Chapter 1
An Introduction to Parametric Resonance

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1.1 Introduction

In many engineering, physical, electrical, chemical, and biological systems, oscillatory behavior of the dynamic system due to periodic excitation is of great interest. Two kinds of oscillatory responses can be distinguished: forced oscillations and parametric oscillations. Forced oscillations appear when the dynamical system is excited by a periodic input. If the frequency of an external excitation is close to the natural frequency of the system, then the system will experience *resonance*, i.e. oscillations with a large amplitude. Parametric oscillations are the result of having time-varying (periodic) parameters in the system. In this case, the system could experience *parametric resonance*, and again the amplitude of the oscillations in the output of the system will be large.

Systems with time-varying parameters are called parametrically excited systems [19]. A very classical example of a parametrically excited system is a swing. To increase the amplitude of the motion, the person must crouch in the extreme position and sit straight up in the middle position. Consequently, the distance between the hanging point and the center of gravity of the person varies periodically. This system (person on a swing) can be seen as pendulum with varying length [24].

In this book, many of the chapters deal with a specific class of dynamical systems, namely with the dynamics describing the motion of a ship sailing in the ocean. Indeed, a ship can be seen as an autoparametric system, where the dynamics corresponding to the roll motion are parametrically excited by the other motions in the ship and under some circumstances, it will experience parametric resonance known as *parametric roll*, which consists of large oscillations in the roll motion, which may become dangerous for the ship, its cargo, and its crew. The phenomenon of

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parametric roll is related to the periodic change of stability as the ship is sailing against the waves (with length close to the length of the ship and height exceeding a critical value) at a speed such that the wave excitation frequency (called encounter frequency) is approximately twice the natural roll frequency and the roll damping of the ship is insufficient to avoid the onset of parametric roll [9, 11].

Perhaps the lead in research on parametric roll was taken by William Froude (1810–1879), who in 1861 discovered that the roll angle can increase rapidly when the period of the ship is in resonance with the period of wave encounter [8]. He also came to the conclusion that the roll motion is not produced by the waves hitting the side of the hull, but rather because of the pressure of the waves acting on the hull. For a historical note on the early days of parametric roll, the reader is referred to [6] and [10]. It is worthwhile mentioning that in 1998 there was an incident with a post-Panamax C11 class container ship, which was caught by a violent storm and experienced parametric roll with roll angles close to 40°. As a consequence, one third of the on-deck containers were lost overboard and a similar amount were severely damaged [9]. This event converted the study of parametric roll into a hot topic for research.

When studying parametric resonance in a dynamical system, at some point in the analysis, a very familiar equation will pop up. We refer to *Mathieu's equation*, which is a special case of second order differential equation with a periodic coefficient. Therefore, this introductory chapter is devoted to present the Mathieu equation and some of its applications. The idea is to motivate the reader to investigate the properties of this equation, its solutions, and the stability of the solutions and then apply it in practical problems. The last part of this chapter presents a short introduction of *autoparametric systems*, which are interconnected systems where parametric resonance appears in one of the constituting subsystems due to the vibrations in one of the other constituting subsystems, which can be externally excited, parametrically excited, or self excited.

1.2 The Mathieu Equation

In 1868, M. Émile Mathieu (1835–1890) published his celebrated work about the vibrational movement of a stretched membrane having an elliptical boundary [16]. By transforming the two dimensional wave equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + c^2 V = 0 \tag{1.1}$$

into elliptical coordinates and separating it into two ordinary differential equations, Mathieu obtained the differential equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + (a - 2q\cos 2t)\, u = 0. \tag{1.2}$$

This equation is called the Mathieu equation. In the physical problem studied by Mathieu the constants a and q are real. This is also true for any practical problem. Some of the properties of this equation are [3]:

- Mathieu's equation always has one odd and one even solution.
- (Floquet's theorem) Mathieu's equation always has at least one solution $x(t)$ such that $x(t+\pi) = \sigma x(t)$, where the constant σ, called the periodicity factor, depends on the parameters a and q and may be real or complex.
- Mathieu's equation always has at least one solution of the form $e^{\mu t}\phi(t)$, where the constant μ is called the periodicity exponent and $\phi(t)$ has period π.
- In the case that the parameters a and q are real, then it holds for the periodicity exponent μ that either $\mathrm{Re}(\mu) = 0$ or $\mathrm{Im}(\mu)$ is an integer.

As a consequence of Floquet's theorem, the general solution of (1.2) has the form

$$u(t) = Ae^{\mu t}\phi(t) + Be^{-\mu t}\phi(-t), \tag{1.3}$$

where A and B are constants of integration and μ and $\phi(t)$ are as described above. In the case that $\mathrm{Re}(\mu) = 0$ and $\mathrm{Im}(\mu) = r/s$, a rational fraction with $s \geq 2$, the general solution (1.3) is *periodic* and remains bounded as $t \longrightarrow \infty$ and therefore (1.3) is called a *stable solution*. If $\mathrm{Re}(\mu) = 0$ and $\mathrm{Im}(\mu)$ is irrational, then solution (1.3) is *aperiodic* and *stable*. In the particular case that $\mathrm{Re}(\mu) = 0$, $\mathrm{Im}(\mu) = n$ with n integer, solution (1.3) is classified as *unstable*. In the same way, if $\mathrm{Re}(\mu) \neq 0$ and $\mathrm{Im}(\mu)$ is an integer then (1.3) is an *unstable solution*. Since the value of μ depends on a and q, it follows that the solution of Mathieu's equation is stable for certain values of a and q, whereas it is unstable for other values. For a complete explanation of solutions of the Mathieu equation and its stability regions in the plane (a,q), the reader is referred to [3, 17, 27].

In the analysis of parametric resonance, it is common to use a generalized form of Mathieu's equation given by:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + F(t)x = 0, \tag{1.4}$$

where $F(t)$ is a periodic function of time. Equation (1.4) is called Hill's equation. Its solution is of the form (1.3) and it has similar stability properties as the Mathieu equation.

By using Floquet's theorem, it follows that if $x(t)$ is an arbitrary solution of (1.4), then

$$x(t+2\pi) = \sigma x(t). \tag{1.5}$$

Three cases are considered for (1.5) in order to determine its stability (see [27]):

$$\begin{cases} 1 \ |\sigma| > 1 & \text{then } x(t) \text{ is unstable} \\ 2 \ |\sigma| < 1 & \text{then } x(t) \text{ is stable} \\ 3 \ |\sigma| = 1 & \text{then } x(t) \text{ is unstable.} \end{cases} \tag{1.6}$$

For a complete analysis about the properties (solutions and their stability) of (1.4), the reader is again referred to [3, 17, 27] and to [7] for a geometrical approach.

1.3 Applications of Mathieu's Equation

The Mathieu equation can be used in practical applications where the problem at hand is either a boundary value problem or an initial value problem. An example of a boundary value problem is the solution of the wave equation expressed in elliptical coordinates. An application illustrating an initial value problem is that one of a pendulum with periodically varying length.

For the sake of clarity, three classical examples where the Mathieu equation appears are presented. The choice of the examples comes from the fact that we want to show the broad application of Mathieu's equation in several disciplines. Examples 1 and 2 correspond to the class of initial value problems, whereas Example 3 belongs to the class of boundary value problems.

1.3.1 Example 1: A Mechanical Mechanism

Figure 1.1 shows a link–mass–spring mechanism which can be viewed as a dynamical system covered by Mathieu's equation [17]. The mass m can move in the vertical direction and the spring and the links are assumed to be massless. The spring is unstretched when the mass m is at D. Moreover, it is assumed that $y/l \ll 1$. The driving force $F_0 = l f_0 \cos 2\omega t$ applied at the pin-joint B can be resolved into components, one along AB and the other along DA. Then, it follows that the force driving the mass in the vertical direction is $F_1 = (f_0 \cos 2\omega t)\, y$.

Using Newton's second law, it follows that the equation of motion of the system in the vertical direction is

$$m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + ky = F_1. \tag{1.7}$$

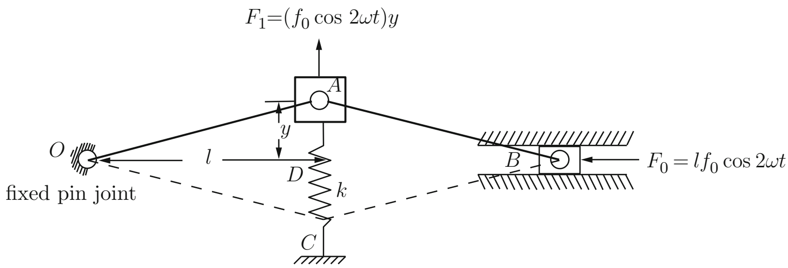

Fig. 1.1 Schematic diagram of a dynamical mechanical system covered by Mathieu's equation

Since $F_1 = (f_0 \cos 2\omega t)\, y$, (1.7) becomes

$$m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (k - f_0 \cos 2\omega t)\, y = 0. \tag{1.8}$$

Introducing a new time variable $\tau = \omega t$ leads to the standard form of the Mathieu equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2} + (a - 2q\cos 2\tau)\, y = 0, \tag{1.9}$$

where $a = k/(\omega^2 m)$ and $q = f_0/(2\omega^2 m)$.

1.3.2 Example 2: An Oscillatory Electrical Circuit

Another interesting application of the Mathieu equation is in the analysis of oscillatory circuits having time varying parameters, which have been highly important in the development of communication systems since the introduction of the super-regenerative receiver by Edwin Howard Armstrong (1890–1954) in 1922 [2]. The example at hand is the RLC circuit depicted in Fig. 1.2, which is described in [4]. The circuit contains a coil with constant inductance L, a capacitor with constant capacitance C and a time varying resistance R which is given by the expression:

$$R = \gamma + \rho(i) + R_m \sin \omega t, \tag{1.10}$$

where γ is an ohmic resistance, $\rho(i)$ is a negative resistance (accounting for regeneration, i.e., supplying energy to the circuit to reinforce the oscillations), which is dependent on the current i and can be either less, equal, or greater than γ. Finally, $R_m \sin \omega t$ is a periodic resistance.

Using Kirchhoff's voltage law it follows that the governing differential equation of the circuit of Fig. 1.2 is

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + (\gamma + \rho(i) + R_m \sin \omega t)\, i + \frac{1}{C}\int i\mathrm{d}t = 0. \tag{1.11}$$

Under the assumption that $\gamma + \rho(i) \ll R_m \sin \omega t$, (1.11) verifies [4]:

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + R_m \sin \omega t i + \frac{1}{C}\int i\mathrm{d}t = 0. \tag{1.12}$$

After taking the time derivative of (1.12), it follows that

$$\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} + \frac{R_m}{L}\sin \omega t \frac{\mathrm{d}i}{\mathrm{d}t} + \left(\omega_0^2 + \frac{\omega R_m}{L}\cos \omega t\right) i = 0, \tag{1.13}$$

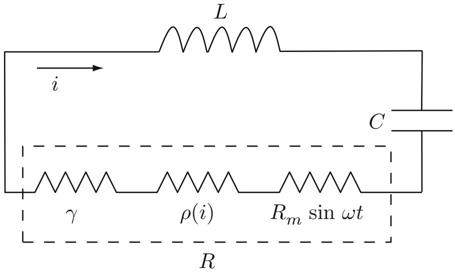

Fig. 1.2 Equivalent circuit of a super-regenerative receiver

where $\omega_0 = 1/\sqrt{LC}$ is the resonance frequency of the circuit. The damping term may be removed by the substitution $i = Iye^{(k/\omega)\cos\omega t}$ with $k = R_m/2L$. With this substitution, (1.13) becomes

$$Ie^{(k/\omega)\cos\omega t}\left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \left(\omega_0^2 + 2k\omega\cos\omega t + \frac{k^2}{2} - \frac{k^2}{2}\cos 2\omega t\right)y\right) = 0. \tag{1.14}$$

By considering the extreme cases $k \ll \omega$ and $k \gg \omega$, (1.14) reduces to

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \left(\omega_0^2 + 2\omega k\cos\omega t\right)y = 0 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \left(\omega_0^2 + \frac{k^2}{2} - \frac{k^2}{2}\cos 2\omega t\right)y = 0, \tag{1.15}$$

respectively. Both equations in (1.15) can be written in the standard form of Mathieu's equation by defining a new time variable. For instance, consider the second equation in (1.15) and define $\tau = \omega t$. This leads to

$$\frac{\mathrm{d}^2y}{\mathrm{d}\tau^2} + (a - 2q\cos 2\tau)y = 0, \tag{1.16}$$

where $a = (2\omega_0^2 + k^2)/2\omega^2$ and $q = k^2/4$. Clearly, (1.16) is the Mathieu equation.

Further applications of Mathieu's equation on oscillatory electrical circuits containing time varying parameters can be found in [5] and [17].

1.3.3 Example 3: Hydrodynamics

A considerable part of this book is dedicated to the study of parametric roll occurring in ships sailing in the sea; therefore, in this introductory chapter, an application of the Mathieu equation related to hydrodynamics is obvious. The example at hand is related with the free oscillations of water in an elliptical lake (see Fig. 1.3), whose eccentricity is close to 1. This example is described in [13]. The position of any point referred to the major and minor axes of the lake is given by the horizontal coordinates x and y with corresponding velocities u and v. Assume that the coordinate system rotates at constant angular velocity ω about the z-axis (earth's

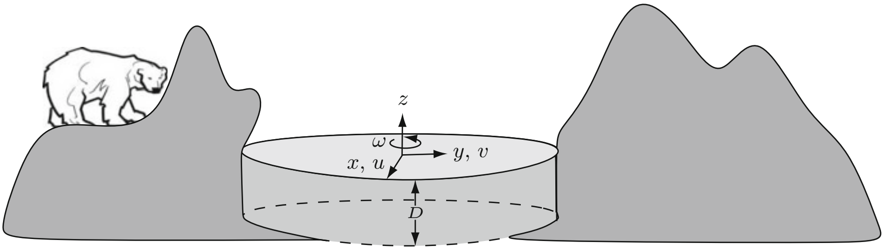

Fig. 1.3 Elliptical lake

rotation axis). It can be shown that the equations of horizontal motion (assuming infinitely small relative motion) are [14]:

$$\frac{\partial u}{\partial t} - 2\omega v = -g\frac{\partial \zeta}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial t} + 2\omega x = -g\frac{\partial \zeta}{\partial y}, \tag{1.17}$$

where g is the acceleration of gravity and ζ is the height of the free surface above its equilibrium position. Next, the equation of continuity is given by the expression

$$-\frac{\partial \zeta}{\partial t} = \frac{\partial (Du)}{\partial x} + \frac{\partial (Dv)}{\partial y}, \tag{1.18}$$

where D is the depth of the lake, which is assumed to be uniform.

Next, explicit expressions for u and v are computed by considering "perturbative" solutions for u, v, and ζ of the form $e^{i\sigma t}$ (see [12, 14]), with $\sigma = 2\pi/T$ with T being the period of tidal oscillation [25]. Hence

$$u = u_1 e^{i\sigma t} = u_1(\cos\sigma t + i\sin\sigma t) \tag{1.19}$$

$$v = v_1 e^{i\sigma t} = v_1(\cos\sigma t + i\sin\sigma t) \tag{1.20}$$

$$\zeta = \zeta_1 e^{i\sigma t} = \zeta_1(\cos\sigma t + i\sin\sigma t). \tag{1.21}$$

Substitution of (1.19)–(1.21) in (1.17)–(1.18) yields

$$i\sigma u - 2\omega v = -g\frac{\partial \zeta}{\partial x} \quad \text{and} \quad i\sigma v + 2\omega u = -g\frac{\partial \zeta}{\partial y} \tag{1.22}$$

and

$$-i\sigma\zeta = D\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \tag{1.23}$$

From (1.22) follows that

$$u = \frac{g}{\sigma^2 - 4\omega^2}\left(i\sigma\frac{\partial \zeta}{\partial x} + 2\omega\frac{\partial \zeta}{\partial y}\right) \quad \text{and} \quad v = \frac{g}{\sigma^2 - 4\omega^2}\left(i\sigma\frac{\partial \zeta}{\partial y} - 2\omega\frac{\partial \zeta}{\partial x}\right). \tag{1.24}$$

By using (1.24), (1.23) is represented as:

$$\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + \frac{\sigma^2 - 4\omega^2}{gD}\zeta = 0. \tag{1.25}$$

Equation (1.25) is the well known two-dimensional wave equation. Since the problem at hand involves a boundary condition, which is elliptical in shape, it is convenient to express (1.25) in elliptical coordinates ξ and η, which are related to x and y in the following manner

$$x + iy = h\cosh(\xi + i\eta), \tag{1.26}$$

or

$$x = h\cosh\xi\cos\eta, \quad y = h\sinh\xi\cos\eta. \tag{1.27}$$

Expressed in elliptical coordinates, (1.25) becomes (for the transformation procedure, see [17] or [28])

$$\frac{\partial^2 \zeta}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial \eta^2} + 2k^2(\cosh 2\xi - \cos 2\eta)\xi = 0, \tag{1.28}$$

where $k^2 = \left(\sigma^2 - 4\omega^2\right)h^2/8gD$.

By substituting in (1.28) a solution of the form

$$\zeta = XY \tag{1.29}$$

with X being a function only depending on ξ and Y a function only depending on η, it follows that

$$-\frac{1}{X}\left(\frac{\mathrm{d}^2 X}{\mathrm{d}\xi^2}\right) + 2k^2\cosh 2\xi = \frac{1}{Y}\left(\frac{\mathrm{d}^2 Y}{\mathrm{d}\eta^2}\right) - 2k^2\cos 2\eta. \tag{1.30}$$

It is straightforward to see that both sides of (1.30) should be equal to the same constant, say R. In consequence, it follows that

$$\frac{\mathrm{d}^2 X}{\mathrm{d}\xi^2} + \left(2k^2\cosh 2\xi - R\right)X = 0 \tag{1.31}$$

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}\eta^2} + \left(R - 2k^2\cos 2\eta\right)Y = 0. \tag{1.32}$$

Equation (1.31) is the so-called *modified* Mathieu equation [17], whereas (1.32) is the standard Mathieu equation. Then, it is clear that the problem of free oscillations of water in an elliptical lake is an example of an application of the Mathieu equation.

1.3.4 Other Modern Applications

Nowadays, the literature is very rich in applications related to the Mathieu equation. For example, this equation is considered in

- the study of the stability of structural elements such as plates and shells, which are widely used in aerospace and mechanical applications [1, 23].
- the analysis of the dynamical behavior of micro- and nano-electromechanical systems, which are very often used in actuators, sensors, and in data and communication applications [18, 21].
- the study of parametric resonance in civil structures like bridges [20].

For more applications of the Mathieu equation the reader is referred to [22] and the references there in.

As a final note, an example of parametric resonance in a biological system is briefly discussed. The squid giant axon membrane is taken as example. This axon controls part of the water jet propulsion system in squid. When considering the membrane capacitance as a periodic time varying parameter, it has been found that the membrane sensitivity to stimulation is increased due to parametric resonance [15]. However, it should be stressed that the equations describing the membrane potential response are parametrically excited first order equations, which can not be written in the form of the Mathieu equation.

1.4 Autoparametric Systems

The phenomenon of parametric resonance also occurs in a special class of nonlinear dynamical systems called *autoparametric* systems. In its simplest form, an autoparametric system consists of two nonlinearly coupled subsystems. One of the subsystems (called primary system) can be externally excited, parametrically excited, or self-excited. Due to the coupling, the other subsystem (called secondary system) can be seen as a parametrically excited system. A particular feature of autoparametric systems is that for certain values of the excitation frequency and/or certain values of the parameters, the primary system will have an oscillating response, whereas the secondary system will be at rest. When this solution becomes unstable, the system will experience autoparametric resonance, i.e., the oscillations of the primary system will produce an oscillating behavior in the secondary subsystem. In some cases, the oscillations of the secondary system will grow unbounded [26].

Perhaps one of the simplest examples of an autoparametric system is given by a pendulum attached to a mass–spring–damper system, where the mass can move in the vertical direction and is driven by a harmonic force. In such case, the primary system is given by the mass–spring–damper subsystem, whereas the pendulum is considered as the secondary subsystem. For certain intervals of the excitation frequency, the behavior of the system is as follows: the mass oscillates in the vertical axis, whereas the pendulum remains at rest. However, there are intervals of the excitation frequency such that the pendulum is parametrically excited by the oscillatory mass–spring–damper system and the pendulum will no longer stay at rest but it will show an oscillating behavior.

In the study of the stability of autoparametric systems, often the Mathieu equation plays a fundamental roll. This is demonstrated by the following academic example where the primary system is assumed to be externally excited. Note that the considered system does not model any physical system; however, it is quite useful in demonstrating some of the basic properties of autoparametric systems. This example has been presented in [26].

Consider an autoparametric system (in dimensionless form) given by the set of equations:

$$\ddot{x} + \beta_1 \dot{x} + x + \alpha_1 y^2 = k\eta^2 \cos \eta t, \tag{1.33}$$

$$\ddot{y} + \beta_2 \dot{y} + q^2 y + \alpha_2 xy = 0, \tag{1.34}$$

where the primary system (1.33) is externally excited by a harmonic force, and the secondary system (1.34) is parametrically excited. The secondary system is coupled to the first system by the term $\alpha_2 xy$, whereas the primary system is coupled to the secondary system with the term $\alpha_1 y^2$. Parameters $\beta_i \in \mathbb{R}^+$ $(i = 1, 2)$ are the damping coefficients, $q = \omega_2/\omega_1$ is the ratio of the natural frequency ω_2 of the secondary system and the natural frequency ω_1 of the primary system. The amplitude of the external excitation is given by k and the driving frequency η is given by the ratio of the (dimensional) excitation frequency ω and the natural frequency of the primary system ω_1, i.e., $\eta = \omega/\omega_1$.

This system has a semi-trivial solution (i.e., a solution where $x(t)$ and $\dot{x}(t)$ have oscillatory behavior and $y(t) = \dot{y}(t) = 0$), which is determined by substituting

$$x(t) = R\cos(\eta t + \psi) \tag{1.35}$$

$$y(t) = 0 \tag{1.36}$$

into (1.33)–(1.34). This yields an expression for R, which is

$$R = \frac{k\eta^2}{\sqrt{(1-\eta^2)^2 + \beta_1^2 \eta^2}}. \tag{1.37}$$

Then, the semi-trivial solution of (1.33)–(1.34) is given by:

$$x(t) = \frac{k\eta^2}{\sqrt{(1-\eta^2)^2 + \beta_1^2 \eta^2}} \cos(\eta t + \psi), \quad y(t) = 0, \tag{1.38}$$

$$\dot{x}(t) = -\frac{k\eta^3}{\sqrt{(1-\eta^2)^2 + \beta_1^2\eta^2}} \sin(\eta t + \psi), \;\; \dot{y}(t) = 0. \tag{1.39}$$

The stability of the semi-trivial solution (1.38)–(1.39) is determined by inserting the expressions

$$x = R\cos(\eta t + \psi_1) + u(t), \quad y = 0 + v(t), \tag{1.40}$$

where $u(t)$ and $v(t)$ are small perturbations, into (1.33)–(1.34). This leads to the linear approximation

$$\ddot{u} + \beta_1 \dot{u} + u = 0 \tag{1.41}$$

$$\ddot{v} + \beta_2 \dot{v} + \left[q^2 + \alpha_2 R\cos(\eta t + \psi_1)\right] v = 0. \tag{1.42}$$

From (1.41) it is clear that u is asymptotically stable. In consequence, the stability of (1.38)–(1.39) is completely determined by (1.42), which indeed is the Mathieu equation (1.2). In [26], it has been found that the main instability domain of (1.42) corresponds to values of $q \approx \frac{1}{2}\eta$. Indeed, by using the averaging method, it is possible to find the boundary of the main instability region and then to determine, for which values of the amplitude of the external excitation, the semi-trivial solution (1.38)–(1.39) becomes unstable, i.e., for which values of k the response $y(t) \neq 0$.

Now, an example of an autoparametric system occurring in engineering applications is considered. When studying parametric roll resonance in ships, the dynamics describing the motions of the ship in the vertical plane (heave, pitch, and roll) can be seen as an autoparametric system, where the primary system, consisting of the dynamics of heave and pitch motion, is externally excited by the ocean waves, whereas the secondary system, corresponding to the roll dynamics, is parametrically excited by the oscillations in heave and pitch. As in the previous example, the heave–pitch–roll system also accepts a semi-trivial solution similar to (1.38)–(1.39), i.e. under certain conditions confer [11], the ship will exhibit oscillations in heave and pitch directions, whereas no oscillations in roll will appear. However, it has been found that this solution is prone to become unstable when the frequency, to which the system approaches waves, is almost twice the value of the roll natural frequency. In such case parametric roll will appear.

1.5 Conclusions

This introductory chapter has been written in order to show the reader that a solid knowledge of the Mathieu equation, its properties, and solutions will facilitate the analysis of the parametric resonance phenomenon occurring in dynamical systems with time-varying parameters. By using some examples from diverse disciplines, it has been shown that since its introduction in 1868, the Mathieu equation has been of paramount importance in the development of the theory of parametrically excited systems.

A short summary about a special class of systems called autoparametric systems has been presented. In this kind of systems, parametric resonance will occur due to the interconnection of the constituting subsystems. Therefore, the theory of autoparametric systems is a very useful framework when analyzing, for instance, the parametric roll effect, because the influence of other motions of the ship can be included into the analysis.

As a final remark, it should be clear to the reader that when analyzing a single degree of freedom dynamical system with time varying parameters, the solutions of the Mathieu equation or Hill's equation are of key importance to analyze the behavior of the system, whereas when analyzing an interconnected system, where one of the subsystems acts as a parametric excitation of the other subsystem, then the theory related to autoparametric systems should be used in order to determine the behavior of the system.

In general, the chapters contained in this book are indeed applications strongly related to the Mathieu equation or Hill's equation and/or applications, where the system under consideration belongs to the class of autoparametric systems.

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