Chapter 9
Another Theorem of Burnside

In this chapter we give another application of representation theory to finite group theory, again due to Burnside. The result is based on a study of real characters and conjugacy classes.

9.1 Conjugate Representations

Recall that if $A = (a_{ij})$ is a matrix, then $\overline{A}$ is the matrix $(\overline{a_{ij}})$. One easily verifies that $\overline{AB} = \overline{A} \cdot \overline{B}$ and that if A is invertible, then so is $\overline{A}$ and moreover $\overline{A}^{-1} = \overline{A^{-1}}$. Hence if $\varphi\colon G \longrightarrow GL_d(\mathbb{C})$ is a representation of G, then we can define the *conjugate representation* $\overline{\varphi}$ by $\overline{\varphi}_g = \overline{\varphi_g}$. If $f\colon G \longrightarrow \mathbb{C}$ is a function, then define $\overline{f}$ by $\overline{f}(g) = \overline{f(g)}$.

Proposition 9.1.1. *Let $\varphi\colon G \longrightarrow GL_d(\mathbb{C})$ be a representation. Then we have $\chi_{\overline{\varphi}} = \overline{\chi_\varphi}$.*

Proof. First note that if $A \in M_d(\mathbb{C})$, then

$$\mathrm{Tr}(\overline{A}) = \overline{a_{11}} + \cdots + \overline{a_{dd}} = \overline{a_{11} + \cdots + a_{dd}} = \overline{\mathrm{Tr}(A)}.$$

Thus $\chi_{\overline{\varphi}}(g) = \mathrm{Tr}(\overline{\varphi_g}) = \overline{\mathrm{Tr}(\varphi_g)} = \overline{\chi_\varphi(g)}$, as required. □

As a consequence, we observe that the conjugate of an irreducible representation is again irreducible.

Corollary 9.1.2. *Let $\varphi\colon G \longrightarrow GL_d(\mathbb{C})$ be irreducible. Then $\overline{\varphi}$ is irreducible.*

Proof. Let $\chi = \chi_\varphi$. We compute

$$\langle \overline{\chi}, \overline{\chi} \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \chi(g) = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi(g)} = \langle \chi, \chi \rangle = 1$$

and so $\overline{\varphi}$ is irreducible by Proposition 9.1.1 and Corollary 4.3.15. □

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Quite often one can use the above corollary to produce new irreducible characters for a group. However, the case when $\overline{\chi} = \chi$ is also of importance.

Definition 9.1.3 (Real character). A character χ of G is called *real*[1] if $\chi = \overline{\chi}$, that is, $\chi(g) \in \mathbb{R}$ for all $g \in G$.

Example 9.1.4. The trivial character of a group is always real. The groups S_3 and S_4 have only real characters. On the other hand, if n is odd then $\mathbb{Z}/n\mathbb{Z}$ has no non-trivial real characters.

Since the number of irreducible characters equals the number of conjugacy classes, there should be a corresponding notion of a "real" conjugacy class. First we make two simple observations.

Proposition 9.1.5. *Let χ be a character of a group G. Then $\chi(g^{-1}) = \overline{\chi(g)}$.*

Proof. Without loss of generality, we may assume that χ is the character of a unitary representation $\varphi\colon G \longrightarrow U_n(\mathbb{C})$. Then

$$\chi(g^{-1}) = \mathrm{Tr}(\varphi_{g^{-1}}) = \mathrm{Tr}(\overline{\varphi_g}^T) = \mathrm{Tr}(\overline{\varphi_g}) = \overline{\mathrm{Tr}(\varphi_g)} = \overline{\chi(g)}$$

as required. □

Proposition 9.1.6. *Let g and h be conjugate. Then g^{-1} and h^{-1} are conjugate.*

Proof. Suppose $g = xhx^{-1}$. Then $g^{-1} = xh^{-1}x^{-1}$. □

So if C is a conjugacy class of G, then $C^{-1} = \{g^{-1} \mid g \in C\}$ is also a conjugacy class of G and moreover if χ is any character then $\chi(C^{-1}) = \overline{\chi(C)}$.

Definition 9.1.7 (Real conjugacy class). A conjugacy class C of G is said to be *real* if $C = C^{-1}$.

The following proposition motivates the name.

Proposition 9.1.8. *Let C be a real conjugacy class and χ a character of G. Then $\chi(C) = \overline{\chi(C)}$, that is, $\chi(C) \in \mathbb{R}$.*

Proof. If C is real then $\chi(C) = \chi(C^{-1}) = \overline{\chi(C)}$. □

An important result of Burnside is that the number of real irreducible characters is equal to the number of real conjugacy classes. The elegant proof we provide is due to R. Brauer and is based on the invertibility of the character table. First we prove a lemma.

Lemma 9.1.9. *Let $\varphi\colon S_n \longrightarrow GL_n(\mathbb{C})$ be the standard representation of S_n and let $A \in M_n(\mathbb{C})$ be a matrix. Then, for $\sigma \in S_n$, the matrix $\varphi_\sigma A$ is obtained from*

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[1] Some authors divide what we call real characters into two subclasses: real characters and quaternionic characters.

A by permuting the rows of A according to σ and $A\varphi_\sigma$ is obtained from A by permuting the columns of A according to σ^{-1}.

Proof. We compute $(\varphi_\sigma A)_{\sigma(i)j} = \sum_{k=1}^{n} \varphi(\sigma)_{\sigma(i)k}A_{kj} = A_{ij}$ since

$$\varphi(\sigma)_{\sigma(i)k} = \begin{cases} 1 & k = i \\ 0 & \text{else.} \end{cases}$$

Thus $\varphi_\sigma A$ is obtained from A by placing row i of A into row $\sigma(i)$. Since the representation φ is unitary and real-valued, $A\varphi_\sigma = (\varphi_\sigma^T A^T)^T = (\varphi_{\sigma^{-1}} A^T)^T$ the second statement follows from the first. □

Theorem 9.1.10 (Burnside). *Let G be a finite group. The number of real irreducible characters of G equals the number of real conjugacy classes of G.*

Proof (Brauer). Let s be the number of conjugacy classes of G. Our standing notation will be that $\chi_1, \dots, \chi_s$ are the irreducible characters of G and $C_1, \dots, C_s$ are the conjugacy classes. Define $\alpha, \beta \in S_s$ by $\overline{\chi_i} = \chi_{\alpha(i)}$ and $C_i^{-1} = C_{\beta(i)}$. Notice that χ_i is a real character if and only if $\alpha(i) = i$ and similarly C_i is a real conjugacy class if and only if $\beta(i) = i$. Therefore, $|\mathrm{Fix}(\alpha)|$ is the number of real irreducible characters and $|\mathrm{Fix}(\beta)|$ is the number of real conjugacy classes. Notice that $\alpha = \alpha^{-1}$ since α swaps the indices of χ_i and $\overline{\chi_i}$, and similarly $\beta = \beta^{-1}$.

Let $\varphi\colon S_s \longrightarrow GL_s(\mathbb{C})$ be the standard representation of S_s. Then we have $\chi_\varphi(\alpha) = |\mathrm{Fix}(\alpha)|$ and $\chi_\varphi(\beta) = |\mathrm{Fix}(\beta)|$ so it suffices to prove $\mathrm{Tr}(\varphi_\alpha) = \mathrm{Tr}(\varphi_\beta)$. Let X be the character table of G. Then, by Lemma 9.1.9, $\varphi_\alpha \mathsf{X}$ is obtained from X by swapping the rows of X corresponding to χ_i and $\overline{\chi_i}$ for each i. But this means that $\varphi_\alpha \mathsf{X} = \overline{\mathsf{X}}$. Similarly, $\mathsf{X}\varphi_\beta$ is obtained from X by swapping the columns of X corresponding to C_i and C_i^{-1} for each i. Since $\chi(C^{-1}) = \overline{\chi(C)}$ for each conjugacy class C, this swapping again results in $\overline{\mathsf{X}}$. In other words,

$$\varphi_\alpha \mathsf{X} = \overline{\mathsf{X}} = \mathsf{X}\varphi_\beta.$$

But by the second orthogonality relations (Theorem 4.4.12) the columns of X form an orthogonal set of non-zero vectors and hence are linearly independent. Thus X is invertible and so $\varphi_\alpha = \mathsf{X}\varphi_\beta\mathsf{X}^{-1}$. We conclude $\mathrm{Tr}(\varphi_\alpha) = \mathrm{Tr}(\varphi_\beta)$, as was required. □

As a consequence, we see that groups of odd order do not have non-trivial real irreducible characters.

Proposition 9.1.11. *Let G be a group. Then $|G|$ is odd if and only if G does not have any non-trivial real irreducible characters.*

Proof. By Theorem 9.1.10, it suffices to show that $\{1\}$ is the only real conjugacy class of G if and only if $|G|$ is odd. Suppose first G has even order. Then there is an element $g \in G$ of order 2. Since $g = g^{-1}$, if C is the conjugacy class of g, then $C = C^{-1}$ is real.

Suppose conversely that G contains a non-trivial real conjugacy class C. Let $g \in C$ and let $N_G(g) = \{x \in G \mid xg = gx\}$ be the normalizer of g. Then $|C| = [G : N_G(g)]$. Suppose that $hgh^{-1} = g^{-1}$. Then

$$h^2gh^{-2} = hg^{-1}h^{-1} = (hgh^{-1})^{-1} = g$$

and so $h^2 \in N_G(g)$. If $h \in \langle h^2 \rangle$, then $h \in N_G(g)$ and so $g^{-1} = hgh^{-1} = g$. Hence in this case $g^2 = 1$ and so $|G|$ is even. If $h \notin \langle h^2 \rangle$, then h^2 is not a generator of $\langle h \rangle$ and so 2 divides the order of h. Thus $|G|$ is even. This completes the proof. □

From Proposition 9.1.11, we deduce a curious result about groups of odd order that does not seem to admit a direct elementary proof.

Theorem 9.1.12 (Burnside). *Let G be a group of odd order and let s be the number of conjugacy classes of G. Then $s \equiv |G| \bmod 16$.*

Proof. By Proposition 9.1.11, G has the trivial character χ_0 and the remaining characters come in conjugate pairs $\chi_1, \chi_1', \ldots, \chi_k, \chi_k'$ of degrees $d_1, \ldots, d_k$. In particular, $s = 1 + 2k$ and

$$|G| = 1 + \sum_{j=1}^{k} 2d_j^2.$$

Since d_j divides $|G|$ it is odd and so we may write it as $d_j = 2m_j + 1$ for some non-negative integer m_j. Therefore, we have

$$\begin{aligned} |G| &= 1 + \sum_{j=1}^{k} 2(2m_j + 1)^2 = 1 + \sum_{j=1}^{k} (8m_j^2 + 8m_j + 2) \\ &= 1 + 2k + 8\sum_{j=1}^{k} m_j(m_j + 1) = s + 8\sum_{j=1}^{k} m_j(m_j + 1) \\ &\equiv s \bmod 16 \end{aligned}$$

since exactly one of m_j and $m_j + 1$ is even. □

Exercises

Exercise 9.1. Let G be a finite group.

1. Prove that two elements $g, h \in G$ are conjugate if and only if $\chi(g) = \chi(h)$ for all irreducible characters χ.
2. Show that the conjugacy class C of an element $g \in G$ is real if and only if $\chi(g)$ is real for all irreducible characters χ.

Exercise 9.2. Prove that all characters of the symmetric group S_n are real.

Exercise 9.3. Let G be a non-abelian group of order 155. Prove that G has 11 conjugacy classes.

Exercise 9.4. Let G be a group and let α be an automorphism of G.

1. If $\varphi\colon G \longrightarrow GL(V)$ is a representation, show that $\alpha^*(\varphi) = \varphi \circ \alpha$ is a representation of G.
2. If $f \in L(G)$, define $\alpha^*(f) = f \circ \alpha$. Prove that $\chi_{\alpha^*(\varphi)} = \alpha^*(\chi_\varphi)$.
3. Prove that if φ is irreducible, then so is $\alpha^*(\varphi)$.
4. Show that if C is a conjugacy class of G, then $\alpha(C)$ is also a conjugacy class of G.
5. Prove that the number of conjugacy classes C with $\alpha(C) = C$ is equal to the number of irreducible characters χ with $\alpha^*(\chi) = \chi$. (Hint: imitate the proof of Theorem 9.1.10; the key point is that permuting the rows of the character table according to α^* yields the same matrix as permuting the columns according to α.)