Virtual Models of Conic Sections

Pavel Boytchev

Introduction

People know and use conic sections for a long time. They observe these curves in many situations: the parabolic trajectory of a thrown stone, the circular waves when the stone falls in calm water, and the elliptical shadows of round objects during sunsets (see Fig. 1).

Conic sections were and still are one of the most favorite objects of mathematical study and education. Students spend hours in the classroom working with circles, ellipses, parabolas, and hyperbolas. They are presented with a concentrated view about these curves, a view that has been distilled for hundreds of years. Although mathematically correct, this view may not lead to complete rationalization, because it might be hard for students to project mathematical ideas into something more comprehensible from their everyday life.

A preliminary informal inquiry showed that it is difficult for many students to identify conic sections in a non-classroom environment. This triggered the creation of visualization tools that could introduce these curves from various perspectives. These tools are the focus of this paper.

Traditionally, conic sections are described as intersections of a plane and a cone (Downs 1993). Searching the WWW reveals that this is the predominant description of conic sections, independent on whether materials describe mathematical concepts in simple language (like Math2.org's "A conic section is the intersection of a plane and a cone") or use scientific terminology (like Wolfram MathWorld's "The conic sections are the nondegenerate curves generated by the intersections of a plane with one or two nappes of a cone").

P. Boytchev (✉)
Sofia University, Sofia, Bulgaria
e-mail: boytchev@fmi.uni-sofia.bg

A. Jimoyiannis (ed.), *Research on e-Learning and ICT in Education*, 267
DOI 10.1007/978-1-4614-1083-6_20, © Springer Science+Business Media, LLC 2012

Fig. 1 Circular shapes of lamps in Korinthos, Greece, and their elliptical shadows

Unfortunately, "although intuitively and visually appealing, these definitions for the conic sections tell us little about their properties and uses" (Smith 2011). Additionally, these definitions do not always create transferable knowledge – i.e., knowledge that a person can use to bridge concepts from two distinct disciplines. There are observations that connecting is important to understand. According to Wageman (2010), "The more connections students can make the more interesting the topic becomes to them and then deeper understanding can occur."

Some educators are forced to make compromise by choosing only few aspects of the curves, those that "have important applications in the real world" (Demana et al. 2000). A few educational materials based on this traditional approach are being advertised as "the perfect set for teaching a unit on conic sections" (Nasco 2010).

The introduction of Dynamic Geometry Software (DGS) added a lot of expressiveness to the representation of conic sections and provided a playground for interesting explorations. However, DGS is still visually bound to the mathematical representation. DGS uses the conventional geometrical primitives that are not immediately relatable to concepts outside the educational environment. Even advanced DGS tools like Cabri (Schumann 2005) and the Geometer's Sketchpad (Scher 2003) represent conic sections in the traditional way. Although correct, 3D and interactive, these representations are just advanced variations of the schemes found in mathematical textbooks and in online math resources.

The author's own attempt to combine DGS with virtual reality also did not provide any significant impact. Figure 2 represents a snapshot of an interactive 3D application for experimenting with the traditional approach. It is less mathematical and more like a game, but still it is a cone intersected by a plane.

Another application of the author is a microworld developed for the *Developing Active Learning Environment for Stereometry* (DALEST) project which was co-funded by the European Union under the Socrates Program, MINERVA, 2005 Selection (Boytchev 2007). Partners in this project were several educational

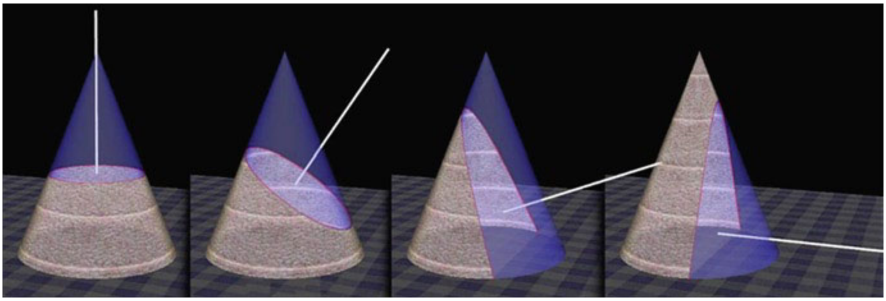

Fig. 2 Interactive 3D application for exploration of the traditional representation of conic sections

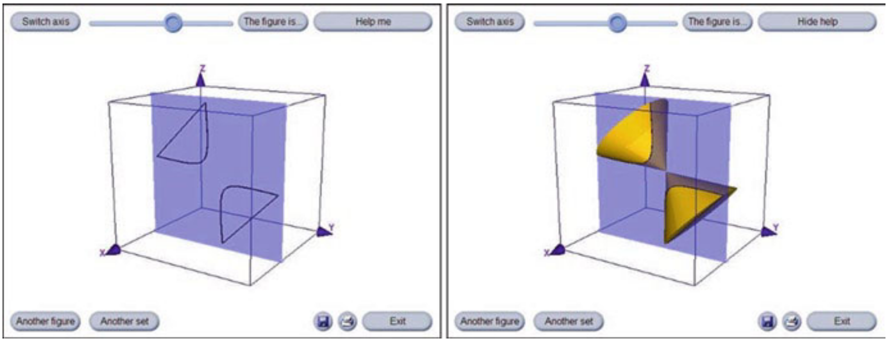

Fig. 3 Snapshots from a DALEST-Elica application for interactive exploration of the intersection of a plane and a solid

institutions across Europe. The Slider application represents the intersection of an object with a plane. By moving the plane and studying the intersection, the user has to "guess" the object. One of the subsets of activities is related to studying conic sections (Fig. 3).

Both applications (the ones shown in Figs. 2 and 3) utilized modern visualization technologies by providing the user with a game-like look-and-feel. However, like models developed by other DGS tools, the representation does not help students to resolve practical challenges like:

- Make an ellipse with a reading lamp.
- Construct a rolling mechanism that generates an ellipse.
- Draw a hyperbola using a fixed-length thread.

These applications do not provide assistance in solving the inverse problems too, like "Is this shadow a parabola or a hyperbola?" or "Where are the focal points of this ellipse?"

There are trends of bringing the reality back into Math education by constructing and using mechanical tools that relate to mathematical concepts. And interesting work is the construction of LEGO mechanisms that function as physical representation for mathematics and mathematical inquiry (Isoda et al. 2001)

The idea that initiated the work presented in this paper is to create new tools and models that describe and utilize various properties of conic sections. The main features of these tools are to present the properties of the conic sections in a way that is:

- *Unique*: The tools have to demonstrate conic sections form a perspective that is inherently unavailable in traditional hard-copy textbooks and is still difficult to implement in contemporary systems of dynamic geometry.
- *Attractive*: The tools should use virtual reality, game-like 3D models and interactive interfaces to build and then to support the student's interest in conic sections. Such attention to the visual appearance is important in order to minimize the gap between a "boring" topic in mathematics and the out-of-school entertainment.
- *Natural*: The tools should represent ideas that can be immediately related (and even applied) to real-life situations and at the same time to be still mathematically correct.

Homemade Conic Sections with Light

In 2009 the author completed an artistic project–exhibition based on computer generated images and digital photographs (Boytchev 2009). All posters feature fragments of the Mandelbrot set fractal accompanied by artistic interpretations. One of the posters depicts an area from the fractal that resembles a coordinate system with a pair of hyperbolas and their asymptotes (see Fig. 4).

The description of the poster says: *The hyperbola might have been discovered by Menaechmus, a tutor of Alexander the Great. One hundred years later Apollonius named the ellipse, parabola and hyperbola. There are many ways to construct a conic section, but the easiest one is with a table, a ball and a torch. How?* Apparently, it appears that the ball is not necessary in order to generate all types of conic sections.

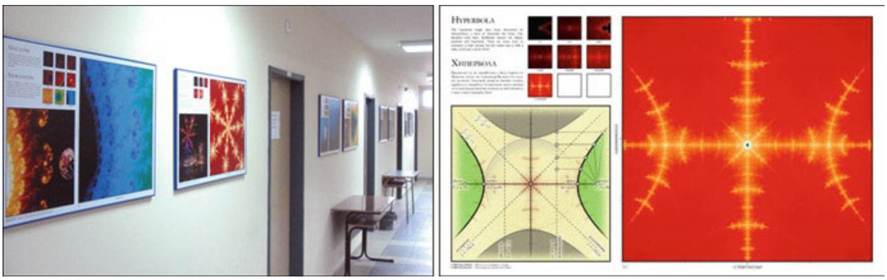

Fig. 4 The fractal exhibition and the hyperbolic shapes in the Mandelbrot set fractal

Fig. 5 All four types of conic sections generated with objects at hand

The question in the poster raises an interesting problem: Is it possible to model all conic sections at home, using only objects from our everyday life? Is it possible to classify conic sections produced in this way? To answer these questions a set of interactive 3D applications are implemented by the author. They are based on real-life "experiments" and some of them were demonstrated at the Spring Conference of the Union of Bulgarian Mathematicians in 2010. Figure 5 shows several "experiments" conducted in a hotel room using available objects and without any preliminary preparation. The "experiments" provide enough data for students to determine why the light reflection in the first photograph is a parabola, while the light in the last one makes a hyperbola.

These experiments inspired the construction of a set of interactive 3D applications. They are designed and developed within the scope of the project InnoMathEd – *Innovations in Mathematics Education on European Level* (http://www.math.uni-augsburg.de/prof/dida/innomath). Partners in this project are University of Augsburg, Bulgarian Academy of Sciences, University of South Bohemia, University of Bayreuth, Projekt Bildung Institut, German School Board Bolzano, University of Cyprus, Tyrolean Educational Service, University of Cambridge, and University of Oslo. The project addresses pupils' mathematical understanding, use of ICT and competences for lifelong learning (Bianco 2009).

The first application in the set re-explores the traditional approach that involves a cone and an intersecting plane. Snapshots of this program are shown in Fig. 2. Although the software provides an intuitive and easy-to-understand way for describing conic sections, it does not help students implement the model in real life. Instead, the model is suitable only for virtual experiments. Most of the other applications, however, are designed in a way that their ideas can be re-implemented and re-acted at home – this is crucial to our goal of having mathematical knowledge that is transferrable and applicable outside the classroom.

The next few applications in the set model just a torch and a table. The light from the torch forms a cone, while the table is an intersecting plane. The image on the table surface is the intersection of the light cone and the plane. It is straightforward to create a circle or an ellipse. However, is it also possible to generate parabolas and hyperbolas, in spite of the fact that they extend to infinity – see Fig. 6.

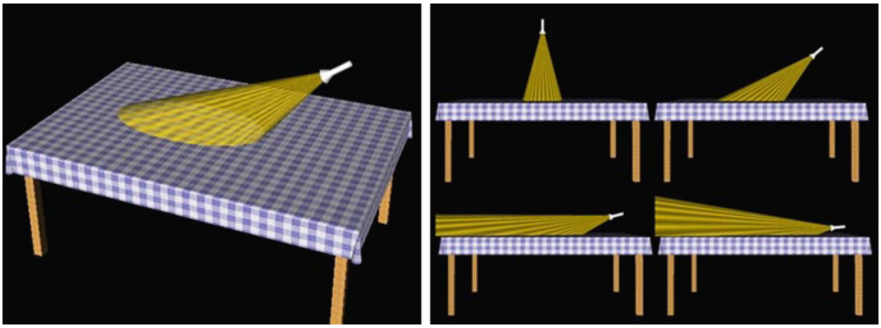

Fig. 6 Making an ellipse (*left*) and all conic sections (*right*) using a torch

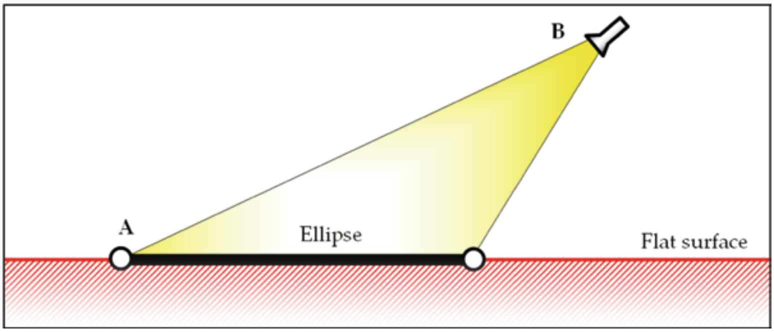

Fig. 7 Generation of an ellipse

There are simple rules that "predict" the type of the curve. These rules have a mathematical background, but they can be understood and applied by students with insufficient mathematical skills. The slope of the upper side of the light cone (segment AB in Fig. 7) determines the type of the curve. It is an ellipse (or a circle) if A is below B, a hyperbola if A is above B, and a parabola if AB is horizontal.

It is possible to rephrase the rules – if point A is below the horizon, we have an ellipse (or a circle), if A is above the horizon – hyperbola, and if it exactly at the horizon – we have generated a parabola.

The torch in the model is used to generate a light cone. If we have a traditional electrical bulb, it emits light in all directions. Yet, we are still able to generate conic sections. Figure 8 (left) shows the elliptical shadow generated by another 3D application. By moving the light, we can generate all conic sections. Again, the rules are simple. A parabola appears when the bulb is at the same level as the top of the ball, a hyperbola – when it is below it.

The position of the ball has an important mathematical meaning. The point of contact with the table is a focal point (focus) and the ball is a Dandelin sphere (Kendig 2005; Weisstein 2010). The right snapshot in Fig. 8 shows a similar application where two balls produce hyperbolic shadow and at the same time they play

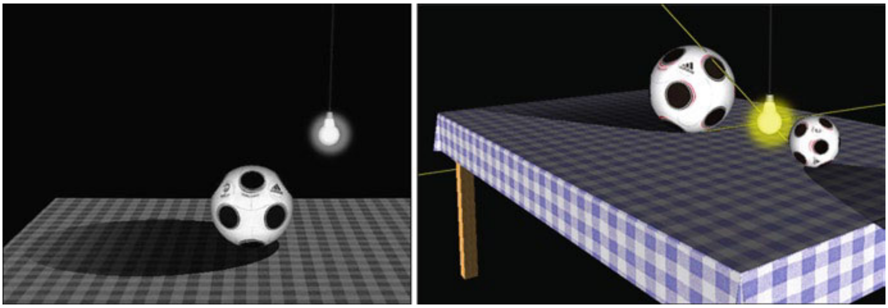

Fig. 8 Using shadows to generate an ellipse (*left*) and hyperbola (*right*)

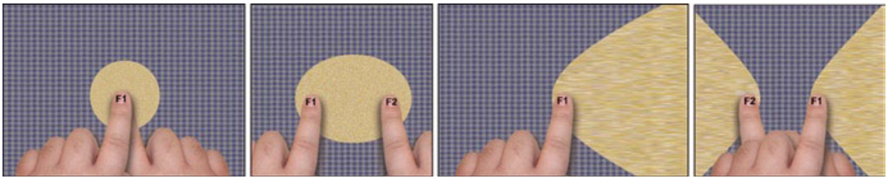

Fig. 9 The interactive pizza application

the role of two Dandelin spheres. A quick examination of the online visual resources shows that the predominant representation of Dandelin spheres is that they touch the foci of an ellipse. However, the application provides all options – students can immediately explore the Dandelin spheres of all types of conic sections.

There are a few other interactive 3D applications in the set. One of them uses just a tube. If a student looks the ground through it, the cone of sight will “cut” a conic section from the ground. The simple identification rules could be based on the horizon. If the student sees only ground – this is an ellipse. If the horizon cuts through the sight – it is a hyperbola.

The last application in the set illustrates the transition from one conic curve to another by using a virtual pizza that can be digitally deformed by pulling the focal points apart. Initially, the pizza is circular and both foci coincide – Fig. 9. If the student moves a finger to the right, the pizza becomes an ellipse. If the focus is dragged to infinity, the pizza will be a parabola. And, finally, when that hand goes beyond infinity, it “wraps” through the other side of the screen, we will produce a hyperbolic pizza.

Virtual Models of Mechanical Devices

Except for the applications using the light or the absence of light to model conic sections, the author has implemented a rich set of non-interactive 3D applications that represent models of mechanical devices drawing conic curves (Boytchev 2010a).

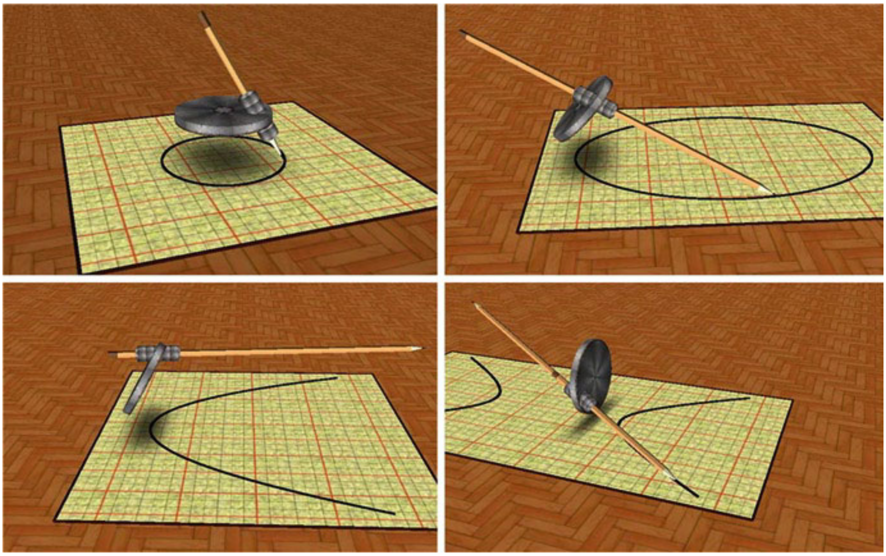

Fig. 10 Pencil attached to a disk can draw any conic section

These devices represent various methods and could foster interesting mathematical research activities for students. All devices are built by a set of primitive elements including a pencil that draws on paper.

It is easy and straightforward to model a device drawing a circle; however we should think of a more enhanced device if we need all conic sections. Figure 10 presents snapshots of such device, where the pencil can slide forward and backward controlled only by gravity.

Some of the mathematical problems occurring during the design of the animations were:

- How long should the pencil be (so that it will not slide off its holder)?
- How to model the infinities of parabolas (using finite objects)?
- How to draw both branches of hyperbolas (within a single device)?

The model in Fig. 10 recreates the conical nature of conic sections – the pencil rolls on the surface of an invisible cone (defined by the angle between the disk and the pencil) while the paper acts as an intersecting plane. The gravity forces the pencil to slide forward as much as needed to reach the paper, while the paper, itself, pushes it back.

As seen in the third snapshot, there is one specific case when the pencil becomes horizontal in its upmost position. This is the case of drawing a parabola. The tip of the pencil points to infinity (or to the horizon, if we consider the model in Figs. 6 and 7).

When the disk is vertical, there are situations when the tip of the pencil points upwards. In such cases the gravity pulls the pencil down and it touches the paper with its “back.” If we use a double-sided pencil, in which both ends can draw, we

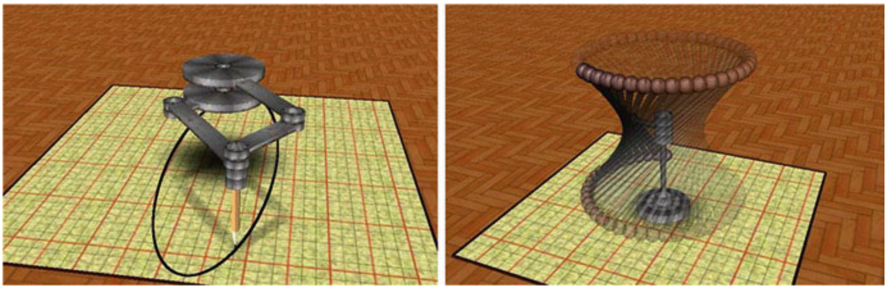

Fig. 11 Other rotational methods to generate some conic sections

will get the pencil to draw one of the hyperbola branches with one of the ends, and the other branch with the other end. This construction is shown in the last snapshot in Fig. 10. Animations of this device can be seen in YouTube playlist "Mathematical devices" (Boytchev 2010a).

The disk rotation used in the model is essential for the formation of the conic curves, but is not the only way to make them with rotation. The sum of two vectors with different lengths rotating at the same angular speed but in opposite directions is a vector that traverses an ellipse; see Fig. 11 (left). The two shorter beams represent one of the vectors; the longer ones represent the other. The construction uses a parallelogram to maintain mechanical stability and to demonstrate the commutative nature of vector addition.

The right snapshot in Fig. 11 represents a ruled surface called hyperboloid. It is created by a tilted line rotated around a vertical axis. The intersection of the hyperboloid and a horizontal plane is a circle, while the intersection with a vertical plane is a hyperbola.

Virtual Models of Existing Mechanical Devices

The models shown in Figs. 10 and 11 are not quite practical in the sense that they are suitable for generation of imaginary curves, but cannot be used for making tangible conic sections. For example, it is hard to use these devices to make elliptical windows. Carpenters have solved this problem by using a simple yet effective device called the Trammel of Archimedes (Apostol and Mnatsakanian 2009). This device can draw an ellipse with predefined major and minor axes. It also provides sufficient precision for carpentry.

Variations of the trammel are shown in Fig. 12. A fixed length segment slides along two perpendicular pairs of rails while the pencil is attached near the segment's midpoint (if the pencil is exactly in the midpoint, it will draw a circle). In reality, the devices used by carpenters are slightly different. For example, the pencil is often

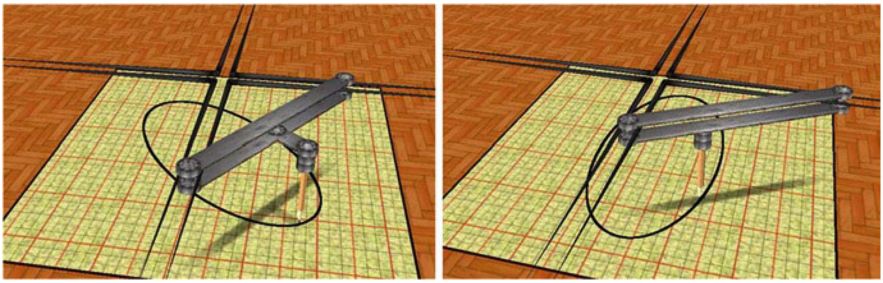

Fig. 12 Variations of the Trammel of Archimedes

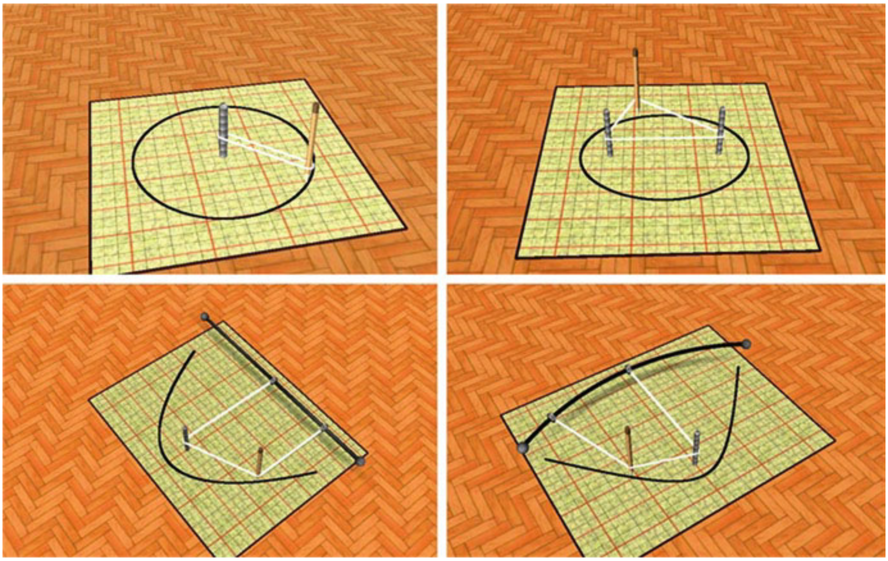

Fig. 13 A thread used to draw conic sections

attached on an extension of the sliding segment. This allows the carpenters to use much shorter rails.

The Trammel of Archimedes can draw any curve from a straight line to a circle, traversing through ellipses with various eccentricities. Unfortunately, it cannot draw a parabola or a hyperbola. Fortunately, there is another ancient method of drawing ellipses based on the property that the sum of distances from any point on the ellipse to its foci is constant. Figure 13 shows snapshots of animations visualizing all conic curves generated with a fixed-length thread. The top two cases are well known and are included just for completeness. An interesting challenge is to design a similar

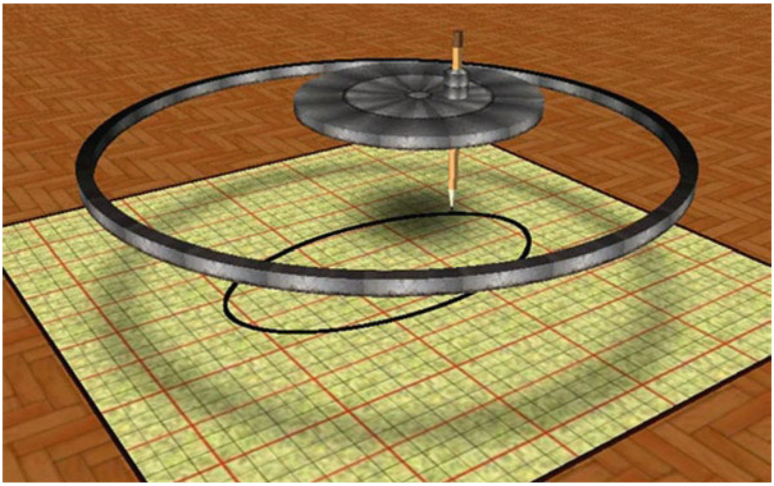

Fig. 14 A hypotrochoidal ellipse

mechanism for parabolas and hyperbolas, as long as almost all hard copy and online textbooks show only the case with an ellipse.

The lower left snapshot shows a parabola. The thread has both of its ends freely attached to a rail that is collinear to the parabola's directrix. These ends can slide along the rail keeping both sides of the thread perpendicular to the rail. The midsection of the thread embraces the pencil and the fixed focus point. A similar device is shown in the last snapshot. It has the same structure, except that the rail is a circular arc. In this case, the pencil draws a hyperbola. It is a nice mathematical exercise to prove that the curve is really a hyperbola.

The topic of modeling the construction of conic intersections is virtually unlimited. Figure 14 represents a hypotrochoidal device, where a disk rolls inside a ring. The radii of the disk and the ring are selected in such a way, that the attached pencil draws an ellipse.

Student Activities

The interactive 3D applications and the 3D animations presented in this paper are included in a set of more than 60 models of devices. Some of them draw mathematical curves, other represent mathematical transformations, construction of 3D surfaces, and even non-geometrical phenomena like normal distribution in statistics.

All these applications are rather new and they are still not used in the classroom or at home. Teaching materials are now being prepared that will utilize the full multidisciplinary power of the models. The rest of the section describes briefly some potential activities.

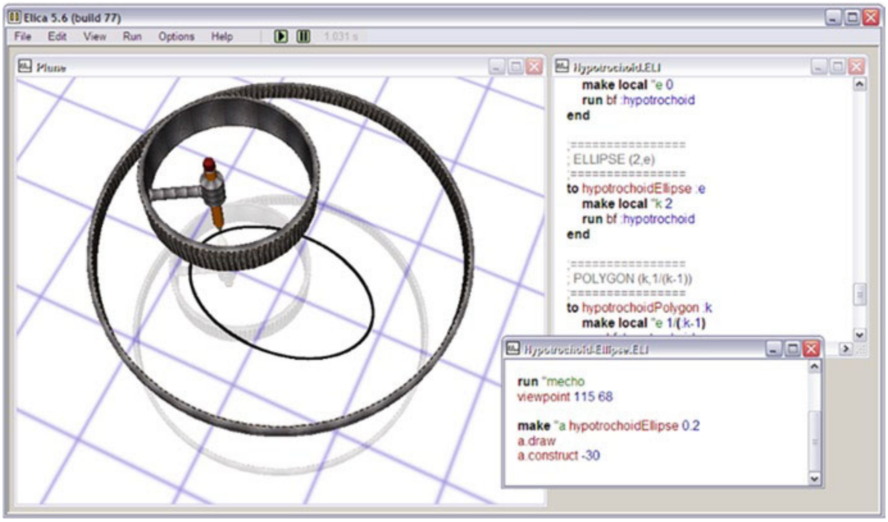

Fig. 15 A snapshot of the environment using the library for virtual mechanics

Computer Science/Computer Graphics

Each model is a program written in Elica Logo programming language. The design and the implementation of such program require skills and knowledge. Some models are very simple from programming point of view, others are quite complex. The building of precisely selected series of models can improve these skills and amass knowledge. The internal structure of models is based on concepts of the Object-oriented programming.

The power of having a complete programming control over the model allows the student to create new virtual mechanical elements. This is something which is impossible in closed software environments and in the LEGO-based models of mechanisms where the user can operate only a limited set of parts.

Physics: Mechanics

The virtual models included in the collection are not just mathematical abstractions. They comply with known physical restrictions and utilize the properties of components made of different materials. A thread can change its shape, while a solid beam cannot.

A library with virtual mechanical parts is now under development (see Fig. 15). Its purpose is simplify the construction of virtual models in non-programming contexts. The parameters of each part are customizable by the students, which only

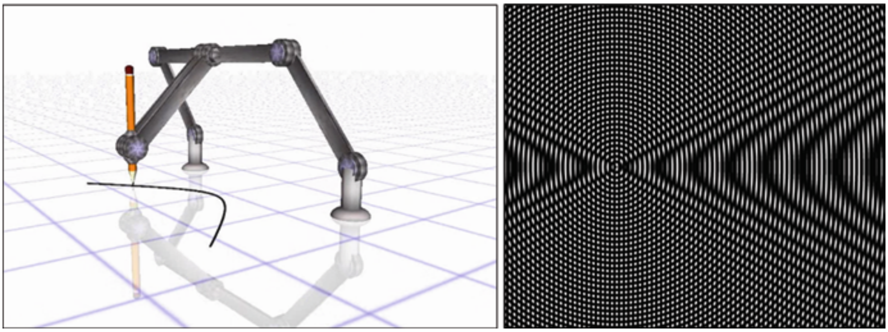

Fig. 16 Are these hyperbolas?

need to pick the desired parts and define their behavior. The first classroom application of this library is in the 2011 spring semester of the undergraduate course "Geometry of Motion" at Sofia University. Students are expected to "invent" new devices and then to implement them virtually using the library.

Mathematics: Geometry

There are many papers and web sites describing the properties of conic sections. However, the construction of virtual devices requires understanding of these properties at a higher level of abstraction. Converting a mathematical model into a mechanical one is a real mathematical and engineering challenge. Besides, mathematical challenges are hidden in the models themselves. They can be used to teach mathematical concepts and be used as an experimental playground for various tasks.

Figure 16 (left) illustrates a mathematical problem – to find whether the curve generated by the device is a hyperbola or it is just looking like a hyperbola. Also, if it draws hyperbola, could we customize the dimensions of the mechanical parts, so that it draws a degenerate hyperbola – a pair of intersecting lines? Figure 16 (right) shows a moiré pattern (Strong 1964). Is it a family of hyperbolas at various eccentricities?

Art: Animation

The collection of animations inspired the author to create a mathematical film that illustrates seven different ways of constructing ellipses (Boytchev 2010b). Although purely artistic, the making of the film was based on solving many geometrical, mechanical, and programming problems. Students may also be engaged in similar multidisciplinary activities, where, they blend scientific and artistic designs while constructing an artifact of their choice.

The InnoMathEd project provided a home for the collection and it is a nice environment to create and evaluate teaching/learning activities. The adoption of techniques from virtual reality and gaming edutainment makes the animations and the applications more appealing.

Conclusion and Future Plans

The set of virtual models and the library described in this paper are still under construction. Most likely they will never be finished as long as new models and devices are continuously being added. All models are available for free both as source codes and as 30-s clips.

The first classroom evaluation of the proposed interactive 3D models will be completed by mid-2011 within the InnoMathEd project and the results will be reported in future papers. The work described in this paper is focused solely on the design and the development of the software tools. The pedagogical aspects of their application will be researched in the next phase of the project.

The software by itself does not imply that it is effective for educational purposes, because technology in education is educationally neutral. Whether its use is effective or not in the classroom, depends on how it is used. In this respect, the presented software provides educational perspectives that are not feasible in the traditional mathematical textbooks and are not used by the modern DGS. Whether these perspectives will be utilized, is related to the actual application in the classroom.

The major expected benefits of using the presented models are: (1) to provide multiple real-life representations of conic sections; (2) to demonstrate the main properties of conics in a clear way; and (3) to build engaging and entertaining virtual environment, which utilizes the power of ICT in areas, where conventional textbooks fail.

Acknowledgments The paper is partly supported by Grant 162/2010 from Sofia University Research Fund.

References

Apostol, T., & Mnatsakanian, M. (2009). A new look at the so-called trammel of Archimedes. *American Mathematical Monthly*, 116(2), 115–133.

Bianco, T. (2009). *Needs analysis report for InnoMathEd - Innovations in Mathematics education*. Augsburg: University of Augsburg.

Boytchev, P. (2007). Enhancing spatial imagination of young students by activities in 3D ELICA applications. In P. Boyvalenkov & P. Rusev (eds.), *Proceedings of the 36th Spring Conference of the Union of Bulgarian Mathematicians* (pp. 109–119), Varna: Union of Bulgarian Mathematicians.

Boytchev, P. (2009). *Exhibition "Seduction"*. Retrieved 23 April 2011 from http://mandelbrot-set.elica.net.

Boytchev, P. (2010a). *Mathematical devices playlist at ElicaTeam's Channel*. Retrieved 23 April 2011 from http://www.youtube.com/elicateam#g/c/6534E936D46257BF.

Boytchev, P. (2010b). *Mathematical film "Ellipses…"*. Retrieved 22 April 2011 from http://www.youtube.com/watch?v=1v5Aqo6PaFw.

Demana, F., Waits, B., Foley, G., & Kennedy, D. (2000). *Precalculus: Functions and Graphs.* Chapter 8, Boston: Addison Wesley Longman.

Downs, J. (1993). *Practical Conic Sections: The Geometric Properties of Ellipses, Parabolas and Hyperbolas.* Palo Alto: Dale Seymour Publications.

Isoda, M., Suzuki, A., Ohneda, Y., Sakamoto, M., Mizutani, N., Kawasaki, N., Morozumi, T., Katajima, S., Hiroi, N., Aoyama, K., Matsuzaki, A. (2001). LEGO project – Mediational means for mathematics by mechanics. *Tsukuba Journal of Educational Study in Mathematics*, 20, 77–92.

Kendig, K. (2005). *Conics*. Cambridge: Cambridge University Press.

Wageman, J. (2010). *Conic Sections.* The North Dakota Curriculum Initiative. Retrieved 22 April 2011 from http://ndcurriculuminitiative.org/index.php/lesson_bank/lesson/conic_sections.

Nasco (2010). *Conic Sections Set ~ Conic Sections*. Product web page. Retrieved 29 December 2010 from http://www.enasco.com/product/TB23712T.

Scher, D. (2003). *Exploring Conic Sections with The Geometer's Sketchpad (Version 4)*. Berkeley: Key Curriculum Press

Schumann, H. (2005). Introduction to Conics with Cabri 3D. In K. Kwok (ed.), *Proceedings of 20th Mathematics Education conference* (pp. 25–38). Hong Kong: Hong Kong Institute of Mathematics Education.

Smith, B. (2011). *Conics*. Math Academy. Retrieved 22 April 2011 from http://www.mathacademy.com/pr/prime/articles/conics/index.asp.

Strong, C. (1964). Moire patterns provide both recreation and some analogues for solving problems. *Scientific American*, 211(5), 134–142.

Weisstein, E. (2010). *Dandelin Spheres*. MathWorld – A Wolfram Web Resource. Retrieved 29 December 2010 from http://mathworld.wolfram.com/DandelinSpheres.html.