Chapter 12
A U-Tank Control System for Ships in Parametric Roll Resonance

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12.1 Introduction

To control parametric resonance, there are two basic approaches; use control force to counter-act the unwanted motion, or ensure that the system's parameters are in such a state that parametric resonance cannot occur. We can call these methods *direct* and *indirect*, respectively. The difference is perhaps best explained by analyzing a differential equation.

A simple model for parametric resonance in ships is the *Mathieu equation*

$$m_{44}\ddot{\phi} + d_{44}\dot{\phi} + \left[k_{44} + k_{\phi t}\cos(\omega_{\mathrm{e}} t + \alpha_{\phi})\right]\phi = u_{\mathrm{c}}$$

where ϕ is the roll angle, u_{c} an externally applied torque and the parameters are constant. The system is known to parametrically resonate when $\omega_{\mathrm{e}} \approx 2\sqrt{k_{44}/m_{44}}$. With indirect control, ω_{e} is dynamically changed so that ϕ will not parametrically resonate. With direct control, u_{c} is used to set up a counter-moment to force the system to zero. Direct and indirect methods can be combined, as seen in [8, 9].

As shown in Chaps. 9 and 10, it is possible to change the encounter frequency ω_{e} (which depends on the ship's speed) and thus control the system indirectly. However, in practice, this depends on very early detection and the ability of the ship to rapidly

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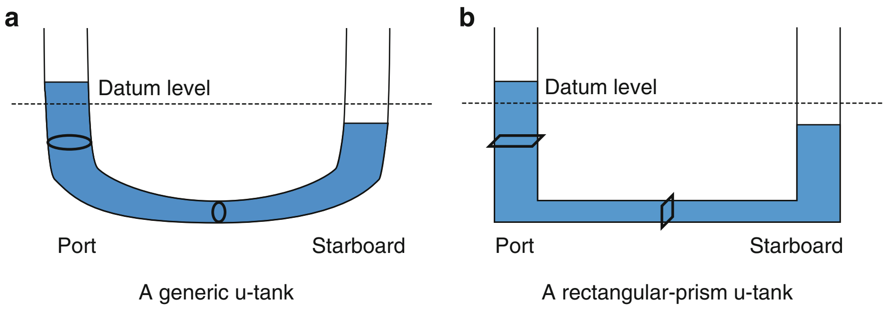

Fig. 12.1 U-tank design

perform a speed change. If the ship has high inertia or is at rest with the engines turned off, it is unlikely that the ship can change its speed fast enough to avoid large roll angles.

There are also some disadvantages associated with direct control. Ships are often not equipped with actuation in roll, as such systems are not necessary for propulsion [5]. Possible actuators include fins, tanks, and gyro stabilizers [21]. In this particular work, we will focus on the use of u-tanks as actuators. These have the advantage that they can be used even if the ship is at rest [18]. As they are internal, they do not increase drag. Unfortunately, they do take up space inside the hull, potentially decreasing the available space for other machinery, cargo, or passengers.

A u-tank (sometimes referred to as u-tube tank or u-shaped anti-roll tank) consists of two reservoirs, one on the starboard side and one on the port side, connected by a duct (see Fig. 12.1a). The basic principle is to use the weight and motion of the fluid to give a direct moment in roll, which can be used to counteract parametric resonance or other unwanted motion.

A disadvantage of u-tanks compared to other potential actuators, is that the roll and tank modes are tightly coupled, and only indirectly give a control moment in roll. Output stabilization (driving roll to zero) tends to not leave the tank in its equilibrium position, as seen in [12].

Most models of u-tanks are derived for tanks shaped like three connected rectangular prisms (see Fig. 12.1b) [14, 15, 17–19, 22], while several actually installed tanks do not match this shape [20–22]. A model for more generic tank shapes is therefore useful. In addition, most models are linear, and technically only valid for small roll angles [6, 11, 15, 17–19]. During parametric resonance, the roll angle can reach 40 to 50° [7, 9, 12, 13].

In this chapter, a novel nonlinear 2-DOF u-tank model is presented for an arbitrarily-shaped u-tank, and a controller that stabilizes parametric roll resonance with the aid of such a tank developed. The model is compared to existing models. The validity of the controller is proved mathematically and tested by simulation.

12.2 Preliminaries

The model will be derived using a combination of Lagrangian (analytical) and Newtonian mechanics. Initially, the tank–ship interaction (which is conservative) will be modeled using Lagrangian mechanics. Forces and moments induced by the surrounding ocean, in addition to friction and other nonconservative forces, will be modeled using Newtonian mechanics and incorporated into the conservative model.

Only roll and the motion of the tank fluid will be modeled in this chapter. We assume that the ship is not translating relative to the inertial frame.

12.2.1 Coordinate Systems

To use the Lagrangian approach, the dynamics have to be derived in an inertial reference frame [3]. The geometry of the vessel is easier to describe in a reference frame fixed to the body, but as the body is rotating, a body-fixed frame is not inertial. Therefore, we define two coordinate systems: an inertial frame fixed to the surface of the Earth,[1] and a noninertial frame fixed to the body.

The origin of the inertial reference can be placed arbitrarily. For simplicity, we let the xy-plane coincide with the mean ocean surface and the z-axis point with the gravity field. The body frame is placed at the transversal center of gravity at the calm-water water plane, with the x-axis pointing forwards, the y-axis pointing starboard and the z-axis pointing downwards, see Fig. 12.2. The ship is assumed symmetric around the xz-plane.

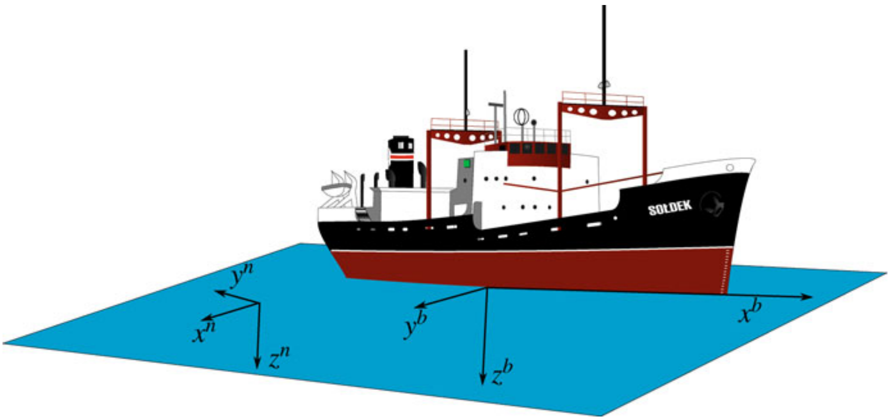

Fig. 12.2 Reference frames used in this chapter

[1]As the Earth is not inertial, clearly an Earth-fixed reference frame is not inertial. However, the effects of the non-inertial nature of the Earth's motion are small for many applications [5].

A vector $\mathbf{r}$ is denoted $\mathbf{r}^{\mathrm{n}}$ in the inertial frame and $\mathbf{r}^{\mathrm{b}}$ in the body-fixed frame. These are related by $\mathbf{r}^{\mathrm{n}} = \mathbf{R}\mathbf{r}^{\mathrm{b}} \Leftrightarrow \mathbf{r}^{\mathrm{b}} = \mathbf{R}^{\top}\mathbf{r}^{\mathrm{n}}$ where $\mathbf{R}$ is a rotation matrix [3].

12.2.2 Modeling Hypothesis

To model the ship–tank system, some assumptions and simplifications have to be made.

12.2.2.1 The Ship

The ship's motion is assumed restricted to a single degree of freedom, namely roll. It can be defined as the number ϕ so that the rotation matrix $\mathbf{R}$ can be written

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix} . \tag{12.1}$$

This is equivalent to having the body-fixed frame (and the ship with it) rotated an angle ϕ about the inertial x-axis [3]. Note that, with the ship restricted to roll, the inertial and body-fixed x-axes are parallel, and without loss of generality can be assumed to be coinciding.

We note that the ship's angular velocity relative to the inertial frame $\boldsymbol{\omega}$ is then given by

$$\boldsymbol{\omega}^n = \boldsymbol{\omega}^b = [\dot{\phi}, 0, 0]^{\top} . \tag{12.2}$$

We also make some assumptions regarding the ship and the ocean:

A12.1. The ship is port–starboard symmetric (i.e., around the body xz-plane) in mass and geometry.
A12.2. The ship is not translating relative to the inertial reference frame.
A12.3. The waves are sinusoidal, planar, and stationary.
A12.4. The wave length is approximately equal to the ship length.
A12.5. There is either head or stern seas.

By the first assumption, the ship's center of gravity (excluding the tank fluid) is given by

$$\mathbf{r}_g^b = \left[x_g^b, 0, z_g^b\right]^{\top} . \tag{12.3}$$

12.2.2.2 The Tank Fluid

A u-tank is simply two reservoirs of water or another liquid, one on the port side and the other at starboard, with a duct in between to allow the passage of liquid. To be able to model this intrinsically complicated behavior, some assumptions have to be made:

A12.6. The surface of the fluid in the tank is perpendicular to the centerline of the tank.
A12.7. The fluid in the tank is incompressible.
A12.8. The flow of fluid in the tank is one-dimensional.
A12.9. Tank fluid memory effects are negligible.
A12.10. The u-tank is placed at the transversal geometrical center of the ship.
A12.11. The tank is symmetrical around the centerline.
A12.12. There are no air bubbles in the tank.
A12.13. The centerline of the tank is smooth.
A12.14. The centerline of the tank runs port–starboard.

Assumption A12.6 is clearly false for a ship in motion; the actual fluid surface in the tank is likely to behave in a complicated and chaotic fashion. Modeling this accurately without resorting to computational fluid dynamics is unfeasible. Assuming the fluid surface to be horizontal would not be much more accurate than Assumption A12.6.

Assumptions A12.6–A12.14 imply that the tank fluid is parameterizable as a tube of varying cross-sectional area. Defining the centerline of the tube of fluid as $\mathbf{r}_\mathrm{t}(\sigma)$ with σ as parameter, $\mathbf{r}_\mathrm{t}^b(\sigma)$ can be written as

$$\mathbf{r}_\mathrm{t}^b(\sigma) = \begin{bmatrix} x_t^b \\ y_t^b(\sigma) \\ z_t^b(\sigma) \end{bmatrix} . \tag{12.4}$$

The parameter σ is defined to have its zero point at the ship centerline and positive in the port direction. The fluid surfaces are located at $\sigma = -\varsigma_\mathrm{s} \leq 0$ (starboard side) and $\sigma = \varsigma_\mathrm{p} \geq 0$ (port side). Thus, $\sigma \in [-\varsigma_\mathrm{s}, \varsigma_\mathrm{p}] \subset \mathbb{R}$ defines the fluid-filled part of the tank. When the water level is equal in both the starboard and port side reservoirs, $\varsigma_\mathrm{p} = \varsigma_\mathrm{s} = \varsigma_0$, and $\sigma \in [-\varsigma_0, \varsigma_0] \subset \mathbb{R}$ defines the fluid-filled part of the tank.

Property 12.1. $\mathbf{r}_t^b$ satisfies the following properties:

- x_t^b is a constant, per Assumption A12.14.
- The functions y_t^b and z_t^b are smooth (specifically, C^1 or greater), per Assumption A12.13.
- y_t^b is odd and lies in the second and fourth quadrants (i.e., $y_t^b(-\sigma) = -y_t^b(\sigma)$, $y_t^b(0) = 0$ and $y_t^b(\sigma) < 0 \,\forall\, \sigma > 0$), per Assumptions A12.10 and A12.11.
- z_t^b is even (i.e., $z_t^b(-\sigma) = z_t^b(\sigma)$), per Assumption A12.11.
- $\max z_t^b = z_t^b(0)$, per Assumptions A12.10 and A12.11.

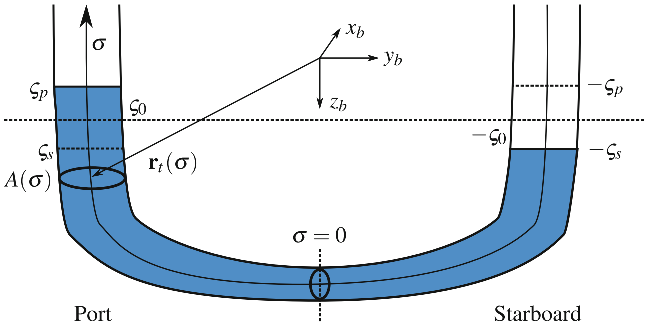

Fig. 12.3 U-tank parameters

To fully describe the tank fluid, the cross-sectional area $A(\sigma)$ is also needed.

Property 12.2. By Assumption A12.12, the fluid fills the entire area $A(\sigma)\forall\sigma \in [-\varsigma_s, \varsigma_p]$. Assumption A12.11 implies that $A(-\sigma) = A(\sigma) > 0$.

See Fig. 12.3 for an illustration of the u-tank and its parameters.

The chief physically measurable states of the system are ς_p, ς_s and the volumetric flow of the tank fluid Q (positive to port). ς_p and ς_s are related to the flow rate by

$$\dot{\varsigma}_p = \frac{Q}{A(\varsigma_p)}\,, \qquad \dot{\varsigma}_s = -\frac{Q}{A(\varsigma_s)}\,.$$

We define the generalized tank coordinate q_t as

$$q_t \triangleq \frac{1}{A_0}\int_{\varsigma_0}^{\varsigma_p} A(\sigma)\,\mathrm{d}\sigma, \tag{12.5}$$

where A_0 is an arbitrary constant with unit m^2.

We note that the total fluid volume in the tank, V_t, is constant. Thus,

$$V_t \triangleq \int_{-\varsigma_0}^{\varsigma_0} A(\sigma)\,\mathrm{d}\sigma = \int_{-\varsigma_s}^{\varsigma_p} A(\sigma)\,\mathrm{d}\sigma = \int_{-\varsigma_s}^{-\varsigma_0} A(\sigma)\,\mathrm{d}\sigma + \int_{-\varsigma_0}^{\varsigma_0} A(\sigma)\,\mathrm{d}\sigma + \int_{\varsigma_0}^{\varsigma_p} A(\sigma)\,\mathrm{d}\sigma$$
$$= \int_{-\varsigma_s}^{-\varsigma_0} A(\sigma)\,\mathrm{d}\sigma + V_t + A_0 q_t.$$

This gives

$$q_t = -\frac{1}{A_0}\int_{-\varsigma_s}^{-\varsigma_0} A(\sigma)\,\mathrm{d}\sigma\,. \tag{12.6}$$

The time derivative of q_t is given by

$$\dot{q}_t = \frac{1}{A_0}A(\varsigma_p)\frac{\mathrm{d}\varsigma_p}{\mathrm{d}t} = \frac{Q}{A_0}. \tag{12.7}$$

By differentiating both sides of (12.5) and (12.6) with respect to q_{t}, it follows that

$$\frac{\mathrm{d}\varsigma_{\mathrm{p}}}{\mathrm{d}q_{\mathrm{t}}} = \frac{A_0}{A(\varsigma_{\mathrm{p}})}\,, \quad \frac{\mathrm{d}\varsigma_{\mathrm{s}}}{\mathrm{d}q_{\mathrm{t}}} = -\frac{A_0}{A(\varsigma_{\mathrm{s}})}\,. \tag{12.8}$$

The speed of the tank fluid relative to the tank walls (i.e., the ship), at any point σ in the tank, is given by

$$\|\mathbf{v}_{\mathrm{t},r}(\sigma,\dot{q}_{\mathrm{t}})\| = \frac{Q}{A(\sigma)} = \frac{A_0\dot{q}_{\mathrm{t}}}{A(\sigma)}\,.$$

From calculus, we know that velocity is tangential to the path, giving

$$\mathbf{v}_{\mathrm{t},r}(\sigma,\dot{q}_{\mathrm{t}}) = \frac{A_0\dot{q}_{\mathrm{t}}}{A(\sigma)}\frac{\mathrm{d}\bar{\mathbf{r}}_t}{\mathrm{d}\sigma}(\sigma)\,, \tag{12.9}$$

where

$$\frac{\mathrm{d}\bar{\mathbf{r}}_t}{\mathrm{d}\sigma} \triangleq \frac{\frac{\mathrm{d}\mathbf{r}_t}{\mathrm{d}\sigma}}{\left\|\frac{\mathrm{d}\mathbf{r}_t}{\mathrm{d}\sigma}\right\|}\,. \tag{12.10}$$

Noting that $\mathrm{d}x_t^b/\mathrm{d}\sigma = 0$, we define

$$\frac{\mathrm{d}\bar{y}_t^b}{\mathrm{d}\sigma} \triangleq [0,1,0]\frac{\mathrm{d}\bar{\mathbf{r}}_t^b}{\mathrm{d}\sigma} = \frac{\frac{\mathrm{d}y_t^b}{\mathrm{d}\sigma}}{\sqrt{\left(\frac{\mathrm{d}y_t^b}{\mathrm{d}\sigma}\right)^2 + \left(\frac{\mathrm{d}z_t^b}{\mathrm{d}\sigma}\right)^2}}, \tag{12.11}$$

$$\frac{\mathrm{d}\bar{z}_t^b}{\mathrm{d}\sigma} \triangleq [0,0,1]\frac{\mathrm{d}\bar{\mathbf{r}}_t^b}{\mathrm{d}\sigma} = \frac{\frac{\mathrm{d}z_t^b}{\mathrm{d}\sigma}}{\sqrt{\left(\frac{\mathrm{d}y_t^b}{\mathrm{d}\sigma}\right)^2 + \left(\frac{\mathrm{d}z_t^b}{\mathrm{d}\sigma}\right)^2}}, \tag{12.12}$$

such that

$$\left(\frac{\mathrm{d}\bar{y}_t^b}{\mathrm{d}\sigma}\right)^2 + \left(\frac{\mathrm{d}\bar{z}_t^b}{\mathrm{d}\sigma}\right)^2 \equiv 1\,.$$

Of course, the ship (and the tank with it) is rotating relative to the inertial frame. Thus, the velocity of the tank fluid relative to the inertial frame, at any point σ in the tank, is given by

$$\mathbf{v}_{\mathrm{t}}(\sigma,\dot{\mathbf{q}}) = \boldsymbol{\omega}\times\mathbf{r}_t(\sigma) + \frac{A_0\dot{q}_{\mathrm{t}}}{A(\sigma)}\frac{\mathrm{d}\bar{\mathbf{r}}_t}{\mathrm{d}\sigma}(\sigma)\,. \tag{12.13}$$

12.3 U-Tank Modeling

According to Lagrangian mechanics, it is necessary to derive the system's kinetic and potential energies.

Define the generalized coordinates $\mathbf{q}$ as

$$\mathbf{q} \triangleq [\phi, q_\mathrm{t}]^\top \in \mathbb{R}^2. \tag{12.14}$$

Proposition 12.1 (Potential energy). *The potential energy of the roll–tank system is given by*

$$\begin{aligned} U(\mathbf{q}) = mgz_g^b + 2g\rho_\mathrm{t}\int_0^{\varsigma_0} z_t^b(\sigma)A(\sigma)\,\mathrm{d}\sigma - g\rho_\mathrm{t}\left[\int_{-\varsigma_\mathrm{s}(q_\mathrm{t})}^{\varsigma_\mathrm{p}(q_\mathrm{t})} y_t^b(\sigma)A(\sigma)\,\mathrm{d}\sigma\right]\sin(\phi) \\ - \left[mgz_g^b + g\rho_\mathrm{t}\int_{-\varsigma_\mathrm{s}(q_\mathrm{t})}^{\varsigma_\mathrm{p}(q_\mathrm{t})} z_t^b(\sigma)A(\sigma)\,\mathrm{d}\sigma\right]\cos(\phi), \end{aligned} \tag{12.15}$$

where m is the mass of the ship (excluding tank fluid), g is the acceleration of gravity and ρ_t is the density of the tank fluid. Note that the first integral is a constant, and that $U(\mathbf{0}) = 0$.

Proof. See Appendix 1. □

Proposition 12.2 (Kinetic energy). *The kinetic energy of the roll–tank system is given by*

$$T(q_\mathrm{t}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{M}_\mathrm{t}(q_\mathrm{t})\dot{\mathbf{q}}, \tag{12.16}$$

where

$$\begin{aligned} \mathbf{M}_\mathrm{t}(q_\mathrm{t}) &= \begin{bmatrix} J_{11} + J_\mathrm{t}(q_\mathrm{t}) & m_{4t}(q_\mathrm{t}) \\ m_{4t}(q_\mathrm{t}) & \bar{m}_t(q_\mathrm{t}) \end{bmatrix} \in \mathbb{R}^{2\times 2} \\ J_\mathrm{t}(q_\mathrm{t}) &= \rho_\mathrm{t}\int_{-\varsigma_\mathrm{s}(q_\mathrm{t})}^{\varsigma_\mathrm{p}(q_\mathrm{t})} A(\sigma)[[y_t^b(\sigma)]^2 + [z_t^b(\sigma)]^2]\,\mathrm{d}\sigma \\ m_{4t}(q_\mathrm{t}) &= \rho_\mathrm{t} A_0 \int_{-\varsigma_\mathrm{s}(q_\mathrm{t})}^{\varsigma_\mathrm{p}(q_\mathrm{t})} \left[y_t^b(\sigma)\frac{\mathrm{d}\bar{z}_t^b}{\mathrm{d}\sigma}(\sigma) - \frac{\mathrm{d}\bar{y}_t^b}{\mathrm{d}\sigma}(\sigma)z_t^b(\sigma)\right]\,\mathrm{d}\sigma \\ \bar{m}_t(q_\mathrm{t}) &= \rho_\mathrm{t} A_0^2 \int_{-\varsigma_\mathrm{s}(q_\mathrm{t})}^{\varsigma_\mathrm{p}(q_\mathrm{t})} \frac{1}{A(\sigma)}\,\mathrm{d}\sigma \end{aligned}$$

and J_{11} is the ship's moment of inertia around the (body) x-axis (excluding the moment of inertia of the tank fluid).

Proof. See Appendix 2. □

Proposition 12.3 (Lagrangian dynamics). *The (lossless) roll–tank dynamics are given by*

$$\mathbf{M}_\mathrm{t}(q_\mathrm{t})\ddot{\mathbf{q}} + \mathbf{C}(q_\mathrm{t}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{k}_\mathrm{t}(\mathbf{q}) = 0, \tag{12.17}$$

where

$$\mathbf{C}(q_\mathrm{t}, \dot{\mathbf{q}}) = \frac{\dot{\phi}}{2}\begin{bmatrix} 0 & \frac{\partial J_\mathrm{t}}{\partial q_\mathrm{t}}(q_\mathrm{t}) \\ -\frac{\partial J_\mathrm{t}}{\partial q_\mathrm{t}}(q_\mathrm{t}) & 0 \end{bmatrix} + \frac{\dot{q}_\mathrm{t}}{2}\begin{bmatrix} \frac{\partial J_\mathrm{t}}{\partial q_\mathrm{t}}(q_\mathrm{t}) & 2\frac{\partial m_{4t}}{\partial q_\mathrm{t}}(q_\mathrm{t}) \\ 0 & \frac{\partial \bar{m}_t}{\partial q_\mathrm{t}}(q_\mathrm{t}) \end{bmatrix}$$

$$\frac{\partial J_\mathrm{t}}{\partial q_\mathrm{t}}(q_\mathrm{t}) = \rho_\mathrm{t} A_0 \left[[y_t^b(\varsigma_\mathrm{p}(q_\mathrm{t}))]^2 - [y_t^b(\varsigma_\mathrm{s}(q_\mathrm{t}))]^2 + [z_t^b(\varsigma_\mathrm{p}(q_\mathrm{t}))]^2 - [z_t^b(\varsigma_\mathrm{s}(q_\mathrm{t}))]^2 \right]$$

$$\begin{aligned} \frac{\partial m_{4t}}{\partial q_\mathrm{t}}(q_\mathrm{t}) = {} & \rho_\mathrm{t}\frac{A_0^2}{A(\varsigma_\mathrm{p}(q_\mathrm{t}))}\left[y_t^b(\varsigma_\mathrm{p}(q_\mathrm{t}))\frac{\mathrm{d}\bar{z}_t^b}{\mathrm{d}\sigma}(\varsigma_\mathrm{p}(q_\mathrm{t})) - \frac{\mathrm{d}\bar{y}_t^b}{\mathrm{d}\sigma}(\varsigma_\mathrm{p}(q_\mathrm{t})) z_t^b(\varsigma_\mathrm{p}(q_\mathrm{t})) \right] \\ & - \rho_\mathrm{t}\frac{A_0^2}{A(\varsigma_\mathrm{s}(q_\mathrm{t}))}\left[y_t^b(\varsigma_\mathrm{s}(q_\mathrm{t}))\frac{\mathrm{d}\bar{z}_t^b}{\mathrm{d}\sigma}(\varsigma_\mathrm{s}(q_\mathrm{t})) - \frac{\mathrm{d}\bar{y}_t^b}{\mathrm{d}\sigma}(\varsigma_\mathrm{s}(q_\mathrm{t})) z_t^b(\varsigma_\mathrm{s}(q_\mathrm{t})) \right] \\ \frac{\partial \bar{m}_t}{\partial q_\mathrm{t}}(q_\mathrm{t}) = {} & \rho_\mathrm{t} A_0^3 \left[\frac{1}{A^2(\varsigma_\mathrm{p}(q_\mathrm{t}))} - \frac{1}{A^2(\varsigma_\mathrm{s}(q_\mathrm{t}))} \right] \\ \mathbf{k}_\mathrm{t}(\mathbf{q}) = {} & \begin{bmatrix} \left(mgz_g^b + g\rho_\mathrm{t}\int_{-\varsigma_\mathrm{s}}^{\varsigma_\mathrm{p}} z_t^b A \,\mathrm{d}\sigma \right)\sin(\phi) - g\rho_\mathrm{t}\int_{-\varsigma_\mathrm{s}}^{\varsigma_\mathrm{p}} y_t^b A \,\mathrm{d}\sigma \cos(\phi) \\ -g\rho_\mathrm{t} A_0 \left[y_t^b(\varsigma_\mathrm{p}) + y_t^b(\varsigma_\mathrm{s}) \right]\sin(\phi) - g\rho_\mathrm{t} A_0 \left[z_t^b(\varsigma_\mathrm{p}) - z_t^b(\varsigma_\mathrm{s}) \right]\cos(\phi) \end{bmatrix} \end{aligned}$$

Proof. The Lagrangian $\mathscr{L}$ of the roll–tank system is given by

$$\begin{aligned} \mathscr{L}(\mathbf{q}, \dot{\mathbf{q}}) &= T(q_\mathrm{t}, \mathbf{q}) - U(\mathbf{q}) \\ &= g\rho_\mathrm{t}\int_{-\varsigma_\mathrm{s}(q_\mathrm{t})}^{\varsigma_\mathrm{p}(q_\mathrm{t})} y_t^b(\sigma)A(\sigma)\,\mathrm{d}\sigma \sin(\phi) - mgz_g^b - 2g\rho_\mathrm{t}\int_0^{\varsigma_0} z_t^b(\sigma)A(\sigma)\,\mathrm{d}\sigma \\ &\quad + \left[mgz_g^b + g\rho_\mathrm{t}\int_{-\varsigma_\mathrm{s}(q_\mathrm{t})}^{\varsigma_\mathrm{p}(q_\mathrm{t})} z_t^b(\sigma)A(\sigma)\,\mathrm{d}\sigma \right]\cos(\phi) + \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{M}_\mathrm{t}(q_\mathrm{t})\dot{\mathbf{q}}\,. \end{aligned} \tag{12.18}$$

The dynamics of the system are then given by the Euler-Lagrange Equation [10]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathscr{L}}{\partial \mathbf{q}} = 0\,. \tag{12.19}$$

It can be shown that

$$\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{q}}} = \mathbf{M}_{\mathrm{t}}(q_{\mathrm{t}})\dot{\mathbf{q}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{q}}} = \dot{\mathbf{M}}_t(q_{\mathrm{t}})\dot{\mathbf{q}} + \mathbf{M}_{\mathrm{t}}(q_{\mathrm{t}})\ddot{\mathbf{q}} = \dot{q}_{\mathrm{t}}\frac{\partial \mathbf{M}_{\mathrm{t}}}{\partial q_{\mathrm{t}}}\dot{\mathbf{q}} + \mathbf{M}_{\mathrm{t}}(q_{\mathrm{t}})\ddot{\mathbf{q}} = \mathbf{C}(q_{\mathrm{t}},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{M}_{\mathrm{t}}(q_{\mathrm{t}})\ddot{\mathbf{q}}$$

$$\frac{\partial \mathscr{L}}{\partial q_{\mathrm{t}}} = -\mathbf{k}_{\mathrm{t}}(\mathbf{q}) \, .$$

Inserting this into (12.19) gives (12.17). □

Proposition 12.4 (Two-DOF u-tank model). *The dynamics of the tank–roll system are given by*

$$\mathbf{M}(q_{\mathrm{t}})\ddot{\mathbf{q}} + \mathbf{C}(q_{\mathrm{t}},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{D}(\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{k}(t,\mathbf{q}) = \mathbf{b}u, \tag{12.20}$$

where

$$\mathbf{M}(q_{\mathrm{t}}) = \begin{bmatrix} m_{a,44} & 0 \\ 0 & 0 \end{bmatrix} + \mathbf{M}_{\mathrm{t}}(q_{\mathrm{t}}) \, , \quad \mathbf{D}(\dot{\mathbf{q}}) = \mathbf{D}_0 + \mathbf{D}_n(\dot{\mathbf{q}}) \, ,$$

$$\mathbf{D}_0 = \begin{bmatrix} d_{44} & 0 \\ 0 & d_{tt} \end{bmatrix} , \quad \mathbf{D}_n(\dot{\mathbf{q}}) = \begin{bmatrix} 0 & 0 \\ 0 & d_{tt,n}|\dot{q}_{\mathrm{t}}| \end{bmatrix} , \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\mathbf{k}(t,\mathbf{q}) = \begin{bmatrix} \bar{k}_{44}\phi + k_{\phi t}\cos(\omega_{\mathrm{e}}t + \alpha_\phi)\phi + k_3\phi^3 \\ 0 \end{bmatrix} + \mathbf{k}_{\mathrm{t}}(\mathbf{q})$$

the other matrices are as in Proposition 12.3 and $u \in \mathbb{R}$ *is the control force on the tank fluid. All the parameters are positive.* $\mathbf{D}$ *satisfies* $y^\top \mathbf{D} y > 0 \; \forall \, y \neq 0$.

Proof. The forces and moments acting on the roll–tank system that are not captured by the Lagrangian modeling are friction and other dissipative forces, added mass, control forces, and pressure torques in roll.

Since nonviscous damping in roll is quite small [4], we model roll damping linearly. Experiments conducted in [14] indicate that quadratic damping is extremely important for the motion of the tank fluid, so this is included. The generalized damping forces are collected in the term $\mathbf{D}(\dot{\mathbf{q}})\dot{\mathbf{q}}$. [14] experimentally investigated the presence of off-diagonal elements in $\mathbf{D}$, but the influence of such terms was found to be negligible. Such terms have therefore been excluded here.

Added mass is a moment proportional to the acceleration of the ship [5], and is caused by interaction with the surrounding ocean. As the tank fluid is not directly in contact with the ocean, the added mass moment $m_{a,44}\ddot{\phi}$ only (directly) affects roll. In general, $m_{a,44}$ is nonconstant [5], but for simplicity we assume it to be constant in this chapter.

The control force u only affects the tank fluid, which implies that $\mathbf{b} = [0,1]^\top$.

As done in [13], the calm-water pressure moment in roll (hydrostatic and -dynamic buoyancy) is modeled as $\bar{k}_{44}\phi + k_3\phi^3$. The parametric excitation is, as in [9, 12] assumed to take place in the linear roll spring term, giving an additional moment of $k_{\phi t}\cos(\omega_e t + \alpha_\phi)\phi$.

Combining all these forces with the Lagrangian-based dynamics of Proposition 12.3 gives (12.20). □

12.3.1 Analysis

It is prudent to ask how the new model (12.20) compares to existing ones. As pointed out in the introduction, most existing models are made for rectangular-prism u-tanks (Fig. 12.1b) [14, 15, 17–19, 22]. Technically, such models cannot fit into the framework developed here, as the tank centerline functions are only C^0 rather than C^1 as required. The integrals that go into the model can still be computed, but the model will technically be invalid.

However, if we ignore this fact, we can explicitly compute the integrals in (12.20). This renders the model identical to the experimentally validated one of [14]. However, that model requires that the duct is always full of water, a constraint not found in the new model.

The linearization of the model is identical to that of [4] and [17, 18],[2] other than that in these works the models take into account sway and yaw in addition to roll and the tank state, and that in [17, 18] $J_t \equiv 0$.

Excluding the tank moment of inertia J_t is likely to have only a small effect, as it is significantly smaller than the moment of inertia of the ship itself, J_{11}. The coupling to the other degrees of freedom might be significant in general, but in this work the ship is assumed not to be maneuvering.

12.4 Control Design

In [14], experiments with a rectangular-prism tank showed that a linearized model was unsuitable to capture the full dynamics of the roll–tank system. However, the experiments also indicated that the full nonlinear model was needlessly complicated, and suggested an alternative model where the dynamics were linearized, with the exception of the damping. This model was an adequate approximation even for relatively high roll and tank state amplitudes. While the results in [14] were for

[2]Note that in [17, 18] the signage is wrong for the tank-induced moment in roll in [17, 18]. [18, (12.54b) p. 266)] reads (neglecting sway and yaw motions) $(I_{44} + a_{44})\ddot{x}_4 + b_{44}\dot{x}_4 + c_{44}x_4 - [a_{4\tau}\ddot{\tau} + c_{4\tau}\tau] = F_{w40}\sin(\omega_e t + \gamma_4)$, but should read $(I_{44} + a_{44})\ddot{x}_4 + b_{44}\dot{x}_4 + c_{44}x_4 + [a_{4\tau}\ddot{\tau} + c_{4\tau}\tau] = F_{w40}\sin(\omega_e t + \gamma_4)$. This error is propagated throughout [17, 18].

a rectangular-prism tank, and not in parametric roll, it seems reasonable that the suggested simplifications would also be applicable in this case. This suggests the model

$$\mathbf{M}_0\ddot{\mathbf{q}} + \mathbf{D}(\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{K}_0(t)\mathbf{q} = \mathbf{b}u, \tag{12.21}$$

where

$$\mathbf{M}_0 \triangleq \mathbf{M}(0)$$
$$\mathbf{K}_0(t) \triangleq \left.\frac{\partial \mathbf{k}}{\partial \mathbf{q}}\right|_{\mathbf{q}=0}$$
$$= \begin{bmatrix} \bar{k}_{44} + mgz_g^b + k_{\phi t}\cos(\omega_e t + \alpha_\phi) + 2g\rho_t \int_0^{\varsigma_0} z_t^b A \,\mathrm{d}\sigma & -2g\rho_t A_0 y_t^b(\varsigma_0) \\ -2g\rho_t A_0 y_t^b(\varsigma_0) & -2g\rho_t \frac{A_0^2}{A(\varsigma_0)} \frac{\mathrm{d}z_t^b}{\mathrm{d}\sigma}(\varsigma_0) \end{bmatrix}.$$

We note that $y_t^b(\varsigma_0) < 0$ and $\frac{\mathrm{d}z_t^b}{\mathrm{d}\sigma}(\varsigma_0) < 0$. We refer to (12.21) as the nominal model.

We define

$$\mathbf{x} \triangleq [\mathbf{q}^\top, \dot{\mathbf{q}}^\top]^\top \in \mathbb{R}^4 \tag{12.22}$$

and rewrite the dynamics as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{G}(t)\mathbf{x} + \mathbf{g}(\dot{\mathbf{q}}), \tag{12.23}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{I}_2 \\ -\mathbf{M}_0^{-1}\mathbf{K}_l & -\mathbf{M}_0^{-1}\mathbf{D}_0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

$$\mathbf{K}_l = \begin{bmatrix} \bar{k}_{44} + mgz_g^b + 2g\rho_t \int_0^{\varsigma_0} z_t^b A \,\mathrm{d}\sigma & -2g\rho_t A_0 y_t^b(\varsigma_0) \\ -2g\rho_t A_0 y_t^b(\varsigma_0) & -2g\rho_t \frac{A_0^2}{A(\varsigma_0)} \frac{\mathrm{d}z_t^b}{\mathrm{d}\sigma}(\varsigma_0) \end{bmatrix} \triangleq \begin{bmatrix} k_{44} & k_{4t} \\ k_{4t} & k_{tt} \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0}_{2\times 1} \\ \mathbf{M}_0^{-1}\mathbf{b} \end{bmatrix} \in \mathbb{R}^{4\times 1}$$

$$\mathbf{G}(t) = \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \\ -\mathbf{M}_0^{-1} \begin{bmatrix} k_{\phi t}\cos(\omega_e t + \alpha_\phi) & 0 \\ 0 & 0 \end{bmatrix} & \mathbf{0}_{2\times 2} \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

$$\mathbf{g}(\dot{\mathbf{q}}) = \begin{bmatrix} \mathbf{0}_{2\times 1} \\ -\mathbf{M}_0^{-1}\mathbf{D}_n(\dot{\mathbf{q}})\dot{\mathbf{q}} \end{bmatrix} \in \mathbb{R}^{4\times 1}.$$

Theorem 12.1 (U-tank control). *The origin of the system (12.23) is globally (uniformly) exponentially stabilized (following [16, Definition 4.5][3]) by the controller*

$$u = d_{tt,n}|\dot{q}_{\mathrm{t}}|\dot{q}_{\mathrm{t}} - \mathbf{K}_p\mathbf{x}, \tag{12.24}$$

where $\mathbf{K}_p = [K_{p,1}, K_{p,2}, K_{p,3}, K_{p,4}] \in \mathbb{R}^{1\times 4}$ *is a matrix such that* $\mathbf{A} - \mathbf{B}\mathbf{K}_p$ *is Hurwitz and the eigenvalues of* $\mathbf{A} - \mathbf{B}\mathbf{K}_p$ *chosen such that*

$$\lambda_{\max}(\mathbf{P}) < \frac{1}{2k_{\phi t}\|\mathbf{M}_0^{-1}\|_2}, \tag{12.25}$$

where $\mathbf{P}$ *is the solution to the Lyapunov equation*

$$\mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K}_p) + (\mathbf{A} - \mathbf{B}\mathbf{K}_p)^\top \mathbf{P} = -\mathbf{I}_4$$

and $\lambda_{\max}(\mathbf{P})$ *is the maximum eigenvalue of* $\mathbf{P}$. *Moreover, a* $\mathbf{K}_p$ *such that (12.25) is satisfied can always be found.*

Proof. See Appendix 3. □

If we take a closer look at the controller (12.24), it cancels the nonlinear tank damping. This damping is "good" damping; in the absence of the time-varying disturbance (setting $k_{\phi t} = 0$) it is fairly straight-forward to show that the origin of the system (12.23) is GAS by using the energy-like Lyapunov function $\bar{V} = \dot{\mathbf{q}}^\top \mathbf{M}_0 \dot{\mathbf{q}} + \mathbf{q}^\top \mathbf{K}_l \mathbf{q}$ (via the Krasowskii–LaSalle theorem [16, Theorem 4.4]; this theorem cannot be used for time-varying systems).

It is therefore reasonable to believe that this damping term is also beneficial in the presence of the time-varying disturbance ($k_{\phi t} \neq 0$). However, proving this has shown itself to be difficult, and the controller is therefore canceling this term.

12.5 Simulation Study

We simulated the full system (12.20) both with and without the controller (12.24) to test the validity and the robustness of the controller. For comparison, we also simulated the controlled nominal system (12.23).

[3] By this definition, exponential stability is stronger than uniform asymptotic stability, and thus the uniformity is implied.

The tank functions y_t^b, z_t^b, and A were given by

$$y_t^b(\sigma) = \begin{cases} \frac{w}{2} & \forall\ \sigma \in (-\infty, -w/2-\varepsilon] \\ -a_0 - a_1\sigma - a_2\sigma^2 & \forall\ \sigma \in [-w/2-\varepsilon, -w/2+\varepsilon] \\ -\sigma & \forall\ \sigma \in [-w/2+\varepsilon, w/2-\varepsilon] \\ a_0 - a_1\sigma + a_2\sigma^2 & \forall\ \sigma \in [w/2-\varepsilon, w/2+\varepsilon] \\ -\frac{w}{2} & \forall\ \sigma \in [w/2+\varepsilon, \infty) \end{cases}, \tag{12.26}$$

$$z_t^b(\sigma) = \begin{cases} r_d + \frac{w}{2} + \sigma & \forall\ \sigma \in (-\infty, -w/2-\varepsilon] \\ -b_0 - b_1\sigma - b_2\sigma^2 & \forall\ \sigma \in [-w/2-\varepsilon, -w/2+\varepsilon] \\ r_d & \forall\ \sigma \in [-w/2+\varepsilon, w/2-\varepsilon] \\ -b_0 + b_1\sigma - b_2\sigma^2 & \forall\ \sigma \in [w/2-\varepsilon, w/2+\varepsilon] \\ r_d + \frac{w}{2} - \sigma & \forall\ \sigma \in [w/2+\varepsilon, \infty) \end{cases}, \tag{12.27}$$

$$A(\sigma) = \begin{cases} A_\mathrm{r} & \forall\ \sigma \in (-\infty, -w/2+\varepsilon] \\ c_0 + c_1\sigma & \forall\ \sigma \in [-w/2-\varepsilon, -w/2+\varepsilon] \\ A_\mathrm{d} & \forall\ \sigma \in [-w/2+\varepsilon, w/2-\varepsilon] \\ c_0 - c_1\sigma & \forall\ \sigma \in [w/2-\varepsilon, w/2+\varepsilon] \\ A_\mathrm{r} & \forall\ \sigma \in [w/2+\varepsilon, \infty) \end{cases} \tag{12.28}$$

with $\varepsilon \ll w/2$ and

$$a_0 = \frac{\left(\varepsilon - \frac{w}{2}\right)^2}{4\varepsilon}, \quad a_1 = \frac{w+2\varepsilon}{4\varepsilon}, \quad a_2 = b_2 = \frac{1}{4\varepsilon}, \quad b_0 = \frac{\left(\varepsilon - \frac{w}{2}\right)^2}{4\varepsilon} - r_d,$$
$$b_1 = \frac{w-2\varepsilon}{4\varepsilon}, \quad c_0 = \frac{2(A_\mathrm{d}+A_\mathrm{r})\varepsilon + w(A_\mathrm{d}-A_\mathrm{r})}{4\varepsilon}, \quad c_1 = \frac{A_\mathrm{d}-A_\mathrm{r}}{2\varepsilon}.$$

Note that this choice of a_i, b_i, c_i ensures that $y_t^b, z_t^b \in C^1$ and that $A \in C^0$. A tank described by these functions has a centerline function describing half a rounded rectangle. The tank state q_t was limited so that $|q_\mathrm{t}| \le q_{\mathrm{t,max}} = V_\mathrm{t}/(2A_0)$ so that there always is tank fluid at the tank center point $\sigma = 0$.

Simulation parameters can be found in Table 12.1. The ship parameters J_{11}, $m_{a,44}$, d_{44}, $\bar{k}_{44}$, $k_{\phi t}$, and k_3 were taken from [13, Experiment 1174]. The tank damping parameters are based on experimental values from [14] and formulas found in [12] and [18]. The encounter frequency ω_e was chosen to be twice the natural roll frequency, when the system is known to parametrically oscillate.

The uncontrolled nominal system (12.23) had eigenvalues

$$\lambda(\mathbf{A}) \approx [-0.0049 \pm 0.3327i, -0.0051 \pm 0.2558i]$$

while the controlled system had eigenvalues

$$\lambda(\mathbf{A} - \mathbf{BK}_p) \approx [-0.0196 \pm 0.3327i, -0.0206 \pm 0.2558i].$$

With the parameters of Table 12.1, V_t was computed to be $V_\mathrm{t} \approx 337.8\ \mathrm{m}^3$. As per the standard rules of u-tank design [18], the tank is dimensioned so that the natural frequency of the tank (here, 0.2978 rad/s) is chosen to be approximately equal to the natural roll frequency (here, 0.2972 rad/s).

Table 12.1 Simulation parameters

Parameter	Value Unit	Parameter	Value Unit
J_{11}	1.4014E10 $\mathrm{kg\,m^2}$	A_r	30 $\mathrm{m^2}$
$m_{a,44}$	2.17E9 $\mathrm{kg\,m^2}$	A_d	3.6145 $\mathrm{m^2}$
d_{44}	3.1951E8 $\mathrm{kg\,m^2/s}$	w	27 m
d_{tt}	2.4618E3 $\mathrm{kg\,m/s}$	ε	1 m
$d_{tt,n}$	2.2742E5 kg m	r_d	2 m
$\bar{k}_{44}$	2.2742E9 $\mathrm{kg\,m^2/s^2}$	ς_0	17.5 m
$k_{\phi t}$	5.0578E8 $\mathrm{kg\,m^2/s^2}$	A_0	30 $\mathrm{m^2}$
k_3	2.974E9 $\mathrm{kg\,m^2/s^2}$	$\phi(t_0)$	5 °
m	7.64688E7 kg	$\dot{\phi}(t_0)$	0 °/s
z_g^b	−1.12 m	$q_\mathrm{t}(t_0)$	0 m
g	9.81 $\mathrm{m/s^2}$	$\dot{q}_\mathrm{t}(t_0)$	0 m/s
ρ_t	1,000 $\mathrm{kg/m^3}$	$K_{p,1}$	3.9935E5 $\mathrm{kg\,m/s^2}$
ω_e	0.594 rad/s	$K_{p,2}$	7.2833E3 $\mathrm{kg/s^2}$
α_ϕ	0 rad	$K_{p,3}$	−4.1664E5 kg m/s
$q_{\mathrm{t,max}}$	5.6307 m	$K_{p,4}$	3.9916E5 kg/s

The results of the simulation study can be seen in Figs. 12.4 and 12.5. We can clearly see that the system trajectory converges to the origin. Note also that the trajectory of the nominal system is almost identical to that of the true system.

From Fig. 12.4, we can also see that a passive (uncontrolled) tank is capable of reducing the roll angle compared to not having a tank at all[4] (a reduction in maximum roll angle of approximately 21° to 7°). However, both roll and the tank fluid will end up in steady-state oscillations.

The issue of correctly tuning the natural frequency of the tank fluid bears some consideration. For a rectangular-prism tank (and a tank like the one used in the simulations), the natural frequency can be changed by adjusting the fluid level ς_0 or the ratio of A_r and A_d (cross-sectional area of the reservoirs and the duct, respectively). The latter can of course only be done when the tank is constructed.

Unfortunately, the natural frequency is quite insensitive to changes in ς_0 [18]. It is therefore almost impossible to change the natural frequency of the tank after it has been built. However, there can be quite some uncertainty in the natural roll frequency, which can also depend on loading conditions [4, 18].

If the natural frequency of the tank is not properly tuned, the effect of a passive tank can be drastically reduced. The more badly tuned it is, the less effective the tank is. However, as proven in Theorem 12.1, an active (controlled) tank will still be able to stabilize the origin of the system.

[4] In [12] it was concluded that using a passive tank did not noticeably reduce the roll angle in parametric resonance, but this tank had a badly tuned natural frequency.

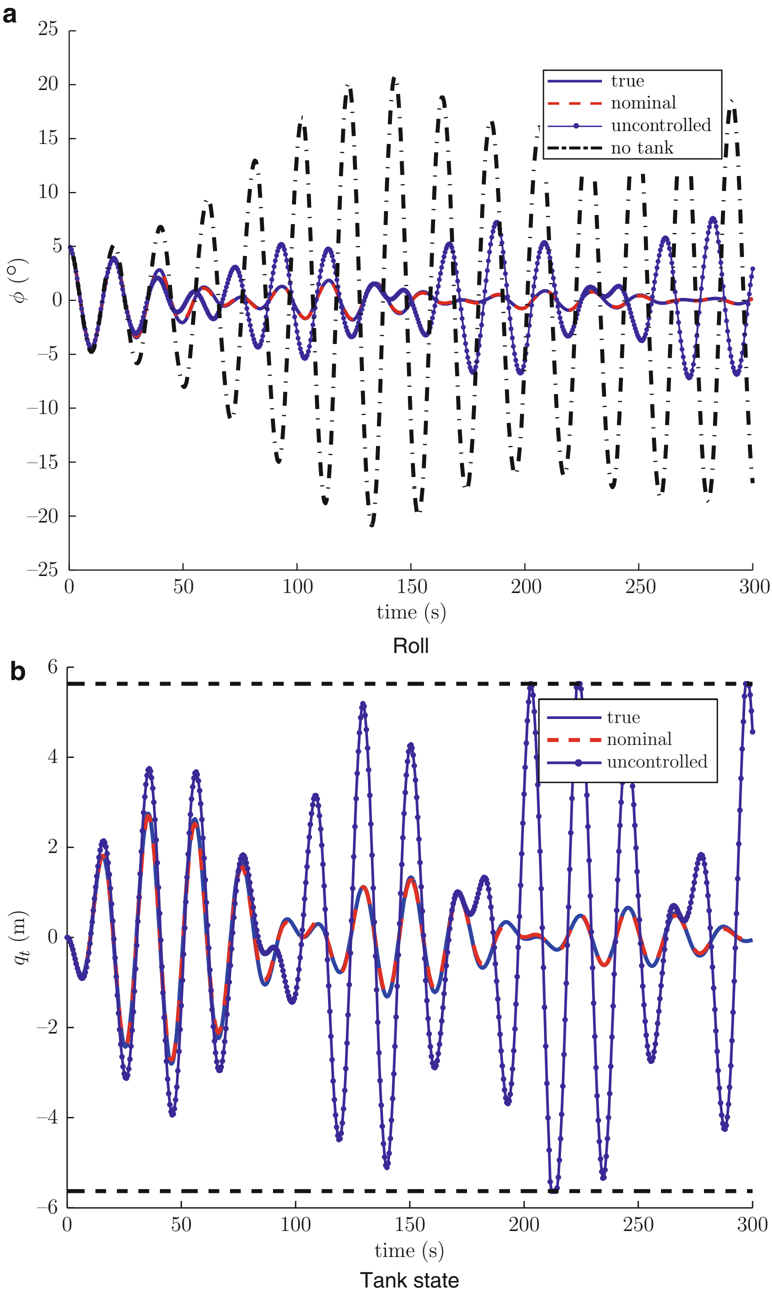

Fig. 12.4 Simulation of the closed- and open-loop system. True and nominal graphs are closed-loop simulations of (12.20) and (12.21), respectively. Uncontrolled is open-loop simulation of (12.20)

The power and energy consumptions of the control system is also worth noting. As can be seen from Fig. 12.5, at peak the controller requires a force of about 250 kN and 210 kW. By integrating the power consumption (over 1,000 s), the total energy use can be found to be approximately 8 MJ.

These numbers require some context. If force on the tank fluid is applied by using high-pressure air in the reservoirs, the pressure difference in the two reservoirs has to be about 8.5 kPa, or 0.085 bar. When considering the maximum power consumption 21 kW, bear in mind that the actuator is moving 337.8 metric tons of fluid, and that the ship itself has a mass of 76,500 metric tons and is likely to have a fairly large power system. The total energy consumption equals about 0.23 liters of gasoline burned in a combustion engine (using 34.8 MJ/liter of gasoline [2]). All in all, the control system if fairly modest in scale.

12.6 Conclusions

This work presents two important contributions: A novel model of u-tanks, suitable for u-tanks of any shape and system response of any magnitude; and a u-tank control system capable of exponentially driving the roll angle to zero and the tank fluid to its equilibrium state during parametric roll resonance.

The proposed model has two degrees of freedom; roll, and one for the motion of the tank fluid. To derive the model, the inherently complex motion of a fluid in a tank was modeled as a one-dimensional flow. While technically not true, the model is only designed to capture the tank fluid's and the ship's mutual effect on each other. Only the macroscopic fluid effects are likely to be relevant in this case.

Unlike most existing u-tank models, which can only be used for tanks of a very specific shape, the new model can describe u-tanks of arbitrary shape. The new model also captures the inherently nonlinear behavior of the system, and is valid for large system motions. Existing models are largely linear and assume small motions.

The control system was developed and its stability properties proven for a simplification of the 2-DOF model. Under the assumption of no constraints on the states or the input, the controller renders the origin of the closed-loop system globally (uniformly) exponentially stable.[5]

The controller was tested in simulation on both the nominal (simplified) system and the full nonlinear system. The responses of these two systems was virtually indistinguishable in simulation, and the origin was stabilized, as shown theoretically. The power and energy consumptions are quite reasonable; total energy required to stabilize parametric roll is equivalent to a quarter of a liter of gasoline, and peak power requirements are quite modest given the size of the ship.

[5] In the presence of limitations on the tank state or the input, the origin of the controlled system might only be locally (uniformly) exponentially stable.

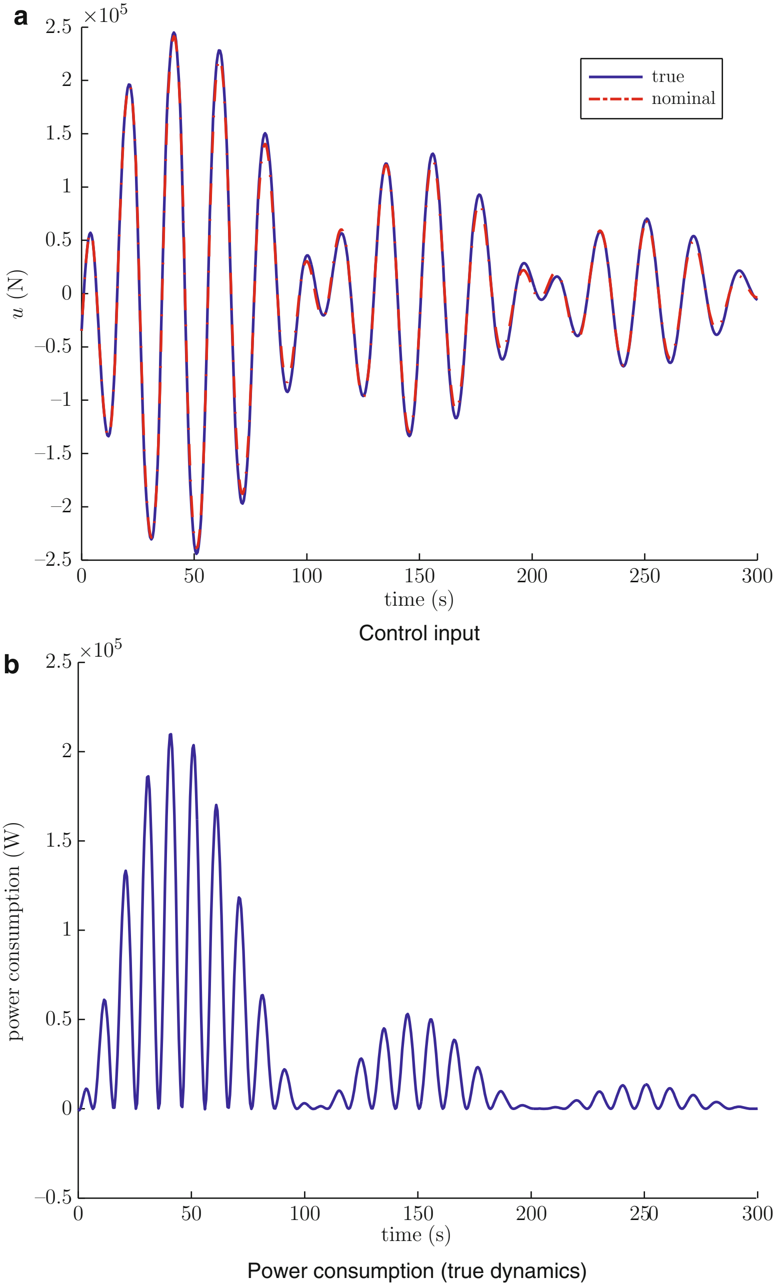

Fig. 12.5 Simulation of the closed-loop system. True and nominal graphs are simulations of (12.20) and (12.21), respectively

Simulations also showed that a passive (uncontrolled) tank would be able to reduce the roll angle significantly (21° to 7°) in the presence of parametric roll, but not to drive the roll angle to zero. It can also be shown that a passive tank only works if it is correctly tuned (i.e., the natural tank frequency is identical to the natural roll frequency). Changing the natural tank frequency without rebuilding the tank is almost impossible. The natural roll frequency can change depending on loading conditions, sailing conditions, and simple discrepancies between theoretical design and practical implementation makes it, to a certain degree, unknown. This makes correctly tuning the tank difficult.

However, the controlled tank would still be able to drive roll to zero, even with a poorly tuned tank.

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Appendix 1

In this appendix, the potential energy of the ship–tank system is derived, proving Proposition 12.1.

An infinitesimal volume block dV of the tank or ship at a position $\mathbf{r}$ has density $\rho(\mathbf{r})$ given by

$$\rho(\mathbf{r}) = \begin{cases} \rho_{\mathrm{t}} & \text{in the tank} \\ \rho_s(\mathbf{r}) & \text{in the ship} \end{cases} \tag{12.29}$$

and is at a height $h(\mathbf{r})$ above some arbitrary zero point. We note that h is the zero level minus the inertial z-component of $\mathbf{r}$, that is,

$$h(\mathbf{r}) = h_0 - [0,0,1]\mathbf{r}^{\mathrm{n}} = h_0 - [0,0,1]\mathbf{R}\mathbf{r}^{\mathrm{b}}. \tag{12.30}$$

The negative signage is because the z-axis has the same direction as the gravity field.

The potential energy dU of the volume block is then given by

$$\mathrm{d}U = g\rho(\mathbf{r})h(\mathbf{r})\mathrm{d}V, \tag{12.31}$$

which, in the body frame, can be written

$$\mathrm{d}U = g\rho(\mathbf{r}^{\mathrm{b}})h(\mathbf{r}^{\mathrm{b}})\,\mathrm{d}V = g\rho(\mathbf{r}^{\mathrm{b}})\left(h_0 - [0,0,1]\mathbf{R}\mathbf{r}^{\mathrm{b}}\right)\,\mathrm{d}V. \tag{12.32}$$

The total potential energy U of the ship and the tank fluid is then given by

$$U = \int_{\text{ship and tank}} \mathrm{d}U = gm_0h_0 - g[0,0,1]\mathbf{R}\left[\int_{\text{ship}} \rho_s(\mathbf{r}^{\mathrm{b}})\mathbf{r}^{\mathrm{b}}\,\mathrm{d}V + \rho_{\mathrm{t}}\int_{\text{tank}} \mathbf{r}_t^b\,\mathrm{d}V\right]$$

$$= gm_0h_0 - g[0,0,1]\mathbf{R}\left[m\mathbf{r}_g^b + \rho_{\mathrm{t}}\int_{-\varsigma_{\mathrm{s}}(q_{\mathrm{t}})}^{\varsigma_{\mathrm{p}}(q_{\mathrm{t}})} A(\sigma)\mathbf{r}_t^b(\sigma)\,\mathrm{d}\sigma\right] \tag{12.33}$$

by the definition of the center of gravity, where m_0 is the combined mass of the ship and tank fluid. From (12.1), $[0,0,1]\mathbf{R} = [0,\sin(\phi),\cos(\phi)]$ and, by assumption, $\mathbf{r}_g^b = [x_g^b, 0, z_g^b]^\top$. This gives

$$U = gm_0h_0 - mgz_g^b\cos(\phi) - g\rho_{\mathrm{t}}\int_{-\varsigma_{\mathrm{s}}(q_{\mathrm{t}})}^{\varsigma_{\mathrm{p}}(q_{\mathrm{t}})} A(\sigma)y_t^b(\sigma)\,\mathrm{d}\sigma\sin(\phi)$$

$$- g\rho_{\mathrm{t}}\int_{-\varsigma_{\mathrm{s}}(q_{\mathrm{t}})}^{\varsigma_{\mathrm{p}}(q_{\mathrm{t}})} A(\sigma)z_t^b(\sigma)\,\mathrm{d}\sigma\cos(\phi). \tag{12.34}$$

A priori, we know that $\mathbf{q} = \mathbf{0}$ is an equilibrium point for the system, so we choose $U(\mathbf{q} = \mathbf{0}) = \mathbf{0}$. This gives

$$U(\mathbf{q} = \mathbf{0}) = gm_0h_0 - mgz_g^b - 2g\rho_{\mathrm{t}}\int_0^{\varsigma_0} A(\sigma)z_t^b(\sigma)\,\mathrm{d}\sigma = 0,$$

since $\varsigma_{\mathrm{s}}(0) = \varsigma_{\mathrm{p}}(0) = \varsigma_0$ and $A(\sigma)z_t^b(\sigma)$ is an even function. We therefore choose

$$gm_0h_0 = mgz_g^b + 2g\rho_{\mathrm{t}}\int_0^{\varsigma_0} A(\sigma)z_t^b(\sigma)\,\mathrm{d}\sigma = 0$$

giving

$$U(\mathbf{q}) = mgz_g^b + 2g\rho_{\mathrm{t}}\int_0^{\varsigma_0} z_t^b(\sigma)A(\sigma)\,\mathrm{d}\sigma - g\rho_{\mathrm{t}}\int_{-\varsigma_{\mathrm{s}}(q_{\mathrm{t}})}^{\varsigma_{\mathrm{p}}(q_{\mathrm{t}})} y_t^b(\sigma)A(\sigma)\,\mathrm{d}\sigma\sin(\phi)$$

$$- \left[mgz_g^b + g\rho_{\mathrm{t}}\int_{-\varsigma_{\mathrm{s}}(q_{\mathrm{t}})}^{\varsigma_{\mathrm{p}}(q_{\mathrm{t}})} z_t^b(\sigma)A(\sigma)\,\mathrm{d}\sigma\right]\cos(\phi), \tag{12.35}$$

which we recognize as (12.15).

Appendix 2

In this appendix, the kinetic energy of the ship–tank system is derived, proving Proposition 12.2.

An infinitesimal volume block dV of the tank or ship at a position $\mathbf{r}$ in the body frame has density $\rho(\mathbf{r})$ given by (12.29) and velocity $\mathbf{v}(\mathbf{r})$ given by

$$\mathbf{v}(\mathbf{r}) = \begin{cases} \boldsymbol{\omega} \times \mathbf{r} + \frac{A_0 \dot{q}_\mathrm{t}}{A(\sigma)} \frac{\mathrm{d}\bar{\mathbf{r}}_t}{\mathrm{d}\sigma}(\sigma) & \text{in the tank} \\ \boldsymbol{\omega} \times \mathbf{r} & \text{in the ship} \end{cases}, \tag{12.36}$$

where $\boldsymbol{\omega}$ is the angular velocity of the ship. The velocity of the tank fluid comes from (12.13).

The volume block has kinetic energy dT given by

$$\mathrm{d}T = \frac{1}{2}\rho(\mathbf{r})\|\mathbf{v}(\mathbf{r})\|_2^2 \mathrm{d}V, \tag{12.37}$$

which, in the body frame, can be written

$$\mathrm{d}T = \frac{1}{2}\rho(\mathbf{r}^\mathrm{b})\|\mathbf{v}^b(\mathbf{r}^\mathrm{b})\|_2^2 \mathrm{d}V. \tag{12.38}$$

The total kinetic energy T of the ship and the tank fluid is then given by

$$\begin{aligned} T &= \frac{1}{2}\int_{\text{ship and tank}} \rho(\mathbf{r}^\mathrm{b})\|\mathbf{v}^b(\mathbf{r}^\mathrm{b})\|_2^2 \,\mathrm{d}V \\ &= \frac{1}{2}\int_{\text{ship}} \rho_s(\mathbf{r}^\mathrm{b})\|\boldsymbol{\omega}^b \times \mathbf{r}^\mathrm{b}\|_2^2 \,\mathrm{d}V + \frac{\rho_\mathrm{t}}{2}\int_{\text{tank}} \left\| \boldsymbol{\omega}^b \times \mathbf{r}^\mathrm{b} + \frac{A_0 \dot{q}_\mathrm{t}}{A(\sigma)} \frac{\mathrm{d}\bar{\mathbf{r}}_t^b}{\mathrm{d}\sigma} \right\|_2^2 \mathrm{d}V \\ &= -\frac{1}{2}\boldsymbol{\omega}^{b\top} \left[\int_{\text{ship}} \rho_s(\mathbf{r}^\mathrm{b}) S^2(\mathbf{r}^\mathrm{b}) \,\mathrm{d}V \right] \boldsymbol{\omega}^b \\ &\quad + \frac{\rho_\mathrm{t}}{2}\int_{-\varsigma_\mathrm{s}(q_\mathrm{t})}^{\varsigma_\mathrm{p}(q_\mathrm{t})} A(\sigma) \left\| \dot{\phi}[0, -z_t^b(\sigma), y_t^b(\sigma)]^\top + \frac{A_0 \dot{q}_\mathrm{t}}{A(\sigma)} \frac{\mathrm{d}\bar{\mathbf{r}}_t^b}{\mathrm{d}\sigma}(\sigma) \right\|_2^2 \mathrm{d}\sigma. \end{aligned}$$

We note that, by definition,

$$\mathbf{J} = -\int_{\text{ship}} \rho_s(\mathbf{r}^\mathrm{b}) S^2(\mathbf{r}^\mathrm{b}) \,\mathrm{d}V \in \mathbb{R}^{3\times 3}$$

is the moment of inertia of the ship and assumed a priori known. Furthermore,

$$\boldsymbol{\omega}^{b\top}\mathbf{J}\boldsymbol{\omega}^b = \dot{\phi}^2[1,0,0]\mathbf{J}[1,0,0]^\top = \dot{\phi}^2 J_{11},$$

where $J_{11} > 0 \in \mathbb{R}$ is the top left element of J, that is, the moment of inertia about the (body) x-axis. Thus,

$$\begin{aligned} T &= \frac{1}{2}J_{11}\dot{\phi}^2 + \frac{\rho_t}{2}\int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} A(\sigma)\left\| \dot{\phi}[0, -z_t^b(\sigma), y_t^b(\sigma)]^\top + \frac{A_0\dot{q}_t}{A(\sigma)}\frac{\mathrm{d}\bar{\mathbf{r}}_t^b}{\mathrm{d}\sigma}(\sigma)\right\|_2^2 \mathrm{d}\sigma \\ &= \frac{1}{2}\left[J_{11} + \rho_t\int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} A(\sigma)\left[[y_t^b(\sigma)]^2 + [z_t^b(\sigma)]^2\right] \mathrm{d}\sigma\right]\dot{\phi}^2 + \frac{\dot{q}_t^2}{2}\int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} \frac{\rho_t A_0^2}{A(\sigma)}\mathrm{d}\sigma \\ &\quad + \dot{\phi}\dot{q}_t\rho_t A_0\int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} \left[y_t^b(\sigma)\frac{\mathrm{d}\bar{z}_t^b}{\mathrm{d}\sigma}(\sigma) - \frac{\mathrm{d}\bar{y}_t^b}{\mathrm{d}\sigma}(\sigma)z_t^b(\sigma)\right] \mathrm{d}\sigma \end{aligned} \tag{12.39}$$

since $\left\|\mathrm{d}\bar{\mathbf{r}}_t^b/\mathrm{d}\sigma\right\|_2^2 = 1$.

Defining

$$\begin{aligned} J_t(q_t) &= \rho_t\int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} A(\sigma)[[y_t^b(\sigma)]^2 + [z_t^b(\sigma)]^2]\,\mathrm{d}\sigma \\ m_{4t}(q_t) &= \rho_t A_0\int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} \left[y_t^b(\sigma)\frac{\mathrm{d}\bar{z}_t^b}{\mathrm{d}\sigma}(\sigma) - \frac{\mathrm{d}\bar{y}_t^b}{\mathrm{d}\sigma}(\sigma)z_t^b(\sigma)\right] \mathrm{d}\sigma \\ \bar{m}_t(q_t) &= \rho_t A_0^2\int_{-\varsigma_s(q_t)}^{\varsigma_p(q_t)} \frac{1}{A(\sigma)}\mathrm{d}\sigma \\ \mathbf{M}_t(q_t) &= \begin{bmatrix} J_{11} + J_t(q_t) & m_{4t}(q_t) \\ m_{4t}(q_t) & \bar{m}_t(q_t) \end{bmatrix} \in \mathbb{R}^{2\times 2}, \end{aligned}$$

we can rewrite (12.39) as

$$T(q_t, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^\top\mathbf{M}_t(q_t)\dot{\mathbf{q}}, \tag{12.40}$$

which we recognize as (12.16).

Appendix 3

This section contains the proof of Theorem 12.1.

We immediately note that by choosing $u = d_{tt,n}|\dot{q}_t|\dot{q}_t + v$, we can transform the dynamics of the system (12.23) into the linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}v + \mathbf{G}(t)\mathbf{x}. \tag{12.41}$$

The term $\mathbf{G}(t)\mathbf{x}$ can be viewed as a time-varying disturbance to the system. We cannot directly cancel it, both because the parameters are unlikely to be known and because roll is not directly actuated.

The unperturbed system has the dynamics

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}v. \tag{12.42}$$

This system is controllable if its controllability matrix $\mathscr{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} \end{bmatrix}$ has full rank (i.e., is nonsingular) [1].

The controllability matrix is given by

$$\mathscr{C} = \begin{bmatrix} \mathbf{M}_0^{-1} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \mathbf{M}_0^{-1} \end{bmatrix} \bar{\mathscr{C}}\,, \quad \bar{\mathscr{C}} \triangleq \begin{bmatrix} 0 & 0 & a_1 & b_1 \\ 0 & 1 & a_2 & b_2 \\ 0 & a_1 & b_1 & c_1 \\ 1 & a_2 & b_2 & c_2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\mathbf{D}_0\mathbf{M}_0^{-1}\mathbf{B}$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = (\mathbf{D}_0\mathbf{M}_0^{-1}\mathbf{D}_0 - \mathbf{K}_l)\mathbf{M}_0^{-1}\mathbf{B}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = (\mathbf{K}_l\mathbf{M}_0^{-1}\mathbf{D}_0 + \mathbf{D}_0\mathbf{M}_0^{-1}\mathbf{K}_l - \mathbf{D}_0\mathbf{M}_0^{-1}\mathbf{D}_0\mathbf{M}_0^{-1}\mathbf{D}_0)\mathbf{M}_0^{-1}\mathbf{B}\,.$$

From [1], we have that $\mathrm{rank}(\mathscr{C}) = \mathrm{rank}(\bar{\mathscr{C}})$ since the matrix $\mathrm{diag}(\mathbf{M}_0^{-1}, \mathbf{M}_0^{-1}) \in \mathbb{R}^{4\times 4}$ is nonsingular.

$\bar{\mathscr{C}}$ has full rank if its determinant is nonzero [1]. Since

$$\det(\bar{\mathscr{C}}) = -\frac{1}{\det(\mathbf{M}_0)^2}\left([k_{44}m_{12} - k_{4t}m_{11}]^2 + d_1^2 k_{4t}m_{12}\right)\,, \tag{12.43}$$

this gives the condition

$$[k_{44}m_{12} - k_{4t}m_{11}]^2 + d_1^2 k_{4t}m_{12} \neq 0\,. \tag{12.44}$$

As long as this condition is satisfied, (12.42) is controllable. Since all the parameters in (12.44) are strictly positive, this condition is always satisfied.

As (12.42) is controllable, we can select a $v = -\mathbf{K}_p\mathbf{x}$, $\mathbf{K}_p \in \mathbb{R}^{1\times 4}$, such that $\mathbf{A} - \mathbf{B}\mathbf{K}_p$ is Hurwitz, and can place the poles arbitrarily far into the left half-plane [1]. The closed-loop system is then given by

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K}_p)\mathbf{x} + \mathbf{G}(t)\mathbf{x}\,. \tag{12.45}$$

From [16], we know that for any positive definite symmetric matrix $\mathbf{N}$, there exists a positive definite symmetric matrix $\mathbf{P}$ such that

$$\mathbf{P}(\mathbf{A} - \mathbf{B}\mathbf{K}_p) + (\mathbf{A} - \mathbf{B}\mathbf{K}_p)^\top\mathbf{P} = -\mathbf{N}\,. \tag{12.46}$$

We choose Lyapunov function candidate

$$V(\mathbf{x}) = \mathbf{x}^\top \mathbf{P}\mathbf{x} \tag{12.47}$$

with $\mathbf{P}$ as the solution to (12.46). We note that it is positive definite and decrescent. Specifically,

$$\lambda_{\min}(\mathbf{P})\|\mathbf{x}\|_2^2 \leq V(\mathbf{x}) \leq \lambda_{\max}(\mathbf{P})\|\mathbf{x}\|_2^2, \tag{12.48}$$

where $\lambda_{\min}(\mathbf{P})$ and $\lambda_{\max}(\mathbf{P})$ are the minimum and maximum eigenvalues of $\mathbf{P}$.

The time derivative of V along the trajectories of the closed-loop system (12.45) is given by

$$\begin{aligned}
\dot{V}(\mathbf{x}) &= \mathbf{x}^\top \left[\mathbf{P}(\mathbf{A}-\mathbf{B}\mathbf{K}_p) + (\mathbf{A}-\mathbf{B}\mathbf{K}_p)^\top \mathbf{P}\right]\mathbf{x} + \mathbf{x}^\top \left[\mathbf{P}\mathbf{G}(t) + \mathbf{G}^\top(t)\mathbf{P}\right]\mathbf{x} \\
&= -\mathbf{x}^\top \mathbf{N}\mathbf{x} + \mathbf{x}^\top \left[\mathbf{P}\mathbf{G}(t) + \mathbf{G}^\top(t)\mathbf{P}\right] x \\
&\leq -\lambda_{\min}(\mathbf{N})\|\mathbf{x}\|_2^2 + 2\lambda_{\max}(\mathbf{P}) \max_t \|\mathbf{G}(t)\|_2 \|\mathbf{x}\|_2^2 \\
&\leq -\lambda_{\min}(\mathbf{N})\|\mathbf{x}\|_2^2 + 2k_{\phi t}\lambda_{\max}(\mathbf{P})\|\mathbf{M}_0^{-1}\|_2\|\mathbf{x}\|_2^2 \\
&= -\left[\lambda_{\min}(\mathbf{N}) - 2k_{\phi t}\lambda_{\max}(\mathbf{P})\|\mathbf{M}_0^{-1}\|_2\right]\|\mathbf{x}\|_2^2,
\end{aligned} \tag{12.49}$$

where we have used that

$$\max_t \|\mathbf{G}(t)\|_2 = \left\| \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \\ -\mathbf{M}_0^{-1}\begin{bmatrix} k_{\phi t} & 0 \\ 0 & 0 \end{bmatrix} & \mathbf{0}_{2\times 2} \end{bmatrix} \right\|_2 \leq k_{\phi t}\|\mathbf{M}_0^{-1}\|_2 \,. \tag{12.50}$$

By [16, Theorem 4.10], the origin of the controlled system (12.45) is globally (uniformly) exponentially stable, as long as $\lambda_{\min}(\mathbf{N}) > 2k_{\phi t}\lambda_{\max}(\mathbf{P})\|\mathbf{M}_0^{-1}\|_2$ or

$$k_{\phi t} < \frac{\lambda_{\min}(\mathbf{N})}{2\lambda_{\max}(P)\|\mathbf{M}_0^{-1}\|_2}.$$

The ratio $\lambda_{\min}(\mathbf{N})/\lambda_{\max}(\mathbf{P})$ is maximized by choosing $\mathbf{N} = \mathbf{I}_4$ [16]. Since we can choose the eigenvalues of $\mathbf{A}-\mathbf{B}\mathbf{K}_p$ arbitrarily far into the left half-plane, we can choose $\lambda_{\max}(\mathbf{P})$ arbitrarily, as this value depends on the eigenvalues of $\mathbf{A}-\mathbf{B}\mathbf{K}_p$ [1].

Thus, for any $k_{\phi t}$, we can find a controller such that the origin of the controlled system (12.45) is globally (uniformly) exponentially stable. □

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