A NOTE ON CONTIGUITY AND HELLINGER DISTANCE*

by

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1. Introduction

For $n = 1, 2, \ldots$ let $(\mathscr{X}_{n1}, \mathscr{A}_{n1}), \ldots, (\mathscr{X}_{nn}, \mathscr{A}_{nn})$ be arbitrary measurable spaces. Let P_{ni} and Q_{ni} be probability measures defined on $(\mathscr{X}_{ni}, \mathscr{A}_{ni})$, $i = 1, \ldots, n$; $n = 1, 2, \ldots$, and let $P_n^{(n)} = \prod_{i=1}^{n} P_{ni}$ and $Q_n^{(n)} = \prod_{i=1}^{n} Q_{ni}$ $(n = 1, 2, \ldots)$ denote the product probability measures. For each i and n let X_{ni} be the identity map from $\mathscr{X}_{ni}$ onto $\mathscr{X}_{ni}$. Then P_{ni} and Q_{ni} represent the two possible distributions of the random element X_{ni} as well as the probability measures of the underlying probability space. Obviously $X_{n1}, \ldots, X_{nn}$ are independent under both $P_n^{(n)}$ and $Q_n^{(n)}$ $(n = 1, 2, \ldots)$.

The sequence $\{Q_n^{(n)}\}$ is said to be contiguous with respect to the sequence $\{P_n^{(n)}\}$ if $\lim_{n\to\infty} P_n^{(n)}(A_n) = 0$ implies $\lim_{n\to\infty} Q_n^{(n)}(A_n) = 0$ for any sequence of measurable sets A_n. This one-sided contiguity notion is denoted by $\{Q_n^{(n)}\} \lhd \{P_n^{(n)}\}$ (the notation is due to H. Witting & G. Nölle [7]). The sequences $\{P_n^{(n)}\}$ and $\{Q_n^{(n)}\}$ are said to be contiguous with respect to each other if both $\{Q_n^{(n)}\} \lhd \{P_n^{(n)}\}$ and $\{P_n^{(n)}\} \lhd \{Q_n^{(n)}\}$. This two-sided contiguity concept we denote by $\{P_n^{(n)}\} \lhd\rhd \{Q_n^{(n)}\}$.

The main purpose of this note is to characterize contiguity of product probability measures in terms of their marginals. To this end we introduce the Hellinger distance $H(P, Q)$ between two probability measures P and Q on the same σ-field, defined by

$$(1.1) \qquad H(P, Q) = \left\{\int (p^{1/2} - q^{1/2})^2 \, d\mu\right\}^{1/2} = \left\{2 - 2\int p^{1/2} q^{1/2} \, d\mu\right\}^{1/2},$$

where $p = dP/d\mu$, $q = dQ/d\mu$ and μ is any σ-finite measure dominating $P + Q$. This metric is independent of the choice of μ and satisfies $0 \leqq H(P, Q) \leqq 2^{1/2}$.

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AMS (MOS) subject classification scheme (1970): 62E20

KEY WORDS & PHRASES: *asymptotic normality, contiguity, Hellinger distance, log likelihood ratio.*

Defining the total variation distance of P and Q by

$$\|P - Q\| = \sup |P(A) - Q(A)|\,, \tag{1.2}$$

where the supremum is taken over all measurable sets A, we have the following inequalities (Le Cam [4])

$$\tfrac{1}{2}H^2(P, Q) \leqq \|P - Q\| \leqq H(P, Q)\,. \tag{1.3}$$

The Hellinger distances of the product measures and of their marginals are connected by the relationship

$$H^2(P_n^{(n)}, Q_n^{(n)}) = 2 - 2\prod_{i=1}^{n}\{1 - \tfrac{1}{2}H^2(P_{ni}, Q_{ni})\}\,. \tag{1.4}$$

For further reference we first mention two easy results, viz.

$$\sum_{i=1}^{n} H^2(P_{ni}, Q_{ni}) = o(1) \quad \text{for} \quad n \to \infty \Rightarrow \{P_n^{(n)}\} \lhd \rhd \{Q_n^{(n)}\}\,, \tag{1.5}$$

and

$$\{Q_n^{(n)}\} \lhd \{P_n^{(n)}\} \Rightarrow \sum_{i=1}^{n} H^2(P_{ni}, Q_{ni}) = O(1) \quad \text{for} \quad n \to \infty\,. \tag{1.6}$$

The proof of (1.5) is an immediate consequence of the string of implications

$$\sum_{i=1}^{n} H^2(P_{ni}, Q_{ni}) = o(1) \Rightarrow \sum_{i=1}^{n} \log\{1 - \tfrac{1}{2}H^2(P_{ni}, Q_{ni})\} = o(1)$$

$$\Rightarrow H^2(P_n^{(n)}, Q_n^{(n)}) = o(1) \Rightarrow \|P_n^{(n)} - Q_n^{(n)}\| = o(1) \Rightarrow \{P_n^{(n)}\} \lhd \rhd \{Q_n^{(n)}\}\,.$$

To prove (1.6) suppose that $\limsup_{n\to\infty} H(P_n^{(n)}, Q_n^{(n)}) = 2^{1/2}$. Then by (1.3) $\limsup_{n\to\infty} \|P_n^{(n)} - Q_n^{(n)}\| = 1$ in contradiction to $\{Q_n^{(n)}\} \lhd \{P_n^{(n)}\}$. Thus $\limsup_{n\to\infty} H^2(P_n^{(n)}, Q_n^{(n)}) < 2$, therefore $\liminf_{n\to\infty} \prod_{i=1}^{n}\{1 - \tfrac{1}{2}H^2(P_{ni}, Q_{ni})\} > 0$ and hence $\limsup_{n\to\infty} \sum_{i=1}^{n} H^2(P_{ni}, Q_{ni}) < \infty$ and the proof is complete.

It can be shown by counterexamples that in (1.5) the condition cannot be weakened to $\sum_{i=1}^{n} H^2(P_{ni}, Q_{ni}) = O(1)$, and that in (1.6) the conclusion cannot be strengthened to $\sum_{i=1}^{n} H^2(P_{ni}, Q_{ni}) = o(1)$, for $n \to \infty$. Hence there remains a gap between the sufficient condition and the necessary condition for contiguity in (1.5) and (1.6) respectively. In section 2 we obtain conditions which are both sufficient and necessary for contiguity of the product measures by adding another condition to $\sum_{i=1}^{n} H^2(P_{ni}, Q_{ni}) = O(1)$.

In many applications asymptotic normality of the log likelihood ratio statistic Λ_n (see (3.1)) plays an important part. Since

$$\mathscr{L}(\Lambda_n \mid P_n^{(n)}) \to_w \mathscr{N}(-\tfrac{1}{2}\sigma^2; \sigma^2) \quad \text{implies} \quad \{P_n^{(n)}\} \lhd \rhd \{Q_n^{(n)}\}$$

(cf. Hájek & Šidák [1], Le Cam [2], [3], [4], Roussas [6]), we have to impose stronger conditions on the marginals P_{ni} and Q_{ni} to ensure the asymptotic normality of Λ_n. Some sufficient (and almost necessary) conditions for the asymptotic normality of Λ_n, which are clearly stronger than those in section 2, are given in section 3. These conditions are closely related to some earlier results of Le Cam [3], [4].

2. Contiguity of product measures

We begin by noting the following useful implication:

$$\text{(2.1)} \qquad \{Q_n^{(n)}\} \lhd \{P_n^{(n)}\} \Rightarrow [\lim_{n\to\infty} \sum_{i=1}^{n} P_{ni}(A_{ni}) = 0 \Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} Q_{ni}(A_{ni}) = 0]$$

for any collection of measurable sets A_{ni}. For suppose $\lim_{n\to\infty} \sum_{i=1}^{n} P_{ni}(A_{ni}) = 0$. Then $\lim_{n\to\infty} P_n^{(n)}(\bigcup_{i=1}^{n} A_{ni}) = 0$, hence by contiguity $\lim_{n\to\infty} Q_n^{(n)}(\bigcup_{i=1}^{n} A_{ni}) = 1 - \lim_{n\to\infty} \prod_{i=1}^{n} (1 - Q_{ni}(A_{ni})) = 0$ and therefore $\lim_{n\to\infty} \sum_{i=1}^{n} Q_{ni}(A_{ni}) = 0$.

Now let μ_{ni} be a σ-finite measure on $(\mathscr{X}_{ni}, \mathscr{A}_{ni})$ dominating $P_{ni} + Q_{ni}$ and write $p_{ni} = \mathrm{d}P_{ni}/\mathrm{d}\mu_{ni}$ and $q_{ni} = \mathrm{d}Q_{ni}/\mathrm{d}\mu_{ni}$ ($i = 1, \ldots, n$; $n = 1, 2, \ldots$). The main result of this section is

Theorem 1. $\{Q_n^{(n)}\} \lhd \{P_n^{(n)}\}$ *iff*

$$\text{(2.2)} \qquad \limsup_{n\to\infty} \sum_{i=1}^{n} H^2(P_{ni}, Q_{ni}) < \infty$$

and

$$\text{(2.3)} \qquad \lim_{n\to\infty} \sum_{i=1}^{n} Q_{ni}(q_{ni}(X_{ni})/p_{ni}(X_{ni}) \geqq c_n) = 0 \quad \textit{whenever} \quad c_n \to \infty .$$

Proof. First assume that (2.2) and (2.3) are satisfied. Write

$$L_{ni} = q_{ni}(X_{ni})/p_{ni}(X_{ni}), \quad i = 1, \ldots, n; \quad n = 1, 2, \ldots,$$

and consider $\prod_{i=1}^{n} L_{ni}$. It is easily shown (cf. Le Cam [4], Roussas [6]) that $\{Q_n^{(n)}\} \lhd \{P_n^{(n)}\}$ is equivalent to tightness of the sequence of distributions $\{\mathscr{L}(\prod_{i=1}^{n} L_{ni} \mid Q_n^{(n)});$

$n = 1, 2, \ldots\}$. The tightness of this set of distributions can also be expressed in the more convenient form

$$\lim_{n\to\infty} Q_n^{(n)}\Big(\prod_{i=1}^{n} L_{ni} \geqq k_n\Big) = 0 \quad \text{whenever} \quad k_n \to \infty\,. \tag{2.4}$$

Hence we have to prove (2.4). Let $0 < k_n \to \infty$. Let $0 < c_n \to \infty$ be real numbers to be chosen in the sequel. If 1_A denotes the indicator function of the set A, we have by (2.3) and Markov's inequality for $n \to \infty$

$$Q_n^{(n)}\Big(\prod_{i=1}^{n} L_{ni} \geqq k_n\Big)$$

$$\leqq Q_n^{(n)}\Big(\prod_{i=1}^{n} L_{ni} \geqq k_n \wedge L_{ni} < c_n \quad \text{for} \quad i = 1, \ldots, n\Big) + Q_n^{(n)}\Big(\bigcup_{i=1}^{n} \{L_{ni} \geqq c_n\}\Big)$$

$$\leqq Q_n^{(n)}\Big(\prod_{i=1}^{n} L_{ni}^{1/2}\, 1_{(0,c_n)}(L_{ni}) \geqq k_n^{1/2}\Big) + \sum_{i=1}^{n} Q_{ni}(L_{ni} \geqq c_n)$$

$$\leqq k_n^{-1/2} \prod_{i=1}^{n} \int_{q_{ni} < c_n p_{ni}} q_{ni}^{3/2} p_{ni}^{-1/2}\, \mathrm{d}\mu_{ni} + o(1)\,.$$

Since for all $c_n \geqq 1$

$$\int_{q_{ni} < c_n p_{ni}} q_{ni}^{3/2} p_{ni}^{-1/2}\, \mathrm{d}\mu_{ni}$$

$$\leqq \int_{q_{ni} < c_n p_{ni}} q_{ni}\, \mathrm{d}\mu_{ni} + \int_{q_{ni} < c_n p_{ni}} q_{ni} p_{ni}^{-1/2}(q_{ni}^{1/2} - p_{ni}^{1/2})\, \mathrm{d}\mu_{ni}$$

$$\leqq 1 + \int_{q_{ni} < c_n p_{ni}} q_{ni}^{1/2} p_{ni}^{-1/2}(q_{ni}^{1/2} - p_{ni}^{1/2})^2\, \mathrm{d}\mu_{ni} + \int_{q_{ni} < c_n p_{ni}} q_{ni}^{1/2}(q_{ni}^{1/2} - p_{ni}^{1/2})\, \mathrm{d}\mu_{ni}$$

$$\leqq 1 + c_n^{1/2} \int (q_{ni}^{1/2} - p_{ni}^{1/2})^2\, \mathrm{d}\mu_{ni} + 1 - \int q_{ni}^{1/2} p_{ni}^{1/2}\, \mathrm{d}\mu_{ni}$$

$$- \int_{q_{ni} \geqq c_n p_{ni}} q_{ni}^{1/2}(q_{ni}^{1/2} - p_{ni}^{1/2})\, \mathrm{d}\mu_{ni} \leqq 1 + (c_n^{1/2} + \tfrac{1}{2})\, H^2(P_{ni}, Q_{ni})\,,$$

it follows that

$$\limsup_{n\to\infty} Q_n^{(n)}\Big(\prod_{i=1}^{n} L_{ni} \geqq k_n\Big)$$

$$\leqq \limsup_{n\to\infty} k_n^{-1/2} \prod_{i=1}^{n} \{1 + (c_n^{1/2} + \tfrac{1}{2})\, H^2(P_{ni}, Q_{ni})\}$$

$$\leqq \limsup_{n\to\infty} k_n^{-1/2} \exp\Big\{(c_n^{1/2} + \tfrac{1}{2}) \sum_{i=1}^{n} H^2(P_{ni}, Q_{ni})\Big\}\,.$$

Choosing c_n in such a way that $c_n = o((\log k_n)^2)$ for $n \to \infty$, (2.2) implies $Q_n^{(n)}(\prod_{i=1}^n L_{ni} \geqq k_n) = o(1)$ for $n \to \infty$ and (2.4) is established.

Conversely, suppose that $\{Q_n^{(n)}\} \lhd \{P_n^{(n)}\}$. Since (1.6) implies that (2.2) is satisfied, it remains to prove (2.3). Let $0 < c_n \to \infty$ and consider the inequality, valid for $c_n \geqq 4$,

$$\sum_{i=1}^n \int_{q_{ni} \geqq c_n p_{ni}} p_{ni} \, d\mu_{ni} \leqq c_n^{-1/2} \sum_{i=1}^n \int_{q_{ni} \geqq c_n p_{ni}} p_{ni}^{1/2} q_{ni}^{1/2} \, d\mu_{ni}$$

$$= c_n^{-1/2} \left\{ \sum_{i=1}^n \int_{q_{ni} \geqq c_n p_{ni}} p_{ni}^{1/2}(q_{ni}^{1/2} - p_{ni}^{1/2}) \, d\mu_{ni} + \sum_{i=1}^n \int_{q_{ni} \geqq c_n p_{ni}} p_{ni} \, d\mu_{ni} \right\}$$

$$\leqq c_n^{-1/2} \left\{ \sum_{i=1}^n \int_{q_{ni} \geqq c_n p_{ni}} (q_{ni}^{1/2} - p_{ni}^{1/2})^2 \, d\mu_{ni} + \sum_{i=1}^n \int_{q_{ni} \geqq c_n p_{ni}} p_{ni} \, d\mu_{ni} \right\}$$

$$\leqq c_n^{-1/2} \left\{ \sum_{i=1}^n H^2(P_{ni}, Q_{ni}) + \sum_{i=1}^n \int_{q_{ni} \geqq c_n p_{ni}} p_{ni} \, d\mu_{ni} \right\}.$$

Since by (2.2) $c_n^{-1/2} \sum_{i=1}^n H^2(P_{ni}, Q_{ni}) \to 0$ for $n \to \infty$, it follows that $\lim_{n\to\infty} \sum_{i=1}^n P_n(L_{ni} \geqq c_n) = 0$. Hence (2.1) implies that $\lim_{n\to\infty} \sum_{i=1}^n Q_{ni}(L_{ni} \geqq c_n) = 0$ and the proof of the theorem is complete. □

Corollary 1. $\{P_n^{(n)}\} \lhd \rhd \{Q_n^{(n)}\}$ *iff* (2.2) *and* (2.3) *are satisfied and*

(2.5) $$\lim_{n\to\infty} \sum_{i=1}^n P_{ni}(p_{ni}(X_{ni})/q_{ni}(X_{ni}) \geqq c_n) = 0 \quad \textit{whenever} \quad c_n \to \infty .$$

In connection with contiguity Hellinger distance seems to be a more appropriate metric than total variation distance. Note that from (1.3) and (1.6) we immediately obtain the implication

(2.6) $$\{Q_n^{(n)}\} \lhd \{P_n^{(n)}\} \Rightarrow \sum_{i=1}^n \|P_{ni} - Q_{ni}\|^2 = O(1) \quad \text{for} \quad n \to \infty ,$$

where again the order term cannot be strenghtened to $o(1)$. However, $\sum_{i=1}^n \|P_{ni} - Q_{ni}\|^2 = O(1)$ is too weak a condition to replace (2.2) in Theorem 1. On the other hand we cannot strengthen this condition to $\sum_{i=1}^n \|P_{ni} - Q_{ni}\|^r = O(1)$ for some $r < 2$, since $\{Q_n^{(n)}\} \lhd \{P_n^{(n)}\}$ does not necessarily imply $\sum_{i=1}^n \|P_{ni} - Q_{ni}\|^r = O(1)$ for any positive $r < 2$. The following example serves to illustrate these points.

Example. Let μ_{ni} denote Lebesgue measure on $(0,1)$, let $p_{ni} = 1_{(0,1)}$ and let $q_{ni} = (1 + n^{-1/2})\, 1_{(0,1-n^{-1/2})} + n^{-1/2}\, 1_{[1-n^{-1/2},1)}$, $i = 1, \ldots, n$; $n = 1, 2, \ldots$. Then $\sum_{i=1}^{n} \|P_{ni} - Q_{ni}\|^2 = (1 - n^{-1/2})^2 \leqq 1$ and (2.3) is trivially satisfied since q_{ni}/p_{ni} is uniformly bounded. But $\{Q_n^{(n)}\} \lhd \{P_n^{(n)}\}$ does not hold because $\sum_{i=1}^{n} H^2(P_{ni}, Q_{ni}) = 2n\{1 - \int q_{ni}^{1/2}\, d\mu_{ni}\} = 2n\{1 - (1 + n^{-1/2})^{1/2}(1 - n^{-1/2}) - n^{-3/4}\} = n^{1/2}(1 + o(1))$ for $n \to \infty$.

Taking $q_{ni} = (1 + n^{-1/2})\, 1_{(0,1/2)} + (1 - n^{-1/2})\, 1_{[1/2,1)}$ for all i and n, we have $\{Q_n^{(n)}\} \lhd \{P_n^{(n)}\}$ since (2.3) is satisfied and $\sum_{i=1}^{n} H^2(P_{ni}, Q_{ni}) = 2n\{1 - \frac{1}{2}(1 + n^{-1/2})^{1/2} - \frac{1}{2}(1 - n^{-1/2})^{1/2}\} = \frac{1}{4} + o(1)$ for $n \to \infty$. However, in this case $\sum_{i=1}^{n} \|P_{ni} - Q_{ni}\|^r = n(\frac{1}{2}n^{-1/2})^r \to \infty$ for $n \to \infty$ if $r < 2$.

3. Asymptotic normality of Λ_n

Define

$$\Lambda_n = \sum_{i=1}^{n} \log \{q_{ni}(X_{ni})/p_{ni}(X_{ni})\}, \quad n = 1, 2, \ldots. \tag{3.1}$$

Note that, with probability one, Λ_n is well-defined under $P_n^{(n)}$, although Λ_n may assume the value $-\infty$ with positive probability under $P_n^{(n)}$.

In our search for necessary and sufficient conditions for the weak convergence $\mathscr{L}(\Lambda_n \mid P_n^{(n)}) \to_w \mathscr{N}(-\frac{1}{2}\sigma^2; \sigma^2)$ in terms of the marginal distributions of the X_{ni} we shall confine ourselves to the case where the summands in (3.1) satisfy the traditional u.a.n. condition (cf. Loève [5]).

Theorem 2. *For any* $\sigma \geqq 0$.

$$\mathscr{L}(\Lambda_n \mid P_n^{(n)}) \to_w \mathscr{N}(-\tfrac{1}{2}\sigma^2; \sigma^2) \tag{3.2}$$

and

$$\lim_{n\to\infty} \max_{1 \leqq i \leqq n} P_{ni}(|\log \{q_{ni}(X_{ni})/p_{ni}(X_{ni})\}| \geqq \varepsilon) = 0 \tag{3.3}$$

for every $\varepsilon > 0$ *iff for every* $\varepsilon > 0$

$$\lim_{n\to\infty} \sum_{i=i}^{n} H^2(P_{ni}, Q_{ni}) = \tfrac{1}{4}\sigma^2, \tag{3.4}$$

(3.5) $$\lim_{n\to\infty} \sum_{i=1}^{n} Q_{ni}(q_{ni}(X_{ni})/p_{ni}(X_{ni}) \geqq 1 + \varepsilon) = 0\,,$$

(3.6) $$\lim_{n\to\infty} \sum_{i=1}^{n} P_{ni}(p_{ni}(X_{ni})/q_{ni}(X_{ni}) \geqq 1 + \varepsilon) = 0\,,$$

or equivalently, iff (3.4) *holds and for every* $\varepsilon > 0$

(3.7) $$\lim_{n\to\infty} \sum_{i=1}^{n} \int_{|q_{ni}-p_{ni}| \geqq \varepsilon p_{ni}} (q_{ni}^{1/2} - p_{ni}^{1/2})^2 \, \mathrm{d}\mu_{ni} = 0\,.$$

Proof. To simplify the notation we write $r_{ni} = q_{ni}/p_{ni}$. We first show that (3.5) and (3.6) are equivalent to (3.7). From

$$\sum_{i=1}^{n} \int_{|q_{ni}-p_{ni}| \geqq \varepsilon p_{ni}} (q_{ni}^{1/2} - p_{ni}^{1/2})^2 \, \mathrm{d}\mu_{ni}$$

$$= \sum_{i=1}^{n} \left\{ \int_{r_{ni} \geqq 1+\varepsilon} q_{ni}(1 - r_{ni}^{-1/2})^2 \, \mathrm{d}\mu_{ni} + \int_{r_{ni} \leqq 1-\varepsilon} p_{ni}(1 - r_{ni}^{1/2})^2 \, \mathrm{d}\mu_{ni} \right\}$$

we obtain the double inequality

$$\{1 - (1+\varepsilon)^{-1/2}\}^2 \sum_{i=1}^{n} Q_{ni}(r_{ni}(X_{ni}) \geqq 1 + \varepsilon)$$

$$+ \{1 - (1-\varepsilon)^{1/2}\}^2 \sum_{i=1}^{n} P_{ni}(r_{ni}^{-1}(X_{ni}) \geqq (1-\varepsilon)^{-1})$$

$$\leqq \sum_{i=1}^{n} \int_{|q_{ni}-p_{ni}| \geqq \varepsilon p_{ni}} (q_{ni}^{1/2} - p_{ni}^{1/2})^2 \, \mathrm{d}\mu_{ni}$$

$$\leqq \sum_{i=1}^{n} Q_{ni}(r_{ni}(X_{ni}) \geqq 1 + \varepsilon) + \sum_{i=1}^{n} P_{ni}(r_{ni}^{-1}(X_{ni}) \geqq (1-\varepsilon)^{-1})$$

and the equivalence of (3.5) and (3.6) to (3.7) is immediate.

Next we note that both (3.2), (3.3) and (3.4), (3.5), (3.6) imply $\{P_n^{(n)}\} \lhd \rhd \{Q_n^{(n)}\}$ (cf. Corollary 1).

The remainder of the proof relies on the normal convergence theorem (cf. Loève [5]). According to an equivalent form of this theorem (3.2) and (3.3) are equivalent to

(3.8) $$\lim_{n\to\infty} \sum_{i=1}^{n} P_{ni}(|\log r_{ni}(X_{ni})| \geqq \delta) = 0 \quad \text{for every} \quad \delta > 0\,,$$

(3.9) $$\lim_{\delta \downarrow 0} \lim_{n\to\infty} \sum_{i=1}^{n} \int_{|\log r_{ni}| \leqq \delta} (\log r_{ni}) \, \mathrm{d}P_{ni} = -\tfrac{1}{2}\sigma^2\,,$$

$$\lim_{\delta \downarrow 0} \lim_{n\to\infty} \sum_{i=1}^{n} \left\{ \int_{|\log r_{ni}| \leqq \delta} (\log r_{ni})^2 \, \mathrm{d}P_{ni} - \left(\int_{|\log r_{ni}| \leqq \delta} (\log r_{ni}) \, \mathrm{d}P_{ni} \right)^2 \right\} = \sigma^2\,.$$
(3.10)

By the contiguity of $\{P_n^{(n)}\}$ and $\{Q_n^{(n)}\}$ and (2.1) the condition (3.8) is equivalent to (3.5) and (3.6) and hence to (3.7). Henceforth we assume (3.7), (3.8) and $\{P_n^{(n)}\} \lhd\rhd$ $\lhd\rhd \{Q_n^{(n)}\}$. We still have to show that (3.4) is equivalent to (3.9) and (3.10).

Let $0 < \delta < 1$. For $|\log r_{ni}| \leqq \delta$ we have the expansion

$$\begin{aligned}\log r_{ni} &= 2 \log \{1 + (q_{ni}^{1/2} - p_{ni}^{1/2})\, p_{ni}^{-1/2}\} \\ &= 2(q_{ni}^{1/2} - p_{ni}^{1/2})\, p_{ni}^{-1/2} - (q_{ni}^{1/2} - p_{ni}^{1/2})^2\, p_{ni}^{-1}(1 + \varrho_{ni\delta})\end{aligned} \tag{3.11}$$

with $|\varrho_{ni\delta}| < 2\delta$. Thus

$$\begin{aligned}&\int_{|\log r_{ni}| \leqq \delta} (\log r_{ni})\, p_{ni}\, \mathrm{d}\mu_{ni} \\ &= -2 \int_{|\log r_{ni}| \leqq \delta} (q_{ni}^{1/2} - p_{ni}^{1/2})^2\, \mathrm{d}\mu_{ni} + \int_{|\log r_{ni}| \leqq \delta} (q_{ni} - p_{ni})\, \mathrm{d}\mu_{ni} \\ &\quad - \int_{|\log r_{ni}| \leqq \delta} \varrho_{ni\delta}(q_{ni}^{1/2} - p_{ni}^{1/2})^2\, \mathrm{d}\mu_{ni}\,.\end{aligned}$$

Since by (3.7)

$$\lim_{n\to\infty} \left\{\sum_{i=1}^{n} \int_{|\log r_{ni}| \leqq \delta} (q_{ni}^{1/2} - p_{ni}^{1/2})^2\, \mathrm{d}\mu_{ni} - \sum_{i=1}^{n} H^2(P_{ni}, Q_{ni})\right\} = 0$$

and by (3.8), $\{P_n^{(n)}\} \lhd\rhd \{Q_n^{(n)}\}$ and (2.1)

$$\sum_{i=1}^{n} \int_{|\log r_{ni}| \leqq \delta} (q_{ni} - p_{ni})\, \mathrm{d}\mu_{ni} = -\sum_{i=1}^{n} \int_{|\log r_{ni}| > \delta} (q_{ni} - p_{ni})\, \mathrm{d}\mu_{ni} \to 0$$

for $n \to \infty$, we have

$$\begin{aligned}&\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \left|\sum_{i=1}^{n} \int_{|\log r_{ni}| \leqq \delta} (\log r_{ni})\, \mathrm{d}P_{ni} + 2\sum_{i=1}^{n} H^2(P_{ni}, Q_{ni})\right| \\ &\leqq \lim_{\delta\downarrow 0} \limsup_{n\to\infty} 2\delta \sum_{i=1}^{n} H^2(P_{ni}, Q_{ni}) = 0\,,\end{aligned} \tag{3.12}$$

where we have used (1.6). Similarly,

$$\begin{aligned}&\lim_{\delta\downarrow 0} \limsup_{n\to\infty} \sum_{i=1}^{n} \left\{\int_{|\log r_{ni}| \leqq \delta} (\log r_{ni})\, \mathrm{d}P_{ni}\right\}^2 \\ &\leqq \lim_{\delta\downarrow 0} \limsup_{n\to\infty} \delta \sum_{i=1}^{n} \left|\int_{|\log r_{ni}| \leqq \delta} (\log r_{ni})\, \mathrm{d}P_{ni}\right| \\ &\leqq \lim_{\delta\downarrow 0} \limsup_{n\to\infty} \delta(2 + 2\delta) \sum_{i=1}^{n} H^2(P_{ni}, Q_{ni}) = 0\,.\end{aligned} \tag{3.13}$$

Finally (3.11) implies that for $|\log r_{ni}| \leqq \delta < 1$

$$(\log r_{ni})^2 = 4(q_{ni}^{1/2} - p_{ni}^{1/2})^2 p_{ni}^{-1} + \bar{\varrho}_{ni\delta}(q_{ni}^{1/2} - p_{ni}^{1/2})^2 p_{ni}^{-1}$$

with $|\bar{\varrho}_{ni\delta}| < 10\delta$. Hence, in view of (3.7) and (1.6),

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \left| \sum_{i=1}^{n} \int_{|\log r_{ni}| \leqq \delta} (\log r_{ni})^2 \, dP_{ni} - 4 \sum_{i=1}^{n} H^2(P_{ni}, Q_{ni}) \right| = 0 . \tag{3.14}$$

The equivalence of (3.4) to (3.9) and (3.10) is now an immediate consequence of (3.12), (3.13) and (3.14). The theorem is proved. □

In the one sample case where, for each n, $X_{n1}, \ldots, X_{nn}$ are identically distributed, condition (3.3) is implied by (3.2) and Theorem 2 slightly simplifies. This remains true in the k sample case ($k \geqq 2$) provided all sample sizes tend to infinity.

The first part of the proof of Theorem 2 also shows that the conditions (2.3) and (2.5) in Corollary 1 may be replaced by the single condition

$$\lim_{n\to\infty} \sum_{i=1}^{n} \int_{|q_{ni} - p_{ni}| \geqq c_n p_{ni}} (q_{ni}^{1/2} - p_{ni}^{1/2})^2 \, d\mu_{ni} = 0 \quad \text{whenever} \quad c_n \to \infty .$$

The proof of Theorem 2 could also be given in a more roundabout way. Introducing the r.v.'s

$$W_{ni} = 2\{q_{ni}(X_{ni})/p_{ni}(X_{ni})\}^{1/2} - 2 , \quad i = 1, \ldots, n ; \quad n = 1, 2, \ldots ,$$

one shows that $\mathscr{L}(\sum_{i=1}^{n} W_{ni} \mid P_n^{(n)}) \to_w \mathscr{N}(-\frac{1}{4}\sigma^2; \sigma^2)$ iff $\mathscr{L}(\Lambda_n \mid P_n^{(n)}) \to_w \mathscr{N}(-\frac{1}{2}\sigma^2; \sigma^2)$, provided the respective u.a.n. conditions are satisfied. It is then not difficult to prove that the weak convergence of $\sum_{i=1}^{n} W_{ni}$ and the u.a.n. condition on the summands are equivalent to (3.4) and (3.7). In this proof (3.7) appears as the Lindeberg condition in the central limit theorem applied to $\sum_{i=1}^{n} W_{ni}$.

The equivalence of both weak convergence results has first been proved by Le Cam ([3], [4]). The initial assumptions $\lim_{n\to\infty} \sup_{1 \leqq i \leqq n} H^2(P_{ni}, Q_{ni}) = 0$ and $\limsup_{n\to\infty} \|P_n^{(n)} - Q_n^{(n)}\| < 1$ made by Le Cam are not restrictive since they are implied by our condition (3.7) and the contiguity of $\{P_n^{(n)}\}$ and $\{Q_n^{(n)}\}$, respectively. One part of this proof is also contained in Hájek & Šidák [1].

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Received October 1975