac{2p}{qN}}$$

where

$$q = p\tfrac{N+2}{N} > 2 \quad \text{ and } \quad b = 2^{2(1+\frac{2p}{qN})}.$$

By Lemma 5.1 of the Preliminaries, $Y_n \to 0$ as $n \to +\infty$, provided k is chosen from

$$Y_o = \iint_{Q_o} u^2\, dx\, d\tau = \gamma\, k^{\frac{N}{p}(q-2)} (1-\sigma)^{N+p} \Big(\frac{t}{\rho^p}\Big)^{\frac{N}{p}}.$$

For this choice

$$M_\sigma \le \frac{\gamma(\text{data})}{(1-\sigma)^{\frac{p(N+p)}{N(q-2)}}} \Big(\frac{\rho^p}{t}\Big)^{\frac{1}{q-2}} \Big(\iint_{Q_o} u^2\, dx\, d\tau\Big)^{\frac{p}{N(q-2)}},$$

$$M_\sigma \le \frac{\gamma(\text{data})}{(1-\sigma)^{\frac{p(N+p)}{N(q-2)}}} \Big(\frac{\rho^p}{t}\Big)^{\frac{1}{q-2}} M^{\frac{p(2-r)}{N(q-2)}} \Big(\iint_{Q_o} u^r\, dx\, d\tau\Big)^{\frac{p}{N(q-2)}}.$$

From this, by Lemma 5.2 of the Preliminaries, and taking into account (A.2.2), we conclude that

$$\sup_{K_{\frac{1}{2}\rho}\times[0,t]} u \le \gamma\Big(\frac{\rho^p}{t}\Big)^{\frac{N}{\lambda_r}} \Big(\iint_{Q_o} u^r\, dx\, d\tau\Big)^{\frac{p}{\lambda_r}} + \gamma\Big(\frac{t}{\rho^p}\Big)^{\frac{1}{2-p}}.$$

A.2.2 Proof of Proposition A.2.1 for p in the Range $1 < p \le \max\{1, \frac{2N}{N+2}\}$

The requirement $\lambda_r > 0$ implies

$$r > 2 \ge q = p\tfrac{N+2}{N}.$$

Estimate

$$\begin{aligned}
\iint_{Q_n} (u-k_{n+1})_+^p dx\, d\tau &\le \gamma \frac{2^{n(r-p)}}{k^{r-p}} \iint_{Q_n} (u-k_n)_+^r dx\, d\tau \\
\iint_{Q_n} (u-k_{n+1})_+^2 dx\, d\tau &\le \gamma \frac{2^{n(r-2)}}{k^{r-2}} \iint_{Q_n} (u-k_n)_+^r dx\, d\tau \\
\iint_{Q_n} \chi_{[u>k_{n+1}]} dx\, d\tau &\le \gamma \frac{2^{rn}}{k^{r}} \iint_{Q_n} (u-k_n)_+^r dx\, d\tau.
\end{aligned}$$

Taking these estimates into account and assuming that condition (A.1.1) is violated, the energy estimates (A.2.1) yield

$$\sup_{t_n<\tau\le t}\int_{K_n}[(u-k_{n+1})_+\zeta]^2(x,\tau)dx+\iint_{Q_n}|D[(u-k_{n+1})_+\zeta]|^p dx\,d\tau$$
$$\le\gamma\frac{2^{nr}}{(1-\sigma)^p t}\Big[\Big(\frac{t}{\rho^p}\Big)k^{p-r}+\frac{1}{k^{r-2}}+\Big(\frac{t}{\rho^p}\Big)^{\frac{2}{2-p}}\frac{1}{k^r}\Big]\iint_{Q_n}(u-k_n)_+^r dx\,d\tau.$$

Assuming (A.2.2) holds, this implies

$$\sup_{t_n<\tau\le t}\int_{K_n}[(u-k_{n+1})_+\zeta]^2(x,\tau)dx+\iint_{Q_n}|D[(u-k_{n+1})_+\zeta]|^p dx\,d\tau$$
$$\le\frac{\gamma 2^{nr}}{(1-\sigma)^p t}\frac{1}{k^{r-2}}\iint_{Q_n}(u-k_n)_+^r dx\,d\tau.$$

Set

$$Y_n=\frac{1}{|Q_n|}\iint_{Q_n}(u-k_n)_+^r dx\,d\tau$$

and estimate

$$Y_{n+1}\le\|u\|_{\infty,Q_o}^{r-q}\Big(\frac{1}{|Q_n|}\iint_{Q_n}(u-k_{n+1})_+^q dx\,d\tau\Big).$$

Applying the embedding Proposition 4.1 of the Preliminaries, the previous inequality can be rewritten as

$$Y_{n+1}\le\gamma\|u\|_{\infty,Q_o}^{r-q}\Big(\frac{\rho^p}{t}\Big)\frac{b^n}{(1-\sigma)^{\frac{p}{N}(N+p)}}\frac{1}{k^{(r-2)\frac{N+p}{N}}}Y_n^{1+\frac{p}{N}},$$

where $b=2^{r\frac{N+p}{N}}$. Apply Lemma 5.1 of the Preliminaries, and conclude that $Y_n\to 0$ as $n\to+\infty$, provided k is chosen to satisfy

$$Y_o=⨌_{Q_o}u^r dx\,d\tau=\gamma(1-\sigma)^{N+p}\|u\|_{\infty,Q_o}^{-(r-q)\frac{N}{p}}\Big(\frac{t}{\rho^p}\Big)^{\frac{N}{p}}k^{(r-2)\frac{N+p}{p}},$$

which yields

$$M_\sigma\le\gamma\frac{M^{\frac{N(r-q)}{(N+p)(r-2)}}}{(1-\sigma)^{\frac{p}{r-2}}}\Big(\frac{\rho^p}{t}\Big)^{\frac{N}{(N+p)(r-2)}}\Big(⨌_{Q_o}u^r\,dx\,d\tau\Big)^{\frac{p}{(r-2)(N+p)}}.$$

The proof is concluded by the interpolation Lemma 5.2 of the Preliminaries.
■

Remark A.2.1 The proof shows that the boundedness of u plays a role only when $1<p\le\frac{2N}{N+2}$, and one does not need to assume it a priori, when $p>\frac{2N}{N+2}$.

A.3 $L^r_{\rm loc}$ Estimates Backward in Time

Proposition A.3.1 *Let u be a locally bounded, local, weak sub(super)-solution to the singular equations (1.1)–(1.2) of Chapter 3, in E_T, for $1<p<2$, and assume that $u\in L^r_{loc}(E_T)$ for some $r>1$. There exists a positive constant γ, depending only on the data $\{p,N,C_o,C_1\}$ and r, such that either*

$$C\rho > \min\{1\,,\,M_r^\pm\} \tag{A.3.1}$$

where

$$M_r^\pm = \Big(\sup_{\tau\le s\le t} \fint_{K_\rho(y)} u_\pm^r(x,s)dx\Big)^{\frac{1}{r}}, \tag{A.3.2}$$

or

$$\sup_{\tau\le s\le t}\int_{K_\rho(y)} u_\pm^r(x,s)dx \le \gamma\Big\{\int_{K_{2\rho}(y)} u_\pm^r(x,\tau)dx + \Big[\frac{(t-\tau)^r}{\rho^{\lambda_r}}\Big]^{\frac{1}{2-p}}\Big\}$$

for all cylinders

$$K_{2\rho}(y)\times[\tau,t]\subset E_T.$$

A.3.1 Proof of Proposition A.3.1

The proof will be given for nonnegative subsolutions, the proof for the remaining cases being similar. Assume $(y,\tau)=(0,0)$, fix $\sigma\in(0,1]$, and choose $\zeta\in C_o^\infty(K_{(1+\sigma)\rho})$ satisfying

$$0\le\zeta\le 1 \text{ in } K_{(1+\sigma)\rho},\quad \zeta=1 \text{ in } K_\rho,\quad |D\zeta|\le\gamma(\sigma\rho)^{-1} \text{ in } K_{(1+\sigma)\rho}$$

for a constant γ depending only on N. Let M be a positive constant to be chosen, and let q be a parameter in the range

$$\max\{r-1\,,\,1\}<q<r.$$

In the weak formulation (1.5) of Chapter 3 take

$$f(u)\zeta^p \quad\text{ with }\quad f(u)=u^{r-1}\Big(\frac{(u-M)_+}{u}\Big)^q \tag{A.3.3}$$

as testing function, modulo a standard Steklov averaging process. One verifies that

$$(r-1)u^{r-2}\Big(\frac{(u-M)_+}{u}\Big)^q \le f'(u)\le q u^{r-2}\Big(\frac{(u-M)_+}{u}\Big)^{q-1}.$$

Set

$$F(u)=\int_M^u f(v)dv$$

and integrate over

$$Q_s = K_{(1+\sigma)\rho} \times (0, s] \quad \text{with} \quad s \in (0, t]$$

to get

$$\begin{aligned}
0 = \iint_{Q_s} F(u)_\tau \zeta^p dx\, d\tau + &\iint_{Q_s} \mathbf{A}(x, \tau, u, Du) \cdot Du f'(u) \zeta^p dx\, d\tau \\
&+ p \iint_{Q_s} \zeta^{p-1} f(u) \mathbf{A}(x, \tau, u, Du) \cdot D\zeta\, dx\, d\tau \\
&- \iint_{Q_s} B(x, \tau, u, Du) f(u) \zeta^p dx\, d\tau \\
&= T_1 + T_2 + T_3 + T_4.
\end{aligned}$$

Since ζ is independent of τ,

$$T_1 = \int_{K_{(1+\sigma)\rho}} F(u)(x, s) \zeta^p(x) dx - \int_{K_{(1+\sigma)\rho}} F(u)(x, 0) \zeta^p(x) dx.$$

Moreover,

$$\begin{aligned}
T_2 &= \iint_{Q_s} f'(u) \zeta^p \mathbf{A}(x, \tau, u, Du) \cdot Du\, dx\, d\tau \\
&\geq C_o(r-1) \iint_{Q_s} \frac{f(u)}{u} |Du|^p \zeta^p dx\, d\tau \\
&\quad - qC^p \iint_{Q_s} u^{r-2} \Big(\frac{(u-M)_+}{u} \Big)^{q-1} \zeta^p dx\, d\tau.
\end{aligned}$$

Next,

$$\begin{aligned}
|T_3| &\leq p \iint_{Q_s} f(u) \big[C_1 |Du|^{p-1} |D\zeta| + C^{p-1} |D\zeta| \big] \zeta^{p-1} dx\, d\tau, \\
|T_4| &\leq C \iint_{Q_s} |Du|^{p-1} f(u) \zeta^p dx\, d\tau + C^p \iint_{Q_s} f(u) \zeta^p dx\, d\tau.
\end{aligned}$$

Combining these remarks,

$$\begin{aligned}
\int_{K_{(1+\sigma)\rho}} F(u)(x, s) \zeta^p dx + (r-1) C_o &\iint_{Q_s} \frac{f(u)}{u} |Du|^p \zeta^p dx\, d\tau \\
&\leq \frac{pC_1}{\sigma\rho} (1 + C\rho) \iint_{Q_s} f(u) |Du|^{p-1} \zeta^{p-1} dx\, d\tau \\
&+ \frac{(C\rho)^{p-1}}{\sigma^p \rho^p} (1 + C\rho) \iint_{Q_s} f(u) \zeta^{p-1} dx\, d\tau \\
&+ qC^p \iint_{Q_s} u^{r-2} \Big(\frac{(u-M)_+}{u} \Big)^{q-1} \zeta^p dx\, d\tau \\
&+ \int_{K_{(1+\sigma)\rho}} F(u)(\cdot, 0) \zeta^p dx.
\end{aligned}$$

If condition (A.3.1) is violated, estimate

$$\begin{aligned}\frac{pC_1}{\sigma\rho}(1+C\rho)\iint_{Q_s} f(u)|Du|^{p-1}\zeta^{p-1}dx\,d\tau \\ \le \frac{r-1}{2}C_o\iint_{Q_s}\frac{f(u)}{u}|Du|^p\zeta^p dx\,d\tau \\ + \frac{\gamma}{\sigma^p\rho^p}\iint_{Q_s} u^{r+p-2}dx\,d\tau\end{aligned}$$

for a constant $\gamma=\gamma(r,p,C_o,C_1)$. These remarks imply

$$\begin{aligned}\int_{K_\rho} F(u)(\cdot,s)dx \le \int_{K_{(1+\sigma)\rho}} F(u)(\cdot,0)\zeta^p dx \\ + \frac{\gamma}{\sigma^p\rho^p}\iint_{Q_s} u^{r+p-2}dx\,d\tau \\ + \frac{\gamma}{\sigma^p\rho^p}\Big((C\rho)^{p-1}+\frac{(C\rho)^p}{M}\Big)\iint_{Q_s} u^{r-1}dx\,d\tau.\end{aligned}$$

By elementary calculations and the Young inequality,

$$\int_{K_\rho\cap[u>M]} u^r(\cdot,s)dx \le 2r\sup_{0\le s\le t}\int_{K_\rho} F(u)(\cdot,s)dx+\bar\gamma M^r|K_\rho|$$

for a constant $\bar\gamma=\bar\gamma(r,p,q,C_o,C_1)$. From this

$$\sup_{0\le s\le t}\fint_{K_\rho} u^r(\cdot,s)dx \le 2r\Big(\sup_{0\le s\le t}\fint_{K_\rho} F(u)(\cdot,s)dx+(1+\bar\gamma)M^r\Big).$$

Choosing

$$M=\frac{1}{[4r(1+\bar\gamma)]^{\frac1r}}\Big(\sup_{0\le s\le t}\fint_{K_\rho} u^r(\cdot,s)ds\Big)^{\frac1r}=\frac{1}{\big[4r(1+\bar\gamma)\big]^{\frac1r}}M_r,$$

with M_r given by (A.3.2), these inequalities yield

$$\begin{aligned}\sup_{0\le s\le t}\int_{K_\rho} u^r(\cdot,s)dx \le 4\int_{K_{(1+\sigma)\rho}} u^r(\cdot,0)dx \\ + \frac{\gamma}{\sigma^p\rho^p}\iint_{Q_s} u^{r+p-2}dx\,d\tau \\ + \frac{\gamma}{\sigma^p\rho^p}\Big((C\rho)^{p-1}+\frac{(C\rho)^p}{M_r}\Big)\iint_{Q_s} u^{r-1}dx\,d\tau.\end{aligned}\tag{A.3.4}$$

Estimate

$$\frac{\gamma}{\sigma^p\rho^p}\iint_{Q_s} u^{r+p-2}dx\,d\tau \le \frac{\gamma}{\sigma^p}\Big(\sup_{0\le s\le t}\int_{K_{(1+\sigma)\rho}} u^r(\cdot,s)dx\Big)^{\frac{r+p-2}{r}}\Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac1r}.$$

To estimate the last term on the right-hand side of (A.3.4) assume that (A.3.1) is violated, so that $C\rho \le M_r$. With this stipulation

$$\begin{aligned}
\frac{\gamma}{\sigma^p \rho^p}\Big((C\rho)^{p-1} + \frac{(C\rho)^p}{M_r}\Big) \iint_{Q_s} u^{r-1} dx\, d\tau \\
\le \frac{2\gamma}{\sigma^p \rho^p} M_r^{p-1} \iint_{Q_s} u^{r-1} dx\, d\tau \\
\le \frac{2\gamma}{\sigma^p}\Big(\sup_{0\le s\le t}\int_{K_{(1+\sigma)\rho}} u^r(\cdot, s)dx\Big)^{\frac{r+p-2}{r}} \Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}}.
\end{aligned}$$

From this

$$\begin{aligned}
\sup_{0\le s\le t}\int_{K_\rho} u^r(\cdot, s)dx \le \gamma \int_{K_{(1+\sigma)\rho}} u^r(\cdot, 0)dx \\
+ \frac{\gamma}{\sigma^p}\Big(\sup_{0\le s\le t}\int_{K_{(1+\sigma)\rho}} u^r(\cdot, s)dx\Big)^{\frac{r+p-2}{r}} \Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}}.
\end{aligned}$$

Fix $R > 0$ and consider the sequence of radii

$$\rho_n = R \sum_{i=1}^{n} 2^{-i},$$

so that

$$\rho_{n+1} = (1 + \sigma_n)\rho_n \qquad \text{for} \qquad \sigma_n = \frac{\rho_{n+1} - \rho_n}{\rho_n} \ge 2^{-n-2}.$$

Setting

$$Y_n = \sup_{0\le s\le t}\int_{K_{\rho_n}} u^r(\cdot, s)dx$$

the previous inequalities yield

$$Y_n \le \gamma \int_{K_{2R}} u^r(\cdot, 0)dx + \gamma 2^n \Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}} Y_{n+1}^{\frac{r+p-2}{r}}.$$

The proposition now follows from the interpolation Lemma 5.2 of the Preliminaries. ∎

Remark A.3.1 The proof shows that the constant γ depends on $(r-1)$ and $\gamma(r) \to \infty$ as $r \to 1$.

Remark A.3.2 Theorems 2.1 and 12.1 of Chapter 6 follow combining Proposition A.2.1 respectively with Propositions A.1.1 and A.3.1.

A.4 Remarks and Bibliographical Notes

The idea of a Harnack-type estimate in the topology of $L^1_{\rm loc}$ appears first in [31]. It is reported in [41] for solutions to the singular $(1 < p < 2)$ prototype equation (1.3) of Chapter 3. A proof for singular equations with the full quasilinear structure (1.2) of Chapter 3 is in [51].

For the prototype singular p-Laplacian equation, the local sup estimates of § A.2 are essentially in [41], taken from [130]. The proof presented here covers equations with the full quasilinear structure (1.2) of Chapter 3.

For homogeneous $(C = 0)$ singular equations, the backward in time estimate of Proposition A.3.1 follows from standard energy estimates obtained by taking the test function $u^{r-1}\zeta^p$ in the weak formulation (1.5) of Chapter 3. For nonhomogeneous structures $(C \neq 0)$, this method fails. The proof we report here essentially follows an idea of Lieberman [106], based on introducing the test function in (A.3.3). The only difference with respect to the approach of [106] lies in the conclusion, where the interpolation Lemma 5.2 was used, instead of the Gronwall Inequality as in [106]. The motivation is in establishing estimates with constants independent of time.

9

Appendix B

B.1 An L^1_{loc}-Form of the Harnack Inequality for All $m \in (0,1)$

Proposition B.1.1 *Let u be a nonnegative, local, weak solution to the singular porous medium type equations (5.1)–(5.2) of Chapter 3, for $0 < m < 1$, in E_T. There exists a positive constant γ depending only on the data $\{m, N, C_o, C_1\}$, such that for all cylinders $K_{2\rho}(y) \times [s,t] \subset E_T$, either $C\rho > 1$, or*

$$\begin{aligned}\sup_{s<\tau<t} \int_{K_\rho(y)} u(x,\tau)dx \leq & \gamma \inf_{s<\tau<t} \int_{K_{2\rho}(y)} u(x,\tau)dx \\ & + \gamma \Big(\frac{t-s}{\rho^\lambda}\Big)^{\frac{1}{1-m}}\end{aligned} \tag{B.1.1}$$

where

$$\lambda = N(m-1) + 2.$$

The constant $\gamma = \gamma(m) \to \infty$ either as $m \to 1$ or as $m \to 0$.

For $\lambda > 0$, the parameter m is in the singular, supercritical range (16.1) of Chapter 6, and if $\lambda \leq 0$, m is in the subcritical range (19.1) of Chapter 6. However, the Harnack-type estimate (B.1.1) in the topology of L^1_{loc}, holds true for all $0 < m < 1$ and accordingly, λ could be of either sign.

B.1.1 An Auxiliary Lemma

The number $0 < m < 1$ being fixed, choose

$$\alpha = \begin{cases} -\frac{1}{2}m & \text{if } 0 < m < \frac{2}{3} \\ -\frac{1}{2}(1-m) & \text{if } \frac{1}{3} < m < 1. \end{cases}$$

One verifies that for such α, the numbers $(m+\alpha)$, $(1+\alpha)$, and $(m-\alpha)$ are all in $(0,1)$.

E. DiBenedetto et al., *Harnack's Inequality for Degenerate and Singular Parabolic Equations*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-1584-8_9,

Lemma B.1.1 *Let u be a nonnegative, local, weak supersolution to the singular equations (5.1)–(5.2) of Chapter 3, for $0 < m < 1$, in E_T. There exists a positive constant γ depending only on the data $\{m, N, C_o, C_1\}$, such that for all cylinders $K_\rho(y) \times [\tau, t] \subset E_T$, all $\sigma \in (0,1)$ such that $K_{(1+\sigma)\rho}(y) \subset E$, either $C\rho > 1$, or*

$$\int_\tau^t \int_{K_\rho(y)} u^{m-1} u^{\alpha-1} |Du|^2 \zeta^2 dx\, d\tau \leq \frac{\gamma(\alpha)}{\sigma^2 \rho^2} \mathcal{S}_\sigma^{m+\alpha} (t-\tau)\, \rho^{N(1-m-\alpha)} + \gamma(\alpha) \mathcal{S}_\sigma^{1+\alpha} \rho^{-\alpha N}$$

where

$$\mathcal{S}_\sigma = \sup_{\tau < s < t} \int_{K_{(1+\sigma)\rho}(y)} u(\cdot, s) dx.$$

The constant $\gamma(m) \to \infty$ either as $m \to 1$ or as $m \to 0$.

Proof Assume $(y, \tau) = (0,0)$, fix $\sigma \in (0,1)$, and let $x \to \zeta(x)$ be a nonnegative piecewise smooth cutoff function in $K_{(1+\sigma)\rho}$ that vanishes outside $K_{(1+\sigma)\rho}$, equals one on K_ρ, and such that

$$|D\zeta| \leq \frac{1}{\sigma\rho}.$$

In the weak formulation (5.5) of Chapter 3 take the test function $\varphi = u^\alpha \zeta^2$, and integrate over $Q = K_{(1+\sigma)\rho} \times (0, t]$, to obtain formally

$$0 \leq \frac{1}{1+\alpha} \iint_Q \frac{\partial}{\partial \tau} u^{1+\alpha} \zeta^2 dx\, d\tau + \iint_Q \mathbf{A}(x, \tau, u, Du) \cdot D(u^\alpha \zeta^2) dx\, d\tau - \iint_{Q_s} B(x, \tau, u, Du) u^\alpha \zeta^2 dx\, d\tau = I_1 + I_2 + I_3.$$

Assume momentarily that $u^\alpha \zeta^2$ is an admissible test function, and proceed to estimating the various terms formally. Since $0 < 1 + \alpha < 1$, estimate

$$|I_1| \leq \frac{2}{1+\alpha} \rho^{-\alpha N} \mathcal{S}_\sigma^{1+\alpha}.$$

Next,

$$\begin{aligned} I_2 &\leq -\frac{|\alpha|}{2} C_o \iint_Q u^{m-1} u^{\alpha-1} |Du|^2 \zeta^2 dx\, d\tau \\ &\quad + \frac{\gamma(\alpha)\big[1 + (C\rho) + (C\rho)^2\big]}{\sigma^2 \rho^2} \iint_Q u^{m+\alpha} dx\, d\tau \\ &\leq -\frac{|\alpha|}{2} C_o \iint_Q u^{m-1} u^{\alpha-1} |Du|^2 \zeta^2 dx\, d\tau + \frac{\gamma(\alpha)}{\sigma^2 \rho^2} \mathcal{S}_\sigma^{m+\alpha} (t\rho^N)^{1-m-\alpha} \end{aligned}$$

where the conditions $C\rho \le 1$ and $0 < m+\alpha < 1$ have been enforced. Finally,

$$|I_3| \le \frac{|\alpha|}{4} C_o \iint_Q u^{m-1} u^{\alpha-1} |Du|^2 \zeta^2 dx\, d\tau + \frac{\gamma_1(\alpha)}{\sigma^2\rho^2} \mathcal{S}_\sigma^{m+\alpha} t \rho^{N(1-m-\alpha)}.$$

The lemma follows by combining the estimates. The use of $u^\alpha \zeta^2$ as test function can be justified using $(u+\epsilon)^\alpha \zeta^2$, and then letting $\epsilon \to 0$. ■

Corollary B.1.1 *Let u be a nonnegative, local, weak supersolution to the singular equations (5.1)–(5.2) of Chapter 3, for $0 < m < 1$, in E_T. There exists a positive constant γ depending only on the data $\{m, N, C_o, C_1\}$, such that for all cylinders $K_\rho(y) \times [s,t] \subset E_T$, all $\sigma \in (0,1)$ such that $K_{(1+\sigma)\rho}(y) \subset E$, either $C\rho > 1$, or*

$$\begin{aligned} \frac{1}{\rho} \int_s^t \int_{K_\rho(y)} \big(|\mathbf{A}(x,\tau,u,Du)| + |B(x,\tau,u,Du)|\,\rho \big) dx\, d\tau \\ \le \frac{\gamma}{\sigma} \mathcal{S}_\sigma^m \Big(\frac{t-s}{\rho^\lambda} \Big) + \gamma \mathcal{S}_\sigma^{\frac{1+m}{2}} \Big(\frac{t-s}{\rho^\lambda} \Big)^{\frac{1}{2}}. \end{aligned}$$

Proof Assume $(y,s) = (0,0)$, and let $Q = K_\rho \times (0,t]$. By the structure conditions of $\mathbf{A}$ and B, and enforcing the requirement $C\rho \le 1$

$$\begin{aligned} \frac{1}{\rho} \int_0^t \int_{K_\rho} \big(|\mathbf{A}(x,\tau,u,Du)| + |B(x,\tau,u,Du)|\,\rho \big) dx\, d\tau \\ \le \frac{\gamma}{\rho} \iint_Q u^{m-1} |Du| dx\, d\tau + \frac{\gamma}{\rho^2} \iint_Q u^m dx\, d\tau \end{aligned}$$

for a constant γ depending only on the data $\{m, N, C_o, C_1\}$. Estimate

$$\frac{\gamma}{\rho^2} \iint_Q u^m dx\, d\tau \le \gamma \mathcal{S}_\sigma^m \Big(\frac{t}{\rho^\lambda} \Big).$$

Next, by the previous lemma

$$\begin{aligned} \frac{\gamma}{\rho} \iint_Q u^{m-1} |Du| dx\, d\tau &\le \frac{\gamma}{\rho} \Big(\iint_Q u^{m-1} u^{\alpha-1} |Du|^2 dx\, d\tau \Big)^{\frac{1}{2}} \Big(\iint_Q u^{m-\alpha} dx\, d\tau \Big)^{\frac{1}{2}} \\ &\le \Big(\frac{\gamma(\alpha)}{\sigma\rho^2} \mathcal{S}_\sigma^{\frac{m+\alpha}{2}} \sqrt{t}\, \rho^{N\frac{1-m-\alpha}{2}} + \frac{\gamma(\alpha)}{\rho} \mathcal{S}_\sigma^{\frac{1+\alpha}{2}} \rho^{-N\frac{\alpha}{2}} \Big) \\ &\quad \times \Big(\sqrt{t} \rho^{N\frac{1-m+\alpha}{2}} \mathcal{S}_\sigma^{\frac{m-\alpha}{2}} \Big). \end{aligned}$$

■

B.1.2 Proof of Proposition B.1.1

Assume $(y,s) = (0,0)$ and $C\rho \le 1$. For $n = 0,1,2\dots$ set

$$\rho_n = \sum_{j=1}^n \frac{1}{2^j} \rho, \quad K_n = K_{\rho_n}; \qquad \tilde\rho_n = \frac{\rho_n + \rho_{n+1}}{2}, \quad \tilde K_n = K_{\tilde\rho_n}$$

and let $x \to \zeta_n(x)$ be a nonnegative, piecewise smooth cutoff function in $\tilde{K}_n$ that equals one on K_n, and such that $|D\zeta_n| \le 2^{n+2}/\rho$. In the weak formulation of (5.1)–(5.2) of Chapter 3 take ζ_n as a test function, to obtain

$$\begin{aligned}\int_{\tilde{K}_n} u(x,\tau_1)\zeta_n dx \le \int_{\tilde{K}_n} u(x,\tau_2)\zeta_n dx&\\ + \frac{2^{n+2}}{\rho}\Big|\int_{\tau_1}^{\tau_2}\int_{\tilde{K}_n}\big(|\mathbf{A}(x,\tau,u,Du)| &+ |B(x,\tau,u,Du)|\rho\big)dx\,d\tau\Big|\\ \le \int_{\tilde{K}_n} u(x,\tau_2)\zeta_n dx + 4^n\gamma\mathcal{S}_{n+1}^m\Big(\frac{t}{\rho^\lambda}\Big) &+ 2^n\gamma\mathcal{S}_{n+1}^{\frac{1+m}{2}}\Big(\frac{t}{\rho^\lambda}\Big)^{\frac{1}{2}},\end{aligned}$$

where

$$\mathcal{S}_n = \sup_{0\le\tau\le t}\int_{K_n} u(\cdot,\tau)dx.$$

Since the time levels τ_1 and τ_2 are arbitrary, choose τ_2 one for which

$$\int_{K_{2\rho}} u(\cdot,\tau_2)dx = \inf_{0\le\tau\le t}\int_{K_{2\rho}} u(\cdot,\tau)dx \overset{\text{def}}{=} \mathcal{I}.$$

With these notation, the previous inequality takes the form

$$\mathcal{S}_n \le \mathcal{I} + \gamma 4^n\mathcal{S}_{n+1}^m\Big(\frac{t}{\rho^\lambda}\Big) + \gamma 2^n\mathcal{S}_{n+1}^{\frac{1+m}{2}}\Big(\frac{t}{\rho^\lambda}\Big)^{\frac{1}{2}}.$$

By Young's inequality, for all $\varepsilon_o \in (0,1)$

$$\mathcal{S}_n \le \varepsilon_o\mathcal{S}_{n+1} + \gamma\big(\text{data},\varepsilon_o\big)b^n\Big[\mathcal{I} + \Big(\frac{t}{\rho^\lambda}\Big)^{\frac{1}{1-m}}\Big],$$

where $b = 4^{\frac{1}{1-m}}$. From this, by iteration

$$\mathcal{S}_o \le \varepsilon_o^n\mathcal{S}_n + \gamma(\text{data},\varepsilon_o)\Big[\mathcal{I} + \Big(\frac{t}{\rho^\lambda}\Big)^{\frac{1}{1-m}}\Big]\sum_{i=1}^{n-1}(\varepsilon_o b)^i.$$

Choose ε_o so that the last term is majorized by a convergent series, and let $n \to \infty$. ∎

B.1.3 An Estimate for Supersolutions

Both the supporting Lemma B.1.1 and Corollary B.1.1 are valid for supersolutions, as stated. However, the proof of Proposition B.1.1 requires the full notion of solution, since the pair (τ_1,τ_2) is arbitrary, and in particular nonordered.

A parallel but weaker statement holds for supersolutions.

Proposition B.1.2 *Let u be a nonnegative, local, weak supersolution to the singular equations (5.1)–(5.2) of Chapter 3, for $0 < m < 1$, in E_T. There exists a positive constant γ depending only upon the data $\{m, N, C_o, C_1\}$, such that for all cylinders $K_{2\rho}(y) \times [s,t] \subset E_T$, either $C\rho > 1$, or*

$$\sup_{s<\tau<t} \int_{K_\rho(y)} u(x,\tau)dx \le \gamma \int_{K_{2\rho}(y)} u(x,t)dx + \gamma\Big(\frac{t-s}{\rho^\lambda}\Big)^{\frac{1}{1-m}} \tag{B.1.2}$$

where

$$\lambda = N(m-1) + 2.$$

The constant $\gamma = \gamma(m) \to \infty$ either as $m \to 1$ or as $m \to 0$.

Proof A standard adaptation of the previous argument. ∎

B.2 Energy Estimates for Sub(Super)-Solutions When $0 < m < 1$

Proposition B.2.1 *Let u be a local, weak sub(super)-solution to the singular equations (5.1)–(5.2) of Chapter 3, for $0 < m < 1$, in E_T and consider the truncations*

$$(u-k)_+ \;\; \textit{for} \;\; k > 0, \qquad \textit{and} \qquad -(u-k)_- \;\; \textit{for} \;\; k < 0. \tag{B.2.1}$$

There exists a positive constant $\gamma = \gamma(m, N, C_o, C_1)$, such that for every cylinder $(y,s) + Q_\rho^-(\theta) \subset E_T$, every k as in (B.2.1), and every nonnegative, piecewise smooth cutoff function ζ vanishing on $\partial K_\rho(y)$,

$$\begin{aligned}
\operatorname*{ess\,sup}_{s-\theta\rho^2<t\le s} &\int_{K_\rho(y)} (u-k)_\pm^2 \zeta^2(x,t)dx \\
&- \int_{K_\rho(y)} (u-k)_\pm^2 \zeta^2(x, s-\theta\rho^2)dx \\
&+ C_o m \iint_{(y,s)+Q_\rho^-(\theta)} |u|^{m-1} |D(u-k)_\pm|^2 \zeta^2 dx\,dt \\
\le \gamma &\iint_{(y,s)+Q_\rho^-(\theta)} (u-k)_\pm^2 \zeta |\zeta_t| dx\,dt \\
&+ \gamma \iint_{(y,s)+Q_\rho^-(\theta)} |u|^{m-1} (u-k)_\pm^2 |D\zeta|^2 dx\,dt \\
&+ \gamma C^2 \iint_{(y,s)+Q_\rho^-(\theta)} |u|^{m+1} \chi_{[(u-k)_\pm>0]} \zeta^2 dx\,dt.
\end{aligned} \tag{B.2.2}$$

Analogous estimates hold in the "forward" cylinder $(y,s) + Q_\rho^+(\theta) \subset E_T$.

Proof We assume $(y, s) = (0, 0)$ and establish the proposition for $(u-k)_+$ for $k > 0$. In (5.1) of Chapter 3 take the testing function

$$\varphi = (u-k)_+ \,\zeta^2$$

over

$$Q^t = K_\rho \times (-\theta\rho^2, t] \quad \text{where} \quad -\theta\rho^2 < t \le 0. \tag{B.2.3}$$

The use of $(u-k)_+$ in this testing function is justified, modulus a standard Steklov averaging process. This gives

$$\begin{aligned}
&\iint_{Q^t} u_\tau (u-k)_+ \zeta^2 dx\, d\tau \\
&\quad + \iint_{Q^t} \mathbf{A}(x, \tau, u, Du) \cdot D(u-k)_+ \zeta^2 dx\, d\tau \\
&\quad + 2 \iint_{Q^t} \mathbf{A}(x, \tau, u, Du) \cdot D\zeta (u-k)_+ \zeta\, dx\, d\tau \\
&= \iint_{Q^t} B(x, \tau, u, Du)(u-k)_+ \zeta^2 dx\, d\tau.
\end{aligned}$$

Transform and estimate these integrals separately, to get

$$\begin{aligned}
&\iint_{Q^t} u_\tau (u-k)_+ \zeta^2 dx\, d\tau \\
&\qquad = \frac{1}{2} \int_{K_\rho} (u-k)_+^2 \zeta^2 (x, t) dx - \frac{1}{2} \int_{K_\rho} (u-k)_+^2 \zeta^2 (x, -\theta\rho^2) dx \\
&\qquad\quad - \iint_{Q^t} (u-k)_+^2 \zeta |\zeta_\tau| dx\, d\tau.
\end{aligned}$$

From the first structure condition (5.2) of Chapter 3 it follows that

$$\begin{aligned}
&\iint_{Q^t} \mathbf{A}(x, \tau, u, Du) \cdot D(u-k)_+ \zeta^2 dx\, d\tau \\
&\qquad \ge C_o m \iint_{Q^t} u^{m-1} |D(u-k)_+|^2 \zeta^2 dx\, d\tau \\
&\qquad\quad - C^2 \iint_{Q^t} u^{m+1} \zeta^2 \chi_{[(u-k)_+ > 0]} dx\, d\tau.
\end{aligned}$$

From the second condition in (5.2) of Chapter 3 and Young's inequality it follows that

$$
\begin{aligned}
2\Big| \iint_{Q^t} & (u-k)_+ \mathbf{A}(x,\tau,u,Du)\cdot D\zeta\,\zeta\,dx\,d\tau \Big| \\
&\le 2C_1 m \iint_{Q^t} u^{m-1}(u-k)_+ |D(u-k)_+| \zeta |D\zeta| dx\,d\tau \\
&\quad + 2C \iint_{Q^t} u^m (u-k)_+ \zeta |D\zeta| \chi_{[(u-k)_+>0]} dx\,d\tau \\
&\le \frac{C_o m}{4} \iint_{Q^t} u^{m-1} |D(u-k)_+|^2 \zeta^2 dx\,d\tau \\
&\quad + \gamma(C_o) \iint_{Q^t} u^{m-1}(u-k)_+^2 |D\zeta|^2 dx\,d\tau \\
&\quad + C^2 \iint_{Q^t} u^{m+1} \zeta^2 \chi_{[(u-k)_+>0]} dx\,d\tau .
\end{aligned}
$$

Finally, the third condition of (5.2) of Chapter 3 implies

$$
\begin{aligned}
\Big| \iint_{Q^t} & B(x,\tau,u,Du)(u-k)_+ \zeta^2 dx\,d\tau \Big| \\
&\le \frac{C_o m}{4} \iint_{Q^t} u^{m-1} |D(u-k)_+|^2 \zeta^2 dx\,d\tau \\
&\quad + \gamma(C_o) C^2 \iint_{Q^t} u^{m-1} (u-k)_+^2 dx\,d\tau \\
&\quad + \bar\gamma(C_o) C^2 \iint_{Q^t} u^{m+1} \chi_{[(u-k)_+>0]} \zeta^2 dx\,d\tau .
\end{aligned}
$$

Combining these estimates, and taking the supremum over $t \in (-\theta\rho^2, 0]$ proves the proposition. ∎

Remark B.2.1 The constant $\gamma = \gamma(m, N, C_o, C_1)$ is stable as $m \to 1$, but it tends to infinity, as $m \to 0$.

B.3 A Different Type of Energy Estimates for Sub(Super)-Solutions When $0 < m < 1$

Proposition B.3.1 *Let u be a local, weak subsolution to the singular equations (5.1)–(5.2) of Chapter 3, for $0 < m < 1$, in E_T and consider the truncations*

$$
(u^m - k^m)_+ \quad \textit{for} \quad k > 0.
$$

There exists a positive constant $\gamma = \gamma(m, N, C_o, C_1)$, such that for every cylinder

$$
(y,s) + Q_\rho^-(\theta) \subset E_T,
$$

every $k > 0$, and every nonnegative, piecewise smooth cutoff function ζ vanishing on $\partial K_\rho(y)$,

$$
\begin{aligned}
\operatorname*{ess\,sup}_{s-\theta\rho^2<t\le s} \frac{1}{m+1}\int_{K_\rho(y)} & (u^m-k^m)_+^{\frac{m+1}{m}}\zeta^2(x,t)dx \\
& - \int_{K_\rho(y)}\int_k^u (s^m-k^m)ds\zeta^2(x,s-\theta\rho^2)dx \\
& + \frac{C_o}{4}\iint_{(y,s)+Q_\rho^-(\theta)} |D(u^m-k^m)_+|^2\zeta^2 dx\,dt \\
\le \gamma & \iint_{(y,s)+Q_\rho^-(\theta)} u^{m+1}\chi_{[(u-k)_+>0]}\zeta|\zeta_t|dx\,dt \\
& + \gamma\iint_{(y,s)+Q_\rho^-(\theta)} (u^m-k^m)_+^2|D\zeta|^2 dx\,dt \\
& + \gamma C^2\iint_{(y,s)+Q_\rho^-(\theta)} u^{2m}\chi_{[(u-k)_+>0]}\zeta^2 dx\,dt.
\end{aligned}
\tag{B.3.1}
$$

Analogous estimates hold in "forward" cylinders $(y,s)+Q_\rho^+(\theta)\subset E_T$.

Proof Assume $(y,s)=(0,0)$ and in the weak formulation (5.1) of Chapter 3 take the testing function

$$\varphi = (u^m-k^m)_+\,\zeta^2$$

over Q^t, defined as in (B.2.3). The use of $(u^m-k^m)_+$ in this testing function is justified, modulus a standard Steklov averaging process. This gives

$$
\begin{aligned}
& \iint_{Q^t} u_\tau(u^m-k^m)_+\zeta^2 dx\,d\tau \\
& \quad + \iint_{Q^t} \mathbf{A}(x,\tau,u,Du)\cdot D(u^m-k^m)_+\zeta^2 dx\,d\tau \\
& \quad + 2\iint_{Q^t} \mathbf{A}(x,\tau,u,Du)\cdot D\zeta\,(u^m-k^m)_+\zeta\,dx\,d\tau \\
& = \iint_{Q^t} B(x,\tau,u,Du)(u^m-k^m)_+\zeta^2 dx\,d\tau.
\end{aligned}
$$

Transform and estimate these integrals separately, to get

$$
\begin{aligned}
&\iint_{Q^t} u_\tau (u^m - k^m)_+ \zeta^2 dx\, d\tau \\
&\qquad \geq \int_{K_\rho} \int_0^{(u^m - k^m)_+^{\frac{1}{m}}} s^m ds \zeta^2(x,t) dx \\
&\qquad - \int_{K_\rho} \int_k^u (s^m - k^m) ds \zeta^2(x, -\theta\rho^2) dx \\
&\qquad\quad - \frac{2}{m} \iint_{Q^t} (u^m - k^m)_+ u \zeta |\zeta_\tau| dx\, d\tau \\
&\qquad = \frac{1}{m+1} \int_{K_\rho} (u^m - k^m)_+^{\frac{m+1}{m}} \zeta^2(x,t) dx \\
&\qquad - \int_{K_\rho} \int_k^u (s^m - k^m) ds \zeta^2(x, -\theta\rho^2) dx \\
&\qquad\quad - \frac{2}{m} \iint_{Q^t} (u^m - k^m)_+ u \zeta |\zeta_\tau| dx\, d\tau.
\end{aligned}
$$

From the first structure condition (5.2) of Chapter 3

$$
\begin{aligned}
&\iint_{Q^t} \mathbf{A}(x,\tau,u,Du) \cdot D(u^m - k^m)_+ \zeta^2 dx\, d\tau \\
&\qquad \geq C_o \iint_{Q^t} |D(u^m - k^m)_+|^2 \zeta^2 dx\, d\tau \\
&\qquad\quad - mC^2 \iint_{Q^t} u^{2m} \zeta^2 \chi_{[(u-k)_+ > 0]} dx\, d\tau.
\end{aligned}
$$

From the second condition in (5.2) of Chapter 3 and Young's inequality it follows that

$$
\begin{aligned}
&2\Big| \iint_{Q^t} \mathbf{A}(x,\tau,u,Du) \cdot D\zeta\, (u^m - k^m)_+ \zeta\, dx\, d\tau \Big| \\
&\quad \leq 2C_1 \iint_{Q^t} |D(u^m - k^m)_+| \zeta (u^m - k^m)_+ |D\zeta| dx\, d\tau \\
&\quad + 2C \iint_{Q^t} u^m (u^m - k^m)_+ \zeta |D\zeta| dx\, d\tau \\
&\quad \leq \frac{C_o}{4} \iint_{Q^t} |D(u^m - k^m)_+|^2 \zeta^2 dx\, d\tau \\
&\quad + \gamma(C_o) \iint_{Q^t} (u^m - k^m)_+^2 |D\zeta|^2 dx\, d\tau \\
&\quad + \gamma C^2 \iint_{Q^t} u^{2m} \zeta^2 \chi_{[(u-k)_+ > 0]} dx\, d\tau.
\end{aligned}
$$

Finally, the third condition of (5.2) of Chapter 3 implies

$$
\begin{aligned}
\Big| \iint_{Q^t} & B(x,\tau,u,Du)(u^m-k^m)_+\zeta^2 dx\,d\tau \Big| \\
&\le \frac{C_o}{4}\iint_{Q^t} |D(u^m-k^m)_+|^2\zeta^2 dx\,d\tau \\
&\quad + \gamma(C_o)C^2 \iint_{Q^t} (u^m-k^m)_+^2\zeta^2 dx\,d\tau \\
&\quad + \bar\gamma(C_o)C^2 \iint_{Q^t} u^{2m}\chi_{[(u-k)_+>0]}\zeta^2 dx\,d\tau.
\end{aligned}
$$

Combining these estimates proves the proposition. ■

Remark B.3.1 The constant $\gamma=\gamma(m,N,C_o,C_1)$ is stable as $m\to 1$, but it tends to infinity as $m\to 0$.

B.4 L^r_{loc}–L^∞_{loc} Estimates

Proposition B.4.1 *Let u be a locally bounded, local, weak sub(super)-solution to (5.1)–(5.2) of Chapter 3 for $0<m<1$, and let $r\ge 1$ be such that*

$$
\lambda_r = N(m-1)+2r>0. \tag{B.4.1}
$$

There exists a positive constant γ depending only on the data $\{m,N,C_o,C_1\}$, such that for all cylinders

$$
K_\rho(y)\times[2s-t,t]\subset E_T
$$

either $C\rho>1$, or

$$
\begin{aligned}
\sup_{K_{\frac12\rho}(y)\times[s,t]} u_\pm \le \gamma\Big(\frac{\rho^2}{t-s}\Big)^{\frac{N}{\lambda_r}}\Big(\frac{1}{\rho^N(t-s)}\int_{2s-t}^t\int_{K_\rho(y)} u_\pm^r dx\,d\tau\Big)^{\frac{2}{\lambda_r}} \\
+\left(\frac{t-s}{\rho^2}\right)^{\frac{1}{1-m}}.
\end{aligned}
$$

Proof The proof will be given for nonnegative weak subsolutions, the proof for the remaining cases being identical. Assume $(y,s)=(0,0)$ and for fixed $\sigma\in(0,1)$ and $n=0,1,2,\dots$ set

$$
\rho_n=\sigma\rho+\frac{1-\sigma}{2^n}\rho, \qquad t_n=-\sigma t-\frac{1-\sigma}{2^n}t,
$$

$$
K_n=K_{\rho_n}, \qquad Q_n=K_n\times(t_n,t).
$$

This is a family of nested and shrinking cylinders with common “vertex” at $(0,t)$, and by construction

$$Q_o = K_\rho \times (-t, t) \quad \text{and} \quad Q_\infty = K_{\sigma\rho} \times (-\sigma t, t).$$

Having assumed that u is locally bounded in E_T, set

$$M = \operatorname*{ess\,sup}_{Q_o} \max\{u, 0\}, \quad M_\sigma = \operatorname*{ess\,sup}_{Q_\infty} \max\{u, 0\}.$$

We first find a relationship between M and M_σ. Denote by ζ a nonnegative, piecewise smooth cutoff function in Q_n that equals one on Q_{n+1}, and has the form $\zeta(x,t) = \zeta_1(x)\zeta_2(t)$, where

$$\zeta_1 = \begin{cases} 1 & \text{in } K_{n+1} \\ 0 & \text{in } \mathbb{R}^N - K_n \end{cases} \quad |D\zeta_1| \le \frac{2^{n+1}}{(1-\sigma)\rho},$$

$$\zeta_2 = \begin{cases} 0 & \text{for } t \le t_n \\ 1 & \text{for } t \ge t_{n+1} \end{cases} \quad 0 \le \zeta_{2,t} \le \frac{2^{n+1}}{(1-\sigma)t};$$

introduce the increasing sequence of levels

$$k_n = k - \tfrac{1}{2^n} k$$

where $k > 0$ is to be chosen. Estimates (B.3.1) with $(u^m - k_{n+1}^m)_+$ yield

$$\begin{aligned}
\sup_{t_n \le \tau \le t} \int_{K_n} &[(u^m - k_{n+1}^m)_+ \zeta]^{\frac{m+1}{m}}(x,\tau)dx \\
&+ \frac{C_o}{4} \iint_{Q_n} |D[(u^m - k_{n+1}^m)_+ \zeta]|^2 dx\, d\tau \\
\le \gamma &\iint_{Q_n} u^{m+1} \chi_{[(u^m - k_{n+1}^m)_+ > 0]} \zeta\zeta_\tau dx\, d\tau \\
&+ \gamma \iint_{Q_n} (u^m - k_{n+1}^m)_+^2 |D\zeta|^2 dx\, d\tau \\
&+ \gamma C^2 \iint_{Q_n} (u^m - k_{n+1}^m)_+^2 \zeta^2 dx\, d\tau \\
&+ \gamma C^2 \iint_{Q_n} u^{2m} \chi_{[(u^m - k_{n+1}^m)_+ > 0]} \zeta^2 dx\, d\tau.
\end{aligned} \tag{B.4.2}$$

B.4.1 Proof of Proposition B.4.1 for $\frac{(N-2)_+}{N+2} < m < 1$

This amounts to taking $\lambda_r > 0$ with $r \in [1, \frac{2N}{N+2}]$. In the estimations below repeated use is made of the inequality

$$|[u > k_{n+1}] \cap Q_n| \le \gamma \frac{2^{(n+1)s}}{k^s} \iint_{Q_n} (u - k_n)_+^s dx\, d\tau$$

valid for all $s > 0$. Then estimate

$$\iint_{Q_n} (u^m - k_{n+1}^m)_+^{\frac{m+1}{m}} \zeta\zeta_\tau dx\, d\tau \le \gamma \frac{2^{2n}}{(1-\sigma)t} \iint_{Q_n} (u^m - k_n^m)_+^{\frac{m+1}{m}} dx\, d\tau$$

$$\iint_{Q_n} u^{m+1} \chi_{[(u^m - k_{n+1}^m)_+ > 0]} \zeta\zeta_\tau dx\, d\tau \le \gamma \frac{2^{(2+\frac{m+1}{m})n}}{(1-\sigma)t} \iint_{Q_n} (u^m - k_n^m)_+^{\frac{m+1}{m}} dx\, d\tau$$

$$\iint_{Q_n} (u^m - k_{n+1}^m)_+^2 |D\zeta|^2 dx\, d\tau \le \gamma \frac{2^{(2+\frac{m+1}{m})n}}{(1-\sigma)^2\rho^2} \frac{1}{k^{1-m}} \iint_{Q_n} (u^m - k_n^m)_+^{\frac{m+1}{m}} dx\, d\tau$$

$$C^2 \iint_{Q_n} u^{2m} \chi_{[(u^m - k_{n+1}^m)_+ > 0]} dx\, d\tau \le \gamma \frac{2^{\frac{m+1}{m}n} C^2}{k^{1-m}} \iint_{Q_n} (u^m - k_n^m)_+^{\frac{m+1}{m}} dx\, d\tau.$$

Combining these estimates, and stipulating that $C\rho < 1$, (B.4.2) yields

$$\begin{aligned} &\sup_{t_n \le \tau \le t} \int_{K_n} [(u^m - k_{n+1}^m)_+ \zeta]^{\frac{m+1}{m}} (x,\tau) dx \\ &\qquad + \iint_{Q_n} |D[(u^m - k_{n+1}^m)_+ \zeta]|^2 dx\, d\tau \\ &\qquad \le \frac{\gamma 2^{\frac{2(m+1)}{m}n}}{(1-\sigma)^2 t} \Big[1 + \Big(\frac{t}{\rho^2}\Big) k^{m-1}\Big] \iint_{Q_n} (u^m - k_n^m)_+^{\frac{m+1}{m}} dx\, d\tau. \end{aligned}$$

The last term in $[\cdots]$ is estimated by stipulating to take

$$k \ge \left(\frac{t}{\rho^2}\right)^{\frac{1}{1-m}}. \tag{B.4.3}$$

With these stipulations, the previous inequality implies

$$\begin{aligned} &\sup_{t_n \le \tau \le t} \int_{K_n} [(u^m - k_{n+1}^m)_+ \zeta]^{\frac{m+1}{m}} (x,\tau) dx \\ &\qquad + \iint_{Q_n} |D[(u^m - k_{n+1}^m)_+ \zeta]|^2 dx\, d\tau \\ &\qquad \le \frac{\gamma 2^{\frac{2(m+1)}{m}n}}{(1-\sigma)^2 t} \iint_{Q_n} (u^m - k_n^m)_+^{\frac{m+1}{m}} dx\, d\tau. \end{aligned} \tag{B.4.4}$$

By the Hölder inequality and the embedding Proposition 4.1 of the Preliminaries

$$\begin{aligned} &\iint_{Q_{n+1}} (u^m - k_{n+1}^m)_+^{\frac{m+1}{m}} dx\, d\tau \le \Big[\sup_{t_n \le \tau \le t} \int_{K_n} [(u^m - k_{n+1}^m)_+ \zeta]^{\frac{m+1}{m}} (x,\tau) dx\Big]^{\frac{2}{N}\frac{m+1}{qm}} \\ &\times \Big(\iint_{Q_n} |D(u^m - k_{n+1}^m)_+|^2 \zeta_n^2 dx\, d\tau + \iint_{Q_n} (u^m - k_{n+1}^m)_+ 2|D\zeta_n|^2\, dx\, d\tau\Big)^{\frac{m+1}{qm}} \\ &\times |Q_n|^{1-\frac{m+1}{qm}} \Big(\gamma \frac{2^{\frac{m+1}{m}n}}{k^{m+1}} \frac{1}{|Q_n|} \iint_{Q_n} (u^m - k_n^m)_+^{\frac{m+1}{m}} dx\, d\tau\Big)^{1-\frac{m+1}{qm}} \end{aligned}$$

where

$$q = \frac{2(Nm+m+1)}{Nm}.$$

Now set

$$Y_n = \frac{1}{|Q_n|}\iint_{Q_n} (u^m - k_n^m)_+^{\frac{m+1}{m}}\,dx\,d\tau.$$

Taking into account (B.4.4), in terms of Y_n the previous inequality becomes

$$Y_{n+1} \le \gamma \frac{b^n}{(1-\sigma)^{\frac{2(m+1)(N+2)}{Nqm}} k^{\frac{(m+1)(mq-m-1)}{qm}}} \left(\frac{\rho^2}{t}\right)^{\frac{m+1}{qm}} Y_n^{1+\frac{2(m+1)}{Nqm}},$$

where

$$b = 2^{\frac{2(m+1)}{m}(1+\frac{2(m+1)}{Nqm})}.$$

Now $Y_n \to 0$ as $n \to +\infty$, provided k is chosen such that

$$Y_o = \iint_{Q_o} u^{m+1} dx\,d\tau = \gamma(1-\sigma)^{N+2}\left(\frac{t}{\rho^2}\right)^{\frac{N}{2}} k^{\frac{N(m-1)+2m+2}{2}}.$$

With this choice

$$\begin{aligned} M_\sigma \le{}& \gamma \frac{1}{(1-\sigma)^{\frac{2(N+2)}{N(m-1)+2m+2}}}\left(\frac{\rho^2}{t}\right)^{\frac{N}{N(m-1)+2m+2}} \\ &\times \left(\iint_{Q_o} u^{m+1} dx\,d\tau\right)^{\frac{2}{N(m-1)+2m+2}}. \end{aligned} \tag{B.4.5}$$

Set

$$\rho_n = \sigma\rho + (1-\sigma)\rho\sum_{i=1}^n 2^{-i}, \qquad t_n = -\sigma t - (1-\sigma)t\sum_{i=1}^n 2^{-i},$$

$$Q_n = K_{\rho_n} \times (t_n, t], \quad \Rightarrow Q_\infty = K_\rho \times (-t, t], \quad Q_o = K_{\sigma\rho} \times (-\sigma t, t].$$

Writing (B.4.5) over the pair of cubes Q_n and Q_{n+1} gives

$$\begin{aligned} M_n \le \frac{\gamma 2^{\frac{2(N+2)n}{N(m-1)+2m+2}}}{(1-\sigma)^{\frac{2(N+2)}{N(m-1)+2m+2}}} M_{n+1}^{\frac{2(m+1-r)}{N(m-1)+2m+2}} \left(\frac{\rho^2}{t}\right)^{\frac{N}{N(m-1)+2m+2}} \\ \times \left(\iint_{Q_o} u^r\,dx\,d\tau\right)^{\frac{2}{N(m-1)+2m+2}}. \end{aligned}$$

By Lemma 5.2 of the Preliminaries, we conclude that

$$\sup_{K_{\sigma\rho}\times(-\sigma t, t]} u \le \frac{\gamma}{(1-\sigma)^{\frac{2(N+2)}{\lambda_r}}}\left(\frac{\rho^2}{t}\right)^{\frac{N}{\lambda_r}}\left(\iint_{K_\rho\times(-t,t]} u^r\,dx\,d\tau\right)^{\frac{2}{\lambda_r}} + \left(\frac{t}{\rho^2}\right)^{\frac{1}{1-m}}.$$

B.4.2 Proof of Proposition B.4.1 for $0 < m \le \frac{(N-2)_+}{N+2}$

The requirement $\lambda_r > 0$ implies

$$r > \frac{2N}{N+2} \ge qm = 2\frac{Nm+m+1}{N}.$$

Estimate

$$\iint_{Q_n} (u^m - k_{n+1}^m)_+^{\frac{m+1}{m}} dx\, d\tau \le \gamma \frac{2^{n\frac{r-(m+1)}{m}}}{k^{r-(m+1)}} \iint_{Q_n} (u^m - k_n^m)_+^{\frac{r}{m}} dx\, d\tau$$

$$\iint_{Q_n} (u^m - k_{n+1}^m)_+^2 dx\, d\tau \le \gamma \frac{2^{n\frac{r-2m}{m}}}{k^{r-2m}} \iint_{Q_n} (u^m - k_n^m)_+^{\frac{r}{m}} dx\, d\tau$$

$$\iint_{Q_n} u^{m+1}\chi_{[(u^m-k_{n+1}^m)_+>0]} dx\, d\tau \le \gamma \frac{2^{n\frac{r-(m+1)}{m}}}{k^{r-(m+1)}} \iint_{Q_n} (u^m - k_n^m)_+^{\frac{r}{m}} dx\, d\tau$$

$$\iint_{Q_n} u^{2m}\chi_{[(u^m-k_{n+1}^m)_+>0]} dx\, d\tau \le \gamma \frac{2^{n\frac{r-2m}{m}}}{k^{r-2m}} \iint_{Q_n} (u^m - k_n^m)_+^{\frac{r}{m}} dx\, d\tau.$$

Taking these estimates into account and assuming that $C\rho < 1$, the energy estimates (B.4.2) yield

$$\begin{aligned} &\sup_{t_n<\tau\le t} \int_{K_n} [(u^m - k_{n+1}^m)_+\zeta]^{\frac{m+1}{m}}(x,\tau)dx \\ &\qquad + \iint_{Q_n} |D[(u^m - k_{n+1}^m)_+\zeta]|^2 dx\, d\tau \\ &\quad \le \gamma \frac{2^{n\frac{r}{m}}}{(1-\sigma)^2 t}\Big[\Big(\frac{t}{\rho^2}\Big)\frac{1}{k^{r-2m}} + \frac{1}{k^{r-(m+1)}}\Big] \iint_{Q_n} (u^m - k_n^m)_+^{\frac{r}{m}} dx\, d\tau. \end{aligned}$$

Assuming (B.4.3) holds, this implies

$$\begin{aligned} &\sup_{t_n<\tau\le t} \int_{K_n} [(u^m - k_{n+1}^m)_+\zeta]^{\frac{m+1}{m}}(x,\tau)dx \\ &\qquad + \iint_{Q_n} |D[(u^m - k_{n+1}^m)_+\zeta]|^2 dx\, d\tau \\ &\quad \le \frac{\gamma 2^{n\frac{r}{m}}}{(1-\sigma)^2 t}\frac{1}{k^{r-(m+1)}} \iint_{Q_n} (u^m - k_n^m)_+^{\frac{r}{m}} dx\, d\tau. \end{aligned}$$

Set

$$Y_n = \frac{1}{|Q_n|}\iint_{Q_n} (u^m - k_n^m)_+^{\frac{r}{m}} dx\, d\tau$$

and estimate

$$Y_{n+1} \le \|u\|_{\infty,Q_o}^{r-mq}\Big(\frac{1}{|Q_n|}\iint_{Q_n} (u^m - k_{n+1}^m)_+^q dx\, d\tau\Big).$$

Applying the embedding Proposition 4.1 of the Preliminaries, the previous inequality can be rewritten as

$$Y_{n+1} \le \gamma \|u\|^{r-qm}_{\infty,Q_o} \Big(\frac{\rho^2}{t}\Big) \frac{b^n}{(1-\sigma)^{\frac{2}{N}(N+2)}} \frac{1}{k^{(r-(m+1))\frac{N+2}{N}}} Y_n^{1+\frac{2}{N}},$$

where $b = 2^{\frac{r(N+2)}{Nm}}$. Apply Lemma 5.1 of the Preliminaries, and conclude that $Y_n \to 0$ as $n \to +\infty$, provided k is chosen to satisfy

$$Y_o = \iint_{Q_o} u^r \, dx\, d\tau = \gamma(1-\sigma)^{N+2} \|u\|^{-(r-qm)\frac{N}{2}}_{\infty,Q_o} \Big(\frac{t}{\rho^2}\Big)^{\frac{N}{2}} k^{(r-(m+1))\frac{N+2}{2}},$$

which yields

$$M_\sigma \le \gamma \frac{M^{\frac{N(r-qm)}{(N+2)(r-(m+1))}}}{(1-\sigma)^{\frac{2}{r-(m+1)}}} \Big(\frac{\rho^2}{t}\Big)^{\frac{N}{(N+2)(r-(m+1))}} \times \Big(\iint_{Q_o} u^r \, dx\, d\tau\Big)^{\frac{2}{(r-(m+1))(N+2)}}.$$

The proof is concluded by a further application of the interpolation Lemma 5.2 of the Preliminaries. ■

Remark B.4.1 The proof shows that the boundedness of u plays a role only when $0 < m \le \frac{(N-2)_+}{N+2}$, and one does not need to assume it a priori, when $m > \frac{(N-2)_+}{N+2}$.

B.5 $L^r_{\rm loc}$ Estimates Backward in Time

Proposition B.5.1 *Let u be a nonnegative, local, weak solution to the singular equations (5.1)–(5.2) of Chapter 3, for $0 < m < 1$ in E_T, and assume that $u \in L^r_{loc}(E_T)$ for some $r > 1$. There exists a positive constant γ depending only on the data $\{m, N, C_o, C_1\}$ and r, such that either $C\rho > 1$, or*

$$\sup_{\tau \le s \le t} \int_{K_\rho(y)} u^r(x,s)dx \le \gamma \int_{K_{2\rho}(y)} u^r(x,\tau)dx + \gamma \Big[\frac{(t-\tau)^r}{\rho^{\lambda_r}}\Big]^{\frac{1}{1-m}}$$

for all cylinders

$$K_{2\rho}(y) \times [\tau, t] \subset E_T.$$

The constant $\gamma \to \infty$ as $r \to 1$.

Proof Assume $(y,\tau) = (0,0)$, fix $\sigma \in (0,1]$, and choose $\zeta \in C^\infty_o(K_{(1+\sigma)\rho})$ satisfying

$$0 \le \zeta \le 1 \;\text{ in }\; K_{(1+\sigma)\rho}, \quad \zeta = 1 \;\text{ in }\; K_\rho, \quad |D\zeta| \le \gamma(\sigma\rho)^{-1} \;\text{ in }\; K_{(1+\sigma)\rho}.$$

In the weak formulation of (5.1) of Chapter 3, take $u^{r-1}\zeta^2$ as a test function, modulo a standard Steklov averaging process. Integrating over $Q_s = K_{(1+\sigma)\rho} \times (0,s]$ with $s \in (0,t]$, gives

$$\begin{aligned} 0 &= \frac{1}{r}\iint_{Q_s} \zeta^2 \frac{d}{d\tau} u^r dx\, d\tau \\ &\quad + (r-1)\iint_{Q_s} \mathbf{A}(x,\tau,u,Du)\cdot Du\, u^{r-2}\zeta^2 dx\, d\tau \\ &\quad + 2\iint_{Q_s} \mathbf{A}(x,\tau,u,Du)\cdot D\zeta\,\zeta\, u^{r-1} dx\, d\tau \\ &\quad - \iint_{Q_s} B(x,\tau,u,Du) u^{r-1}\zeta^2 dx\, d\tau \\ &= \frac{1}{r}T_1 + (r-1)T_2 + T_3 + T_4. \end{aligned}$$

Since ζ is independent of t,

$$T_1 = \int_{K_{(1+\sigma)\rho}} u^r(x,s)\zeta^2(x)dx - \int_{K_{(1+\sigma)\rho}} u^r(x,0)\zeta^2(x)dx.$$

Next,

$$\begin{aligned} T_2 &\ge C_o m \iint_{Q_s} u^{r+m-3}|Du|^2\zeta^2 dx\, d\tau - C^2 \iint_{Q_s} u^{r+m-1}\zeta^2 dx\, d\tau \\ |T_3| &\le 2\iint_{Q_s} u^{r-1}[C_1 m u^{m-1}|Du||D\zeta| + C u^m |D\zeta|]\zeta\, dx\, d\tau \\ |T_4| &\le Cm \iint_{Q_s} |Du| u^{r+m-2}\zeta^2 dx\, d\tau + C^2 \iint_{Q_s} u^{r+m-1}\zeta^2 dx\, d\tau. \end{aligned}$$

Combining these estimates,

$$\begin{aligned} \int_{K_{(1+\sigma)\rho}} u^r(x,s)\zeta^2 dx &+ (r-1)C_o m \iint_{Q_s} u^{r+m-3}|Du|^2\zeta^2 dx\, d\tau \\ &\le m \iint_{Q_s} u^{r+m-2}|Du|\zeta(2C_1|D\zeta| + C\zeta) dx\, d\tau \\ &\quad + \iint_{Q_s} u^{r+m-1}\zeta(2C|D\zeta| + rC^2\zeta) dx\, d\tau \\ &\quad + \int_{K_{(1+\sigma)\rho}} u^r(x,0)\zeta^2 dx. \end{aligned}$$

By Young's inequality, and assuming that $C\rho \le 1$,

$$
\begin{aligned}
m\iint_{Q_s} u^{r+m-2}|Du|\zeta(2C_1|D\zeta|+C\zeta)dx\,d\tau
&\le (r-1)C_o m\iint_{Q_s} u^{r+m-3}|Du|^2\zeta^2 dx\,d\tau\\
&\quad+\frac{\gamma(r,m,C_o,C_1)}{\sigma^2\rho^2}\iint_{Q_s} u^{r+m-1}dx\,d\tau.
\end{aligned}
$$

Therefore

$$
\sup_{0\le s\le t}\int_{K_\rho} u^r(x,s)dx \le \int_{K_{(1+\sigma)\rho}} u^r(x,0)dx + \frac{\gamma}{\sigma^2\rho^2}\iint_{Q_s} u^{r+m-1}dx\,d\tau,
$$

where $\gamma=\gamma(N,r,m,C_o,C_1)$. Estimate

$$
\iint_{Q_s} u^{r+m-1}dx\,d\tau \le \gamma t\Big(\sup_{0\le s\le t}\int_{K_{(1+\sigma)\rho}} u^r(x,s)dx\Big)^{\frac{r+m-1}{r}}\rho^{\frac{N(1-m)}{r}}.
$$

Hence

$$
\begin{aligned}
\sup_{0\le s\le t}\int_{K_\rho} u^r(x,s)dx &\le \int_{K_{(1+\sigma)\rho}} u^r(x,0)dx\\
&\quad+\frac{\gamma}{\sigma^2}\Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}}\Big(\sup_{0\le s\le t}\int_{K_{(1+\sigma)\rho}} u^r dx\Big)^{\frac{r+m-1}{r}}.
\end{aligned}
$$

Now fix $R>0$ and consider the sequence of radii

$$
\rho_n = R\sum_{i=1}^n 2^{-i} \qquad \text{so that} \qquad \sigma_n = \frac{\rho_{n+1}-\rho_n}{\rho_n} \ge 2^{-n-2}.
$$

Setting

$$
Y_n = \sup_{0\le s\le t}\int_{K_{\rho_n}} u^r(x,s)dx,
$$

the previous estimates yield the recursive inequalities

$$
Y_n \le \int_{K_{2R}} u^r(x,0)dx + \gamma 2^n\Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}} Y_{n+1}^{\frac{r+m-1}{r}}.
$$

The proof is concluded by Lemma 5.2 of the Preliminaries. ■

Remark B.5.1 The proof shows that $\gamma(r)\to\infty$ as $r\to 1$ and that the condition $r>1$ cannot be relaxed to $r\ge 1$.

Remark B.5.2 Theorems 17.1 and 20.1 of Chapter 6 follow combining Proposition B.4.1 respectively with Propositions B.1.1 and B.5.1.

B.6 A DeGiorgi-Type Lemma for Nonnegative Subsolutions to Singular Porous Medium Type Equations

For a cylinder $(y,s)+Q^-_{2\rho}(\theta)\subset E_T$ denote by $\mu_\pm$ and ω numbers satisfying

$$\mu_+\ge \operatorname*{ess\,sup}_{[(y,s)+Q^-_{2\rho}(\theta)]} u,\qquad \mu_-\le \operatorname*{ess\,inf}_{[(y,s)+Q^-_{2\rho}(\theta)]} u,\qquad \omega=\mu_+-\mu_-.$$

Denote by ξ and a fixed numbers in $(0,1)$.

Lemma B.6.1 *Let u be a nonnegative, locally bounded, local, weak subsolution to the singular equations (5.1)–(5.2) of Chapter 3, for $0<m<1$, in E_T. Assume that*

$$\omega\ge \tfrac{12}{13}\mu_+. \tag{B.6.1}$$

There exists a positive number ν_*, depending on ω, θ, ξ, a, and the data $\{m,N,C_o,C_1\}$, such that if

$$\big|[u\ge\mu_+-\xi\omega]\cap[(y,s)+Q^-_{2\rho}(\theta)]\big|\le\nu_*|Q^-_{2\rho}(\theta)|,$$

then either $C\rho>\xi$ or

$$u\le\mu_+-a\xi\omega\qquad \textit{a.e. in }\ (y,s)+Q^-_\rho(\theta).$$

Proof Assume $(y,s)=(0,0)$ and introduce the sequence of cubes $\{K_n\}$ and cylinders $\{Q_n\}$ as in (3.6) of Chapter 3 with $p=2$, and a nonnegative, piecewise smooth cutoff function on Q_n of the form $\zeta(x,t)=\zeta_1(x)\zeta_2(t)$ defined as in (3.8) of Chapter 3 with $p=2$. Introduce the sequence of truncating levels

$$k_n=\mu_+-\xi_n\omega\qquad\text{with}\qquad \xi_n=a\xi+\frac{1-a}{2^n}\xi,$$

and write down the energy estimates (B.2.2) over the cylinder Q_n, for the truncated function $(u-k_n)_+$. Taking also into account (B.6.1), this gives

$$\begin{aligned}
&\sup_{-\theta\rho_n^2<t\le0}\int_{K_n}(u-k_n)_+^2\zeta^2(x,t)dx\\
&\qquad+C_om\iint_{Q_n}u^{m-1}|D[(u-k_n)_+\zeta]|^2dx\,d\tau\\
&\qquad\le\gamma\frac{2^{2n}}{\rho^2}(\xi\omega)^2\iint_{Q_n}\Big(\frac{\omega^{m-1}}{(1-\xi)^{1-m}}+\frac1\theta\Big)\chi_{[u>k_n]}dx\,d\tau\\
&\qquad\quad+\gamma\Big(\frac{C}{\xi}\Big)^2(\xi\omega)^2\omega^{m-1}\iint_{Q_n}\chi_{[u>k_n]}dx\,d\tau\\
&\qquad\le\gamma\frac{2^{2n}}{\rho^2}(\xi\omega)^2\frac{\omega^{m-1}}{(1-\xi)^{1-m}}\Big(1+\frac{1}{\theta\omega^{m-1}}\Big)|[u>k_n]\cap Q_n|\\
&\qquad\quad+\gamma\frac{1}{\rho^2}\Big(\frac{C\rho}{\xi}\Big)^2(\xi\omega)^2\omega^{m-1}\big|[u>k_n]\cap Q_n|.
\end{aligned}$$

Therefore, if $C\rho \le \xi$,

$$
\begin{aligned}
&\sup_{-\theta\rho_n^2<t\le 0}\int_{K_n}(u-k_n)_+^2\zeta^2(x,t)dx \\
&\qquad + C_o m\iint_{Q_n} u^{m-1}|D[(u-k_n)_+\zeta]|^2 dx\,d\tau \\
&\qquad \le \gamma\frac{2^{2n}}{\rho^2}(\xi\omega)^2\frac{\omega^{m-1}}{(1-\xi)^{1-m}}\Big(1+\frac{1}{\theta\omega^{m-1}}\Big)|[u>k_n]\cap Q_n|.
\end{aligned}
$$

To estimate below the second integral on the left-hand side, take into account that $u\le\mu_+$ and (B.6.1). This gives

$$
\begin{aligned}
&\iint_{Q_n} u^{m-1}|D[(u-k_n)_+\zeta]|^2 dx\,d\tau \\
&\qquad \ge \big(\tfrac{13}{12}\omega\big)^{m-1}\iint_{Q_n}|D[(u-k_n)_+\zeta]|^2 dx\,d\tau.
\end{aligned}
$$

Setting

$$
A_n=[u>k_n]\cap Q_n \quad\text{and}\quad Y_n=\frac{|A_n|}{|Q_n|},
$$

and combining these estimates gives

$$
\begin{aligned}
&\sup_{-\theta\rho_n^2<t\le 0}\int_{K_n}(u-k_n)_+^2\zeta^2(x,t)dx \\
&\qquad + \omega^{m-1}\iint_{Q_n}|D[(u-k_n)_+\zeta]|^2 dx\,d\tau \\
&\qquad \le \gamma\frac{2^{2n}}{\rho^2}(\xi\omega)^2\frac{\omega^{m-1}}{(1-\xi)^{1-m}}\Big(1+\frac{1}{\theta\omega^{m-1}}\Big)|A_n|.
\end{aligned}
\tag{B.6.2}
$$

Apply Hölder's inequality and the embedding Proposition 4.1 of the Preliminaries, and recall that $\zeta=1$ on Q_{n+1}, to get

$$
\begin{aligned}
\Big(\frac{1-a}{2^{n+1}}\Big)^2(\xi\omega)^2|A_{n+1}| &\le \iint_{Q_{n+1}}(u-k_n)_+^2 dx\,d\tau \\
&\le \Big(\iint_{Q_n}[(u-k_n)_+\zeta]^{2\frac{N+2}{N}}dx\,d\tau\Big)^{\frac{N}{N+2}}|A_n|^{\frac{2}{N+2}} \\
&\le \gamma\Big(\iint_{Q_n}|D[(u-k_n)_+\zeta]|^2 dx\,d\tau\Big)^{\frac{N}{N+2}} \\
&\quad\times\Big(\sup_{-\theta\rho_n^2<t\le 0}\int_{K_n}[(u-k_n)_+\zeta]^2(x,t)dx\Big)^{\frac{2}{N+2}}|A_n|^{\frac{2}{N+2}}
\end{aligned}
$$

for a constant γ depending only on N. Combine this with (B.6.2) to get

$$|A_{n+1}| \le \frac{\gamma\, 2^{4n}}{(1-a)^2\rho^2} \frac{\omega^{\frac{2(m-1)}{N+2}}}{(1-\xi)^{1-m}} \Big(1 + \frac{1}{\theta\omega^{m-1}}\Big) |A_n|^{1+\frac{2}{N+2}}.$$

In terms of $Y_n = \frac{|A_n|}{|Q_n|}$ this can be rewritten as

$$Y_{n+1} \le \frac{\gamma 2^{4n}}{(1-a)^2(1-\xi)^{1-m}} \frac{\big(1+\theta\omega^{m-1}\big)}{\big(\theta\omega^{m-1}\big)^{\frac{N}{N+2}}} Y_n^{1+\frac{2}{N+2}}.$$

By Lemma 5.1 of the Preliminaries, $\{Y_n\} \to 0$ as $n \to \infty$, provided

$$Y_o = \frac{|A_o|}{|Q_o|} \le \Big[\frac{(1-a)^2(1-\xi)^{1-m}}{\gamma 4^{N+2}}\Big]^{\frac{N+2}{2}} \frac{\big(\theta\omega^{m-1}\big)^{\frac{N}{2}}}{\big(1+\theta\omega^{m-1}\big)^{\frac{N+2}{2}}} \stackrel{\text{def}}{=} \nu_*. \qquad \blacksquare$$

When

$$\theta = \nu\, \omega^{1-m}, \tag{B.6.3}$$

for some $\nu \in (0,1)$, that is, when the relative length of the cylinders $(y,s) + Q_\rho^\pm(\theta)$ is of the order of $\nu\omega^{1-m}$, then the functional dependence of ν_*, on ν, ξ, and a is

$$\nu_* = \gamma^{-1}(1-a)^{N+2}(1-\xi)^{(1-m)\frac{N+2}{2}}\, \nu^{\frac{N}{2}} \tag{B.6.4}$$

for a quantitative constant $\gamma = \gamma(m, N, C_o, C_1) > 1$, independent of ν, ξ, and a.

B.7 A Logarithmic Estimate for Nonnegative Subsolutions to Singular Porous Medium Type Equations

Introduce the logarithmic function

$$\psi(u) = \ln^+ \Big[\frac{H}{H - (u-k)_+ + c}\Big] \tag{B.7.1}$$

where

$$H = \operatorname*{ess\,sup}_{(y,s)+Q_\rho^-(\theta)} (u-k)_+, \qquad \text{and} \qquad 0 < c < \min\{1\,,\, H\}$$

and for $s > 0$

$$\ln^+ s = \max\{\ln s, 0\}.$$

In the cylinder $(y,s) + Q_\rho^-(\theta)$ take a nonnegative, piecewise smooth cutoff function ζ independent of t.

Proposition B.7.1 *Let u be a nonnegative, locally bounded, local, weak subsolution to singular porous medium type equations (5.1)–(5.2) of Chapter 3, for $0 < m < 1$ in E_T. There exists a constant γ, depending only on the data $\{m, N, C_o, C_1\}$, such that for every cylinder*

$$(y, s) + Q_\rho^-(\theta) \subset E_T$$

and for every level $k \geq 0$,

$$\begin{aligned}
\sup_{s-\theta\rho^2<t<s} \int_{K_\rho(y)} & \psi^2(u)(x,t)\zeta^2(x)dx \\
&\leq \int_{K_\rho(y)} \psi^2(u)(x, s-\theta\rho^2)\zeta^2(x)dx \\
&+ \gamma \iint_{(y,s)+Q_\rho^-(\theta)} u^{m-1}\psi(u)\left|D\zeta\right|^2 dx\,dt \\
&+ \gamma C^2 \iint_{(y,s)+Q_\rho^-(\theta)} u^{m-1}\psi(u)\zeta^2 dx\,dt \\
&+ \frac{\gamma C^2}{c^2}\left(1 + \ln\frac{H}{c}\right) \iint_{(y,s)+Q_\rho^-(\theta)} u^{m+1}\chi_{[(u-k)_+>0]}\zeta^2 dx\,dt.
\end{aligned} \tag{B.7.2}$$

Analogous estimates hold for "forward" cylinders $(y, s) + Q_\rho^+(\theta)$.

Proof Take $(y, s) = (0, 0)$ and work within the cylinder Q^t introduced in (B.2.3). In the weak formulation of (5.1) of Chapter 3 take the testing function

$$\varphi = \frac{\partial}{\partial u}\left[\psi^2(u)\right]\zeta^2 = 2\psi\psi'\zeta^2.$$

By direct calculation

$$\left[\psi^2(u)\right]'' = 2(1+\psi)\psi'^2 \in L^\infty_{loc}(E_T)$$

which implies that such a φ is an admissible testing function, modulo a Steklov averaging process. Since $\psi(u)$ vanishes on the set where $(u-k)_+ = 0$,

$$\iint_{Q^t} u_\tau[\psi^2]'\zeta^2 dx\,d\tau = \int_{K_\rho} \psi^2(x,t)\zeta^2 dx - \int_{K_\rho} \psi^2(x, -\theta\rho^2)\zeta^2 dx.$$

The remaining terms are estimated by using the structure conditions (5.2) of Chapter 3.

$$\begin{aligned}
\iint_{Q^t} \mathbf{A}(x,\tau,u,Du)\cdot D\varphi\,dx\,d\tau \\
&\ge 2mC_o \iint_{Q^t} (1+\psi)\psi'^2 u^{m-1}|Du|^2\zeta^2 dx\,d\tau \\
&\quad - 2C^2 \iint_{Q^t} (1+\psi)\psi'^2 u^{m+1}\zeta^2 dx\,d\tau \\
&\quad - 4mC_1 \iint_{Q^t} u^{m-1}|Du|\psi\psi'\zeta|D\zeta| dx\,d\tau \\
&\quad - 4C \iint_{Q^t} \psi\psi' u^m \zeta|D\zeta| dx\,d\tau.
\end{aligned}$$

From this, by repeated application of Young's inequality

$$\begin{aligned}
\iint_{Q^t} \mathbf{A}(x,\tau,u,Du)\cdot D\varphi\,dx\,d\tau \\
&\ge mC_o \iint_{Q^t} (1+\psi)\psi'^2 u^{m-1}|Du|^2\zeta^2 dx\,d\tau \\
&\quad - \gamma C^2 \iint_{Q^t} (1+\psi)\psi'^2 u^{m+1}\zeta^2 dx\,d\tau \\
&\quad - \gamma \iint_{Q^t} \psi u^{m-1}|D\zeta|^2 dx\,d\tau.
\end{aligned}$$

The forcing terms are estimated as

$$\begin{aligned}
2\iint_{Q^t} |B(x,\tau,u,Du)\psi\psi'\zeta^2| dx\,d\tau \\
&\le 2mC \iint_{Q^t} u^{m-1}|Du|(1+\psi)\psi'\zeta^2 dx\,d\tau \\
&\quad + 2C^2 \iint_{Q^t} u^m \psi\psi'\zeta^2 dx\,d\tau \\
&\le mC_o \iint_{Q^t} (1+\psi)\psi'^2 u^{m-1}|Du|^2\zeta^2 dx\,d\tau \\
&\quad + \gamma C^2 \iint_{Q^t} u^{m-1}\psi\zeta^2 dx\,d\tau \\
&\quad + \gamma C^2 \iint_{Q^t} (1+\psi)\psi'^2 u^{m+1}\zeta^2 dx\,d\tau.
\end{aligned}$$

By the definition of $\psi(u)$, estimate

$$\psi \le \ln\left(\frac{H}{c}\right), \qquad \psi' \le \frac{1}{c}.$$

Collecting these estimates establishes the proposition. ■

B.8 Hölder Continuity

Let $u \in L^\infty(E_T)$, and denote by $\varGamma$ the parabolic boundary of E_T,

$$\varGamma = \partial E_T - \bar{E} \times \{T\}.$$

For compact set $K \subset E_T$ introduce the parabolic m-distance from K to $\varGamma$, intrinsic to u, by

$$m - \operatorname{dist}(K;\varGamma) \overset{\text{def}}{=} \inf_{\substack{(x,t)\in K\\(y,s)\in\varGamma}} \Big(|x-y| + \|u\|_{\infty,E_T}^{\frac{m-1}{2}} |t-s|^{\frac{1}{2}} \Big).$$

Theorem B.8.1 *Let u be a nonnegative, bounded, local, weak solution to the singular porous medium type equations (5.1)–(5.2) of Chapter 3, for $0 < m < 1$ in E_T. Then u is locally Hölder continuous in E_T, and there exist constants $\gamma > 1$ and $\alpha \in (0,1)$ that can be determined a priori only in terms of the data $\{m, N, C_o, C_1\}$ and C, such that for every compact set $K \subset E_T$,*

$$|u(x_1,t_1) - u(x_2,t_2)| \le \gamma \|u\|_{\infty,E_T} \left(\frac{|x_1 - x_2| + \|u\|_{\infty,E_T}^{\frac{m-1}{2}} |t_1 - t_2|^{\frac{1}{2}}}{m - \operatorname{dist}(K;\varGamma)} \right)^\alpha$$

for every pair of points (x_1,t_1), and $(x_2,t_2) \in K$.

B.8.1 Proof of Theorem B.8.1. Preliminaries

For a fixed $(x_o,t_o) \in E_T$ and fixed numbers

$$\delta \in (0,1), \qquad b > 1, \qquad R,\, \omega > 0,$$

construct the sequences

$$R_o = R, \quad R_n = \frac{R}{b^n}; \qquad \omega_o = \omega, \quad \omega_{n+1} = \delta\omega_n \quad \text{for } n = 0,1,2,\dots$$

and the cylinders

$$Q_n = K_{R_n}(x_o) \times \big(t_o - \omega_n^{1-m} R_n^2, t_o\big] \quad \text{for } n = 1,2,\dots.$$

The function u is Hölder continuous at $(x_o,t_o) \in E_T$ if the constants $\delta \in (0,1)$ and $b > 1$ can be determined, independent of u and (x_o,t_o), such that

$$Q_{n+1} \subset Q_n \subset Q_o \subset E_T \quad \text{and} \quad \operatorname*{ess\,osc}_{Q_n} u \le \omega_n \tag{B.8.1}$$

for all $n = 0,1,\dots$. Having fixed $(x_o,t_o) \in E_T$ assume it coincides with the origin of $\mathbb{R}^{N+1}$ and for $R > 0$ set

$$Q_{R_o} = K_{R_o} \times (-R_o^2, 0], \tag{B.8.2}$$

where R_o is so small that $Q_{R_o} \subset E_T$. Set also

$$\mu_o^+ = \operatorname*{ess\,sup}_{Q_{R_o}} u, \quad \mu_o^- = \operatorname*{ess\,inf}_{Q_{R_o}} u, \quad \omega_o = \mu_o^+ - \mu_o^- = \operatorname*{ess\,osc}_{Q_{R_o}} u. \tag{B.8.3}$$

Since u is locally bounded in E_T, without loss of generality we may assume that $\omega_o \le 1$ so that

$$Q_o = K_{R_o} \times (-\omega_o^{1-m} R_o^2, 0] \subset Q_{R_o} \subset E_T \tag{B.8.4}$$

and

$$\operatorname*{ess\,osc}_{Q_o} u \le \omega_o.$$

Thus (B.8.1) holds for $n = 0$. We will determine numbers $b > 1$ and $\delta \in (0,1)$ depending only on the set of data $\{m, N, C_o, C_1\}$ and C, and independent of u and (x_o, t_o) for which (B.8.1) holds inductively for all n.

B.9 The Induction Argument. Two Alternatives

Assuming (B.8.1) holds for n, remove the index n by setting $\omega_n = \omega$, and

$$2r = R_n, \qquad Q_n = Q_{2r}(\theta) = K_{2r} \times (-\theta 4r^2, 0], \qquad \theta = \omega^{1-m},$$

and

$$\mu_n^+ = \mu^+ = \operatorname*{ess\,sup}_{Q_{2r}(\theta)} u, \qquad \mu_n^- = \mu^- = \operatorname*{ess\,inf}_{Q_{2r}(\theta)} u.$$

Lemma B.9.1 *There exists a number ν depending on the data $\{m, N, C_o, C_1\}$, such that if*

$$|[u \le \tfrac{1}{2}\omega] \cap Q_r(\theta)| \le \nu |Q_r(\theta)|, \tag{B.9.1}$$

then either $Cr > 1$, or

$$u \ge \tfrac{1}{4}\omega \qquad a.e.\ in \quad Q_{\frac{1}{2}r}(\theta). \tag{B.9.2}$$

Proof This is the content of Lemma 10.1 of Chapter 3 applied for $\xi = a = \frac{1}{2}$. The number ν is determined in (10.5) of Chapter 3, and because of the choice of θ, it is determined a priori only in terms of the data $\{m, N, C_o, C_1\}$. ■

The proof unfolds along two alternatives. The first is that (B.9.1) holds, thereby providing the lower bound (B.9.2) for u, away from the singularity. The second is that (B.9.1) fails, whose consequences are examined in § B.10–§ B.12.

Taking into account the definition of μ^+ and ω, from here on we assume that

$$\tfrac{1}{2}\omega < \mu^+ - \tfrac{1}{4}\omega < \tfrac{5}{6}\omega. \tag{B.9.3}$$

The left-hand inequality can be taken as holding in all cases. Indeed if it did not, in any smaller cylinder contained in $Q_{2r}(\theta)$ the oscillation would be reduced by a factor $\frac{3}{4}$ and there would be nothing to prove. The right-hand inequality coincides with (B.6.1), which we assume. The case when it fails will be examined later.

Lemma B.9.2 *Let the assumption (B.9.1) of Lemma B.9.1 be violated. There exists a time level s, in the interval*

$$-\theta r^2 \le s \le -\tfrac{1}{2}\nu\theta r^2 \tag{B.9.4}$$

such that

$$|[u(\cdot,s) < \tfrac{1}{2}\omega] \cap K_r| > \tfrac{1}{2}\nu|K_r|. \tag{B.9.5}$$

This in turn implies

$$|[u(\cdot,s) > \mu^+ - \tfrac{1}{4}\omega] \cap K_r| \le (1 - \tfrac{1}{2}\nu)|K_r|. \tag{B.9.6}$$

Proof If (B.9.5) does not hold for any s in the range (B.9.4), then

$$\begin{aligned} |[u < \tfrac{1}{2}\omega] \cap Q_r(\theta)| &= \int_{-\theta r^2}^{-\frac{1}{2}\nu\theta r^2} |[u\cdot, s) < \tfrac{1}{2}\omega] \cap K_r| ds \\ &\quad + \int_{-\frac{1}{2}\nu\theta r^2}^{0} |[u\cdot, s) < \tfrac{1}{2}\omega] \cap K_r| ds \\ &\le \nu|Q_r(\theta)|. \end{aligned}$$

This proves (B.9.5), and also (B.9.6), due to (B.9.3). ■

B.10 A Uniform Time Control on the Measure of the Level Sets

The next lemma asserts that a property similar to (B.9.6) continues to hold for all time levels from s up to 0.

Lemma B.10.1 *There exists a positive integer n_* depending only on the data $\{m, N, C_o, C_1\}$ and the number ν claimed by Lemma B.9.1, such that either*

$$2^{n_*} Cr > 1$$

or

$$\left|[u(\cdot,t) > \mu^+ - \frac{\omega}{2^{n_*}}] \cap K_r\right| < \left(1 - \tfrac{1}{4}\nu^2\right)|K_r| \tag{B.10.1}$$

for all times $s < t < 0$.

Proof Consider the logarithmic estimates (B.7.2), written over the cylinder $K_r \times (s, 0)$, for the function $(u-k)_+$ and for the level

$$k = \mu^+ - \tfrac{1}{4}\omega.$$

The number c in the definition (B.7.1) is taken as

$$c = \frac{\omega}{2^{n+2}},$$

where n is a positive number to be chosen. Thus we take

$$\psi(u) = \ln^+ \left\{ \frac{H}{H - \left[u - (\mu^+ - \frac{1}{4}\omega)\right]_+ + \frac{1}{2^{n+2}}\omega} \right\}$$

where

$$H = \operatorname*{ess\,sup}_{K_r \times (s,0)} \left[u - (\mu^+ - \tfrac{1}{4}\omega)\right]_+ .$$

The cutoff function $x \to \zeta(x)$ is taken so that

$$\zeta = 1 \ \text{ on } \ K_{(1-\sigma)r}, \ \text{ for } \ \sigma \in (0,1), \qquad \text{and} \qquad |D\zeta| \le (\sigma r)^{-1}.$$

With these choices, the inequalities (B.7.2) of Proposition B.7.1 yield

$$\begin{aligned} \int_{K_{(1-\sigma)r}} \psi^2(u)(x,t)dx \le \int_{K_r} &\psi^2(u)(x,s)dx \\ &+ \frac{\gamma}{(\sigma r)^2} \int_s^0 \int_{K_r} u^{m-1}\psi(u)dx\,d\tau \\ &+ \gamma \frac{(Cr)^2}{(\sigma r)^2} \int_s^0 \int_{K_r} u^{m-1}\psi(u)dx\,d\tau \\ &+ \gamma C^2 \left(\frac{\omega}{2^{n+2}}\right)^{-2} \left[1 + \ln H \left(\frac{\omega}{2^{n+2}}\right)^{-1}\right] \\ &\quad \times \int_s^0 \int_{K_r} u^{m+1}\chi_{[(u-k)_+ > 0]}dx\,d\tau \end{aligned} \tag{B.10.2}$$

for all $s < t < 0$. Estimate

$$\psi \le n \ln 2, \qquad \left[1 + \ln H \left(\frac{\omega}{2^{n+2}}\right)^{-1}\right] \le \gamma n \ln 2.$$

To estimate the first integral on the right-hand side of (B.10.2), observe that ψ vanishes on the set $[u < \mu^+ - \frac{1}{4}\omega]$. Therefore using (B.9.6) of Lemma B.9.2,

$$\int_{K_r} \psi^2(u)(x,s)dx \le n^2 \ln^2 2\left(1 - \tfrac{1}{2}\nu\right)|K_r|.$$

The remaining integrals are estimated as

$$\begin{aligned} &\frac{\gamma}{(\sigma r)^2} \int_s^0 \int_{K_r} u^{m-1}\psi\,dx\,d\tau \le \frac{\gamma n}{(\sigma r)^2}\omega^{1-m} r^2 \omega^{m-1}|K_r| \le \frac{\gamma n}{\sigma^2}|K_r|; \\ &\gamma C^2 \int_s^0 \int_{K_r} u^{m-1}\psi\,dx\,d\tau \le (Cr)^2 \frac{\gamma n}{\sigma^2}|K_r|; \\ &\gamma C^2 \left(\frac{2^{n+2}}{\omega}\right)^2 \left[1 + \ln H \left(\frac{2^{n+2}}{\omega}\right)\right] \int_s^0 \int_{K_r} u^{m+1}\chi_{[(u-k)_+ > 0]}dx\,d\tau \\ &\qquad\qquad \le (2^n Cr)^2 \frac{\gamma n}{\sigma^2}|K_r|. \end{aligned}$$

Assume momentarily that n has been chosen, and stipulate that

$$2^n C\, r < 1.$$

Then combining these estimates gives

$$\int_{K_{(1-\sigma)r}} \psi^2(u)(x,t)dx \le n^2 \ln^2 2\big(1 - \tfrac{1}{2}\nu\big)|K_r| + \frac{\gamma n}{\sigma^2}|K_r| \tag{B.10.3}$$

for all $s < t < 0$. The left-hand side of (B.10.3) is estimated below by integrating over the smaller set

$$\big[u(\cdot,t) > \mu^+ - \tfrac{1}{2^{n+2}}\omega\big] \cap K_{(1-\sigma)r}.$$

On such a set, since ψ is a decreasing function of H, estimate below

$$\psi^2 \ge \ln^2\left(\frac{\frac{\omega}{4}}{\frac{\omega}{2^{n+1}}}\right) = (n-1)^2 \ln^2 2.$$

Carry this in (B.10.3) and divide through by $(n-1)^2 \ln^2 2$, to obtain

$$\Big|\Big[u(\cdot,t) > \mu^+ - \frac{\omega}{2^{n+2}}\Big] \cap K_{(1-\sigma)r}\Big| \le \Big(\frac{n}{n-1}\Big)^2\big(1 - \tfrac{1}{2}\nu\big)|K_r| + \frac{\gamma}{n\sigma^2}|K_r|$$

for all $s < t < 0$. On the other hand,

$$\begin{aligned}
&\Big|\Big[u(\cdot,t) > \mu^+ - \frac{\omega}{2^{n+2}}\Big] \cap K_r\Big| \\
&\le \Big|\Big[u(\cdot,t) > \mu^+ - \frac{\omega}{2^{n+2}}\Big] \cap K_{(1-\sigma)r}\Big| + |K_r - K_{(1-\sigma)r}| \\
&\le \Big|\Big[u(\cdot,t) > \mu^+ - \frac{\omega}{2^{n+2}}\Big] \cap K_{(1-\sigma)r}\Big| + N\sigma|K_r|.
\end{aligned}$$

Therefore for all $s < t < 0$,

$$\Big|\Big[u(\cdot,t) > \mu^+ - \frac{\omega}{2^{n+2}}\Big] \cap K_r\Big| \le \Big[\Big(\frac{n}{n-1}\Big)^2\big(1 - \tfrac{1}{2}\nu\big) + \frac{\gamma}{n\sigma^2} + N\sigma\Big]|K_r|.$$

To prove the lemma choose σ so small, and then n so large, as to satisfy (B.10.1) with $n_* = n + 2$. ∎

Remark B.10.1 Since the number ν is independent of ω, r, and s, also n_* is independent of all these parameters.

Corollary B.10.1 *Let ν and n_* be the numbers determined by Lemmas B.9.1 and B.10.1, respectively, depending only on the data $\{m, N, C_o, C_1\}$. Then either $2^{n_*}Cr > 1$, or*

$$\Big|\Big[u(\cdot,t) > \mu^+ - \frac{\omega}{2^j}\Big] \cap K_r\Big| < \big(1 - \tfrac{1}{4}\nu^2\big)|K_r| \tag{B.10.4}$$

for all $j \ge n_*$, and for all times

$$-\tfrac{1}{2}\nu\omega^{1-m}r^2 < t < 0. \tag{B.10.5}$$

B.11 The Second Alternative Continued

Motivated by the time range (B.10.5), introduce the cylinder

$$Q_r(\theta_*) = K_r \times (-\theta_* r^2, 0], \qquad \theta_* = \tfrac{1}{2}\nu\omega^{1-m}. \tag{B.11.1}$$

Lemma B.11.1 *For every $\nu_* \in (0,1)$ there exists a positive integer q_*, depending only on the data $\{m, N, C_o, C_1\}$ and ν_*, such that either*

$$2^{n_*+q_*} Cr > 1$$

or

$$\Big|\Big[u > \mu^+ - \frac{\omega}{2^{n_*+q_*}}\Big] \cap Q_r(\theta_*)\Big| < \nu_* |Q_r(\theta_*)|. \tag{B.11.2}$$

Proof Set $Q_r(\theta_*) = Q$, and write down the energy estimates (B.2.2) for the truncated functions

$$(u - k_j)_+ \quad \text{where} \quad k_j = \mu^+ - \frac{\omega}{2^j} \quad \text{for} \quad j = n_*, \dots, n_* + q_*,$$

over the pair of cylinders

$$Q \quad \text{and} \quad Q' = K_{2r} \times (-\nu\omega^{1-m} r^2, 0].$$

The cutoff function ζ is taken to be one on Q, vanishing on the parabolic boundary of Q', and such that

$$|D\zeta| \le \frac{1}{r} \qquad \text{and} \qquad 0 \le \zeta_t \le \frac{2}{\nu\theta r^2}, \qquad \theta = \omega^{1-m}.$$

With these stipulations, and taking also into account (B.9.3), the energy estimates (B.2.2) take the form

$$\omega^{m-1} \iint_Q |D(u - k_j)_+|^2 dx\, dt \le \frac{\gamma}{\nu} \frac{\omega^{m-1}}{r^2} \big[1 + (2^{n_*+q_*} Cr)^2\big] \Big(\frac{\omega}{2^j}\Big)^2 |Q|$$

for a constant γ depending only on the data $\{m, N, C_o, C_1\}$. Therefore if $2^{n_*+q_*} Cr < 1$, then

$$\iint_Q |D(u - k_j)_+|^2 dx\, dt \le \frac{\gamma}{\nu r^2} \Big(\frac{\omega}{2^j}\Big)^2 |Q|. \tag{B.11.3}$$

Next, apply the discrete isoperimetric inequality (2.1) of Lemma 2.2 of Chapter 3 to the function $u(\cdot, t)$, for t in the range (B.10.5), over the cube K_r, for the levels

$$k = k_j < \ell = k_{j+1} \quad \text{so that} \quad (\ell - k) = \frac{\omega}{2^{j+1}}.$$

Taking also into account (B.10.4) this gives

$$
\begin{aligned}
&\frac{\omega}{2^{j+1}}|[u(\cdot,t) > k_{j+1}] \cap K_r| \\
&\qquad \le \frac{\gamma r^{N+1}}{|[u(\cdot,t) < k_j] \cap K_r|} \int_{[k_j < u(\cdot,t) < k_{j+1}] \cap K_r} |Du| dx \\
&\qquad \le \frac{\gamma}{\nu^2} r \Big(\int_{[k_j < u(\cdot,t) < k_{j+1}] \cap K_r} |Du(\cdot,t)|^2 dx \Big)^{\frac{1}{2}} \\
&\qquad\quad \times \Big| \big([u(\cdot,t) > k_j] - [u(\cdot,t) > k_{j+1}] \big) \cap K_r \Big|^{\frac{1}{2}}.
\end{aligned}
$$

Set

$$
A_j = [u > k_j] \cap Q = \int_{-\frac{1}{2}\nu\theta r^2}^{0} \Big| \big[u(\cdot,t) > k_j \big] \cap K_r \Big| dt
$$

and integrate in dt over the time interval (B.10.5). This gives

$$
\frac{\omega}{2^j}|A_{j+1}| \quad \le \frac{\gamma}{\nu^2} r \Big(\iint_Q |D(u-k_j)_+|^2 dx\, dt \Big)^{\frac{1}{2}} \big(|A_j| - |A_{j+1}| \big)^{\frac{1}{2}}.
$$

Square both sides of this inequality and estimate above, the term containing $D(u-k_j)_+$ by the energy inequality (B.11.3), to obtain

$$
|A_{j+1}|^2 \le \frac{\gamma}{\nu^5} |Q| \big(|A_j| - |A_{j+1}| \big).
$$

Add these recursive inequalities for

$$
j = n_* + 1, n_* + 2, \dots, n_* + q_* - 1
$$

where q_* is to be chosen. Majorizing the right-hand side with the corresponding telescopic series gives

$$
(q_* - 2)|A_{n_*+q_*}|^2 \le \sum_{j=n_*+1}^{n_*+q_*-1} |A_{j+1}|^2 \le \frac{\gamma}{\nu^5}|Q|^2.
$$

From this

$$
|A_{n_*+q_*}| \le \frac{1}{\sqrt{q_* - 2}} \sqrt{\frac{\gamma}{\nu^5}}\, |Q|.
$$

The number ν_* being fixed choose q_* from

$$
\frac{1}{\sqrt{q_* - 2}} \sqrt{\frac{\gamma}{\nu^5}} = \nu_*, \tag{B.11.4}
$$

in order to satisfy the thesis. ■

To proceed, apply Lemma B.6.1 to the cylinder $Q_r(\theta_*)$, with θ_* given in (B.11.1) and for the choices

$$
r = 2\rho, \qquad \xi = \frac{1}{2^{n_*+q_*}}, \qquad a = \frac{1}{2}.
$$

The conclusion is that either

$$2^{n_*+q_*}Cr > 1$$

or

$$u \le \mu^+ - \frac{\omega}{2^{n_*+q_*+1}} \qquad \text{in} \qquad Q_{\frac{1}{2}r}(\theta_*) \tag{B.11.5}$$

provided ν_* is chosen from (B.6.3)–(B.6.4) and then, in turn, q_* is chosen from (B.11.4).

B.12 The Induction Argument Concluded-(i)

The numbers ν, ν_*, n_*, and q_* being determined by the previous procedure, we assume that the radius R_o in (B.8.2) that starts the sequence of the R_n is so small that

$$2^{n_*+q_*}CR_o < 1$$

so that this inequality continues to hold for all $R \le R_o$. If the assumption (B.9.1) is violated, then by the arguments of the second alternative, (B.11.5) holds true, which implies

$$\operatorname*{ess\,sup}_{Q_{\frac{1}{2}r}(\theta_*)} u \le \mu^+ - \frac{\omega}{2^{n_*+q_*+1}}$$

provided (B.9.3) is in force. Since the left-hand inequality in (B.9.3) can be always assumed, the right-hand inequality holds true if

$$\mu^- < \tfrac{1}{12}\omega. \tag{B.12.1}$$

This in turn coincides with (B.6.1) and guarantees that the arguments of the second alternative are in force. Assuming (B.12.1) for the moment, subtracting

$$\operatorname*{ess\,inf}_{Q_{\frac{1}{2}r}(\theta_*)} u$$

from the left-hand side, and μ^- from the right-hand side, of the previous oscillation inequality, gives

$$\operatorname*{ess\,osc}_{Q_{\frac{1}{2}r}(\theta_*)} u \le \left(1 - \frac{1}{2^{n_*+q_*+1}}\right)\omega.$$

Recalling the definition of θ_* in (B.11.1), that $2r = R_n$, and that $\omega = \omega_n$, this implies

$$\omega_{n+1} = \operatorname*{ess\,osc}_{Q_{n+1}} u \le \delta\omega_n \qquad \text{where} \qquad \delta = 1 - \frac{1}{2^{n_*+q_*+1}} \tag{B.12.2}$$

and

$$Q_{n+1} = K_{R_{n+1}} \times (-\omega_{n+1}^{1-m} R_{n+1}^2, 0]$$
$$\text{for} \quad R_{n+1} = \frac{1}{b} R_n \quad \text{with} \quad b = \sqrt{\frac{8}{\nu}}. \tag{B.12.3}$$

Assume next that the assumptions (B.9.1) of Lemma B.9.1 are verified, and that (B.12.1) continues to hold. Then (B.9.2) implies

$$- \operatorname*{ess\,inf}_{Q_{\frac{1}{2}r}(\theta)} u < -\tfrac{1}{4}\omega.$$

Adding

$$\operatorname*{ess\,sup}_{Q_{\frac{1}{2}r}(\theta)} u$$

on the left-hand side and μ^+ on the right-hand side gives

$$\operatorname*{ess\,osc}_{Q_{\frac{1}{2}r}(\theta)} u \le \mu^+ - \tfrac{1}{4}\omega \le \tfrac{5}{6}\omega.$$

Thus recalling the definition (B.12.3) of Q_{n+1}, one finds that (B.12.2) continues to hold also in this case. Thus the induction argument is completed provided μ^- satisfies the restriction (B.12.1).

B.13 The Induction Argument Concluded-(ii)

If (B.12.1) is violated for some index n, then

$$\mu_n^- \ge \tfrac{1}{12}\omega_n, \quad \text{and} \quad \mu_n^+ > \tfrac{13}{12}\omega_n. \tag{B.13.1}$$

The first of these implies that u is bounded below in Q_n and therefore the singular porous medium type equations (5.1)–(5.2) of Chapter 3 are nonsingular in Q_n. Thus u is Hölder continuous in Q_n by classical theory ([101]).

To make this quantitative, assume first that (B.13.1) occurs for $n = 0$. Then with $\mu_o^\pm$ and ω_o defined by (B.8.3), modify the construction of Q_o in (B.8.4) into

$$Q_{R_o}(\theta) = K_{R_o} \times (-\theta R_o^2, 0] \quad \text{where} \quad \theta = \mu_o^{+(1-m)}$$

since without loss of generality we may assume $\mu_o^+ \le 1$. Introduce the change of time variable and unknown function

$$\tau = \mu_o^{+(m-1)} t \quad \text{and} \quad v(\cdot, \tau) = \frac{u(\cdot, t)}{\mu_o^+}. \tag{B.13.2}$$

This transforms $Q_{R_o}(\theta)$ into

$$Q_{R_o} = K_{R_o} \times (-R_o^2, 0]$$

and transforms the equations into

$$v_\tau - \operatorname{div}\tilde{\mathbf{A}}(x,\tau,v,Dv) = \tilde{B}(x,\tau,v,Dv) \quad \text{weakly in } Q_{R_o}$$

where the functions $\tilde{\mathbf{A}}$ and $\tilde{B}$ satisfy the structure conditions

$$\begin{cases} \tilde{\mathbf{A}}(x,\tau,v,Dv)\cdot Dv \ge C_o m|Dv|^2 - C^2 \\ |\tilde{\mathbf{A}}(x,\tau,v,Dv)| \le 13^{1-m}C_1 m|Dv| + C \quad \text{a.e. in } E_T \\ |\tilde{B}(x,\tau,v,Dv)| \le 13^{1-m}Cm|Dv| + C^2 \end{cases}$$

for the same constants $m>0$, C_o, C_1, and C as in the structure conditions (5.2) of Chapter 3. Therefore the equation is nonsingular in Q_{R_o} and by classical theory, there exist $\delta_o \in (0,1)$ that can be determined a priori only in terms of the data $\{m,N,C_o,C_1\}$ and independent of R_o and μ_o^+, and a sequence of radii $R_n = 4^{-n}R_o$ such that either

$$C\mu_o^+ R_n > 1$$

or

$$\operatorname*{ess\,osc}_{Q_{R_{n+1}}} v \le \delta_o \operatorname*{ess\,osc}_{Q_{R_n}} v.$$

Returning to the function u and the cylinder Q_{R_o}, this establishes the induction argument for the sequence of cylinders as in (B.8.1) for such a sequence $\{R_n\}$, since $\omega_n \le \mu_o$.

If (B.13.1) holds for some $n>0$, then it is violated for the index $n-1$, and hence

$$\omega_n = \delta\omega_{n-1} > \tfrac{12}{13}\delta\mu_{n-1}^+ \ge \tfrac{12}{13}\delta\mu_n^+.$$

Introduce then the cylinder

$$Q_n \supset \tilde{Q}_n = K_n \times (-\theta_n R_n^2, 0] \quad \text{where} \quad \theta_n = \tfrac{12}{13}\delta\mu_n^{+(1-m)}.$$

Redefining $\mu_n^\pm$ for such a cylinder either (B.13.1) fails or it does hold true. If it fails, the induction argument is carried as in the previous sections. If it continues to hold true, then introduce the change of variables similar to (B.13.2) to transform the equation into a nonsingular one, to which classical methods can be applied.

B.14 Remarks and Bibliographical Notes

The Harnack-type inequality (B.1.1) in the topology of L^1_{loc}, was first established in [84], in the context of nonnegative solutions to the Cauchy problem for the prototype porous medium equation (5.3) of Chapter 3. The proof presented in § B.1, in the generality of the quasilinear equations (5.1)–(5.2) for $0<m<1$, is new. The approach is significantly different from the one in

[84], and that for the analogous result of § A.1 for the singular p-Laplacian equations.

The $L^r_{\rm loc}$–$L^\infty_{\rm loc}$ estimates of § B.4, while essentially known as analogues of the singular p-Laplacian equation ([41], Chapter IV), were not explicitly present in the literature (see [151] for the case of homogeneous doubly nonlinear singular equations). The one we present in § B.4 is the first complete formal proof, in the context of the singular equations (5.1) with the full quasilinear structure (5.2) of Chapter 3.

Analogous comments apply to the $L^r_{\rm loc}$ estimates backward in time of § B.5. While essentially known they have not appeared formally in the literature, and the one we present is the first formal proof in the quasilinear context.

The proof of the Hölder continuity of nonnegative, locally bounded, local, weak solutions to the singular porous medium type equations (5.1)–(5.2) of Chapter 3, follows a modified version of the arguments of [31] reported in [41] Chapter IV. While used in several contexts ([59, 60]) a formal, independent proof does not appear in the literature. The proof of § B.6–§ B.13 fills this gap.

10

Appendix C

C.1 More General Structures

The theory developed in the previous chapters extends to equations (1.1) of Chapter 3, with structure conditions more general than (1.2). For example, these could be replaced by

$$\begin{cases} \mathbf{A}(x,t,u,Du)\cdot Du \ge C_o|Du|^p - C^p u^p - \tilde{C}^p \\ |\mathbf{A}(x,t,u,Du)| \le C_1|Du|^{p-1} + C^{p-1}u^{p-1} + \tilde{C}^{p-1} \quad \text{a.e. in } E_T \\ |B(x,t,u,Du)| \le C|Du|^{p-1} + C^p u^{p-1} + C\tilde{C}^{p-1} \end{cases}$$

where $p>1$, C_o and C_1 are given positive constants, and C and $\tilde{C}$ are given nonnegative constants. With these conditions, all results continue to hold except that the alternatives in the statements of the main theorems take a new aspect, namely,

$$\dots \quad \text{either} \quad C\rho > 1 \quad \text{and} \quad \tilde{C}\rho > u(x_o,t_o)$$

$$\text{or the main result holds true.}$$

The proof is almost identical except for the indicated alternative changes.

More significant are generalizations to equations where the constant C in the structure conditions (1.2) of Chapter 3 is replaced by suitable integrable functions. Specifically consider parabolic equations in divergence form of the type

$$u_t - \operatorname{div}\mathbf{A}(x,t,u,Du) = B(x,t,u,Du) \quad \text{weakly in } E_T \tag{C.1.1}$$

where the functions $\mathbf{A}$ and B are measurable and satisfy

$$\begin{cases} \mathbf{A}(x,t,u,Du)\cdot Du \ge C_o|Du|^p - C^p\varphi_o(x,t) \\ |\mathbf{A}(x,t,u,Du)| \le C_1|Du|^{p-1} + C^{p-1}\varphi_1(x,t) \quad \text{a.e. in } E_T \\ |B(x,t,u,Du)| \le C|Du|^{p-1} + C^p\varphi_2(x,t) \end{cases} \tag{C.1.2}$$

E. DiBenedetto et al., *Harnack's Inequality for Degenerate and Singular Parabolic Equations*,
Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-1584-8_10,

where $p > 1$, and C_o and C_1 are positive constants, and C is a nonnegative constant. The nonnegative functions φ_i for $i = 0, 1, 2$ are defined in E_T and satisfy

$$\varphi_o, \quad \varphi_1^{\frac{p}{p-1}}, \quad \varphi_2^{\frac{p}{p-1}} \in L^q(E_T), \tag{C.1.3}$$

where

$$\frac{1}{q} = (1-\kappa_o)\frac{p}{N+p}, \qquad \kappa_o \in (0,1]. \tag{C.1.4}$$

Notice that κ_o can be equal to one. In such a case the functions φ_i all belong to $L^\infty(E_T)$ and we are back to the framework studied in the previous chapters.

The notion of weak solution is the same as in § 1 of Chapter 3. The statement that a constant $\gamma = \gamma(\text{data})$ depends only on the data, means that it can be determined a priori only in terms of

$$\Big\{N, p, \kappa_o, C_o, C_1, \|\varphi_o, \varphi_1^{\frac{p}{p-1}}, \varphi_2^{\frac{p}{p-1}}\|_{q,E_T}\Big\}. \tag{C.1.5}$$

The constant C appears only in the alternatives, but plays no role in determining the value of the various constants γ.

C.2 Energy Estimates for $(u-k)_\pm$ on Cylinders $(y,s) + Q_\rho^\pm(\theta) \subset E_T$

The notation is the same as in § 2 of Chapter 3.

Proposition C.2.1 *Let u be a local, weak sub(super)-solution to (C.1.1)–(C.1.4) in E_T. There exists a positive constant $\gamma = \gamma(data)$, such that for every cylinder*

$$(y,s) + Q_\rho^-(\theta) \subset E_T,$$

every $k \in \mathbb{R}$, and every nonnegative, piecewise smooth function ζ vanishing on $\partial K_\rho(y)$

$$\begin{aligned}
&\operatorname*{ess\,sup}_{s-\theta\rho^p<t<s} \int_{K_\rho(y)} (u-k)_\pm^2 \zeta^p(x,t)dx \\
&\qquad - \int_{K_\rho(y)} (u-k)_\pm^2 \zeta^p(x, s-\theta\rho^p)dx \\
&\qquad + C_o \iint_{(y,s)+Q_\rho^-(\theta)} |D(u-k)_\pm \zeta|^p dx\, d\tau \\
&\quad \le \gamma \iint_{(y,s)+Q_\rho^-(\theta)} \big[(u-k)_\pm^p |D\zeta|^p + (u-k)_\pm^2 |\zeta_\tau|\big]\, dx\, d\tau \\
&\qquad + \gamma C^p \iint_{(y,s)+Q_\rho^-(\theta)} (u-k)_\pm^p \zeta^p dx\, d\tau \\
&\qquad + \gamma C^p \Big(\int_{s-\theta\rho^p}^s |A_{k,\rho}^\pm(\tau)| d\tau\Big)^{\frac{N(1+\kappa)}{N+p}}
\end{aligned} \tag{C.2.1}$$

where C_o and C are the constants appearing in the structure conditions (C.1.2),

$$\kappa = \frac{p}{N}\kappa_o,$$

and

$$A^{\pm}_{k,\rho}(t) = [(u(\cdot,t)-k)_{\pm} > 0] \cap K_\rho.$$

Analogous energy estimates hold for "forward" cylinders $(y,s)+Q^+_\rho(\theta) \subset E_T$.

The proof is a straightforward generalization of the one given for Proposition 2.1 of Chapter 3.

C.3 DeGiorgi-Type Lemmas

Local, weak sub(super)-solutions to (C.1.1)–(C.1.3) in E_T are locally bounded above(below) in E_T ([41], Chapter V). For a fixed cylinder

$$(y,s)+Q^-_{2\rho}(\theta) \subset E_T$$

denote by $\mu_\pm$ and ω, nonnegative numbers such that

$$\mu_+ \geq \operatorname*{ess\,sup}_{(y,s)+Q^-_{2\rho}(\theta)} u, \qquad \mu_- \leq \operatorname*{ess\,inf}_{(y,s)+Q^-_{2\rho}(\theta)} u, \qquad \omega \geq \mu_+ - \mu_-.$$

Denote by $\xi \in (0,1]$ and $a \in (0,1)$ fixed numbers.

Lemma C.3.1 *Let u be a locally bounded, local, weak subsolution to (C.1.1)–(C.1.4) in E_T. There exists a number ν_+ depending on the data in (C.1.5), and on the parameters θ, ξ, ω, a, such that if*

$$\big|[u \geq \mu_+ - \xi\omega] \cap (y,s)+Q^-_{2\rho}(\theta)\big| \leq \nu_+|Q^-_{2\rho}(\theta)|,$$

then either

$$C > \min\left\{\frac{1}{\rho}\,,\ \frac{(\xi\omega)^{\frac{N+p-(p-2)(1-\kappa_o)}{N+p}}}{\rho^{\kappa_o}}\right\} \tag{C.3.1}$$

or

$$u \leq \mu_+ - a\xi\omega \quad \text{a.e. in } (y,s)+Q^-_\rho(\theta).$$

Likewise, if u is a locally bounded, local, weak supersolution to (C.1.1)–(C.1.4) in E_T, there exists a constant $\nu_- \in (0,1)$ dependent on the same data and parameters such that if

$$\big|[u \leq \mu_- + \xi\omega] \cap (y,s)+Q^-_{2\rho}(\theta)\big| \leq \nu_-|Q^-_{2\rho}(\theta)|,$$

then either (C.3.1) holds true, or

$$u \geq \mu_- + a\xi\omega \quad \text{a.e. in } (y,s)+Q^-_\rho(\theta).$$

The constants $\nu_\pm$ are independent of C, and the latter enters into the statement only via the alternative (C.3.1). Their functional dependence on the indicated parameters can be explicitly calculated as

$$\nu_\pm = \frac{\gamma^{-\frac{N}{p\kappa_o}} b^{-\left(\frac{N}{p\kappa_o}\right)^2} (1-a)^{\frac{N+2}{\kappa_o}} \dfrac{(\xi\omega)^{2-p}}{\theta}}{\left[1+\left(\dfrac{\theta}{(\xi\omega)^{2-p}}\right)^{\frac{p(1-\kappa_o)}{N}} \cdot \left(1+\dfrac{(\xi\omega)^{2-p}}{\theta}\right)^{\frac{N+p}{N}}\right]^{\frac{N}{p\kappa_o}}}.$$

When $\kappa_o = 1$, this coincides with the same form as (3.12) of Chapter 3.

C.3.1 A Variant of DeGiorgi-Type Lemma, Involving "Initial Data"

Assume now that some information is available on the "initial data" relative to the cylinder $(y,s)+Q^+_{2\rho}(\theta)$, say for example

$$u(x,s) \ge \xi M \quad \text{for a.e. } x \in K_{2\rho}(y) \tag{C.3.2}$$

for some $M>0$ and $\xi \in (0,1]$. Then

Lemma C.3.2 *Let u be a nonnegative, local, weak supersolution to (C.1.1)–(C.1.4) in E_T. Let M and ξ be positive numbers such that (C.3.2) holds. There exists a constant ν_o depending only on the data in (C.1.5) and a, and independent of ξ and M, such that if*

$$\begin{aligned}&\frac{\left|[u \le \mu_- + \xi\omega] \cap (y,s)+Q^+_{2\rho}(\theta)\right|}{|Q^+_{2\rho}(\theta)|} \\ &\qquad \le \frac{\nu_o}{(\xi M)^{p-2}\theta\left[1+[(\xi M)^{p-2}\theta]^{\frac{p(1-\kappa_o)}{N}}\right]^{\frac{N}{p\kappa_o}}},\end{aligned}$$

then either

$$C > \min\left\{\frac{1}{\rho}, \frac{(\xi M)^{\frac{N+p-(p-2)(1-\kappa_o)}{N+p}}}{\rho^{\kappa_o}}\right\}$$

or

$$u \ge a\xi M \quad \textit{a.e. in } K_\rho(y)\times(s, s+\theta(2\rho)^p].$$

The proof of these two lemmas is analogous to Lemmas 3.1 and 4.1 of Chapter 3. In those proofs use is made of the iteration Lemma 5.1 of the Preliminaries. In the present context, because of the last term in the energy estimates (C.2.1), this lemma has to be generalized into the following.

C.3.2 A Technical Lemma

Lemma C.3.3 *Let $\{Y_n\}$ and $\{Z_n\}$ be sequences of positive numbers, satisfying the recursive inequalities*

$$\begin{aligned} Y_{n+1} &\le Cb^n(Y_n^{1+\alpha} + Z_n^{1+\kappa}Y_n^\alpha) \\ Z_{n+1} &\le Cb^n(Y_n + Z_n^{1+\kappa}) \end{aligned} \tag{C.3.3}$$

where $C, b > 1$ *and* $\kappa, \alpha > 0$ *are given numbers. If*

$$Y_o + Z_o^{1+\kappa} \le (2C)^{-\frac{1+\kappa}{\sigma}} b^{-\frac{1+\kappa}{\sigma^2}} \qquad where \qquad \sigma = \min\{\kappa\,,\,\alpha\},$$

then $\{Y_n\}$ *and* $\{Z_n\}$ *tend to zero as* $n \to \infty$.

Proof Set $M_n = Y_n + Z_n^{1+\kappa}$ and rewrite the second of (C.3.3) as

$$Z_{n+1}^{1+\kappa} \le C^{1+\kappa} b^{(1+\kappa)n} M_n^{1+\kappa}. \tag{C.3.4}$$

Consider the term in braces in the first of (C.3.3). If $Z_n^{1+\kappa} \le Y_n$, such a term is majorized by $2M_n^{1+\alpha}$. If $Z_n^{1+\kappa} \ge Y_n$, then the same term can be majorized by

$$Y_n^{1+\alpha} + (Z_n^{1+\kappa})^{1+\alpha} \le M_n^{1+\alpha}.$$

Combining this with (C.3.4), we deduce that in either case

$$M_{n+1} \le 2C^{1+\kappa} b^{(1+\kappa)n} M_n^{1+\min\{\kappa\,,\,\alpha\}}.$$

The proof is concluded by induction. ■

C.4 Expansion of Positivity for Nonnegative Solutions to (C.1.1)–(C.1.4)

In this more general context, Lemma 1.1 of Chapter 4 becomes:

Lemma C.4.1 *Assume that for some* $(y, s) \in E_T$ *and some* $\rho > 0$

$$\big|[u(\cdot, s) \ge M] \cap K_\rho(y)\big| \ge \alpha\big|K_\rho(y)\big| \tag{C.4.1}$$

for some $M > 0$ *and some* $\alpha \in (0, 1)$. *There exist* δ *and* ϵ *in* $(0, 1)$, *depending only on the data in (C.1.5), and* α, *and independent of* M, *such that either*

$$C > \min\left\{\frac{1}{\rho}\,,\,\frac{M^{1-\frac{(p-2)(1-\kappa_o)}{N+p}}}{\rho^{\kappa_o}}\right\}$$

or

$$|[u(\cdot, t) > \epsilon M] \cap K_\rho(y)| \ge \tfrac{1}{2}\alpha|K_\rho(y)| \qquad for\ all\ \ t \in \Big(s, s + \frac{\delta\rho^p}{M^{p-2}}\Big].$$

This is the starting point for the proof of the expansion of positivity. Working as in Chapter 4, with the obvious modifications, we have the corresponding statements, both for $p > 2$ and for $1 < p < 2$.

Assume first that u is a nonnegative, local, weak supersolution to (C.1.1)–(C.1.4) in E_T, for $p > 2$. For $(y,s) \in E_T$, and some given positive number M, consider the cylinder

$$K_{8\rho}(y) \times \Big(s, s + \frac{b^{p-2}}{(\eta M)^{p-2}}\delta\rho^p\Big],$$

where δ, η, b are the constants given by Proposition C.4.1, and $\rho > 0$ is so small that such a cylinder is included in E_T.

Proposition C.4.1 *Assume that (C.4.1) holds for some $M > 0$ and some $\alpha \in (0,1)$. There exist constants η and δ in $(0,1)$ and γ, $b > 1$ depending only on the data in (C.1.5) and α, such that either*

$$\gamma C > \min\left\{\frac{1}{\rho}, \frac{M}{\rho^{\kappa_o + \frac{(1-\kappa_o)p}{(N+p)(p-2)}}}, \frac{M^{1-\frac{(p-2)(1-\kappa_o)}{N+p}}}{\rho^{\kappa_o}}\right\},$$

or

$$u(\cdot,t) \ge \eta M \qquad a.e.\ in\ K_{2\rho}(y)$$

for all times

$$s + \frac{b^{p-2}}{(\eta M)^{p-2}}\tfrac{1}{2}\delta\rho^p \le t \le s + \frac{b^{p-2}}{(\eta M)^{p-2}}\delta\rho^p.$$

Now let u be a nonnegative, local, weak supersolution to (C.1.1)–(C.1.4) with $1 < p < 2$, and let cylinder

$$K_{16\rho}(y) \times (s, s + \delta M^{2-p}\rho^p]$$

be contained in E_T.

Proposition C.4.2 *Assume that (C.4.1) holds for some $M > 0$ and some $\alpha \in (0,1)$. There exist constants η, δ, and ε in $(0,1)$, and $\gamma > 1$ depending only on the data in (C.1.5) and α, such that either*

$$\gamma C > \min\left\{\frac{1}{\rho}, \frac{M}{\rho}, \frac{M^{1-\frac{(p-2)(1-\kappa_o)}{N+p}}}{\rho^{\kappa_o}}\right\},$$

or

$$u(\cdot,t) \ge \eta M \qquad a.e.\ in\ K_{2\rho}(y)$$

for all times

$$s + (1-\varepsilon)\delta M^{2-p}\rho^p \le t \le s + \delta M^{2-p}\rho^p.$$

When $\kappa_o = 1$ these statements coincide with the analogous ones in Chapter 4. Relying on these propositions, the Harnack estimates of Chapters 5 and 6 can be extended to nonnegative solutions to (C.1.1)–(C.1.4), both in the degenerate $p > 2$, and in the singular $1 < p < 2$, range. The only significant difference is in the alternatives, where one needs to take into account the presence of the parameter $\kappa_o \in (0,1]$.

C.5 Equations of Porous Medium Type

Consider quasilinear evolution equations of the type

$$u_t - \operatorname{div}\mathbf{A}(x,t,u,Du) = B(x,t,u,Du) \quad \text{weakly in } \; E_T. \tag{C.5.1}$$

The functions $\mathbf{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ and $B : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}$ are measurable and satisfy

$$\begin{cases} \mathbf{A}(x,t,u,Du)\cdot Du \ge mC_o|u|^{m-1}|Du|^2 - C^2|u|^{m+1}\varphi_o(x,t) \\ |\mathbf{A}(x,t,u,Du)| \le mC_1|u|^{m-1}|Du| + C|u|^m\varphi_1(x,t) \\ |B(x,t,u,Du)| \le mC|u|^{m-1}|Du| + C^2|u|^m\varphi_2(x,t) \end{cases} \tag{C.5.2}$$

for $m>0$ and a.e. $(x,t)\in E_T$. Here C_o and C_1 are positive constants, and C is a nonnegative constant. The nonnegative functions φ_i, $i=0,1,2$, are defined in E_T and satisfy

$$\varphi_o, \quad \varphi_1^2, \quad \varphi_2^2 \;\in\; L^q(E_T)$$

where

$$\frac{1}{q} = (1-\kappa_o)\frac{2}{N+2}, \qquad \kappa_o\in(0,1]. \tag{C.5.3}$$

Notice that κ_o can be equal to one. In such a case the functions φ_i all belong to $L^\infty(E_T)$ and we are back to the framework studied in the previous chapters.

The notion of weak solution is the same one, as discussed in § 5 of Chapter 3. In the following the statement that a constant $\gamma=\gamma(\text{data})$ depends only on the data, means that it can be determined a priori only in terms of

$$\Big\{N, m, \kappa_o, C_o, C_1, \|\varphi_o, \varphi_1^2, \varphi_2^2\|_{q,E_T}.\Big\} \tag{C.5.4}$$

The constant C appears only in the alternatives, but plays no role in determining the value of the various constants γ.

C.6 Energy Estimates for $(u-k)_\pm$ on Cylinders $(y,s)+Q_\rho^\pm(\theta)\subset E_T$

The notation is the same as in § 2 of Chapter 3. The next two propositions discriminate between $m>1$ and $0<m<1$.

Proposition C.6.1 *Let u be a local, weak sub(super)-solution to (C.5.1)–(C.5.3) in E_T with $m>1$. There exists a positive constant $\gamma=\gamma(data)$, such that for every cylinder*

$$(y,s)+Q_\rho^-(\theta)\subset E_T,$$

every $k\in\mathbb{R}$, and every nonnegative, piecewise smooth function ζ vanishing on $\partial K_\rho(y)$

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{s-\theta\rho^2<t<s} \int_{K_\rho(y)} (u-k)_\pm^2 \zeta^2(x,t)dx \\
&\quad - \int_{K_\rho(y)} (u-k)_\pm^2 \zeta^2(x, s-\theta\rho^2)dx \\
&\quad + mC_o \iint_{(y,s)+Q_\rho^-(\theta)} |u|^{m-1} |D(u-k)_\pm \zeta|^2 dx\, d\tau \\
&\le \gamma \iint_{(y,s)+Q_\rho^-(\theta)} \left[|u|^{m-1}(u-k)_\pm^2 |D\zeta|^2 + (u-k)_\pm^2 |\zeta_\tau| \right] dx\, d\tau \\
&\quad + \gamma C^2 \iint_{(y,s)+Q_\rho^-(\theta)} |u|^{m-1}(u-k)_\pm^2 \zeta^2 dx\, d\tau \\
&+ \gamma C^2 \iint_{(y,s)+Q_\rho^-(\theta)} |u|^{m+1} [\varphi_o + \varphi_1^2 + \varphi_2^2] \chi_{[(u-k)_\pm>0]} \zeta^2 dx\, d\tau
\end{aligned}
$$

where C_o and C are the constants appearing in the structure conditions (C.5.1). Analogous energy estimates hold for "forward" cylinders $(y,s) + Q_\rho^+(\theta) \subset E_T$.

Proposition C.6.2 *Let u be a nonnegative, local, weak supersolution to (C.5.1)–(C.5.3) in E_T with $0 < m < 1$. There exists a positive constant $\gamma = \gamma(data)$, such that for every cylinder*

$$(y,s) + Q_\rho^-(\theta) \subset E_T,$$

every $k > 0$, and every nonnegative, piecewise smooth function ζ vanishing on $\partial K_\rho(y)$,

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{s-\theta\rho^2<t<s} \int_{K_\rho(y)} (u-k)_-^2 \zeta^2(x,t)dx \\
&\qquad - \gamma k \int_{K_\rho(y)} (u-k)_- \zeta^2(x, s-\theta\rho^2)dx \\
&\qquad + C_o k^{m-1} \iint_{(y,s)+Q_\rho^-(\theta)} |D(u-k)_- \zeta|^2 dx\, d\tau \\
&\quad \le \gamma k^2 \iint_{(y,s)+Q_\rho^-(\theta)} \chi_{[(u-k)_->0]} \zeta |\zeta_\tau| dx\, d\tau \\
&\qquad + \gamma k^{m+1} \iint_{(y,s)+Q_\rho^-(\theta)} [C^2\zeta^2 + |D\zeta|^2] \chi_{[(u-k)_->0]} dx\, d\tau \\
&\qquad + \gamma C^2 k^{m+1} \left[\int_{s-\theta\rho^2}^s |A_{k,\rho}^-(\tau)| d\tau \right]^{\frac{N(1+\kappa)}{N+2}}
\end{aligned}
$$

where C_o and C are the constants appearing in the structure conditions (C.5.2),

$$\kappa = \frac{2}{N}\kappa_o,$$

and

$$A^-_{k,\rho}(t) = [(u(\cdot,t) - k)_- > 0] \cap K_\rho.$$

Analogous energy estimates hold for "forward" cylinders $(y,s) + Q^+_\rho(\theta) \subset E_T$.

The proofs are a straightforward adaptation of those given for Propositions 6.1 and 9.1 of Chapter 3.

C.7 DeGiorgi-Type Lemmas

Local, weak sub(super)-solutions to (C.5.1)–(C.5.3) in E_T are locally bounded above(below) in E_T ([7]). For a fixed cylinder

$$(y,s) + Q^-_{2\rho}(\theta) \subset E_T$$

denote by $\mu_\pm$ and ω, nonnegative numbers such that

$$\mu_+ \geq \operatorname*{ess\,sup}_{(y,s)+Q^-_{2\rho}(\theta)} u, \qquad \mu_- \leq \operatorname*{ess\,inf}_{(y,s)+Q^-_{2\rho}(\theta)} u, \qquad \omega = \mu_+ - \mu_-.$$

Since the degeneracy or singularity occurs at $u = 0$, we will assume at the outset that

$$\mu_- = \operatorname*{ess\,inf}_{(y,s)+Q^-_{2\rho}(\theta)} u = 0 \quad \text{so that} \quad \omega = \mu_+.$$

Denote by $\xi \in (0,1]$ and $a \in (0,1)$ fixed numbers.

Lemma C.7.1 *Let u be a nonnegative, locally bounded, local, weak subsolution to (C.5.1)–(C.5.3) in E_T for $m > 1$. There exists a number ν_+ depending on the data in (C.5.4), and the parameters θ, ξ, ω, a, such that if*

$$\big|[u \geq \mu_+ - \xi\omega] \cap [(y,s) + Q^-_{2\rho}(\theta)]\big| \leq \nu_+ |Q^-_{2\rho}(\theta)|,$$

then either

$$C > \min\left\{\frac{1}{\rho}\,,\ \frac{(\xi\omega)^{\frac{(1-m)(1-\kappa_o)}{N+2}}}{\rho^{\kappa_o}}\right\} \tag{C.7.1}$$

or

$$u \leq \mu_+ - a\xi\omega \qquad \textit{a.e. in } \ (y,s) + Q^-_\rho(\theta).$$

Lemma C.7.2 *Let u be a nonnegative, locally bounded, local, weak supersolution to (C.5.1)–(C.5.3) in E_T, for $m > 0$. There exists a number ν_- depending on the data in (C.5.4), and the parameters θ, ξ, ω, a, such that if*

$$\big|[u \leq \xi\omega] \cap (y,s) + Q^-_{2\rho}(\theta)\big| \leq \nu_- |Q^-_{2\rho}(\theta)|,$$

then either (C.7.1) holds, or

$$u \geq a\xi\omega \qquad \textit{a.e. in } \ \big[(y,s) + Q^-_\rho(\theta)\big].$$

The constants $\nu_\pm$ are independent of C, and the latter enters into the statement only via the alternative (C.7.1). The functional dependence of $\nu_\pm$ on these parameters can be explicitly calculated by setting

$$\nu_* = \frac{2^{-(\frac{N+2}{\kappa_o})^2}\Big[\frac{(1-a)^2}{\gamma^2}\Big]^{\frac{N+2}{2\kappa_o}}\Big[\frac{(\xi\omega)^{1-m}}{\theta}\Big]^{\frac{1}{\kappa_o}}}{\Big[1+\frac{1}{\theta(\xi\omega)^{m-1}}+\Big(\frac{1}{\theta(\xi\omega)^{m-1}}\Big)^{\frac{2(1-\kappa_o)}{N+2}}\Big]^{\frac{N+2}{2\kappa_o}}}.$$

Then

$$\nu_+ = \Big[\frac{\xi^{2(N+m+1)}}{(1-\xi)^{m-1}}\Big]^{\frac{N+2}{2\kappa_o}}\nu_*, \qquad \text{and} \qquad \nu_- = \Lambda(a)^{\frac{N+2}{2\kappa_o}}\nu_*,$$

where

$$\Lambda(a) = \begin{cases} 1 & \text{if } m\in(0,1) \\ \big(\frac{1}{2}a\big)^{1-m} & \text{if } m>1. \end{cases}$$

When $\kappa_o = 1$ these reduce to the functional dependences of § 7 of Chapter 3.

C.7.1 A Variant of DeGiorgi-Type Lemma, Involving "Initial Data"

Assume now in addition that some information is available on the "initial data" relative to the cylinder $(y,s)+Q^+_{2\rho}(\theta)$, say for example

$$u(x,s) \ge \xi M \qquad \text{for a.e. } x\in K_{2\rho}(y) \tag{C.7.2}$$

for some $M>0$ and $\xi\in(0,1]$.

Lemma C.7.3 *Let u be a nonnegative, locally bounded, local, weak supersolution to (C.5.1)–(C.5.3) in E_T, for $m>0$. Let M and ξ be positive numbers such that (C.7.2) holds. There exists a constant $\nu_o\in(0,1)$ depending only on the data in (C.5.4) and a, and independent of ξ and M, such that if*

$$\frac{\big|[u\le\mu_-+\xi\omega]\cap(y,s)+Q^+_{2\rho}(\theta)\big|}{|Q^+_{2\rho}(\theta)|} \le \frac{\nu_o}{(\theta(\xi M)^{m-1})^{\frac{1}{\kappa_o}}\Big[1+\Big(\frac{1}{(\xi M)^{m-1}\theta}\Big)^{\frac{2(1-\kappa_o)}{N+2}}\Big]^{\frac{N+2}{2\kappa_o}}},$$

then either (C.7.1) holds, or

$$u \ge a\xi M \qquad \textit{a.e. in } K_\rho(y)\times(s,s+\theta(2\rho)^p].$$

C.8 Expansion of Positivity for Solutions to (C.5.1)–(C.5.3)

In this more general context, Lemma 7.1 of Chapter 4 becomes:

Lemma C.8.1 *Assume that for some* $(y,s)\in E_T$ *and some* $\rho>0$

$$\big|[u(\cdot,s)\ge M]\cap K_\rho(y)\big|\ge\alpha\big|K_\rho(y)\big| \tag{C.8.1}$$

for some $M>0$ *and some* $\alpha\in(0,1)$*. There exist* δ *and* ϵ *in* $(0,1)$*, depending only on the data in (C.5.4) and* α*, and independent of* M*, such that either*

$$C>\min\left\{\frac{1}{\rho}\,,\ \frac{M^{\frac{(m-1)(\kappa_o-1)}{N+2}}}{\rho^{\kappa_o}}\right\}, \tag{C.8.2}$$

or

$$|[u(\cdot,t)>\epsilon M]\cap K_\rho(y)|\ge\tfrac{1}{2}\alpha|K_\rho(y)|\quad \textit{for all}\ \ t\in\Big(s,s+\frac{\delta\rho^2}{M^{m-1}}\Big].$$

This is the starting point for the proof of the expansion of positivity. Working as in Chapter 4, with the obvious modifications, we have the corresponding statements, both for $m>1$ and for $0<m<1$.

Assume first that u is a nonnegative, local, weak supersolution to (C.5.1)–(C.5.3) in E_T, for $m>1$. For $(y,s)\in E_T$, and some given positive number M, consider the cylinder

$$K_{8\rho}(y)\times\Big(s,s+\frac{b^{m-1}}{(\eta M)^{m-1}}\delta\rho^2\Big],$$

where δ, η, b are the constants given by Proposition C.8.1, and $\rho>0$ is so small that such a cylinder is included in E_T.

Proposition C.8.1 *Assume that for some* $(y,s)\in E_T$ *and some* $\rho>0$ *(C.8.1) holds for some* $M>0$ *and some* $\alpha\in(0,1)$*. There exist constants* η *and* δ *in* $(0,1)$ *and* γ*,* $b>1$ *depending only on the data in (C.5.4) such that either*

$$\gamma C>\min\left\{\frac{1}{\rho}\,,\ \frac{1}{\rho^{\kappa_o+\frac{2(1-\kappa_o)}{(N+2)(m-1)}}}\,,\ \frac{M^{\frac{(m-1)(\kappa_o-1)}{N+2}}}{\rho^{\kappa_o}}\right\}$$

or

$$u(\cdot,t)\ge\eta M\qquad \textit{a.e. in}\ \ K_{2\rho}(y)$$

for all times

$$s+\frac{b^{m-1}}{(\eta M)^{m-1}}\tfrac{1}{2}\delta\rho^2\le t\le s+\frac{b^{m-1}}{(\eta M)^{m-1}}\delta\rho^2.$$

Let now u be a nonnegative, local, weak supersolution to (C.5.1)–(C.5.3) with $0 < m < 1$, and let the cylinder

$$K_{16\rho}(y) \times (s, s + \delta M^{1-m}\rho^2]$$

be contained in E_T.

Proposition C.8.2 *Assume that for some* $(y, s) \in E_T$ *and some* $\rho > 0$ *(C.8.1) holds for some* $M > 0$ *and some* $\alpha \in (0,1)$. *There exist constants* η, δ, *and* ε *in* $(0,1)$, *and* $\gamma > 1$ *depending only upon the data in (C.5.4) and* α, *such that either (C.8.2) holds, or*

$$u(\cdot, t) \geq \eta M \qquad a.e.\ in\ \ K_{2\rho}(y)$$

for all times

$$s + (1-\varepsilon)\delta M^{1-m}\rho^2 \leq t \leq s + \delta M^{1-m}\rho^2.$$

When $\kappa_o = 1$, we recover the statements of Chapter 4 for both Propositions C.8.1 and C.8.2.

Relying on these propositions, the Harnack estimates of Chapters 5 and 6 can be extended to nonnegative solutions to equations (C.5.1)–(C.5.3), both in the degenerate $m > 1$, and in the singular $0 < m < 1$ range. The only significant difference is in the alternatives, where one needs to take into account the presence of the parameter $\kappa_o \in (0, 1]$.

C.9 Remarks and Bibliographical Notes

Parabolic equations with integrable lower order terms were extensively studied in [101], in the case of $p = 2$, $m = 1$. For degenerate ($p > 2$) and singular ($1 < p < 2$) equations the theory of local regularity is in [41]. As discussed in Chapter V of [41], the assumptions (C.1.3)–(C.1.4) are natural, to insure both boundedness and local Hölder regularity of solutions. If one knew that solutions are already bounded, and does not need a quantitative statement about their boundedness, more general assumptions could be made. In particular, the functions φ_i could be taken in proper $L^{q,r}(E_T)$ spaces, as discussed in Chapters II–IV of [41]. The expansion of positivity of Chapter 4 and the Harnack inequalities of Chapters 5 and 6, could be extended to this more general setting, the difficulty being only technical. Analogous considerations apply to porous medium type equations (C.5.1)–(C.5.3).

There is some dissymmetry between the structure conditions of p-Laplacian type equations and porous medium type equations. The structure conditions one gets by letting $p \to 2$ in (C.1.2)–(C.1.4) coincide with those given in [101], for nondegenerate equations. The latter, in the context of nondegenerate equations, in [101] are shown to be optimal for the local boundedness and the local Hölder continuity of solutions.

This is not the case, however, if one lets $m \to 1$ in (C.5.2)–(C.5.3), and it raises the question of whether a refinement of conditions (1.2) and (5.2) of Chapter 3 is possible, so that they coincide when $p \to 2$ and $m \to 1$.

Suitable structure conditions for degenerate ($m > 1$) porous medium type equations are

$$\begin{cases} \mathbf{A}(x,t,u,Du) \cdot Du \geq mC_o|u|^{m-1}|Du|^2 - C^2\varphi_o(x,t) \\ |\mathbf{A}(x,t,u,Du)| \leq mC_1|u|^{m-1}|Du| + C|u|^{\frac{m-1}{2}}\varphi_1(x,t) \\ |B(x,t,u,Du)| \leq mC|u|^{m-1}|Du| + C^2|u|^{\frac{m-1}{2}}\varphi_2(x,t), \end{cases} \tag{C.9.1}$$

where the functions φ_i satisfy (C.1.3)–(C.1.4). It turns out that nonnegative local, weak solutions to these degenerate ($m > 1$) porous medium type equations satisfy the expansion of positivity of Chapter 4 and the Harnack inequality of Chapter 5. The proof is almost identical with the minor modifications due to the structure (C.9.1).

For singular ($0 < m < 1$) porous medium type equations, it is not known whether the structural conditions (5.2) of Chapter 3, or (C.5.2)–(C.5.3) are optimal for a Harnack estimate to hold. For $0 < m < 1$ the conditions on the lower order terms (those terms involving the alternative constant C) are crucial in the proof of Proposition B.1.1 of Appendix B, which seems to be the key point.

Rather than discriminating between $m > 1$ and $0 < m < 1$ we have chosen to use a unified approach.

References

1. A. Abbott, Cell Biology: Hopping Fences, *Nature,* **433***, (2005), 680–683.*
2. E. Acerbi, G. Mingione and G. Seregin, Regularity results for parabolic systems related to a class of non-Newtonian fluids, *Ann. Inst. H. Poincaré Anal. Non Linéaire,* **21***(1), (2004), 25–60.*
3. H. Aimar, L. Forzani and R. Toledano, Hölder regularity of solutions of PDE's: a geometrical view, *Comm. Partial Differential Equations, 26(7–8), (2001), 1145–1173.*
4. P.F. Almeida, Lateral Diffusion in Membranes, *Handbook of Biological Physics, Vol. 1-Structure and Dynamics of Membranes, 1995; R. Lipowsky and E. Sackmann Eds., 305–307.*
5. R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
6. D. Andreucci and E. DiBenedetto, A new approach to initial traces in nonlinear filtration, *Ann. Inst. H. Poincaré Anal. Non Linéaire,* **7***(4), (1990), 305–334.*
7. D. Andreucci, Apriori Bounds for Weak Solutions of the Filtration Equation, *SIAM J. Math. Anal.,* **22***(1), (1991), 138–145.*
8. S.N. Antonsev, Axially symmetric problems of gas dynamics with free boundaries, *Dokl. Akad. Nauk SSSR,* **216***(3), (1974), 473–476.*
9. D.G. Aronson and Ph. Bénilan, Régularité des solutions de l'équation des milieux poreux dans $\mathbb{R}^N$, *C. R. Acad. Sci. Paris Sér. A-B* **228***, (1979), A103–A105.*
10. D.G. Aronson and J. Serrin, Local Behaviour of Solutions of Quasilinear Parabolic Equations, *Arch. Rational Mech. Anal.,* **25***, (1967), 81–122.*
11. G. Auchmuty and D. Bao, Harnack–Type Inequalities for Evolution Equations, *Proc. Amer. Math. Soc.,* **122***(1), (1994), 117–129.*
12. D.S. Banks and C. Fradin, Anomalous Diffusion of Proteins Due to Molecular Crowding, *Biophys. J.,* **89***, (2005), 2960–2971.*
13. G.I. Barenblatt, On some unsteady motions of a liquid or a gas in a porous medium, *Prikl. Mat. Mech.,* **16***, (1952), 67–78.*
14. T. Baumgart, A. T. Hammond, P. Sengupta, S.T. Hess, D.A. Holowka, B.A. Baird and W.W. Webb, Large-Scale Fluid/Fluid Phase Separation of Proteins and Lipids in Giant Plasma Membrane Vesicles, *PNAS,* **104***, (2007), 3165–3170.*
15. J. Bear, *Dynamics of Fluids in Porous Media*, Dover Publications, New York, 1972.

E. DiBenedetto et al., *Harnack's Inequality for Degenerate and Singular Parabolic Equations*, Springer Monographs in Mathematics, DOI 10.1007/978-1-4614-1584-8,

16. J. Bear, *Hydraulics of Groundwater*, McGraw Hill, New York, 1979.
17. S.N. Bernstein, *Collected works. III Differential equations, calculus of variations and geometry (1903–1947)*, Izd. Akad. Nauk SSSR, Moscow, 1960 (Russian).
18. J.G. Berryman, Evolution of a stable profile for a class of nonlinear diffusion equations with fixed boundaries, *J. Math. Phys.,* **18***(11), (1977), 2108–2115.*
19. J.G. Berryman and C.J. Holland, Stability of the separable solution for fast diffusion equation, *Arch. Rational Mech. Anal.,* **74***(4), (1980), 379–388.*
20. R.B. Bird, Useful Non-Newtonian Models, *Ann. Rev. Fluid Mech.,* **8***, (1976), 13–34.*
21. M. Bonforte and J.L. Vázquez, Positivity, Local Smoothing and Harnack Inequalities for Very Fast Diffusion Equations, *Adv. Math.,* **223***, (2010), 529–578.*
22. M. Bonforte, R.G. Iagar and J.L. Vázquez, Local smoothing effects, positivity, and Harnack inequalities for the fast p-Laplacian equation, *Adv. Math.,* **224***, (2010), 2151–2215.*
23. I.A. Brigadnov and A. Dorfmann, Mathematical Modeling of Magnetorheological Fluids, *Contin. Mech. Thermodyn.,* **17***, (2005), 29–42.*
24. S.J. Bussell, D.L. Koch and D.A. Hammer, The Effect of Hydrodynamic Interactions on the Diffusion of Integral Membrane Proteins: Diffusion in Plasma Membranes, *Biophys. J.,* **68***, (1995), 1836–1849.*
25. L.A. Caffarelli and S. Salsa, *A geometric approach to free boundary problems*, Graduate Studies in Mathematics, 68, American Mathematical Society, Providence, RI, 2005.
26. S. Campanato, Equazioni ellittiche del II^o ordine e spazi $\mathcal{L}^{2,\lambda}$, *Ann. Mat. Pura Appl.,* **69***, (1965), 321–381.*
27. S. Campanato, Equazioni paraboliche del secondo ordine e spazi $\mathcal{L}^{p,\theta}(\Omega, \delta)$, *Ann. Mat. Pura Appl.,* **73***, (1966), 55–102.*
28. Y. Chen, Hölder continuity of the gradient of the solutions of certain degenerate parabolic equations, *Chinese Ann. Math. Ser. B,* **8***(3), (1987), 343–356.*
29. Y.Z. Chen and E. DiBenedetto, On the Local Behaviour of Solutions of Singular Parabolic Equations, *Arch. Rational Mech. Anal.,* **103***(4), (1988), 319–346.*
30. Y.Z. Chen and E. DiBenedetto, On the Harnack Inequality for Non-Negative Solutions of Singular Parabolic Equations, *Proceedings of Non-Linear Diffusion; in Honour of J. Serrin, Minneapolis, May 1990.*
31. Y.Z. Chen and E. DiBenedetto, Hölder Estimates of Solutions of Singular Parabolic Equations with Measurable Coefficients, *Arch. Rational Mech. Anal.,* **118***(3), (1992), 257–271.*
32. H.J. Choe and J.H. Lee, Cauchy problem for nonlinear parabolic equations, *Hokkaido Math. J.,* **27***(1), (1998), 51–75.*
33. I. Chopra, Review of State of Art of Smart Structures and Integrated Systems, *AIAA J.,* **40***, (2002), 2145–2187.*
34. R.E. Collins, *Flow of Fluids Through Porous Materials*, Reinhold Publishing, 1961.
35. S. Davis, E. DiBenedetto and D. Diller, Some A-Priori Estimates for a Singular Evolution Equation Arising in Thin Film Dynamics, *SIAM J. Math. Anal.,* **27***(3), (1996), 638–660.*
36. E. De Giorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3),* **3***, (1957), 25–43.*

37. E. DiBenedetto, Continuity of Weak Solutions to a General Porous Medium Equation, *Indiana Univ. Math. J.,* **32***(1), (1983), 83–118.*
38. E. DiBenedetto, On the Local Behavior of Solutions of Degenerate Parabolic Equations with Measurable Coefficients, *Ann. Sc. Norm. Super. Pisa Cl. Sc. Ser. IV,* **XIII***(3), (1986), 487–535.*
39. E. DiBenedetto, Intrinsic Harnack type inequalities for solutions of certain degenerate parabolic equations, *Arch. Rational Mech. Anal.,* **100***(2), (1988), 129–147.*
40. E. DiBenedetto, Harnack Estimates in Certain Function Classes, *Atti Sem. Mat. Fis. Univ. Modena,* **37***, (1989), 173–182.*
41. E. DiBenedetto, *Degenerate Parabolic Equations*, Universitext, Springer-Verlag, New York, 1993.
42. E. DiBenedetto, *Real Analysis*, Birkhäuser, Boston, 2002.
43. E. DiBenedetto, *Partial Differential Equations*, 2nd ed., Birkhäuser, Boston, 2009.
44. E. DiBenedetto and D. Diller, About a Singular Parabolic Equation arising in Thin Film Dynamics and in the Ricci Flow for Complete $\mathbb{R}^2$, *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, New York, Vol. 177, 1995, 103–120. Special Volume for the 70$^{\text{th}}$ birthday of C. Pucci. Edited by G. Talenti, E. Vesentini and P. Marcellini.
45. E. DiBenedetto and D. Diller, Singular semilinear elliptic equations in $L^1(\mathbb{R}^N)$, *Topics in Nonlinear Analysis, 143–181, Progr. Nonlinear Differential Equations Appl., 35, Birkhäuser, Basel, 1999.*
46. E. DiBenedetto and D. Diller, On the Asymptotic Behavior of Drying a Photographic Film, *Adv. Differential Equations,* **1***(6), (1996), 989–1003.*
47. E. DiBenedetto and A. Friedman, Hölder Estimates for Non-Linear Degenerate Parabolic Systems, *J. Reine Angew. Math.,* **357***, (1985), 1–22.*
48. E. DiBenedetto, U. Gianazza and V. Vespri, Local Clustering on the Non-Zero Set of Functions in $W^{1,1}(E)$, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.,* **17***, (2006), 223–225.*
49. E. DiBenedetto, U. Gianazza and V. Vespri, Harnack Estimates for Quasi-Linear Degenerate Parabolic Differential Equation, *Acta Math.,* **200***, (2008), 181–209.*
50. E. DiBenedetto, U. Gianazza and V. Vespri, Sub-Potential Lower Bounds for Non-Negative Solutions to Certain Quasi-Linear Degenerate Parabolic Differential Equations, *Duke Math. J.,* **143***(1), (2008), 1–15.*
51. E. DiBenedetto, U. Gianazza and V. Vespri, Forward, Backward and Elliptic Harnack Inequalities for Non-Negative Solutions to Certain Singular Parabolic Partial Differential Equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5),* **IX***, (2010), 385–422.*
52. E. DiBenedetto, U. Gianazza and V. Vespri, Alternative Forms of the Harnack Inequality for Non-Negative Solutions to Certain Degenerate and Singular Parabolic Equations, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.,* **20***, (2009), 369–377.*
53. E. DiBenedetto, U. Gianazza and V. Vespri, Harnack Estimates and Hölder Continuity for Solutions to Singular Parabolic Partial Differential Equations in the Sub-critical Range, *Manuscripta Math.,* **131***, (2010), 231–245.*
54. E. DiBenedetto, U. Gianazza and V. Vespri, A New Approach to the Expansion of Positivity Set of Non-Negative Solutions to Certain Singular Parabolic

Partial Differential Equations, *Proc. Amer. Math. Soc.,* **138***(10), (2010), 3521–3529.*
55. E. DiBenedetto, U. Gianazza and V. Vespri, Continuity of the Saturation in the Flow of Two Immiscible Fluids in a Porous Medium, *Indiana Univ. Math. J. (in press).*
56. E. DiBenedetto, U. Gianazza and V. Vespri, Liouville-Type Theorems for Certain Degenerate and Singular Parabolic Equations, *C. R. Acad. Sci. Paris Ser. I,* **348***, (2010), 873–877.*
57. E. DiBenedetto and M.A. Herrero, On the Cauchy Problem and Initial Traces for a Degenerate Parabolic Equation, *Trans. Amer. Math. Soc.,* **314***(1), (1989), 187–224.*
58. E. DiBenedetto and M.A. Herrero, Non Negative Solutions of the Evolution p-Laplacian Equation. Initial Traces and Cauchy Problem When $1 < p < 2$, *Arch. Rational Mech. Anal.,* **111***(3), (1990), 225–290.*
59. E. DiBenedetto and Y.C. Kwong, Intrinsic Harnack Estimates and Extinction Profile for Certain Singular Parabolic Equations, *Trans. Amer. Math. Soc.,* **330***(2), (1992), 783–811.*
60. E. DiBenedetto, Y.C. Kwong and V. Vespri, Local Space-analiticity of Solutions of Certain Singular Parabolic Equations, *Indiana Univ. Math. J.,* **40***(2), (1991), 741–765.*
61. E. DiBenedetto and N.S. Trudinger, Harnack Inequalities for Quasi-Minima of Variational Integrals, *Ann. Inst. H. Poincaré Anal. Non Linéaire,* **1***(4), (1984), 295–308.*
62. E. DiBenedetto, J.M. Urbano and V. Vespri, Current Issues on Singular and Degenerate Evolution Equations, *Handbook of Differential Equations, Vol. II, Evolutionary Equations, 169–286; North-Holland, Amsterdam, 2004.*
63. E. DiBenedetto and V. Vespri, On the singular equation $\beta(u)_t = \Delta u$, *Arch. Rational Mech. Anal.,* **132***(3), (1995), 247–309.*
64. D.T. Egorov, Sur les suites des fonctions mesurables, *C. R. Acad. Sci.,* **152***, (1911), 244.*
65. L.C. Evans and R.F. Gariepy, Wiener's Criterion for the Heat Equation, *Arch. Rational Mech. Anal.,* **78***(4), (1982), 293–314.*
66. E.B. Fabes and D.W. Stroock, A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash, *Arch. Rational Mech. Anal.,* **96***(4), (1986), 327–338.*
67. W.H. Fleming and R. Rishel, An integral formula for total gradient variation, *Arch. Math.,* **XI***, (1960), 218–222.*
68. S. Fornaro and V. Vespri, Harnack estimates for non negative weak solutions of singular parabolic equations satisfying the comparison principle, *(in preparation).*
69. A. Friedman, *Variational Principles and Free Boundary Problems*, John Wiley & Sons, New York, 1982.
70. A. Friedman, On the regularity of the solutions of nonlinear elliptic and parabolic systems of partial differential equations, *J. Math. Mech.,* **7***, (1958), 43–59.*
71. E. Gagliardo, Proprietà di alcune funzioni in n variabili, *Ric. Mat.,* **7***, (1958), 102–137.*
72. R. Gariepy and W.P. Ziemer, A Regularity Condition at the Boundary for Solutions of Quasilinear Elliptic Equations, *Arch. Rational Mech. Anal.,* **67***(1), (1977), 25–39.*

73. J. Garcia-Azorero and I. Peral Alonso, Existence and Nonuniqueness for the p-Laplacian: nonlinear eigenvalues, *Comm. Partial Differential Equations,* **12***(12), (1987), 1389–1430.*
74. U. Gianazza and S. Polidoro, Lower bounds for solutions of degenerate parabolic equations, *Subelliptic PDE's and applications to geometry and finance, Lect. Notes Semin. Interdiscip. Mat.,* **6**, *(2007), 157–162.*
75. U. Gianazza, M. Surnachev and V. Vespri, On a new proof of Hölder continuity of solutions of p-Laplace type parabolic equations, *Adv. Calc. Var.,* **3**, *(2010), no. 3, 263–278.*
76. D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Die Grundlehren der Mathematischen Wissenschaften, No. 224, Springer-Verlag, Berlin, 1983.
77. B.H. Gilding, On a class of similarity solutions of the porous media equation III, *J. Math. Anal. Appl.,* **77** *(1980), 381–402.*
78. R. Ya. Glagoleva, Liouville theorems for the solution of second order linear parabolic equations with discontinuous coefficients, *Math. Zametki,* **5**, *(1969), 599–606.*
79. R. Ya. Glagoleva, Phragmén-Lindelöf type theorems and Liouville theorems for a linear parabolic equation, *Math. Zametki,* **37**, *(1985), 119–124.*
80. J. Hadamard, Extension à l'équation de la chaleur d'un théorème de A. Harnack, *Rend. Circ. Mat. Palermo (2),* **3**, *(1954), 337–346.*
81. T.C. Halsey, Electrorheological Fluids, *Science,* **258**, *(1992), 761–766.*
82. C.-G. Axel Harnack, *Die Grundlagen der Theorie des logaritmischen Potentiales und der eindeutigen Potentialfunktion in der Ebene*, Teubner, Leipzig, 1887.
83. J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, Oxford University Press, New York, 1993.
84. M.A. Herrero and M. Pierre, The Cauchy problem for $u_t = \Delta u^m$ when $0 < m < 1$, *Trans. Amer. Math. Soc.,* **291***(1), (1985), 145–158.*
85. R.D. James and D.S. Kinderlehrer, Mathematical approaches to the study of smart materials, *SPIE,* **1919**, *(1993), 2–18.*
86. A.S. Kalashnikov, Some problems of the qualitative theory of second-order nonlinear degenerate parabolic equations, *Russian Math. Surv.,* **42***(2), (1987), 169–222.*
87. A. Kar, J.C. Jacquier, D. J. Morgan, J.G. Lyng and B.M. McKenna, Influence of Lipid Extraction Process on the Rheological Characteristics, Swelling Power, and Granule Size of Rice Starches in Excess Water, *J. Agric. Food Chem.,* **53**, *(2005), 8259–8264.*
88. M. Kassmann, Harnack Inequalities: An Introduction, *Boundary Value Problem, Volume 2007, Article ID 81415, 21 pages.*
89. T. Kilpeläinen and J. Malý, The Wiener Test and Potential Estimates for Quasi-linear Elliptic Equations, *Acta Math.,* **172**, *(1994), 137–161.*
90. D. Kinderlehrer and L. Nirenberg, Analyticity at the boundary of solutions of nonlinear second-order parabolic equations, *Comm. Pure Appl. Math.,* **31***(3), (1978), 283–338.*
91. A.E. Kogoj and E. Lanconelli, Liouville theorems in halfspaces for parabolic hypoelliptic equations, *Ric. Mat.,* **55**, *(2006), 267–282.*
92. R. Korte, T. Kuusi and M. Parviainen, A connection between a general class of super-parabolic functions and super-solutions, *J. Evol. Equ.,* **10**, *(2010), 1–20.*

93. N.V. Krylov, *Nonlinear elliptic and parabolic equations of the second order*, Mathematics and its Applications (Soviet Series), 7, D. Reidel Publishing Co., Dordrecht, 1987.
94. N.V. Krylov and M.V. Safonov, A property of the solutions of parabolic equations with measurable coefficients, *Izv. Akad. Nauk SSSR Ser. Mat.,* **44***(1), (1980), 161–175, 239.*
95. S.N. Kruzkov, On the apriori estimation of solutions of linear parabolic equations and of solutions of boundary value problems for a certain class of quasilinear parabolic equations, *Dokl. Akad. Nauk SSSR,* **138***, (1961), 1005–1008*; (Engl. transl.: *Soviet Math. Dokl.,* **2***, (1961), 764–767*).
96. S.N. Kruzkov, A priori estimates and certain properties of the solutions of elliptic and parabolic equations of second order, *Mat. Sb.,* **65***(107), (1964), 522–570*; (Engl. transl.: *Amer. Math. Soc. Transl.,* **2***(68), (1968), 169–220*).
97. S.N. Kruzkov, Results concerning the nature of the continuity of solutions of parabolic equations and some of their applications, *Mat. Zametki,* **6***, (1969), 97–108 (Russian).*
98. T. Kuusi, The weak Harnack estimate for weak supersolutions to nonlinear degenerate parabolic equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5),* **7***(4), (2008), 673–716.*
99. T. Kuusi and M. Parviainen, Existence of solutions to degenerate Cauchy problems, *Manuscripta Math.,* **128***(2), (2009), 213–249.*
100. O.A. Ladyzenskaya, New equations for the description of the motions of viscous incompressible fluids, and global solvability for their boundary value problems, *Trudy Math. Inst. Steklov,* **102***, (1967), 85–104*; (Engl. transl.: *Proc. Steklov Inst. Math.,* **102***, (1967), 95–118*).
101. O.A. Ladyzenskaya, N.A. Solonnikov and N.N. Ural'tzeva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, 23, American Mathematical Society, Providence, RI, 1967.
102. O.A. Ladyzenskaya and N.N. Ural'tzeva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.
103. E. Lanconelli, Bruno Pini and the parabolic Harnack inequality: the dawning of Parabolic Potential Theory, *preprint, (2011), 1–13.*
104. E.M. Landis, *Second Order Equations of Elliptic and Parabolic Type*, Transl. Math. Monographs, #171, AMS, Providence, RI, 1997. Nauka, Moscow, 1971.
105. M.C. Leverett, Capillary Behaviour in Porous Solids, *Trans. Amer. Inst. Mining Metall. Eng.,* **142***, (1941), 151–169.*
106. G. Lieberman, A new regularity estimate for solutions of singular parabolic equations, *Discrete Contin. Dyn. Syst., (2005), suppl., 605–610.*
107. P. Lindqvist, On the time derivative in a quasilinear equation, *Trans. R. Norw. Soc. Sci. Lett.,* **2***, (2008), 1–7.*
108. J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.
109. V. Liskevich and I.I. Skrypnik, Harnack inequality and continuity of solutions to quasi-linear degenerate parabolic equations with coefficients from Kato-type classes, *J. Differential Equations,* **247***, (2009), 2740–2777.*
110. J. Malý and W.P. Ziemer, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, American Mathematical Society, Providence, RI, 1997.
111. D. Marguet, P.F. Lenne, H. Rigneault and H.T. He, Dynamics in the Plasma Membrane: How to Combine Fluidity and Order, *EMBO J.,* **25***, (2006), 3446–3457.*

112. L.K. Martinson and K.B. Paplov, The effect of magnetic plasticity in non-Newtonian fluids, *Magnit. Gidrodinamika,* **3**, *(1969), 69–75.*
113. L.K. Martinson and K.B. Paplov, Unsteady shear flows of a conducting fluid with a rheological power law, *Magnit. Gidrodinamika,* **2**, *(1970), 50–58.*
114. V.G. Mazja, *Sobolev Spaces*, Springer-Verlag, New York, 1985.
115. V.G. Maz'ya, On the Continuity at a Boundary Point of Solutions of Quasi-linear Elliptic Equations, *Vestnik Leningrad Univ. Math.,* **3**, *(1976), 225–242.*
116. G. Minty, Monotone (non linear) operators in Hilbert spaces, *Duke Math. J.,* **29**, *(1962), 341–346.*
117. M. Miranda, Sul minimo dell'integrale del gradiente di una funzione, *Ann. Scuola Norm. Sup. Pisa (3),* **19**, *(1965), 626–665.*
118. F.A. Morrison, *Understanding Rheology*, Oxford University Press, Oxford, 2001.
119. J. Moser, A new proof of DeGiorgi's theorem concerning the regularity problem for elliptic differential equations, *Comm. Pure Appl. Math.,* **13**, *(1960), 457–468.*
120. J. Moser, On Harnack's theorem for elliptic differential equations, *Comm. Pure Appl. Math.,* **14**, *(1961), 577–591.*
121. J. Moser, A Harnack Inequality for Parabolic Differential Equations, *Comm. Pure Appl. Math.,* **17**, *(1964), 101–134.*
122. J. Moser, On a pointwise estimate for parabolic differential equations, *Comm. Pure Appl. Math.,* **24**, *(1971), 727–740.*
123. J. Nash, Continuity of solutions of parabolic and elliptic equations, *Amer. J. Math.,* **80**, *(1958), 931–954.*
124. L. Nirenberg, On Elliptic Partial Differential Equations, *Ann. Sc. Norm. Sup. Pisa (3),* **13**, *(1959), 115–162.*
125. R.E. Pattle, Diffusion from an instantaneous point source with a concentration-dependent coefficient, *Q. J. Appl. Math.,* **12**, *(1959), 407–409.*
126. M.A. Peletier and H. Zhang, Self-Similar solutions of a fast diffusion equation that do not conserve mass, *Diff. Int. Eqns.,* **8**, *(1995), 2045–2064.*
127. B. Pini, Sulla soluzione generalizzata di Wiener per il primo problema di valori al contorno nel caso parabolico, *Rend. Sem. Mat. Univ. Padova,* **23**, *(1954), 422–434.*
128. H. Poincaré, Sur Les Equations aux Dérivées Partielles de la Physique Mathématique, *Amer. J. Math.,* **12***(3), (1890), 211–294.*
129. H. Poincaré, Sur les équations de la physique mathématique, *Rend. Circ. Mat. Palermo (1),* **8**, *(1894), 57–156.*
130. M.M. Porzio, $L^\infty_{\rm loc}$-estimates for degenerate and singular parabolic equations, *Nonlinear Anal.,* **17***(11), (1991), 1093–1107.*
131. M.M. Porzio and V. Vespri, Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations, *J. Differential Equations,* **103***(1), (1993), 146–178.*
132. T.V. Ratto and M.L. Longo, Anomalous Subdiffusion in Heterogeneous Lipid Bilayers, *Langmuir,* **19**, *(2003), 1788–1793.*
133. P.A. Raviart, Sur la résolution de certaines équations paraboliques non linéaires, *J. Functional Analysis,* **5**, *(1970), 299–328.*
134. M. Ružička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics, # 1748, Springer, Berlin, 2000.

135. M.V. Safonov, Harnack's inequality for elliptic equations and Hölder property of their solutions, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI),* **96***, (1980), 272–287, 312.*
136. L. Saloff-Coste, *Aspects of Sobolev Type Inequalities*, London Mathematical Society, Lecture Notes # 289, London, 2002.
137. S. Salsa, Some properties of nonnegative solutions of parabolic differential operators, *Ann. Mat. Pura Appl.,* **128***(4), (1981), 193–206.*
138. M.P. Saxton, Lateral Diffusion of Lipids and Proteins, *Curr. Top. Membranes,* **48***, (1999), 229–282.*
139. B.A. Scalettar and J.R. Abney, Molecular Crowding and Protein Diffusion in Biological Membranes, *Comm. Mol. Cell. Biophys.,* **7***, (1991), 79–107.*
140. A.E. Scheidegger, *The Physics of Flow Through Porous Media*, University of Toronto Press, Toronto, 1974.
141. J. Serrin, Local behavior of solutions of quasi-linear equations, *Acta Math.,* **111***, (1964), 247–302.*
142. C. Severini, Sulle successioni di funzioni ortogonali, *Atti dell'Accademia Gioenia, 3 (5) Memoria XIII, 1ª (1910).*
143. S.L. Sobolev and S.M. Nikol'skii, Embedding Theorems, *Izdat. Akad. Nauk SSSR, Leningrad, (1963), 227–242.*
144. G. Stampacchia, *Equations elliptiques du second ordre a coefficients discontinues*, Sém. Math. Sup. **16**, Les Presses de l'Université de Montréal, Montréal, 1966.
145. M. Surnachev, A Harnack inequality for weighted degenerate parabolic equations, *J. Differential Equations,* **248***, (2010), 2092–2129.*
146. G. Talenti, Best Constants in Sobolev Inequalities, *Ann. Mat. Pura Appl.,* **110***, (1976), 353–372.*
147. N.S. Trudinger, On Harnack type inequalities and their application to quasilinear elliptic partial differential equations, *Comm. Pure Appl. Math.,* **20***, (1967), 721–747.*
148. N.S. Trudinger, Pointwise Estimates and Quasilinear Parabolic Equations, *Comm. Pure Appl. Math.,* **21***, (1968), 205–226.*
149. J.M. Urbano, *The Method of Intrinsic Scaling*, Lecture Notes in Mathematics, # 1930, Springer, Berlin, 2008.
150. J.L. Vázquez, *Smoothing and decay estimates for nonlinear diffusion equations*, Oxford Lecture Series in Mathematics and its Applications, 33, Oxford University Press, Oxford, 2006.
151. V. Vespri, Harnack type inequalities for solutions of certain doubly nonlinear parabolic equations, *J. Math. Anal. Appl.,* **181***(1), (1994), 104–131.*
152. D.V. Widder, *The Heat Equation*, Academic Press, New York, 1975.

Index

$L^{q,r}$ spaces, 18
$V^{m,p}$ spaces, 19, 20

Analyticity of solutions to the porous medium equation, 172, 184

Barenblatt solution
  p-Laplacian, 11, 63, 94, 108, 179, 182, 196
  Porous medium equation, 128, 184, 203

Comparison Principle
  for the p-Laplacian, 179, 180, 189, 195
  for the porous medium equation, 183, 201, 203
Counterexamples
  to a nonintrinsic expansion of positivity, 62
  to a nonintrinsic Harnack inequality, 94, 128
  to the Harnack inequality when $0 < m \le \frac{(N-2)_+}{N}$, 168
  to the Harnack inequality when $1 < p \le \frac{2N}{N+1}$, 140

DeGiorgi-type lemma
  A variant for the p-Laplacian, 32, 62, 77, 258
  A variant for the porous medium equation when $0 < m < 1$, 52, 87, 264
  A variant for the porous medium equation when $m > 1$, 45, 264
  for nonnegative subsolutions to the porous medium equation when $0 < m < 1$, 238, 249
  for nonnegative supersolutions to the porous medium equation when $0 < m < 1$, 50, 87, 166, 244, 263
  for the p-Laplacian, 29, 71, 76, 94, 97, 139, 257
  for the porous medium equation when $m > 1$, 39, 128, 129, 263

Expansion of positivity, 88, 135
  for nondegenerate equations, 59
  for the p-Laplacian when $1 < p < 2$, 72–77, 142, 146, 159, 193–196, 260
  for the p-Laplacian when $p > 2$, 66–72, 94, 97, 112, 115, 260
  for the porous medium equation when $0 < m < 1$, 83–88, 171, 178, 203, 266
  for the porous medium equation when $m > 1$, 82, 128, 129, 265
  in some intrinsic geometry, 66
Explicit solutions
  to the prototype p-Laplacian, 63, 65, 89, 94, 108, 140, 196
  to the prototype porous medium equation, 128, 168, 203

Harnack implies Hölder, 102, 153, 160, 180
Harnack inequality

Classical form for harmonic functions, 1
Elliptic form for the p-Laplacian, 140
Elliptic form for the porous medium equation, 168
Form when $0 < m \leq \frac{(N-2)_+}{N}$, 176
Form when $1 < p \leq \frac{2N}{N+1}$, 155
Mean value form for harmonic functions, 2
Mean value form when $\frac{(N-2)_+}{N} < m < 1$, 166, 202
Mean value form when $\frac{2N}{N+1} < p < 2$, 138
Mean value form when $m > 1$, 127
Mean value form when $p = 2$, 3, 91, 134
Mean value form when $p > 2$, 92
Time-insensitive mean value form when $\frac{(N-2)_+}{N} < m < 1$, 167, 202
Time-insensitive mean value form when $\frac{2N}{N+1} < p < 2$, 139, 190
Hölder continuity for the porous medium equation, 130, 135, 172–174, 183, 243–253

Lemma
Fast geometric convergence, 20, 258
Interpolation, 21, 214, 218, 219, 233, 235
Measure-theoretical, 16, 113
Liouville-type theorems
for the p-Laplacian, 105–108, 135, 153
for the porous medium equation, 131, 175
Logarithmic estimate, 240, 245

Monotonicity of the principal part, 55, 135, 187, 199

Notion of solution
for the p-Laplacian, 23, 256
for the porous medium equation, 34, 261

Parabolic embedding, 18, 30, 33, 41, 44, 46, 51, 81, 118, 120, 181, 185, 212, 214, 232, 235, 239
Parabolicity, 24, 35, 55
Poincaré inequality, 13, 15, 22

Regularity of u_t
for the p-Laplacian, 24, 54
for the porous medium equation, 34, 55

Sobolev
exponent, 11
inequality, 13, 22
spaces, 13
Stability of constants, 26, 29, 36, 38, 46, 47, 50, 54, 59, 67, 78, 79, 82, 84, 91, 93, 97, 112, 128, 137, 138, 148, 150, 165, 166, 205, 206, 208, 210, 218, 222, 225, 227, 230, 237
Stability of the expansion of positivity
for the p-Laplacian, 78
for the porous medium equation, 82
Steklov averages, 21, 22, 24, 34, 47, 54, 206, 215, 226, 228, 236, 241
Structure conditions
for the p-Laplacian, 23, 68, 73, 95, 143, 266
for the porous medium equation, 33, 84, 129, 171, 252, 261, 266, 267
Subpotential lower bounds
for the p-Laplacian, 108, 196
for the porous medium equation, 132, 203

Weak Harnack inequality
for $0 < m < 1$, 181
for $1 < p < 2$, 181
for $m > 1$, 133
for $p > 2$, 112–127