Chapter 5
Altitude and Yaw Control of Helicopters with Uncertain Dynamics

5.1 Introduction

In Chap. 4, a robust adaptive neural network (NN) control is presented for helicopters in vertical flight, with dynamics in single-input single-output (SISO) nonlinear nonaffine form. By limiting the scope to the vertical flight regime, SISO models can be used to yield useful results, since the coupling between longitudinal and lateral-directional equations in this flight regime is weak [84]. While the proposed controller handles vertical flight, other flight regimes can be handled by other control modules. Evidently, SISO control designs have limited practical use, and many more investigations are needed in the control of multi-input multi-output (MIMO) helicopter dynamics for generality in applications.

Practical helicopter motion governed by a MIMO model has an underactuated configuration, i.e., the number of control inputs is less than the number of degrees of freedom to be stabilized, which makes it difficult to apply the conventional robotics approach for controlling Euler–Lagrange systems. Thus, some flight control techniques need to be further developed for the nonlinear MIMO helicopter dynamics. In [104], model-based control was applied to an autonomous scale MIMO model helicopter mounted in a 2-degree-of-freedom (2DOF) platform. Since helicopter control applications are characterized by unknown aerodynamical disturbances, they are generally difficult to model accurately. The presence of modeling errors, in the form of parametric and functional uncertainties, and unmodeled dynamics and disturbances from the environment, is a common problem. In this context, model-based control, such as the aforementioned schemes, tends to be susceptible to uncertainties and disturbances that cause performance degradation.

In this chapter, altitude and yaw angle tracking are considered for a scale MIMO model helicopter [104] in the presence of model uncertainties, which may be caused by unmodeled dynamics, sensor errors or aerodynamical disturbances from the environment. To deal with the presence of model uncertainties, approximation-based techniques using a NN have been proposed. In particular, two commonly used NNs, namely the multilayer neural network (MNN) and the radial basis

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function neural network (RBFNN) are adopted in control design and stability analysis. Based on Lyapunov synthesis, the proposed adaptive NN control ensures that both the altitude and the yaw angle track the given bounded reference signals to a small neighborhood of zero, and guarantees the Semi-Globally Uniformly Ultimate Boundedness (SGUUB) of all the closed-loop signals at the same time. The effectiveness of the proposed control is illustrated through extensive simulations. Compared with the model-based control used in [104], approximation-based control using NN, proposed in this chapter, can accommodate the presence of model uncertainties, reduce the dependence on accurate model building, and thus lead to the tracking performance improvement.

5.2 Problem Formulation and Preliminaries

We consider the VARIO scale model helicopter [104], however, the functions and parameters involved in the model are unknown. For clarity, we restate the helicopter dynamics here, which are described by Lagrangian formulation in the following:

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + F(\dot{q}) + G(q) + \Delta(q,\dot{q}) = B(\dot{q})\tau \tag{5.1}$$

where $q, \dot{q}$, and $\ddot{q}$ are referred as the vectors of generalized coordinates, generalized velocities, and generalized accelerations, respectively. In particular, $q = [q_1, q_2, q_3]^{\mathrm{T}} = [z, \phi, \gamma]^{\mathrm{T}}$ with z as the attitude ($z > 0$ downwards), ϕ as the yaw angle, and γ as the main rotor azimuth angle; $\dot{q} = [\dot{q}_1, \dot{q}_2, \dot{q}_3]^{\mathrm{T}} = [\dot{z}, \dot{\phi}, \dot{\gamma}]^{\mathrm{T}}$ with $\dot{z}$ as the vertical velocity, $\dot{\phi}$ as the yaw rate, and $\dot{\gamma}$ as the main rotor angular velocity; $\ddot{q} = [\ddot{q}_1, \ddot{q}_2, \ddot{q}_3]^{\mathrm{T}} = [\ddot{z}, \ddot{\phi}, \ddot{\gamma}]^{\mathrm{T}}$ with $\ddot{z}$ as the vertical acceleration, $\ddot{\phi}$ as the yaw acceleration, and $\ddot{\gamma}$ as the main rotor angular acceleration; $D(q) \in R^{3\times 3}$ is the inertia matrix; $C(q,\dot{q})\dot{q} \in R^3$ is the vector of Coriolis and centrifugal forces; $F(\dot{q}) \in R^3$ is the vector of friction forces; $G(q) \in R^3$ is the vector of gravitational forces; $\Delta(q,\dot{q}) \in R^3$ is the vector of the model uncertainties, which may be caused by unmodeled dynamics, sensor errors or aerodynamical disturbances from the environment; $B(\dot{q}) \in R^{3\times 2}$ is the matrix of control coefficients; and the control inputs $\tau = [\tau_1, \tau_2]^{\mathrm{T}} \in R^2$ are the main and tail rotor collectives (swash plate displacements), respectively. By exploiting the physical properties of the helicopter, e.g., how the control inputs are distributed to the helicopter dynamics, or the coupling relationship between the states, better performance can be achieved. To this end, we assume partial knowledge of the structure of the dynamics [104], although the functions and parameters involved are unknown:

$$D(q) = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22}(q_3) & d_{23} \\ 0 & d_{23} & d_{33} \end{bmatrix} \quad C(q,\dot{q}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_{22}(q_3,\dot{q}_3) & c_{23}(q_3,\dot{q}_2) \\ 0 & c_{32}(q_3,\dot{q}_2) & 0 \end{bmatrix}$$

$$F(\dot{q}) = \begin{bmatrix} f_1(\dot{q}_3) \\ 0 \\ f_3(\dot{q}_3) \end{bmatrix} \quad G(q) = \begin{bmatrix} g_1 \\ 0 \\ g_3 \end{bmatrix} \quad \Delta(q,\dot{q}) = \begin{bmatrix} \Delta_1(q,\dot{q}) \\ \Delta_2(q,\dot{q}) \\ \Delta_3(q,\dot{q}) \end{bmatrix}$$

$$B(\dot{q}) = \begin{bmatrix} b_{11}(\dot{q}_3) & 0 \\ 0 & b_{22}(\dot{q}_3) \\ b_{31}(\dot{q}_3) & 0 \end{bmatrix} \tag{5.2}$$

where $d_{11},\ d_{23},\ d_{33},\ g_1,\ g_3$ are unknown constants, $d_{22}(q_3),\ c_{22}(q_3,\dot{q}_3), c_{23}(q_3,\dot{q}_2), c_{32}(q_3,\dot{q}_2),\ \ f_1(\dot{q}_3), f_3(\dot{q}_3),\ \ b_{11}(\dot{q}_3),\ \ b_{22}(\dot{q}_3),\ \ b_{31}(\dot{q}_3), \Delta_1(q,\dot{q}),\ \ \Delta_2(q,\dot{q})$ and $\Delta_3(q,\dot{q})$ are unknown functions.

To facilitate control design in Sect. 5.3, the following assumptions are in order:

Assumption 5.1. The terms d_{11} and $\frac{d_{22}(q_3)d_{33}-d_{23}^2}{2d_{33}}$ are positive.

Assumption 5.2. The following equation $\dot{d}_{22}(q_3) - 2c_{22}(q_3,\dot{q}_3) = 0$ holds.

Remark 5.1. It is easy to know that the helicopter model in (5.1) with the parameters given in [104], which will be used in the subsequent simulation section, satisfies both Assumptions 5.1 and 5.2.

Assumption 5.3. The signs of $b_{11}(\dot{q}_3)$ and $b_{22}(\dot{q}_3)$ are known. Without losing generality, assume that $b_{11}(\dot{q}_3)$ is positive and $b_{22}(\dot{q}_3)$ is negative. There exist positive constants $\underline{b}_{11}$ and $\underline{b}_{22}$, such that $0 \le \underline{b}_{11} \le |b_{11}(\dot{q}_3)|$ and $0 \le \underline{b}_{22} \le |b_{22}(\dot{q}_3)|$.

Remark 5.2. In this section, the vertical flight mode after take-off is considered. From physical analysis, to lift the helicopter up for flight operation, $|\dot{q}_3|$ has to be larger than some certain positive value (e.g., c_0) to overcome the gravity. It is noted that in the specific helicopter model given in (5.1), $b_{11}(\dot{q}_3) = 3.411\dot{q}_3^2 \ge 3.411c_0^2 > 0$. Therefore, there always exist some positive constants $\underline{b}_{11}$ such that $0 \le \underline{b}_{11} \le |b_{11}(\dot{q}_3)|$ during the vertical flight mode. Similar analysis can be applied to $b_{22}(\dot{q}_3)$ as in Assumption 5.3.

Assumption 5.4. There exist positive constants $\underline{d}_{22}$ and $\bar{d}_{22}$, such that $\underline{d}_{22} \le |d_{22}(q_3)| \le \bar{d}_{22}$.

Remark 5.3. Assumption 5.4 is reasonable due to $d_{22}(q_3) = 0.4305 + 0.0003 \cos^2(-4.143q_3)$ in the specific helicopter model given in (5.1), which will be used in the subsequent simulation section.

The control objective is to ensure that the tracking errors for the altitude $q_1(t)$ and yaw angle $q_2(t)$ from their respective desired trajectories $q_{1d}(t)$ and $q_{2d}(t)$, are driven to a small neighborhood of zero, i.e., $|q_i(t) - q_{id}(t)| \le \epsilon_i$, where $\epsilon_i > 0$, $i = 1, 2$; at the same time, the main rotor angular velocity $\dot{q}_3(t)$ is stable.

Assumption 5.5. The desired trajectories $q_{1d}(t)$ and $q_{2d}(t)$ and their time derivatives up to the 3rd order are continuously differentiable and bounded for all $t \ge 0$.

The following technical lemma is required in the subsequent control design and stability analysis.

Lemma 5.4. *For $a, b \in R^+$, the following inequality holds*

$$\frac{ab}{a+b} \leq a \tag{5.3}$$

5.3 Control Design

In this section, we will design an adaptive neural control to accommodate the presence of uncertainties in the dynamics (5.1), appearing in the functions $D(q)$, $C(q, \dot{q})$, $F(\dot{q})$, $G(q)$, $\Delta(q, \dot{q})$ and $B(\dot{q})$. After some simple manipulations on (5.1) and (5.2), we can obtain three subsystems: q_1-subsystem (5.4), q_2-subsystem (5.5) and q_3-subsystem (5.6) as follows:

$$d_{11}\ddot{q}_1 + f_1(\dot{q}_3) + g_1 + \Delta_1(q, \dot{q}) = b_{11}(\dot{q}_3)\tau_1 \tag{5.4}$$

$$\begin{aligned} &\frac{d_{22}(q_3)d_{33} - d_{23}^2}{d_{33}}\ddot{q}_2 + c_{22}(q_3, \dot{q}_3)\dot{q}_2 + c_{23}(q_3, \dot{q}_2)\dot{q}_3 + \Delta_2(q, \dot{q}) + \frac{d_{23}}{d_{33}}(b_{31}(\dot{q}_3)\tau_1 \\ &\quad - c_{32}(q_3, \dot{q}_2)\dot{q}_2 - f_3(\dot{q}) - g_3 - \Delta_3(q, \dot{q})) = b_{22}(\dot{q}_3)\tau_2 \end{aligned} \tag{5.5}$$

$$\begin{aligned} &\frac{d_{22}(q_3)d_{33} - d_{23}^2}{d_{22}(q_3)}\ddot{q}_3 + c_{32}(q_3, \dot{q}_2)\dot{q}_2 + f_3(\dot{q}_3) + g_3 + \Delta_3(q, \dot{q}) \\ &\quad + \frac{d_{23}}{d_{22}(q_3)}(b_{22}(\dot{q}_3)\tau_2 - c_{22}(q_3, \dot{q}_3)\dot{q}_2 - c_{23}(q_3, \dot{q}_2)\dot{q}_3 - \Delta_2(q, \dot{q})) = b_{31}(\dot{q}_3)\tau_1 \end{aligned} \tag{5.6}$$

In the following, we will analyze and design a control for each subsystem. For clarity, define the tracking errors and the filtered tracking errors as

$$e_i = q_i - q_{id}, \quad r_i = \dot{e}_i + \lambda_i e_i \tag{5.7}$$

where λ_i is a positive number, $i = 1, 2$. Then, the boundedness of r_i guarantees the boundedness of e_i and $\dot{e}_i$ [10, 71–94]. To study the stability of e_i and $\dot{e}_i$, we only need to study the properties of r_i. In addition, the following computable signals are defined:

$$\dot{q}_{ir} = \dot{q}_{id} - \lambda_i e_i, \quad \ddot{q}_{ir} = \ddot{q}_{id} - \lambda_i \dot{e}_i$$

5.3.1 RBFNN-Based Control

In this section, we will investigate the RBFNN based control design by Lyapunov synthesis to achieve the control objective. Regarding to the obtained three subsystems (5.4)–(5.6), our control design consists of three steps: First, we will design control τ_1 based on the q_1-subsystem (5.4); Second, design τ_2 based on the q_2-subsystem (5.5) and τ_1; finally, analyze the stability of the internal dynamics of q_3-subsystem (5.6).

□ q_1*-subsystem*

Since $\dot{q}_1 = \dot{q}_{1r} + r_1, \ddot{q}_1 = \ddot{q}_{1r} + \dot{r}_1$, (5.4) becomes

$$d_{11}\dot{r}_1 = b_{11}(\dot{q}_3)\tau_1 - f_{S1,1} \tag{5.8}$$

where

$$f_{S1,1} = d_{11}\ddot{q}_{1r} + f_1(\dot{q}_3) + g_1 + \Delta_1(q, \dot{q}) \tag{5.9}$$

is an unknown continuous function, which is approximated by RBFNN to arbitrarily any accuracy as

$$f_{S1,1} = W_1^{*\mathrm{T}} S_1(Z_1) + \varepsilon_1(Z_1) \tag{5.10}$$

where the input vector $Z_1 = [q_1, \dot{q}_1, q_2, \dot{q}_2, q_3, \dot{q}_3, \dot{q}_{1d}, \ddot{q}_{1d}]^{\mathrm{T}} \in \Omega_{Z1} \subset R^8$; $\varepsilon_1(Z_1)$ is the approximation error satisfying $|\varepsilon_1(Z_1)| \leq \bar{\varepsilon}_1$, where $\bar{\varepsilon}_1$ is a positive constant; W_1^* are ideal constant weights satisfying $\|W_1^*\| \leq w_{1m}$, where w_{1m} is a positive constant; and $S_1(Z_1)$ are the basis functions. By using $\hat{W}_1$ to approximate W_1^*, the error between the actual and the ideal RBFNNs can be expressed as

$$\hat{W}_1^{\mathrm{T}} S_1(Z_1) - W_1^{*\mathrm{T}} S_1(Z_1) = \tilde{W}_1^{\mathrm{T}} S_1(Z_1) \tag{5.11}$$

where $\tilde{W}_1 = \hat{W}_1 - W_1^*$.

Consider the following Lyapunov function candidate

$$V_1 = \frac{1}{2} d_{11} r_1^2 + \frac{1}{2} \tilde{W}_1^{\mathrm{T}} \Gamma_1^{-1} \tilde{W}_1 \tag{5.12}$$

The time derivative of (5.12) along (5.8) and (5.10) is given by

$$\begin{aligned}\dot{V}_1 &= d_{11} r_1 \dot{r}_1 + \tilde{W}_1^{\mathrm{T}} \Gamma_1^{-1} \dot{\tilde{W}}_1 \\ &= r_1 \left[b_{11}(\dot{q}_3)\tau_1 - W_1^{*\mathrm{T}} S_1(Z_1) - \varepsilon_1(Z_1) \right] + \tilde{W}_1^{\mathrm{T}} \Gamma_1^{-1} \dot{\tilde{W}}_1 \end{aligned} \tag{5.13}$$

As W_1^* is a constant vector, we know that $\dot{\tilde{W}}_1 = \dot{\hat{W}}_1$. Therefore, (5.13) becomes

$$\dot{V}_1 = r_1\left[b_{11}(\dot{q}_3)\tau_1 - W_1^{*\mathrm{T}} S_1(Z_1) - \varepsilon_1(Z_1)\right] + \tilde{W}_1^{\mathrm{T}} \Gamma_1^{-1} \dot{\hat{W}}_1 \tag{5.14}$$

Consider the following RBFNN-based control law and RBFNN weight adaptation law:

$$\tau_1 = -k_1 r_1 - \frac{r_1\left(\hat{W}_1^{\mathrm{T}} S_1(Z_1)\right)^2}{\underline{b}_{11}\left(|r_1 \hat{W}_1^{\mathrm{T}} S_1(Z_1)| + \delta_1\right)} \tag{5.15}$$

$$\dot{\hat{W}}_1 = -\Gamma_1\left[S_1(Z_1) r_1 + \sigma_1 \hat{W}_1\right] \tag{5.16}$$

where $k_1 > 0$, $\delta_1 > 0$, $\Gamma_1 = \Gamma_1^{\mathrm{T}} > 0$, and $\sigma_1 > 0$.

Remark 5.5. The above σ-modification adaptation law (5.16) can be replaced by e-modification adaptation law like $\dot{\hat{W}}_1 = -\Gamma_1\left[S_1(Z_1) r_1 + \sigma_1 |r_1| \hat{W}_1\right]$ easily. The control design based on σ-modification adaptation law in this chapter can be extended to the case based on e-modification adaptation law without any difficulty.

Substituting (5.15) and (5.16) into (5.14), we have

$$\begin{aligned}\dot{V}_1 = &-k_1 b_{11}(\dot{q}_3) r_1^2 - \frac{b_{11}(\dot{q}_3)}{\underline{b}_{11}} \frac{r_1^2\left(\hat{W}_1^{\mathrm{T}} S_1(Z_1)\right)^2}{|r_1 \hat{W}_1^{\mathrm{T}} S_1(Z_1)| + \delta_1} - r_1 W_1^{*\mathrm{T}} S_1(Z_1) - r_1 \varepsilon_1(Z_1) \\ &- r_1 \tilde{W}_1^{\mathrm{T}} S_1(Z_1) - \sigma_1 \tilde{W}_1^{\mathrm{T}} \hat{W}_1\end{aligned} \tag{5.17}$$

According to Assumption 5.3 and (5.11), we can rewrite (5.17) as

$$\begin{aligned}\dot{V}_1 \le &-k_1 \underline{b}_{11} r_1^2 - \frac{r_1^2\left(\hat{W}_1^{\mathrm{T}} S_1(Z_1)\right)^2}{|r_1 \hat{W}_1^{\mathrm{T}} S_1(Z_1)| + \delta_1} - r_1 \hat{W}_1^{\mathrm{T}} S_1(Z_1) - r_1 \varepsilon_1(Z_1) \\ &- \sigma_1 \tilde{W}_1^{\mathrm{T}} \hat{W}_1 \\ \le &-k_1 \underline{b}_{11} r_1^2 - \frac{r_1^2\left(\hat{W}_1^{\mathrm{T}} S_1(Z_1)\right)^2}{|r_1 \hat{W}_1^{\mathrm{T}} S_1(Z_1)| + \delta_1} + |r_1 \hat{W}_1^{\mathrm{T}} S_1(Z_1)| + |r_1||\varepsilon_1(Z_1)| \\ &- \sigma_1 \tilde{W}_1^{\mathrm{T}} \hat{W}_1\end{aligned} \tag{5.18}$$

Noting that

$$-\frac{r_1^2\left(\hat{W}_1^{\mathrm{T}} S_1(Z_1)\right)^2}{|r_1 \hat{W}_1^{\mathrm{T}} S_1(Z_1)| + \delta_1} + |r_1 \hat{W}_1^{\mathrm{T}} S_1(Z_1)| = \frac{|r_1 \hat{W}_1^{\mathrm{T}} S_1(Z_1)| \delta_1}{|r_1 \hat{W}_1^{\mathrm{T}} S_1(Z_1)| + \delta_1} \tag{5.19}$$

According to Lemma 5.4, we can obtain from (5.19) that

$$-\frac{r_1^2\left(\hat{W}_1^{\mathrm{T}} S_1(Z_1)\right)^2}{|r_1\hat{W}_1^{\mathrm{T}} S_1(Z_1)| + \delta_1} + |r_1\hat{W}_1^{\mathrm{T}} S_1(Z_1)| \le \delta_1 \tag{5.20}$$

By completion of squares and using Young's inequality, the following inequalities hold:

$$-\sigma_1\tilde{W}_1^{\mathrm{T}}\hat{W}_1 \le -\frac{\sigma_1}{2}\|\tilde{W}_1\|^2 + \frac{\sigma_1}{2}\|W_1^*\|^2 \tag{5.21}$$

$$|r_1||\varepsilon_1(Z_1)| \le \frac{r_1^2}{2c_1} + \frac{c_1\varepsilon_1^2(Z_1)}{2} \le \frac{r_1^2}{2c_1} + \frac{c_1\bar{\varepsilon}_1^2}{2} \tag{5.22}$$

where c_1 is a positive constant. Substituting the above inequalities (5.20)–(5.22) into (5.18) leads to

$$\begin{aligned}\dot{V}_1 &\le -\left(k_1\underline{b}_{11} - \frac{1}{2c_1}\right)r_1^2 - \frac{\sigma_1}{2}\|\tilde{W}_1\|^2 + \delta_1 + \frac{c_1}{2}\bar{\varepsilon}_1^2 + \frac{\sigma_1}{2}w_{1m}^2\\ &\le -\lambda_{10}V_1 + \mu_{10}\end{aligned} \tag{5.23}$$

where $\lambda_{10} = \min\left\{(2k_1\underline{b}_{11} - 1/c_1)/d_{11}, \sigma_1/\lambda_{\max}(\Gamma_1^{-1})\right\}$, $\mu_{10} = \delta_1 + \frac{c_1}{2}\bar{\varepsilon}_1^2 + \frac{\sigma_1}{2}w_{1m}^2$.

□ *q_2-subsystem*

Similar to Sect. 5.3.1, since $\dot{q}_2 = \dot{q}_{2r} + r_2$, $\ddot{q}_2 = \ddot{q}_{2r} + \dot{r}_2$, (5.5) becomes

$$\frac{d_{22}(q_3)d_{33} - d_{23}^2}{d_{33}}\dot{r}_2 + c_{22}(q_3, \dot{q}_3)r_2 = b_{22}(\dot{q}_3)\tau_2 - f_{S2,1} \tag{5.24}$$

where

$$\begin{aligned}f_{S2,1} = &\frac{d_{22}(q_3)d_{33} - d_{23}^2}{d_{33}}\ddot{q}_{2r} + c_{22}(q_3, \dot{q}_3)\dot{q}_{2r} + c_{23}(q_3, \dot{q}_2)\dot{q}_3 + \Delta_2(q, \dot{q})\\ &+\frac{d_{23}}{d_{33}}(b_{31}(\dot{q}_3)\tau_1 - c_{32}(q_3, \dot{q}_2)\dot{q}_2 - f_3(\dot{q}_3) - g_3 - \Delta_3(q, \dot{q}))\end{aligned}$$

is an unknown function, which is approximated by RBFNN to arbitrarily any accuracy as

$$f_{S2,1} = W_2^{*\mathrm{T}} S_2(Z_2) + \varepsilon_2(Z_2) \tag{5.25}$$

where the input vector $Z_2 = [\tau_1, q_1, \dot{q}_1, q_2, \dot{q}_2, q_3, \dot{q}_3, q_{2d}, \dot{q}_{2d}, \ddot{q}_{2d}]^{\mathrm{T}} \in \Omega_{Z_2} \subset R^{10}$, $\varepsilon_2(Z_2)$ is the approximation error satisfying $|\varepsilon_2(Z_2)| \le \bar{\varepsilon}_2$, where $\bar{\varepsilon}_2$ is an unknown positive constant; W_2^* are unknown ideal constant weights satisfying $\|W_2^*\| \le w_{2m}$, where w_{2m} is an unknown positive constant; and $S_2(Z_2)$ are the basis functions. By using $\hat{W}_2$ to approximate W_2^*, the error between the actual and the ideal RBFNNs can be expressed as

$$\hat{W}_2^{\mathrm{T}} S_2(Z_2) - W_2^{*\mathrm{T}} S_2(Z_2) = \tilde{W}_2^{\mathrm{T}} S_2(Z_2) \tag{5.26}$$

where $\tilde{W}_2 = \hat{W}_2 - W_2^*$.

To analyze the closed loop stability for the q_2-subsystem, let

$$V_2 = \frac{1}{2}\frac{d_{22}(q_3)d_{33} - d_{23}^2}{d_{33}} r_2^2 + \frac{1}{2}\tilde{W}_2^{\mathrm{T}} \Gamma_2^{-1} \tilde{W}_2 \tag{5.27}$$

Lemma 5.6. *The function V_2 (5.27) is positive definite and decrescent, in the sense that there exist two time-invariant positive definite functions $\underline{V}_2(r_2, \tilde{W}_2)$ and $\bar{V}_2(r_2, \tilde{W}_2)$, such that*

$$\underline{V}_2(r_2, \tilde{W}_2) \le V_2 \le \bar{V}_2(r_2, \tilde{W}_2)$$

Proof. Noting that the particular choice of V_2 in (5.27), a function of $r_2, \tilde{W}_2$ and $d_{22}(q_3)$, is to establish the stability for r_2 and $\tilde{W}_2$ only, therefore, we regard $d_{22}(q_3)$ as a function of time. From Assumptions 5.1 and 5.4, we know that

$$0 < \frac{\left|\underline{d}_{22}|d_{33}| - d_{23}^2\right|}{|d_{33}|} < \left|\frac{d_{22}(q_3)d_{33} - d_{23}^2}{d_{33}}\right| \le \frac{\bar{d}_{22}|d_{33}| + d_{23}^2}{|d_{33}|} \tag{5.28}$$

Therefore, there also exist time-invariant positive definite functions $\underline{V}_2(r_2, \tilde{W}_2)$ and $\bar{V}_2(r_2, \tilde{W}_2)$, such that $\underline{V}_2(r_2, \tilde{W}_2) \le V_2 \le \bar{V}_2(r_2, \tilde{W}_2)$, which implies that V_2 is also positive definite and decrescent, according to [94]. This completes the proof. □

The time derivative of (5.27) is given as

$$\dot{V}_2 = \frac{1}{2}\dot{d}_{22}(q_3) r_2^2 + \frac{d_{22}(q_3)d_{33} - d_{23}^2}{d_{33}} r_2 \dot{r}_2 + \tilde{W}_2^{\mathrm{T}} \Gamma_2^{-1} \dot{\tilde{W}}_2 \tag{5.29}$$

According to Assumption 5.2, (5.29) becomes

$$\dot{V}_2 = r_2 \left[\frac{d_{22}(q_3)d_{33} - d_{23}^2}{d_{33}} \dot{r}_2 + c_{22}(q_3, \dot{q}_3) r_2\right] + \tilde{W}_2^{\mathrm{T}} \Gamma_2^{-1} \dot{\tilde{W}}_2 \tag{5.30}$$

As W_2^* is a constant vector, it is easy to obtain that

$$\dot{\tilde{W}}_2 = \dot{\hat{W}}_2 \tag{5.31}$$

Substituting (5.24), (5.25) and (5.31) into (5.30), we have

$$\dot{V}_2 = r_2 \left[b_{22}(\dot{q}_3)\tau_2 - W_2^{*\mathrm{T}} S_2(Z_2) - \varepsilon_2(Z_2) \right] + \tilde{W}_2^{\mathrm{T}} \Gamma_2^{-1} \dot{\hat{W}}_2 \tag{5.32}$$

Consider the following RBFNN-based control law and RBFNN weight adaption law:

$$\tau_2 = k_2 r_2 + \frac{r_2 \left(\hat{W}_2^{\mathrm{T}} S_2(Z_2) \right)^2}{\underline{b}_{22} \left(|r_2 \hat{W}_2^{\mathrm{T}} S_2(Z_2)| + \delta_2 \right)} \tag{5.33}$$

$$\dot{\hat{W}}_2 = -\Gamma_2 \left[S_2(Z_2) r_2 + \sigma_2 \hat{W}_2 \right] \tag{5.34}$$

where $k_2 > 0$, $\delta_2 > 0$, $\Gamma_2 = \Gamma_2^{\mathrm{T}} > 0$ and $\sigma_2 > 0$. Substituting (5.33) and (5.34) into (5.32), we have

$$\begin{aligned} \dot{V}_2 = k_2 b_{22}(\dot{q}_3) r_1^2 + \frac{b_{22}(\dot{q}_3)}{\underline{b}_{22}} \frac{r_2^2 \left(\hat{W}_2^{\mathrm{T}} S_2(Z_2) \right)^2}{|r_2 \hat{W}_2^{\mathrm{T}} S_2(Z_2)| + \delta_2} - r_2 W_2^{*\mathrm{T}} S_2(Z_2) - r_2 \varepsilon_2(Z_2) \\ - r_2 \tilde{W}_2^{\mathrm{T}} S_2(Z_2) - \sigma_2 \tilde{W}_2^{\mathrm{T}} \hat{W}_2 \end{aligned} \tag{5.35}$$

According to Assumption 5.3 and (5.26), we can rewrite (5.35) as

$$\begin{aligned} \dot{V}_2 &\le -k_2 \underline{b}_{22} r_2^2 - \frac{r_2^2 \left(\hat{W}_2^{\mathrm{T}} S_2(Z_2) \right)^2}{|r_2 \hat{W}_2^{\mathrm{T}} S_2(Z_2)| + \delta_2} - r_2 \hat{W}_2^{\mathrm{T}} S_2(Z_2) - r_2 \varepsilon_2(Z_2) \\ &\quad - \sigma_2 \tilde{W}_2^{\mathrm{T}} \hat{W}_2 \\ &\le -k_2 \underline{b}_{22} r_2^2 - \frac{r_2^2 \left(\hat{W}_2^{\mathrm{T}} S_2(Z_2) \right)^2}{|r_2 \hat{W}_2^{\mathrm{T}} S_2(Z_2)| + \delta_2} + |r_2 \hat{W}_2^{\mathrm{T}} S_2(Z_2)| + |r_2||\varepsilon_2(Z_2)| \\ &\quad - \sigma_2 \tilde{W}_2^{\mathrm{T}} \hat{W}_2 \end{aligned} \tag{5.36}$$

Similar to (5.20), we have

$$- \frac{r_2^2 \left(\hat{W}_2^{\mathrm{T}} S_2(Z_2) \right)^2}{|r_2 \hat{W}_2^{\mathrm{T}} S_2(Z_2)| + \delta_2} + |r_2 \hat{W}_2^{\mathrm{T}} S_2(Z_2)| \le \delta_2 \tag{5.37}$$

By completion of squares and using Young's inequality, the following inequalities hold:

$$-\sigma_2 \tilde{W}_2^{\mathrm{T}} \hat{W}_2 \le -\frac{\sigma_2}{2}\|\tilde{W}_2\|^2 + \frac{\sigma_2}{2}\|W_2^*\|^2 \tag{5.38}$$

$$|r_2||\varepsilon_2(Z_2)| \le \frac{r_2^2}{2c_2} + \frac{c_2\varepsilon_2^2(Z_2)}{2} \le \frac{r_2^2}{2c_2} + \frac{c_2\bar{\varepsilon}_2^2}{2} \tag{5.39}$$

where c_2 is a positive constant. Substituting the above inequalities (5.37)–(5.39) into (5.36) leads to

$$\begin{aligned}\dot{V}_2 &\le -\left(k_2\underline{b}_{22} - \frac{1}{2c_2}\right) r_2^2 - \frac{\sigma_2}{2}\|\tilde{W}_2\|^2 + \delta_2 + \frac{c_2}{2}\bar{\varepsilon}_2^2 + \frac{\sigma_2}{2}w_{2m}^2 \\ &\le -\lambda_{20}V_2 + \mu_{20}\end{aligned} \tag{5.40}$$

where $\lambda_{20} = \min\left\{(2k_2\underline{b}_{22} - 1/c_2)|d_{33}|/(\bar{d}_{22}|d_{33}| + d_{23}^2), \sigma_2/\lambda_{\max}(\Gamma_2^{-1})\right\}$, $\mu_{20} = \delta_2 + \frac{c_2}{2}\bar{\varepsilon}_2^2 + \frac{\sigma_2}{2}w_{2m}^2$.

□ <u>*q_3-subsystem*</u>

Finally, using the designed control laws (5.15) and (5.33), the q_3-subsystem (5.6) can be rewritten as

$$\dot{\eta} = \psi(\xi, \eta, u) \tag{5.41}$$

where $\eta = [q_3, \dot{q}_3]^{\mathrm{T}}$, $\xi = [q_1, q_2, \dot{q}_1, \dot{q}_2]^{\mathrm{T}}$, $u = [\tau_1, \tau_2]^{\mathrm{T}}$.

Then, the zero dynamics can be addressed as [35]

$$\dot{\eta} = \psi(0, \eta, u^*(0, \eta)) \tag{5.42}$$

where $u^* = [\tau_1^*, \tau_2^*]^{\mathrm{T}}$.

Assumption 5.6. [35] System (5.4)–(5.6) is hyperbolically minimum-phase, i.e., zero dynamics (5.42) is exponentially stable. In addition, assume that the control input u is designed as a function of the states (ξ, η) and the reference signal satisfying Assumption 5.5, and the function $f(\xi, \eta, u)$ is Lipschitz in ξ, i.e., there exist constants L_ξ and L_f for $f(\xi, \eta, u)$ such that

$$\|f(\xi, \eta, u) - f(0, \eta, u_\eta)\| \le L_\xi\|\xi\| + L_f \tag{5.43}$$

where $u_\eta = u^*(0, \eta)$.

Under Assumption 5.6, by the Converse Theorem of Lyapunov [52], there exists a Lyapunov function $V_0(\eta)$ which satisfies

$$\gamma_a\|\eta\|^2 \le V_0(\eta) \le \gamma_b\|\eta\|^2 \tag{5.44}$$

$$\frac{\partial V_0}{\partial \eta} f(0, \eta, u_\eta) \leq -\lambda_a \|\eta\|^2 \tag{5.45}$$

$$\|\frac{\partial V_0}{\partial \eta}\| \leq \lambda_b \|\eta\| \tag{5.46}$$

where γ_a, γ_b, λ_a and λ_b are positive constants.

Lemma 5.7. *[35] For the internal dynamics $\dot{\eta} = f(\xi, \eta, u)$ of the system, if Assumption 5.6 is satisfied, and the states ξ are bounded by a positive constant $\|\xi\|_{\max}$, i.e., $\|\xi\| \leq \|\xi\|_{\max}$, then there exist positive constants L_η and T_0, such that*

$$\|\eta(t)\| \leq L_\eta, \quad \forall t > T_0 \tag{5.47}$$

Proof. According to Assumption 5.6, there exists a Lyapunov function $V_0(\eta)$. Differentiating $V_0(\eta)$ along (5.4)–(5.6) yields

$$\begin{aligned}\dot{V}_0(\eta) &= \frac{\partial V_0}{\partial \eta} f(\xi, \eta, u) \\ &= \frac{\partial V_0}{\partial \eta} f(0, \eta, u_\eta) + \frac{\partial V_0}{\partial \eta}\left[f(\xi, \eta, u) - f(0, \eta, u_\eta)\right]\end{aligned} \tag{5.48}$$

Noting (5.43)–(5.46), (5.48) can be written as

$$\begin{aligned}\dot{V}_0(\eta) &\leq -\lambda_a \|\eta\|^2 + \lambda_b \|\eta\| (L_\xi \|\xi\| + L_f) \\ &\leq -\lambda_a \|\eta\|^2 + \lambda_b \|\eta\| (L_\xi \|\xi\|_{\max} + L_f)\end{aligned}$$

Therefore, $\dot{V}_0(\eta) \leq 0$, whenever

$$\|\eta\| \geq \frac{\lambda_b}{\lambda_a}(L_\xi \|\xi\|_{\max} + L_f)$$

By letting $L_\eta = \frac{\lambda_b}{\lambda_a}(L_\xi \|\xi\|_{\max} + L_f)$, we conclude that there exists a positive constant T_0, such that (5.47) holds. □

The following Theorem shows the stability and control performance of the closed loop system.

Theorem 5.8. *Consider the closed-loop system consisting of the subsystems (5.4)–(5.6), the control laws (5.15), (5.33) and adaptation laws (5.16), (5.34). Under Assumptions 5.1–5.6, the overall closed-loop neural control system is Semi-Globally Uniformly Ultimately Bounded (SGUUB) in the sense that all of the signals in the closed-loop system are bounded, and the tracking errors and neural weights converge to the following regions,*

$$
\begin{aligned}
|e_1| &\le |e_1(0)| + \frac{1}{\lambda_1}\sqrt{\frac{2\mu_1}{d_{11}}} \\
\|\hat{W}_1\| &\le \sqrt{\frac{2\mu_1}{\lambda_{\min}(\Gamma_1^{-1})}} + w_{1m} \\
|e_2| &\le |e_2(0)| + \frac{1}{\lambda_2}\sqrt{\frac{2|d_{33}|\mu_2}{\left|\underline{d}_{22}|d_{33}| - d_{23}^2\right|}} \\
\|\hat{W}_2\| &\le \sqrt{\frac{2\mu_2}{\lambda_{\min}(\Gamma_2^{-1})}} + w_{2m}
\end{aligned}
\tag{5.49}
$$

with

$$
\begin{aligned}
\mu_i &= \frac{\mu_{i0}}{\lambda_{i0}} + V_i(0), \quad \mu_{i0} = \delta_i + \frac{1}{2}\bar{\varepsilon}_i^2 + \frac{\sigma_i}{2}w_{im}^2, \quad i = 1, 2 \\
\lambda_{10} &= \min\left\{(2k_1\underline{b}_{11} - 1/c_1)/d_{11}, \sigma_1/\lambda_{\max}(\Gamma_1^{-1})\right\} \\
\lambda_{20} &= \min\left\{(2k_2 - 1/c_2)|d_{33}|/(\bar{d}_{22}|d_{33}| + d_{23}^2), \sigma_2/\lambda_{\max}(\Gamma_2^{-1})\right\}
\end{aligned}
$$

where $e_i(0)$ and $V_i(0)$ are initial values of $e_i(t)$ and $V_i(t)$, respectively.

Proof. Based on the previous analysis, the proof proceeds by studying each subsystem in order. First, the closed loop stability analysis of the q_1-subsystem (5.4) with control τ_1 (5.15) and adaptation law (5.16) is made by use of Lyapunov synthesis. Second, the similar closed loop stability will be achieved on the q_2-subsystem (5.5) with τ_2 (5.33) and adaptation law (5.34). Finally, the stability analysis of internal dynamics of the q_3-subsystem (5.6) is made based on the stability of the previous two subsystems.

q_1- subsystem:

Solving the inequality (5.23), we have $0 \le V_1(t) \le \mu_1$ with $\mu_1 = \frac{\mu_{10}}{\lambda_{10}} + V_1(0)$. Then, from the definition of $V_1(t)$ (5.12), we can obtain

$$
|r_1| \le \sqrt{\frac{2\mu_1}{d_{11}}}, \quad \|\tilde{W}_1\| \le \sqrt{\frac{2\mu_1}{\lambda_{\min}(\Gamma_1^{-1})}} \tag{5.50}
$$

Since $\dot{e}_1 = -\lambda_1 e_1 + r_1$, solving this equation results in

$$
e_1 = \mathrm{e}^{-\lambda_1 t} e_1(0) + \int_0^t \mathrm{e}^{-\lambda_1 (t-\tau)} r_1 \mathrm{d}\tau \tag{5.51}
$$

According to (5.50) and (5.51), we have

$$|e_1| \leq |e_1(0)| + \frac{1}{\lambda_1}\sqrt{\frac{2\mu_1}{d_{11}}} \tag{5.52}$$

Noting $q_1 = e_1 + q_{1d}$, $\hat{W}_1 = \tilde{W}_1 + W_1^*$, $\|W_1^*\| \leq w_{1m}$ and Assumption 5.5, we obtain

$$|q_1| \leq |e_1| + |q_{1d}| \leq |e_1(0)| + \frac{1}{\lambda_1}\sqrt{\frac{2\mu_1}{d_{11}}} + |q_{1d}| \in L_\infty$$

$$\|\hat{W}_1\| \leq \|\tilde{W}_1\| + \|W_1^*\| \leq \sqrt{\frac{2\mu_1}{\lambda_{\min}(\Gamma_1^{-1})}} + w_{1m} \in L_\infty$$

Since the control τ_1 is a function of r_1 and $\hat{W}_1$, its boundedness is also assured.

q_2- subsystem:

Similar to the analysis of q_1- subsystem, we have

$$|r_2| \leq \sqrt{\frac{2|d_{33}|\mu_2}{\left|\underline{d}_{22}|d_{33}| - d_{23}^2\right|}}, \quad \|\tilde{W}_2\| \leq \sqrt{\frac{2\mu_2}{\lambda_{\min}(\Gamma_2^{-1})}} \tag{5.53}$$

Furthermore, we obtain

$$|e_2| \leq |e_2(0)| + \frac{1}{\lambda_2}\sqrt{\frac{2|d_{33}|\mu_2}{\left|\underline{d}_{22}|d_{33}| - d_{23}^2\right|}}$$

$$|q_2| \leq |e_2| + |q_{2d}| \leq |e_2(0)| + \frac{1}{\lambda_2}\sqrt{\frac{2|d_{33}|\mu_2}{\left|\underline{d}_{22}|d_{33}| - d_{23}^2\right|}} + |q_{2d}| \in L_\infty$$

$$\|\hat{W}_2\| \leq \|\tilde{W}_2\| + \|W_2^*\| \leq \sqrt{\frac{2\mu_2}{\lambda_{\min}(\Gamma_2^{-1})}} + w_{2m} \in L_\infty \tag{5.54}$$

and thus the boundedness of control τ_2.

q_3- subsystem:

From the previous stability analysis about the q_1-subsystem and the q_2-subsystem, we know that q_1, q_2, $\dot{q}_1$, $\dot{q}_2$ are bounded. Accordingly, ξ are bounded. According to Lemma 5, we know that the internal dynamics are stable, i.e., η (q_3 and $\dot{q}_3$) are bounded. All the signals in the closed-loop system are bounded. This completes the proof. □

5.3.2 MNN-Based Control

Nonlinearly parameterized approximators, such as the MNN, can be linearized by Taylor series expansions, with the higher order terms being taken as part of the modeling error. Due to the nonlinear parameterizations, the control design and stability analysis involving the MNN is more complex than the previous one based on the linearly parameterized network, i.e., the RBFNN, but still can follow the similar procedures as the afore-mentioned RBFNN-based one.

□ q_1*-subsystem*

Similar to the RBFNN case in Sect. 5.3.1, (5.4) is written as

$$d_{11}\dot{r}_1 = b_{11}(\dot{q}_3)\tau_1 - f_{S1,1} \tag{5.55}$$

where the unknown continuous function

$$f_{S1,1} = d_{11}\ddot{q}_{1r} + f_1(\dot{q}_3) + g_1 + \varDelta_1(q,\dot{q}) \tag{5.56}$$

is approximated by MNN to arbitrarily any accuracy as

$$f_{S1,1} = W_1^{*\mathrm{T}} S_1(V_1^{*\mathrm{T}} Z_1) + \varepsilon_1(Z_1) \tag{5.57}$$

where the input vector $Z_1 = [q_1,\ \dot{q}_1,\ q_2,\ \dot{q}_2,\ q_3,\ \dot{q}_3,\ \dot{q}_{1d},\ \ddot{q}_{1d},\ 1]^{\mathrm{T}} \in \varOmega_{Z_1} \subset R^9$; $\varepsilon_1(Z_1)$ is the approximation error satisfying $|\varepsilon_1(Z_1)| \leq \bar{\varepsilon}_1$, where $\bar{\varepsilon}_1$ is a positive constant; W_1^* and V_1^* are unknown ideal constant weights satisfying $\|W_1^*\| \leq w_{1m}, \|V_1^*\|_F \leq v_{1m}$, which are positive constants. By using $\hat{W}_1^{\mathrm{T}} S_1(\hat{V}_1^{\mathrm{T}} Z_1)$ to approximate $W_1^{*\mathrm{T}} S_1(V_1^{*\mathrm{T}} Z_1)$, the error between the actual and the ideal MNN can be expressed as

$$\hat{W}_1^{\mathrm{T}} S(\hat{V}_1^{\mathrm{T}} Z_1) - W_1^{*\mathrm{T}} S(V_1^{*\mathrm{T}} Z_1) = \tilde{W}_1^{\mathrm{T}}(\hat{S}_1 - \hat{S}_1'\hat{V}_1^{\mathrm{T}} Z_1) + \hat{W}_1^{\mathrm{T}}\hat{S}_1'\tilde{V}_1^{\mathrm{T}} Z_1 + d_{u1} \tag{5.58}$$

where $\hat{S}_1 = S(\hat{V}_1^{\mathrm{T}} Z_1)$, $\hat{S}_1' = \mathrm{diag}\,\{\hat{s}_1', \hat{s}_2', ..., \hat{s}_l'\}$ with

$$\hat{s}_i' = s'(\hat{v}_i^{\mathrm{T}} Z) = \frac{\mathrm{d}[s(z_a)]}{\mathrm{d}z_a}|_{z_a = \hat{v}_i^{\mathrm{T}} Z}$$

the residual term d_{u1} is bounded by

$$|d_{u1}| \leq \|V_1^*\|_F \|Z_1 \hat{W}_1^{\mathrm{T}} \hat{S}_1'\|_F + \|W_1^*\| \|\hat{S}_1' \hat{V}_1^{\mathrm{T}} Z_1\| + |W_1^*|_1 \tag{5.59}$$

and the weight estimation errors $\tilde{W}_1 = \hat{W}_1 - W_1^*$, $\tilde{V}_1 = \hat{V}_1 - V_1^*$.

Consider the following Lyapunov function candidate

$$V_1(r_1, \tilde{W}_1, \tilde{V}_1) = \frac{1}{2}d_{11}r_1^2 + \frac{1}{2}\tilde{W}_1^{\mathrm{T}}\Gamma_{W1}^{-1}\tilde{W}_1 + \frac{1}{2}\mathrm{tr}\left\{\tilde{V}_1^{\mathrm{T}}\Gamma_{V1}^{-1}\tilde{V}_1\right\} \tag{5.60}$$

The time derivative of (5.60) along (5.55) and (5.57) is given by

$$\begin{aligned}\dot{V}_1 &= r_1\left[b_{11}(\dot{q}_3)\tau_1 - W_1^{*\mathrm{T}}S_1(V_1^{*\mathrm{T}}Z_1) - \varepsilon_1(Z_1)\right] + \tilde{W}_1^{\mathrm{T}}\Gamma_{W1}^{-1}\dot{\tilde{W}}_1 \\ &\quad + \mathrm{tr}\left\{\tilde{V}_1^{\mathrm{T}}\Gamma_{V1}^{-1}\dot{\tilde{V}}_1\right\}\end{aligned} \tag{5.61}$$

As W_1^*, V_1^* are constant vectors, it is easy to obtain that

$$\dot{\tilde{W}}_1 = \dot{\hat{W}}_1, \quad \dot{\tilde{V}}_1 = \dot{\hat{V}}_1 \tag{5.62}$$

Substituting (5.62) into (5.61), we have

$$\begin{aligned}\dot{V}_1 &= r_1\left[b_{11}(\dot{q}_3)\tau_1 - W_1^{*\mathrm{T}}S_1(V_1^{*\mathrm{T}}Z_1) - \varepsilon_1(Z_1)\right] + \tilde{W}_1^{\mathrm{T}}\Gamma_{W1}^{-1}\dot{\hat{W}}_1 \\ &\quad + \mathrm{tr}\{\tilde{V}_1^{\mathrm{T}}\Gamma_{V1}^{-1}\dot{\hat{V}}_1\}\end{aligned} \tag{5.63}$$

Consider the following MNN-based control law and MNN weight adaption laws:

$$\begin{aligned}\tau_1 &= -k_1r_1 - \frac{r_1\left(\hat{W}_1^{\mathrm{T}}S(\hat{V}_1^{\mathrm{T}}Z_1)\right)^2}{\underline{b}_{11}\left(|r_1\hat{W}_1^{\mathrm{T}}S(\hat{V}_1^{\mathrm{T}}Z_1)| + \delta_1\right)} - \frac{k_1r_1}{\underline{b}_{11}}(\|Z_1\hat{W}_1^{\mathrm{T}}\hat{S}_1'\|_F^2 \\ &\quad + \|\hat{S}_1'\hat{V}_1^{\mathrm{T}}Z_1\|^2)\end{aligned} \tag{5.64}$$

$$\dot{\hat{W}}_1 = -\Gamma_{W1}[(\hat{S}_1 - \hat{S}_1'\hat{V}_1^{\mathrm{T}}Z_1)r_1 + \sigma_{W1}\hat{W}_1] \tag{5.65}$$

$$\dot{\hat{V}}_1 = -\Gamma_{V1}[Z_1\hat{W}_1^{\mathrm{T}}\hat{S}_1'r_1 + \sigma_{V1}\hat{V}_1] \tag{5.66}$$

where $k_1 > 0$, $\delta_1 > 0$, $\Gamma_{W1} = \Gamma_{W1}^{\mathrm{T}} > 0$, $\Gamma_{V1} = \Gamma_{V1}^{\mathrm{T}} > 0$, $\sigma_{W1} > 0$, $\sigma_{V1} > 0$.

Substituting (5.64)–(5.66) in (5.63), we have

$$\begin{aligned}\dot{V}_1 &= -k_1b_{11}(\dot{q}_3)r_1^2 - \frac{b_{11}(\dot{q}_3)}{\underline{b}_{11}}\frac{r_1^2\left(\hat{W}_1^{\mathrm{T}}S(\hat{V}_1^{\mathrm{T}}Z_1)\right)^2}{\left(|r_1\hat{W}_1^{\mathrm{T}}S(\hat{V}_1^{\mathrm{T}}Z_1)| + \delta_1\right)} \\ &\quad - \frac{b_{11}(\dot{q}_3)}{\underline{b}_{11}}k_1r_1^2\left(\|Z_1\hat{W}_1^{\mathrm{T}}\hat{S}_1'\|_F^2 + \|\hat{S}_1'\hat{V}_1^{\mathrm{T}}Z_1\|^2\right) - r_1W_1^{*\mathrm{T}}S_1(V_1^{*\mathrm{T}}Z_1) \\ &\quad - r_1\varepsilon_1(Z_1) - r_1\tilde{W}_1^{\mathrm{T}}(\hat{S}_1 - \hat{S}_1'\hat{V}_1^{\mathrm{T}}Z_1) - \sigma_{W1}\tilde{W}_1^{\mathrm{T}}\hat{W}_1 \\ &\quad - \mathrm{tr}\{\tilde{V}_1^{\mathrm{T}}Z_1\hat{W}_1^{\mathrm{T}}\hat{S}_1'r_1\} - \sigma_{V1}\mathrm{tr}\{\tilde{V}_1^{\mathrm{T}}\hat{V}_1\}\end{aligned} \tag{5.67}$$

Noting Assumption 5.3 and the fact that $\mathrm{tr}\{\tilde{V}_1^{\mathrm{T}} Z_1 \hat{W}_1^{\mathrm{T}} \hat{S}_1' r_1\} = r_1 \hat{W}_1^{\mathrm{T}} \hat{S}_1' \tilde{V}_1^{\mathrm{T}} Z_1$, (5.67) becomes

$$
\begin{aligned}
\dot{V}_1 \le\ & -k_1 \underline{b}_{11} r_1^2 - \frac{r_1^2 \Big(\hat{W}_1^{\mathrm{T}} S(\hat{V}_1^{\mathrm{T}} Z_1) \Big)^2}{\Big(|r_1 \hat{W}_1^{\mathrm{T}} S(\hat{V}_1^{\mathrm{T}} Z_1)| + \delta_1 \Big)} \\
& -k_1 r_1^2 \Big(\|Z_1 \hat{W}_1^{\mathrm{T}} \hat{S}_1'\|_F^2 + \|\hat{S}_1' \hat{V}_1^{\mathrm{T}} Z_1\|^2 \Big) \\
& +|r_1||\varepsilon_1(Z_1)| - r_1 W_1^{*\mathrm{T}} S_1(V_1^{*\mathrm{T}} Z_1) - r_1 \tilde{W}_1^{\mathrm{T}} (\hat{S}_1 - \hat{S}_1' \hat{V}_1^{\mathrm{T}} Z_1) \\
& -r_1 \hat{W}_1^{\mathrm{T}} \hat{S}_1' \tilde{V}_1^{\mathrm{T}} Z_1 - \sigma_{W1} \tilde{W}_1^{\mathrm{T}} \hat{W}_1 - \sigma_{V1} \mathrm{tr}\{\tilde{V}_1^{\mathrm{T}} \hat{V}_1\}
\end{aligned} \tag{5.68}
$$

From (5.58) and (5.59), we know

$$
\begin{aligned}
& -r_1 W_1^{*\mathrm{T}} S_1(V_1^{*\mathrm{T}} Z_1) - r_1 \tilde{W}_1^{\mathrm{T}} (\hat{S}_1 - \hat{S}_1' \hat{V}_1^{\mathrm{T}} Z_1) - r_1 \hat{W}_1^{\mathrm{T}} \hat{S}_1' \tilde{V}_1^{\mathrm{T}} Z_1 \\
=\ & -r_1 \hat{W}_1^{\mathrm{T}} S(\hat{V}_1^{\mathrm{T}} Z_1) - r_1 d_{u1} \\
\le\ & |r_1 \hat{W}_1^{\mathrm{T}} S(\hat{V}_1^{\mathrm{T}} Z_1)| + |r_1| \|V_1^*\|_F \|Z_1 \hat{W}_1^{\mathrm{T}} \hat{S}_1'\|_F + |r_1| \|W_1^*\| \|\hat{S}_1' \hat{V}_1^{\mathrm{T}} Z_1\| \\
& +|r_1||W_1^*|_1
\end{aligned} \tag{5.69}
$$

Substituting (5.69) in (5.68) leads to

$$
\begin{aligned}
\dot{V}_1 \le\ & -k_1 \underline{b}_{11} r_1^2 - \frac{r_1^2 \Big(\hat{W}_1^{\mathrm{T}} S(\hat{V}_1^{\mathrm{T}} Z_1) \Big)^2}{\Big(|r_1 \hat{W}_1^{\mathrm{T}} S(\hat{V}_1^{\mathrm{T}} Z_1)| + \delta_1 \Big)} + |r_1 \hat{W}_1^{\mathrm{T}} S(\hat{V}_1^{\mathrm{T}} Z_1)| \\
& -k_1 r_1^2 \Big(\|Z_1 \hat{W}_1^{\mathrm{T}} \hat{S}_1'\|_F^2 + \|\hat{S}_1' \hat{V}_1^{\mathrm{T}} Z_1\|^2 \Big) + |r_1||\varepsilon_1(Z_1)| \\
& +|r_1| \|V_1^*\|_F \|Z_1 \hat{W}_1^{\mathrm{T}} \hat{S}_1'\|_F + |r_1| \|W_1^*\| \|\hat{S}_1' \hat{V}_1^{\mathrm{T}} Z_1\| \\
& +|r_1||W_1^*|_1 - \sigma_{W1} \tilde{W}_1^{\mathrm{T}} \hat{W}_1 - \sigma_{V1} \mathrm{tr}\{\tilde{V}_1^{\mathrm{T}} \hat{V}_1\}
\end{aligned} \tag{5.70}
$$

According to Lemma 5.4,

$$
-\frac{r_1^2 \Big(\hat{W}_1^{\mathrm{T}} S(\hat{V}_1^{\mathrm{T}} Z_1) \Big)^2}{|r_1 \hat{W}_1^{\mathrm{T}} S(\hat{V}_1^{\mathrm{T}} Z_1)| + \delta_1} + |r_1 \hat{W}_1^{\mathrm{T}} S(\hat{V}_1^{\mathrm{T}} Z_1)| = \frac{|r_1 \hat{W}_1^{\mathrm{T}} S(\hat{V}_1^{\mathrm{T}} Z_1)| \delta_1}{|r_1 \hat{W}_1^{\mathrm{T}} S(\hat{V}_1^{\mathrm{T}} Z_1)| + \delta_1} \le \delta_1 \tag{5.71}
$$

By completion of squares and using Young's inequality, the following inequalities hold:

$$|r_1||\varepsilon_1(Z_1)| \le \frac{r_1^2}{2c_{11}} + \frac{c_{11}\bar{\varepsilon}_1^2}{2} \tag{5.72}$$

$$|r_1|\|V_1^*\|_F\|Z_1\hat{W}_1^{\mathrm{T}}\hat{S}_1'\|_F \le k_1 r_1^2\|Z_1\hat{W}_1^{\mathrm{T}}\hat{S}_1'\|_F^2 + \frac{1}{4k_1}\|V_1^*\|_F^2 \tag{5.73}$$

$$|r_1|\|W_1^*\|\|\hat{S}_1'\hat{V}_1^{\mathrm{T}}Z_1\| \le k_1 r_1^2\|\hat{S}_1'\hat{V}_1^{\mathrm{T}}Z_1\|^2 + \frac{1}{4k_1}\|W_1^*\|^2 \tag{5.74}$$

$$|r_1||W_1^*|_1 \le \frac{r_1^2}{2c_{12}} + \frac{c_{12}|W_1^*|_1^2}{2} \tag{5.75}$$

$$-\sigma_{W1}\tilde{W}_1^{\mathrm{T}}\hat{W}_1 \le -\frac{\sigma_{W1}}{2}\|\tilde{W}_1\|^2 + \frac{\sigma_{W1}}{2}\|W_1^*\|^2 \tag{5.76}$$

$$-\sigma_{V1}\mathrm{tr}\{\tilde{V}_1^{\mathrm{T}}\hat{V}_1\} \le -\frac{\sigma_{V1}}{2}\|\tilde{V}_1\|_F^2 + \frac{\sigma_{V1}}{2}\|V_1^*\|_F^2 \tag{5.77}$$

Substituting (5.71)–(5.77) into (5.70), we have

$$\begin{aligned}\dot{V}_1 &\le -\left(k_1\underline{b}_{11} - \frac{1}{2c_{11}} - \frac{1}{2c_{12}}\right)r_1^2 - \frac{\sigma_{W1}}{2}\|\tilde{W}_1\|^2 - \frac{\sigma_{V1}}{2}\|\tilde{V}_1\|_F^2 + \delta_1 \\ &\quad + \left(\frac{\sigma_{W1}}{2} + \frac{1}{4k_1}\right)\|W_1^*\|^2 + \left(\frac{\sigma_{V1}}{2} + \frac{1}{4k_1}\right)\|V_1^*\|_F^2 + \frac{c_{11}}{2}\bar{\varepsilon}_1^2 + \frac{c_{12}|W_1^*|_1^2}{2} \\ &\le -\lambda_{10}V_1 + \mu_{10}\end{aligned} \tag{5.78}$$

where $\lambda_{10} = \min\left\{(2k_1\underline{b}_{11} - 1/c_{11} - 1/c_{12})/d_{11}, \sigma_{W1}/\lambda_{\max}(\Gamma_{W1}^{-1}), \sigma_{V1}/\lambda_{\max}(\Gamma_{V1}^{-1})\right\}$, $\mu_{10} = \delta_1 + (\frac{\sigma_{W1}}{2} + \frac{1}{4k_1})\|W_1^*\|^2 + (\frac{\sigma_{V1}}{2} + \frac{1}{4k_1})\|V_1^*\|_F^2 + \frac{c_{11}}{2}\bar{\varepsilon}_1^2 + \frac{c_{12}|W_1^*|_1^2}{2}$.

□ *q_2-subsystem*

Similar to Sect. 5.3.1, (5.5) becomes

$$\frac{d_{22}(q_3)d_{33} - d_{23}^2}{d_{33}}\dot{r}_2 + c_{22}(q_3, \dot{q}_3)r_2 = b_{22}(\dot{q}_3)\tau_2 - f_{S2,1} \tag{5.79}$$

where the unknown function

$$\begin{aligned}f_{S2,1} = &\frac{d_{22}(q_3)d_{33} - d_{23}^2}{d_{33}}\ddot{q}_{2r} + c_{22}(q_3, \dot{q}_3)\dot{q}_{2r} + c_{23}(q_3, \dot{q}_2)\dot{q}_3 + \Delta_2(q, \dot{q}) \\ &+ \frac{d_{23}}{d_{33}}(b_{31}(\dot{q}_3)\tau_1 - c_{32}(q_3, \dot{q}_2)\dot{q}_2 - f_3(\dot{q}_3) - g_3 - \Delta_3(q, \dot{q}))\end{aligned}$$

is approximated by MNN to arbitrarily any accuracy as

$$f_{S2,1} = W_2^{*\mathrm{T}}S_2(V_2^{*\mathrm{T}}Z_2) + \varepsilon_2(Z_2)$$

where the input vector $Z_2 = [\tau_1, q_1, \dot{q}_1, q_2, \dot{q}_2, q_3, \dot{q}_3, q_{2d}, \dot{q}_{2d}, \ddot{q}_{2d}, 1]^{\mathrm{T}} \in \Omega_{Z_2} \subset R^{11}$, $\varepsilon_2(Z_2)$ is the approximation error satisfying $|\varepsilon_2(Z_2)| \le \bar{\varepsilon}_2$, where $\bar{\varepsilon}_2$ is a positive constant; W_2^* and V_2^* are ideal constant weights satisfying $\|W_2^*\| \le w_{2m}, \|V_2^*\|_F \le v_{2m}$, which are positive constants. By using $\hat{W}_2^{\mathrm{T}} S_2(\hat{V}_2^{\mathrm{T}} Z_2)$ to approximate $W_2^{*\mathrm{T}} S_2(V_2^{*\mathrm{T}} Z_2)$, the error between the actual and the ideal MNN can be expressed as

$$\hat{W}_2^{\mathrm{T}} S(\hat{V}_2^{\mathrm{T}} Z_2) - W_2^{*\mathrm{T}} S(V_2^{*\mathrm{T}} Z_2) = \tilde{W}_2^{\mathrm{T}} (\hat{S}_2 - \hat{S}_2' \hat{V}_2^{\mathrm{T}} Z_2) + \hat{W}_2^{\mathrm{T}} \hat{S}_2' \tilde{V}_2^{\mathrm{T}} Z_2 + d_{u2} \tag{5.80}$$

where $\hat{S}_2 = S(\hat{V}_2^{\mathrm{T}} Z_2)$, $\hat{S}_2' = \mathrm{diag}\{\hat{s}_1', \hat{s}_2', ..., \hat{s}_l'\}$ with

$$\hat{s}_i' = s'(\hat{v}_i^{\mathrm{T}} Z_2) = \frac{\mathrm{d}[s(z_a)]}{\mathrm{d}z_a}\Big|_{z_a = \hat{v}_i^{\mathrm{T}} Z_2}$$

and the residual term d_{u2} is bounded by

$$|d_{u2}| \le \|V_2^*\|_F \|Z_2 \hat{W}_2^{\mathrm{T}} \hat{S}_2'\|_F + \|W_2^*\| \|\hat{S}_2' \hat{V}_2^{\mathrm{T}} Z_2\| + |W_2^*|_1 \tag{5.81}$$

and the weight estimation errors $\tilde{W}_2 = \hat{W}_2 - W_2^*$, $\tilde{V}_2 = \hat{V}_2 - V_2^*$.

To analyze the closed loop stability for the q_2-subsystem, consider the following Lyapunov function candidate

$$V_2(r_2, \tilde{W}_2, \tilde{V}_2) = \frac{1}{2} \frac{d_{22}(q_3) d_{33} - d_{23}^2}{d_{33}} r_2^2 + \frac{1}{2} \tilde{W}_2^{\mathrm{T}} \Gamma_{W2}^{-1} \tilde{W}_2 + \frac{1}{2} \mathrm{tr}\{\tilde{V}_2^{\mathrm{T}} \Gamma_{V2}^{-1} \tilde{V}_2\} \tag{5.82}$$

Lemma 5.9. *The function V_2 (5.82) is positive definite and decrescent, in the sense that there exist two time-invariant positive definite functions $\underline{V}_2(r_2, \tilde{W}_2, \tilde{V}_2)$ and $\bar{V}_2(r_2, \tilde{W}_2, \tilde{V}_2)$, such that*

$$\underline{V}_2(r_2, \tilde{W}_2, \tilde{V}_2) \le V_2 \le \bar{V}_2(r_2, \tilde{W}_2, \tilde{V}_2)$$

Proof. The proof follows the same approach as Lemma 5.6 and is omitted here for conciseness. □

The time derivative of (5.82) is given as

$$\dot{V}_2 = \frac{1}{2} \dot{d}_{22}(q_3) r_2^2 + \frac{d_{22}(q_3) d_{33} - d_{23}^2}{d_{33}} r_2 \dot{r}_2 + \tilde{W}_2^{\mathrm{T}} \Gamma_2^{-1} \dot{\hat{W}}_2 + \mathrm{tr}\{\tilde{V}_2^{\mathrm{T}} \Gamma_{V2}^{-1} \dot{\hat{V}}_2\} \tag{5.83}$$

According to Assumption 5.2, (5.83) becomes

$$\dot{V}_2 = r_2\left[\frac{d_{22}(q_3)d_{33} - d_{23}^2}{d_{33}}\dot{r}_2 + c_{22}(q_3, \dot{q}_3)r_2\right] + \tilde{W}_2^{\mathrm{T}}\Gamma_2^{-1}\dot{\tilde{W}}_2 + \mathrm{tr}\{\tilde{V}_2^{\mathrm{T}}\Gamma_{V2}^{-1}\dot{\tilde{V}}_2\} \tag{5.84}$$

As W_2^*, V_2^* are constant vectors, it is easy to obtain that

$$\dot{\tilde{W}}_2 = \dot{\hat{W}}_2, \qquad \dot{\tilde{V}}_2 = \dot{\hat{V}}_2 \tag{5.85}$$

Substituting (5.79),(5.80), and (5.85) into (5.89), we have

$$\begin{aligned}\dot{V}_2 &= r_2\left[b_{22}(\dot{q}_3)\tau_2 - W_2^{*\mathrm{T}}S_2(V_2^{*\mathrm{T}}Z_2) - \varepsilon_2(Z_2)\right] \\ &\quad + \tilde{W}_2^{\mathrm{T}}\Gamma_{W2}^{-1}\dot{\hat{W}}_2 + \mathrm{tr}\{\tilde{V}_2^{\mathrm{T}}\Gamma_{V2}^{-1}\dot{\hat{V}}_2\}\end{aligned} \tag{5.86}$$

Consider the following MNN-based control law and MNN weight adaption laws:

$$\begin{aligned}\tau_2 &= k_2 r_2 + \frac{r_2\left(\hat{W}_2^{\mathrm{T}}S(\hat{V}_2^{\mathrm{T}}Z_2)\right)^2}{\underline{b}_{22}\left(|r_2\hat{W}_2^{\mathrm{T}}S(\hat{V}_2^{\mathrm{T}}Z_2)| + \delta_2\right)} \\ &\quad + \frac{k_2 r_2}{\underline{b}_{22}}(\|Z_2\hat{W}_2^{\mathrm{T}}\hat{S}_2'\|_F^2 + \|\hat{S}_2'\hat{V}_2^{\mathrm{T}}Z_2\|^2)\end{aligned} \tag{5.87}$$

$$\dot{\hat{W}}_2 = -\Gamma_{W2}[(\hat{S}_2 - \hat{S}_2'\hat{V}_2^{\mathrm{T}}Z_2)r_2 + \sigma_{W2}\hat{W}_2] \tag{5.88}$$

$$\dot{\hat{V}}_2 = -\Gamma_{V2}[Z_2\hat{W}_2^{\mathrm{T}}\hat{S}_2'r_2 + \sigma_{V2}\hat{V}_2] \tag{5.89}$$

where $k_2 > 0$, $\delta_2 > 0$, $\Gamma_{W2} = \Gamma_{W2}^{\mathrm{T}} > 0$, $\Gamma_{V2} = \Gamma_{V2}^{\mathrm{T}} > 0$, $\sigma_{W2} > 0$, $\sigma_{V2} > 0$.

Substituting (5.87)–(5.89) into (5.86), we have

$$\begin{aligned}\dot{V}_2 &= k_2 b_{22}(\dot{q}_3)r_2^2 + \frac{b_{22}(\dot{q}_3)}{\underline{b}_{22}}\frac{r_2^2\left(\hat{W}_2^{\mathrm{T}}S(\hat{V}_2^{\mathrm{T}}Z_2)\right)^2}{\left(|r_2\hat{W}_2^{\mathrm{T}}S(\hat{V}_2^{\mathrm{T}}Z_2)| + \delta_2\right)} \\ &\quad + \frac{b_{22}(\dot{q}_3)}{\underline{b}_{22}}k_2 r_2^2\left(\|Z_2\hat{W}_2^{\mathrm{T}}\hat{S}_2'\|_F^2 + \|\hat{S}_2'\hat{V}_2^{\mathrm{T}}Z_2\|^2\right) - r_2 W_2^{*\mathrm{T}}S_2(V_2^{*\mathrm{T}}Z_2) \\ &\quad - r_2\varepsilon_2(Z_2) - r_2\tilde{W}_2^{\mathrm{T}}(\hat{S}_2 - \hat{S}_2'\hat{V}_2^{\mathrm{T}}Z_2) - \sigma_{W2}\tilde{W}_2^{\mathrm{T}}\hat{W}_2 \\ &\quad - \mathrm{tr}\{\tilde{V}_2^{\mathrm{T}}Z_2\hat{W}_2^{\mathrm{T}}\hat{S}_2'r_2\} - \sigma_{V2}\mathrm{tr}\{\tilde{V}_2^{\mathrm{T}}\hat{V}_2\}\end{aligned} \tag{5.90}$$

Noting Assumption 5.3 and the fact that $\text{tr}\{\tilde{V}_2^{\text{T}} Z_2 \hat{W}_2^{\text{T}} \hat{S}_2' r_2\} = r_2 \hat{W}_2^{\text{T}} \hat{S}_2' \tilde{V}_2^{\text{T}} Z_2$, (5.90) becomes

$$\dot{V}_2 \leq -k_2 \underline{b}_{22} r_1^2 - \frac{r_2^2 \Big(\hat{W}_2^{\text{T}} S(\hat{V}_2^{\text{T}} Z_2)\Big)^2}{\Big(|r_2 \hat{W}_2^{\text{T}} S(\hat{V}_2^{\text{T}} Z_2)| + \delta_2\Big)} - k_2 r_2^2 \Big(\|Z_2 \hat{W}_2^{\text{T}} \hat{S}_2'\|_F^2 + \|\hat{S}_2' \hat{V}_2^{\text{T}} Z_2\|^2\Big)$$
$$+|r_2||\varepsilon_2(Z_2)| - r_2 W_2^{*\text{T}} S_2(V_2^{*\text{T}} Z_2) - r_2 \tilde{W}_2^{\text{T}} (\hat{S}_2 - \hat{S}_2' \hat{V}_2^{\text{T}} Z_2)$$
$$-r_2 \hat{W}_2^{\text{T}} \hat{S}_2' \tilde{V}_2^{\text{T}} Z_2 - \sigma_{W2} \tilde{W}_2^{\text{T}} \hat{W}_2 - \sigma_{V2} \text{tr}\{\tilde{V}_2^{\text{T}} \hat{V}_2\} \tag{5.91}$$

From (5.80) and (5.81), we know that

$$-r_2 W_2^{*\text{T}} S_2(V_2^{*\text{T}} Z_2) - r_2 \tilde{W}_2^{\text{T}} (\hat{S}_2 - \hat{S}_2' \hat{V}_2^{\text{T}} Z_2) - r_2 \hat{W}_2^{\text{T}} \hat{S}_2' \tilde{V}_2^{\text{T}} Z_2$$
$$= -r_2 \hat{W}_2^{\text{T}} S(\hat{V}_2^{\text{T}} Z_2) - r_2 d_{u2}$$
$$\leq |r_2 \hat{W}_2^{\text{T}} S(\hat{V}_2^{\text{T}} Z_2)| + |r_2| \|V_2^*\|_F \|Z_2 \hat{W}_2^{\text{T}} \hat{S}_2'\|_F + |r_2| \|W_2^*\| \|\hat{S}_2' \hat{V}_2^{\text{T}} Z_2\|$$
$$+|r_2||W_2^*|_1 \tag{5.92}$$

Substituting (5.92) into (5.91) leads to

$$\dot{V}_2 \leq -k_2 \underline{b}_{22} r_2^2 - \frac{r_2^2 \Big(\hat{W}_2^{\text{T}} S(\hat{V}_2^{\text{T}} Z_2)\Big)^2}{\Big(|r_2 \hat{W}_2^{\text{T}} S(\hat{V}_2^{\text{T}} Z_2)| + \delta_2\Big)} + |r_2 \hat{W}_2^{\text{T}} S(\hat{V}_2^{\text{T}} Z_2)|$$
$$-k_2 r_2^2 \Big(\|Z_2 \hat{W}_2^{\text{T}} \hat{S}_2'\|_F^2 + \|\hat{S}_2' \hat{V}_2^{\text{T}} Z_2\|^2\Big) + |r_2||\varepsilon_2(Z_2)|$$
$$+|r_2| \|V_2^*\|_F \|Z_2 \hat{W}_2^{\text{T}} \hat{S}_2'\|_F + |r_2| \|W_2^*\| \|\hat{S}_2' \hat{V}_2^{\text{T}} Z_2\| + |r_2||W_2^*|_1$$
$$-\sigma_{W2} \tilde{W}_2^{\text{T}} \hat{W}_2 - \sigma_{V2} \text{tr}\{\tilde{V}_2^{\text{T}} \hat{V}_2\} \tag{5.93}$$

According to Lemma 5.4,

$$-\frac{r_2^2 \Big(\hat{W}_2^{\text{T}} S(\hat{V}_2^{\text{T}} Z_2)\Big)^2}{\Big(|r_2 \hat{W}_2^{\text{T}} S(\hat{V}_2^{\text{T}} Z_2)| + \delta_2\Big)} + |r_2 \hat{W}_2^{\text{T}} S(\hat{V}_2^{\text{T}} Z_2)| = \frac{|r_2 \hat{W}_2^{\text{T}} S(\hat{V}_2^{\text{T}} Z_2)| \delta_2}{|r_2 \hat{W}_2^{\text{T}} S(\hat{V}_2^{\text{T}} Z_2)| + \delta_2} \leq \delta_2 \tag{5.94}$$

By completion of squares and using Young's inequality, the following inequalities hold:

$$|r_2||\varepsilon_2(Z_2)| \leq \frac{r_2^2}{2c_{21}} + \frac{c_{21} \bar{\varepsilon}_2^2}{2} \tag{5.95}$$

$$|r_2| \|V_2^*\|_F \|Z_2 \hat{W}_2^{\text{T}} \hat{S}_2'\|_F \leq k_2 r_2^2 \|Z_2 \hat{W}_2^{\text{T}} \hat{S}_2'\|_F^2 + \frac{1}{4k_2} \|V_2^*\|_F^2 \tag{5.96}$$

$$|r_2|\|W_2^*\|\|\hat{S}_2'\hat{V}_2^{\mathrm{T}}Z_2\| \le k_2 r_2^2\|\hat{S}_2'\hat{V}_2^{\mathrm{T}}Z_2\|^2 + \frac{1}{4k_2}\|W_2^*\|^2 \tag{5.97}$$

$$|r_2||W_2^*|_1 \le \frac{r_2^2}{2c_{22}} + \frac{c_{22}|W_2^*|_1^2}{2} \tag{5.98}$$

$$-\sigma_{W2}\tilde{W}_2^{\mathrm{T}}\hat{W}_2 \le -\frac{\sigma_{W2}}{2}\|\tilde{W}_2\|^2 + \frac{\sigma_{W2}}{2}\|W_1^*\|^2 \tag{5.99}$$

$$-\sigma_{V2}\mathrm{tr}\{\tilde{V}_2^{\mathrm{T}}\hat{V}_2\} \le -\frac{\sigma_{V2}}{2}\|\tilde{V}_2\|_F^2 + \frac{\sigma_{V2}}{2}\|V_2^*\|_F^2 \tag{5.100}$$

Substituting (5.94)–(5.100) into (5.93), we have

$$\begin{aligned}\dot{V}_2 \le& -\left(k_2\underline{b}_{22} - \frac{1}{2c_{21}} - \frac{1}{2c_{22}}\right)r_2^2 - \frac{\sigma_{W2}}{2}\|\tilde{W}_2\|^2 - \frac{\sigma_{V2}}{2}\|\tilde{V}_2\|_F^2 + \delta_2\\ &+\left(\frac{\sigma_{W2}}{2} + \frac{1}{4k_2}\right)\|W_2^*\|^2 + \left(\frac{\sigma_{V2}}{2} + \frac{1}{4k_2}\right)\|V_2^*\|_F^2 + \frac{c_{21}}{2}\bar{\varepsilon}_2^2 + \frac{c_{22}|W_2^*|_1^2}{2}\\ \le& -\lambda_{20}V_2 + \mu_{20}\end{aligned} \tag{5.101}$$

where $\lambda_{20} = \min\Big\{(2k_2\underline{b}_{22} - 1/c_{21} - 1/c_{22})|d_{33}|/(\bar{d}_{22}|d_{33}| + d_{23}^2), \sigma_{W2}/\lambda_{\max}(\Gamma_{W2}^{-1}), \sigma_{V2}/\lambda_{\max}(\Gamma_{V2}^{-1})\Big\}$, $\mu_{20} = \delta_2 + (\frac{\sigma_{W2}}{2} + \frac{1}{4k_2})\|W_2^*\|^2 + (\frac{\sigma_{V2}}{2} + \frac{1}{4k_21})\|V_2^*\|_F^2 + \frac{c_{21}}{2}\bar{\varepsilon}_2^2 + \frac{c_{22}|W_2^*|_1^2}{2}$.

□ <u>*q_3-subsystem*</u>

Finally, for the system (5.4)–(5.6) under control laws (5.64) and (5.87), we can obtain similar internal dynamics to Sect. 5.3.1.

The main result in this section can be summarized as the following theorem:

Theorem 5.10. *Consider the closed-loop system consisting of the subsystems (5.4)–(5.6), the control laws (5.64), (5.87), and adaptation laws (5.65)–(5.66), (5.88)–(5.89). Under Assumptions 5.1–5.6, the overall closed-loop neural control system is SGUUB in the sense that all of the signals in the closed-loop system are bounded, and the tracking errors and weights converge to the following regions,*

$$|e_1| \le |e_1(0)| + \frac{1}{\lambda_1}\sqrt{\frac{2\mu_1}{d_{11}}},\quad |e_2| \le |e_2(0)| + \frac{1}{\lambda_2}\sqrt{\frac{2|d_{33}|\mu_2}{\left|\underline{d}_{22}|d_{33}| - d_{23}^2\right|}},$$

$$\|\hat{V}_1\|_F \le \sqrt{\frac{2\mu_1}{\lambda_{\min}(\Gamma_1^{-1})}} + v_{1m},\quad \|\hat{V}_2\|_F \le \sqrt{\frac{2\mu_2}{\lambda_{\min}(\Gamma_2^{-1})}} + v_{2m}$$

$$\|\hat{W}_1\| \le \sqrt{\frac{2\mu_1}{\lambda_{\min}(\Gamma_1^{-1})}} + w_{1m},\quad \|\hat{W}_2\| \le \sqrt{\frac{2\mu_2}{\lambda_{\min}(\Gamma_2^{-1})}} + w_{2m},$$

with

$$\mu_i = \frac{\mu_{i0}}{\lambda_{i0}} + V_i(0)$$

$$\mu_{i0} = \delta_i + \left(\frac{\sigma_{Wi}}{2} + \frac{1}{4k_i}\right)\|W_i^*\|^2 + \left(\frac{\sigma_{Vi}}{2} + \frac{1}{4k_i1}\right)\|V_i^*\|_F^2 + \frac{c_{i1}}{2}\bar{\varepsilon}_i^2 + \frac{c_{i2}|W_i^*|_1^2}{2},$$
$$i = 1, 2$$

$$\lambda_{10} = \min\left\{(2k_1\underline{b}_{11} - 1/c_{11} - 1/c_{12})/d_{11}, \sigma_{W1}/\lambda_{\max}(\Gamma_{W1}^{-1}), \sigma_{V1}/\lambda_{\max}(\Gamma_{V1}^{-1})\right\}$$

$$\lambda_{20} = \min\left\{(2k_2\underline{b}_{22} - 1/c_{21} - 1/c_{22})|d_{33}|/(\bar{d}_{22}|d_{33}| + d_{23}^2), \sigma_{W2}/\lambda_{\max}(\Gamma_{W2}^{-1}),\right.$$
$$\left.\sigma_{V2}/\lambda_{\max}(\Gamma_{V2}^{-1})\right\}$$

where $e_i(0)$ *and* $V_i(0)$ *are initial values of* $e_i(t)$ *and* $V_i(t)$*, respectively.*

Proof. The proof of Theorem 5.10 follows the same approach as Theorem 5.8, and will be omitted here for conciseness. □

5.4 Simulation Study

To illustrate the proposed adaptive neural control, we consider the VARIO helicopter mounted on a platform [104], with the dynamic model as (5.1) and the following parameters $d_{11} = 7.5$, $d_{22}(q_3) = 0.4305 + 0.0003\cos^2(-4.143q_3)$, $d_{23} = 0.108$, $d_{33} = 0.4993$, $c_{22}(q_3, \dot{q}_3) = 0.0006214\sin(-8.286q_3)\dot{q}_3$, $c_{23}(q_3, \dot{q}_2) = c_{32}(q_3, \dot{q}_2) = 0.0006214\sin(-8.286q_3)\dot{q}_2$, $g_1 = -77.259$, $g_3 = -2.642$, $f_1(\dot{q}_3) = -0.6004\dot{q}_3$, $f_3(\dot{q}_3) = -0.0001206\dot{q}_3^2$, $b_{11}(\dot{q}_3) = 3.411\dot{q}_3^2$, $b_{22}(\dot{q}_3) = -0.1525\dot{q}_3^2$, $b_{31}(\dot{q}_3) = 12.01\dot{q}_3 + 10^5$, and all quantities are expressed in S.I. units. The control objective is to track the uniformly bounded desired trajectories given in [104] as follows:

$$q_{1d} = \begin{cases} -0.2 & 0 \le t \le 50\,\text{s} \\ 0.3[e^{-(t-50)^2/350} - 1] - 0.2 & 50 < t \le 130\,\text{s} \\ 0.1\cos[(t-130)/10] - 0.6 & 130 < t \le 20\pi + 130 \\ -0.5 & t \ge 20\pi + 130 \end{cases}$$

$$q_{2d} = \begin{cases} 0 & t < 50\,\text{s} \\ 1 - e^{-(t-50)^2/350} & 50 \le t < 120\,\text{s} \\ e^{-(t-120)^2/350} & 120 \le t < 180 \\ -1 + e^{-(t-180)^2/350} & t \ge 180 \end{cases}$$

5.4.1 Internal Dynamics Stability Analysis

In this section, we analyze the stability of the internal dynamics according to the related discussion in [104]. For conciseness, we consider the RBFNN-based control case only, which can be easily extended to the MNN-based control case without any difficulties. For the RBFNN-based control case, we substitute (5.10), (5.15), (5.25) and (5.33) into the q_3-subsystem (5.6). According to the definition of the zero dynamics [35], we set r_1, r_2, $\tilde{W}_1^{\mathrm{T}}$, $\tilde{W}_2^{\mathrm{T}}$, $\varepsilon_1(Z_1)$ and $\varepsilon_2(Z_2)$ to zero, and the desired trajectories and initial data can be chosen in such a way that terms including $\dot{q}_2^2$, $\ddot{q}_{1d}$, $\ddot{q}_{2d}$ can be neglected [104], so we have

$$\ddot{q}_3 = \frac{1}{d_{33}}\left[\frac{b_{31}(\dot{q}_3)}{b_{11}(\dot{q}_3)}(f_1(\dot{q}_3) + g_1) - f_3(\dot{q}_3) - g_3\right] \tag{5.102}$$

Substituting the term values given in the beginning of Sect. 5.4 into (5.102) and analyzing the values of the main rotor angular velocity from which the main rotor angular acceleration is zero, we have

$$4.1137 \times 10^{-4}\dot{q}_3^4 + 1.8011\dot{q}_3^2 - 60968\dot{q}_3 - 7725900 = 0$$

Its solutions are $\dot{q}_3^* = -124.63, -219.5 \pm 468.16\mathrm{i}$ and 563.64 rad/s. Only the first value $\dot{q}_3^* = -124.63$ has a physical meaning for the system. If we linearize (5.102) around the equilibrium point $\dot{q}_3^* = -124.63$, we can obtain an eigenvalue -2.44. Therefore, according to [52], all initials of $\dot{q}_3$ sufficiently near $\dot{q}_3^* = -124.63$ can converge to -124.63. It then follows that the internal dynamics of the helicopter system in (5.1) have a stable behavior.

The simulation result in Fig. 5.1 also shows that the internal dynamics using RBFNN-based control are indeed stable. From Fig. 5.1, we can observe that the main rotor angular velocity $\dot{q}_3$ converges to the nominal value -124.63 rad/s for different initial conditions ranging from -40 rad/s to -150 rad/s, which includes the typical operating values more than sufficiently. These results are expected from the previous stability analysis, and also consistent with the results in [104]. In particular, we also notice that the further the initial condition starts from the nominal value -124.63 rad/s, the longer the settling time takes, and the more serious the transient oscillations become. This is reasonable in practice. If some preliminary knowledge about the nominal value is known in advance, the initial condition can be set closer to achieve better performance.

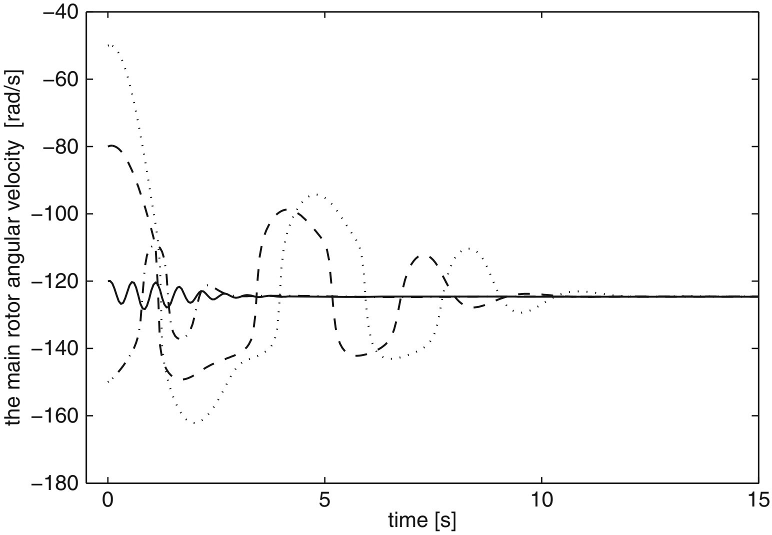

Fig. 5.1 Main rotor angular velocity behavior for different initial conditions using RBFNN-based control

5.4.2 Performance Comparison Results Between Approximation-Based Control and Model-Based Control

In this subsection, we will compare the altitude and yaw angle tracking performance using RBFNN-based control, MNN-based control and the model-based control adopted in [104]. If all the parameters and functions in (5.1) are known exactly, and the unmodeled uncertainties $\Delta(\cdot) = 0$, the perfect tracking performance can be achieved using model-based control, which has been shown in the work [104]. However, in practice, there always exist some model uncertainties, which may be caused by unmodeled dynamics or aerodynamical disturbances from the environment. To this end, we assume $\Delta(\cdot) \neq 0$, in particular, $\Delta(\cdot) = [2.0, \quad 0, \quad 0.0001206\dot{q}_3^2 + 0.142]^{\mathrm{T}}$.

The control parameters for the RBFNN control laws (5.15) (5.33) and adaptation laws (5.16) (5.34) are chosen as follows: $k_1 = 0.000085$, $\Lambda_1 = 0.2$, $k_2 = 0.0002$, $\Lambda_2 = 1.0$, $\Gamma_1 = 0.001I$, $\Gamma_2 = 0.0001I$, $\sigma_1 = 0.001$, $\sigma_2 = 0.001$. NNs $\hat{W}_1^{\mathrm{T}} S_1(Z_1)$ contains 3^8 nodes (i.e., $l_1 = 2187$), with centers $\mu_l (l = 1, ..., l_1)$ evenly spaced in $[-1.0, 1.0] \times [-0.1, 0.1] \times [-10.0, -10.0] \times [-40000.0, 0.0] \times [-1.0, 1.0] \times [-150.0, -40.0] \times [-0.1, 0.1] \times [-0.01, 0.01]$, and widths $\eta_l = 1.0 (l = 1, ..., l_1)$. NNs $\hat{W}_2^{\mathrm{T}} S_2(Z_2)$ contains 3^{10} nodes (i.e., $l_2 = 59049$), with centers $\mu_l (l = 1, ..., l_2)$ evenly spaced in $[-0.005, 0.005] \times [-1.0, 1.0] \times [-0.1, 0.1] \times [-10.0, -10.0] \times [-40000.0, 0.0] \times [-1.0, 1.0] \times [-150.0, -40.0] \times [-10.0, 10.0] \times [-1.0, 1.0] \times [-0.01, 0.01]$, and widths $\eta_l = 1.0 (l = 1, ..., l_2)$. The initial conditions are:

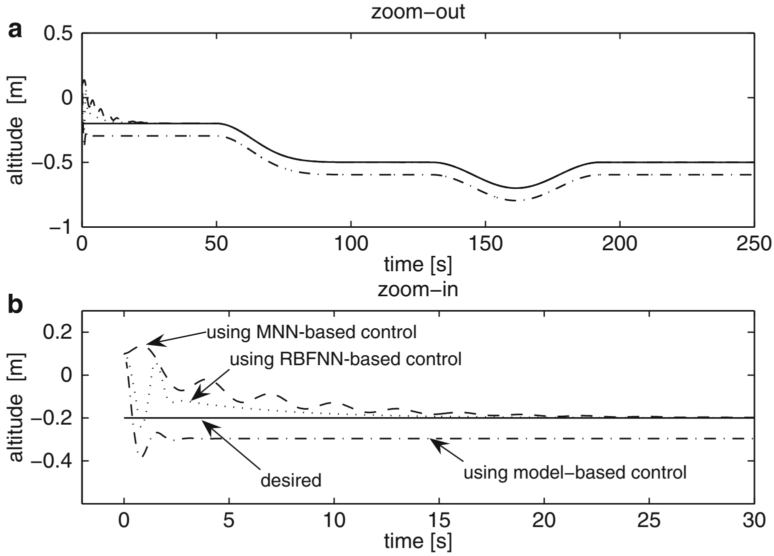

Fig. 5.2 Altitude tracking performance in the presence of model uncertainties

$q_1(0) = 0.1$ m, $\dot{q}_1(0) = 0.0$ m/s, $q_2(0) = -\pi$ rad, $\dot{q}_2(0) = 0.0$ rad/s, $q_3(0) = -\pi$ rad, $\dot{q}_3(0) = -120.0$ rad/s, $\tau_1 = 0.0$ m, $\tau_2 = 0.0$ m, $\hat{W}_1(0) = 0.0$, $\hat{W}_2(0) = 0.0$.

For the MNN control laws (5.64) and (5.87) and adaptation laws (5.65), (5.66), (5.88) and (5.89), the design parameters are chosen as: $k_1 = 0.00016$, $\Lambda_1 = 1.2$, $k_2 = 0.0002$, $\Lambda_2 = 1.0$, $\Gamma_{W1} = 0.0002I$, $\Gamma_{V1} = 0.03I$, $\delta_{W1} = 0.0$, $\sigma_{V1} = 0.0$, $\Gamma_{W2} = 0.0001I$, $\Gamma_{V2} = 0.01I$, $\sigma_{W2} = 0.0$, $\sigma_{V2} = 0.0$. NNs $\hat{W}_1^{\mathrm{T}} S_1(\hat{V}_1^{\mathrm{T}} \bar{z}_1)$ contains five nodes and NNs $\hat{W}_2^{\mathrm{T}} S_2(\hat{V}_2^{\mathrm{T}} \bar{z}_2)$ contains 15 nodes. The initial conditions are: $q_1(0) = 0.1$ m, $\dot{q}_1(0) = 0$ m/s, $q_2(0) = -\pi$ rad, $\dot{q}_2(0) = 0.0$ rad/s, $q_3(0) = -\pi$ rad, $\dot{q}_3(0) = -120.0$ rad/s, $\tau_1 = 0.0$ m, $\tau_2 = 0.0$ m, $\hat{W}_1(0) = 0.0$, $\hat{V}_1(0) = 0.0$, $\hat{W}_2(0) = 0.0$, $\hat{V}_2(0) = 0.0$.

From Figs. 5.2 and 5.3, we can observe that due to the existence of model uncertainties, both the altitude tracking and yaw angle tracking using model-based control have some offsets to the desired trajectories for the whole period. This means that model-based control depends on the accuracy of the model heavily and cannot deal with the uncertainties well. For the tracking performance using the RBFNN-based control and MNN-based control, though there are also some oscillations at the initial period, the tracking errors can converge to a very small neighborhood of desired trajectories in a short time of about 20 s. This is because the model uncertainties can be learnt by RBFNN and MNN during the beginning 25 s. After that period, the uncertainties can be compensated for, and thus, the robustness of uncertainties is improved and good tracking performance is achieved. In addition, Figs. 5.4 and 5.5 indicates norms of neural weights for approximation-based control and control actions for three control methods.

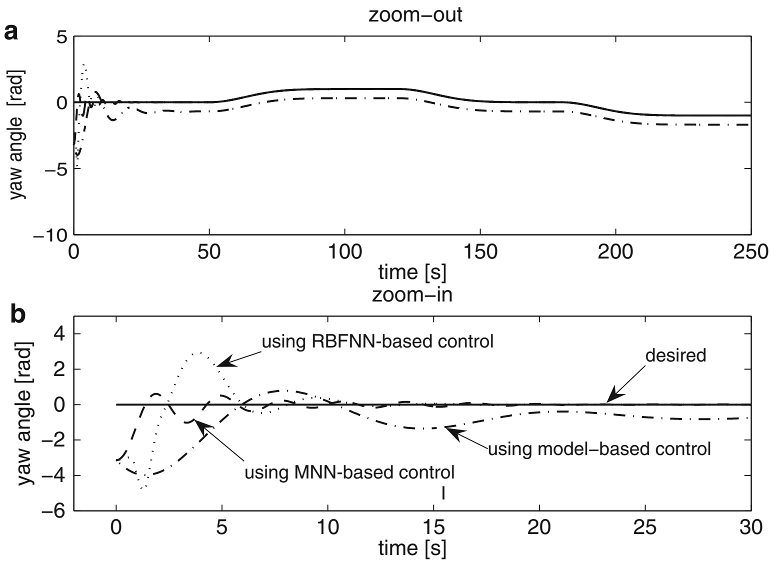

Fig. 5.3 Yaw angle tracking performance in the presence of model uncertainties

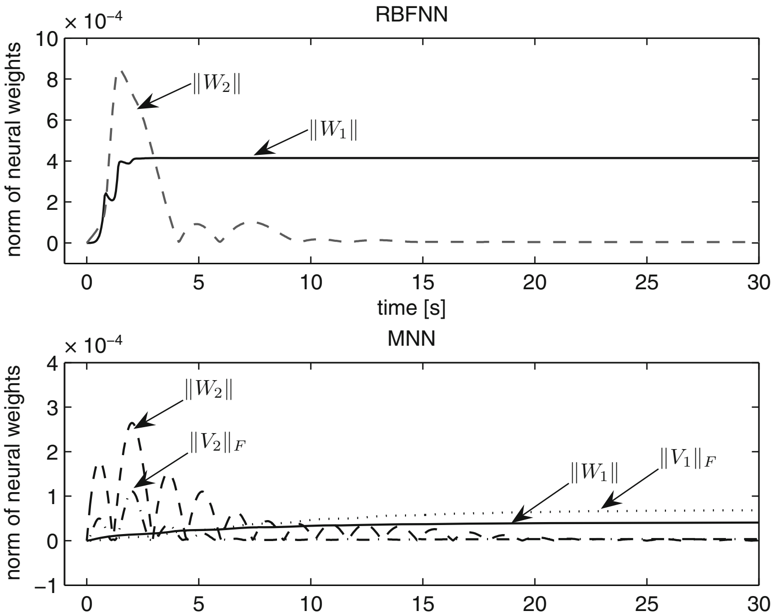

Fig. 5.4 Norm of neural weights

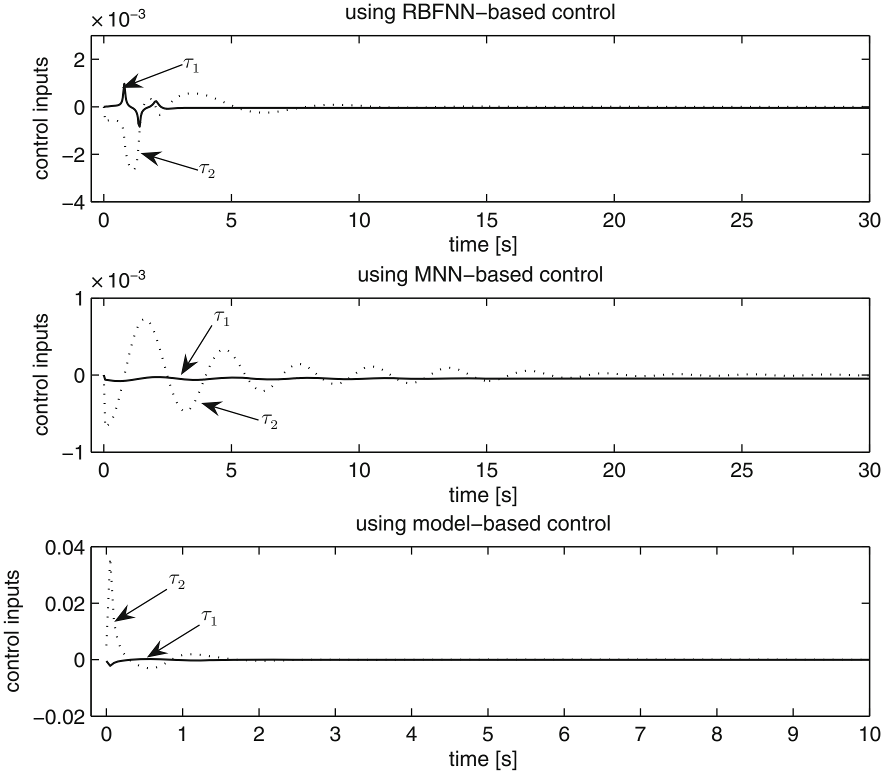

Fig. 5.5 Control inputs for altitude and yaw angle tracking in the presence of model uncertainties

5.5 Conclusion

In this chapter, NN approximation-based control was investigated for the MIMO helicopter altitude and yaw angle tracking in the presence of model uncertainties. Compared with the model-based control, which is sensitive to the accuracy of the model representation, NN approximation-based control is tolerant of model uncertainties, and can be viewed as a key advantage over model-based control of helicopters, for which accurate modeling of helicopter dynamics is difficult, time-consuming and costly. Simulation results demonstrated that the helicopter is able to track altitude and yaw angle reference signals satisfactorily, with all other closed-loop signals bounded.