Chapter 7
Group Actions and Permutation Representations

In this chapter we link representation theory with the theory of group actions and permutation groups. Once again, we are only able to provide a brief glimpse of these connections; see [3] for more. In this chapter all groups are assumed to be finite and all actions of groups are taken to be on finite sets.

7.1 Group Actions

Let us begin by recalling the definition of a group action. If X is a set, then S_X will denote the symmetric group on X. We shall tacitly assume $|X| \geq 2$, as the case $|X| = 1$ is uninteresting.

Definition 7.1.1 (Group action). An *action* of a group G on a set X is a homomorphism $\sigma\colon G \longrightarrow S_X$. We often write σ_g for $\sigma(g)$. The cardinality of X is called the *degree* of the action.

Example 7.1.2 (Regular action). Let G be a group and define $\lambda\colon G \longrightarrow S_G$ by $\lambda_g(x) = gx$. Then λ is called the *regular action* of G on G.

A subset $Y \subseteq X$ is called *G-invariant* if $\sigma_g(y) \in Y$ for all $y \in Y$, $g \in G$. One can always partition X into a disjoint union of minimal G-invariant subsets called orbits.

Definition 7.1.3 (Orbit). Let $\sigma\colon G \longrightarrow S_X$ be a group action. The *orbit* of $x \in X$ under G is the set $G \cdot x = \{\sigma_g(x) \mid g \in G\}$.

Clearly, the orbits are G-invariant. A standard course in group theory proves that distinct orbits are disjoint and the union of all the orbits is X, that is, the orbits form a partition of X (cf. [11]). Of particular importance is the case when there is just one orbit.

B. Steinberg, *Representation Theory of Finite Groups: An Introductory Approach*, Universitext, DOI 10.1007/978-1-4614-0776-8_7,

Definition 7.1.4 (Transitive). A group action $\sigma\colon G \longrightarrow S_X$ is *transitive* if, for all $x, y \in X$, there exists $g \in G$ such that $\sigma_g(x) = y$. Equivalently, the action is transitive if there is one orbit of G on X.

Example 7.1.5 (Coset action). If G is a group and H a subgroup, then there is an action $\sigma\colon G \longrightarrow S_{G/H}$ given by $\sigma_g(xH) = gxH$. This action is transitive.

An even stronger property than transitivity is that of 2-transitivity.

Definition 7.1.6 (2-transitive). An action $\sigma\colon G \longrightarrow S_X$ of G on X is 2-*transitive* if given any two pairs of distinct elements $x, y \in X$ and $x', y' \in X$, there exists $g \in G$ such that $\sigma_g(x) = x'$ and $\sigma_g(y) = y'$.

Example 7.1.7 (Symmetric groups). For $n \geq 2$, the action of S_n on $\{1, \ldots, n\}$ is 2-transitive. Indeed, let $i \neq j$ and $k \neq \ell$ be pairs of elements of X. Let $X = \{1, \ldots, n\} \setminus \{i, j\}$ and $Y = \{1, \ldots, n\} \setminus \{k, \ell\}$. Then $|X| = n - 2 = |Y|$, so we can choose a bijection $\alpha\colon X \longrightarrow Y$. Define $\tau \in S_n$ by

$$\tau(m) = \begin{cases} k & m = i \\ \ell & m = j \\ \alpha(m) & \text{else.} \end{cases}$$

Then $\tau(i) = k$ and $\tau(j) = \ell$. This establishes that S_n is 2-transitive.

Let us put this notion into a more general context.

Definition 7.1.8 (Orbital). Let $\sigma\colon G \longrightarrow S_X$ be a transitive group action. Define $\sigma^2\colon G \longrightarrow S_{X \times X}$ by

$$\sigma_g^2(x_1, x_2) = (\sigma_g(x_1), \sigma_g(x_2)).$$

An orbit of σ^2 is termed an *orbital* of σ. The number of orbitals is called the *rank* of σ.

Let $\Delta = \{(x, x) \mid x \in X\}$. As $\sigma_g^2(x, x) = (\sigma_g(x), \sigma_g(x))$, it follows from the transitivity of G on X that Δ is an orbital. It is called the *diagonal* or *trivial orbital*.

Remark 7.1.9. Orbitals are closely related to graph theory. If G acts transitively on X, then any non-trivial orbital can be viewed as the edge set of a graph with vertex set X (by symmetrizing). The group G acts on the resulting graph as a vertex-transitive group of automorphisms.

Proposition 7.1.10. *Let $\sigma\colon G \longrightarrow S_X$ be a group action (with $X \geq 2$). Then σ is 2-transitive if and only if σ is transitive and* $\operatorname{rank}(\sigma) = 2$.

Proof. First we observe that transitivity is necessary for 2-transitivity since if G is 2-transitive on X and $x, y \in X$, then we may choose $x' \neq x$ and $y' \neq y$.

By 2-transitivity there exists $g \in G$ with $\sigma_g(x) = y$ and $\sigma_g(x') = y'$. This shows that σ is transitive. Next observe that

$$(X \times X) \setminus \Delta = \{(x, y) \mid x \neq y\}$$

and so the complement of Δ is an orbital if and only if, for any two pairs $x \neq y$ and $x' \neq y'$ of distinct elements, there exists $g \in G$ with $\sigma_g(x) = x'$ and $\sigma_g(y) = y'$, that is, σ is 2-transitive. □

Consequently the rank of S_n is 2. Let $\sigma\colon G \longrightarrow S_X$ be a group action. Then, for $g \in G$, we define

$$\mathrm{Fix}(g) = \{x \in X \mid \sigma_g(x) = x\}$$

to be the set of *fixed points* of g. Let $\mathrm{Fix}^2(g)$ be the set of fixed points of g on $X \times X$. The notation is unambiguous because of the following proposition.

Proposition 7.1.11. *Let $\sigma\colon G \longrightarrow S_X$ be a group action. Then the equality*

$$\mathrm{Fix}^2(g) = \mathrm{Fix}(g) \times \mathrm{Fix}(g)$$

holds. Hence $|\mathrm{Fix}^2(g)| = |\mathrm{Fix}(g)|^2$.

Proof. Let $(x, y) \in X \times X$. Then $\sigma_g^2(x, y) = (\sigma_g(x), \sigma_g(y))$ and so $(x, y) = \sigma_g^2(x, y)$ if and only if $\sigma_g(x) = x$ and $\sigma_g(y) = y$. We conclude $\mathrm{Fix}^2(g) = \mathrm{Fix}(g) \times \mathrm{Fix}(g)$. □

7.2 Permutation Representations

Given a group action $\sigma\colon G \longrightarrow S_n$, we may compose it with the standard representation $\alpha\colon S_n \longrightarrow GL_n(\mathbb{C})$ to obtain a representation of G. Let us formalize this.

Definition 7.2.1 (Permutation representation). Let $\sigma\colon G \longrightarrow S_X$ be a group action. Define a representation $\widetilde{\sigma}\colon G \longrightarrow GL(\mathbb{C}X)$ by setting

$$\widetilde{\sigma}_g\left(\sum_{x \in X} c_x x\right) = \sum_{x \in X} c_x \sigma_g(x) = \sum_{y \in X} c_{\sigma_{g^{-1}}(y)} y.$$

One calls $\widetilde{\sigma}$ the *permutation representation* associated to σ.

Remark 7.2.2. Notice that $\widetilde{\sigma}_g$ is the linear extension of the map defined on the basis X of $\mathbb{C}X$ by sending x to $\sigma_g(x)$. Also, observe that the degree of the representation $\widetilde{\sigma}$ is the same as the degree of the group action σ.

Example 7.2.3 (Regular representation). Let $\lambda\colon G \longrightarrow S_G$ be the regular action. Then one has $\widetilde{\lambda} = L$, the regular representation.

The following proposition is proved exactly as in the case of the regular representation (cf. Proposition 4.4.2), so we omit the proof.

Proposition 7.2.4. *Let $\sigma\colon G \longrightarrow S_X$ be a group action. Then the permutation representation $\widetilde{\sigma}\colon G \longrightarrow GL(\mathbb{C}X)$ is a unitary representation of G.*

Next we compute the character of $\widetilde{\sigma}$.

Proposition 7.2.5. *Let $\sigma\colon G \longrightarrow S_X$ be a group action. Then*

$$\chi_{\widetilde{\sigma}}(g) = |\mathrm{Fix}(g)|.$$

Proof. Let $X = \{x_1, \ldots, x_n\}$ and let $[\widetilde{\sigma}_g]$ be the matrix of $\widetilde{\sigma}$ with respect to this basis. Then $\widetilde{\sigma}_g(x_j) = \sigma_g(x_j)$, so

$$[\widetilde{\sigma}_g]_{ij} = \begin{cases} 1 & x_i = \sigma_g(x_j) \\ 0 & \text{else.} \end{cases}$$

In particular,

$$\begin{aligned} [\widetilde{\sigma}_g]_{ii} &= \begin{cases} 1 & x_i = \sigma_g(x_i) \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} 1 & x_i \in \mathrm{Fix}(g) \\ 0 & \text{else} \end{cases} \end{aligned}$$

and so $\chi_{\widetilde{\sigma}}(g) = \mathrm{Tr}([\widetilde{\sigma}_g]) = |\mathrm{Fix}(g)|$. □

Like the regular representation, permutation representations are never irreducible (if $|X| > 1$). To understand better how it decomposes, we first consider the trivial component.

Definition 7.2.6 (Fixed subspace). Let $\varphi\colon G \longrightarrow GL(V)$ be a representation. Then

$$V^G = \{v \in V \mid \varphi_g(v) = v \text{ for all } g \in G\}$$

is the *fixed subspace* of G.

One easily verifies that V^G is a G-invariant subspace and the subrepresentation $\varphi|_{V^G}$ is equivalent to $\dim V^G$ copies of the trivial representation. Let us prove that V^G is the direct sum of all the copies of the trivial representation in φ.

Proposition 7.2.7. *Let $\varphi\colon G \longrightarrow GL(V)$ be a representation and let χ_1 be the trivial character of G. Then $\langle \chi_\varphi, \chi_1 \rangle = \dim V^G$.*

Proof. Write $V = m_1V_1 \oplus \cdots \oplus m_sV_s$ where $V_1, \ldots, V_s$ are irreducible G-invariant subspaces whose associated subrepresentations range over the distinct equivalence classes of irreducible representations of G (we allow $m_i = 0$). Without loss of generality, we may assume that V_1 is equivalent to the trivial representation. Let $\varphi^{(i)}$ be the restriction of φ to V_i. Now if $v \in V$, then $v = v_1 + \cdots + v_s$ with the $v_i \in m_i V_i$ and

$$\varphi_g v = (m_1\varphi^{(1)})_g v_1 + \cdots + (m_s\varphi^{(s)})_g v_s = v_1 + (m_2\varphi^{(2)})_g v_2 + \cdots + (m_s\varphi^{(s)})_g v_s$$

and so $g \in V^G$ if and only if $v_i \in m_i V_i^G$ for all $2 \leq i \leq s$. In other words,

$$V^G = m_1 V_1 \oplus m_2 V_2^G \oplus \cdots \oplus m_s V_s^G.$$

Let $i \geq 2$. Since V_i is irreducible and not equivalent to the trivial representation and V_i^G is G-invariant, it follows $V_i^G = 0$. Thus $V^G = m_1 V_1$ and so the multiplicity of the trivial representation in φ is $\dim V^G$, as required. $\square$

Now we compute $\mathbb{C}X^G$ when we have a permutation representation.

Proposition 7.2.8. *Let $\sigma\colon G \longrightarrow S_X$ be a group action. Let $\mathcal{O}_1, \ldots, \mathcal{O}_m$ be the orbits of G on X and define $v_i = \sum_{x\in\mathcal{O}_i} x$. Then $v_1, \ldots, v_m$ is a basis for $\mathbb{C}X^G$ and hence $\dim \mathbb{C}X^G$ is the number of orbits of G on X.*

Proof. First observe that

$$\widetilde{\sigma}_g v_i = \sum_{x\in\mathcal{O}_i} \sigma_g(x) = \sum_{y\in\mathcal{O}_i} y = v_i$$

as is seen by setting $y = \sigma_g(x)$ and using that σ_g permutes the orbit $\mathcal{O}_i$. Thus $v_1, \ldots, v_m \in \mathbb{C}X^G$. Since the orbits are disjoint, we have

$$\langle v_i, v_j \rangle = \begin{cases} |\mathcal{O}_i| & i = j \\ 0 & i \neq j \end{cases}$$

and so $\{v_1, \ldots, v_s\}$ is an orthogonal set of non-zero vectors and hence linearly independent. It remain to prove that this set spans $\mathbb{C}X^G$.

Suppose $v = \sum_{x\in X} c_x x \in \mathbb{C}X^G$. We show that if $z \in G\cdot y$, then $c_y = c_z$. Indeed, let $z = \sigma_g(y)$. Then we have

$$\sum_{x\in X} c_x x = v = \widetilde{\sigma}_g v = \sum_{x\in X} c_x \sigma_g(x) \tag{7.1}$$

and so the coefficient of z in the left-hand side of (7.1) is c_z, whereas the coefficient of z in the right-hand side is c_y since $z = \sigma_g(y)$. Thus $c_z = c_y$. It follows that there are complex numbers $c_1, \ldots, c_m$ such that $c_x = c_i$ for all $x \in \mathcal{O}_i$. Thus

$$v = \sum_{x\in X} c_x x = \sum_{i=1}^{m} \sum_{x\in\mathcal{O}_i} c_x x = \sum_{i=1}^{m} c_i \sum_{x\in\mathcal{O}_i} x = \sum_{i=1}^{m} c_i v_i$$

and hence $v_1, \ldots, v_m$ span $\mathbb{C}X^G$, completing the proof. $\square$

Since G always has at least one orbit on X, the above result shows that the trivial representation appears as a constituent in $\widetilde{\sigma}$ and so if $|X| > 1$, then $\widetilde{\sigma}$ is not irreducible. As a corollary to the above proposition we prove a useful result, commonly known as *Burnside's lemma*, although it seems to have been known earlier to Cauchy and Frobenius. It has many applications in combinatorics to counting problems. The lemma says that the number of orbits of G on X is the average number of fixed points.

Corollary 7.2.9 (Burnside's lemma). *Let $\sigma\colon G \longrightarrow S_X$ be a group action and let m be the number of orbits of G on X. Then*

$$m = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|.$$

Proof. Let χ_1 be the trivial character of G. By Propositions 7.2.5, 7.2.7 and 7.2.8 we have

$$m = \langle \chi_{\widetilde{\sigma}}, \chi_1\rangle = \frac{1}{|G|}\sum_{g\in G}\chi_{\widetilde{\sigma}}(g)\overline{\chi_1(g)} = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|$$

as required. □

As a corollary, we obtain two formulas for the rank of σ.

Corollary 7.2.10. *Let $\sigma\colon G \longrightarrow S_X$ be a transitive group action. Then the equalities*

$$\mathrm{rank}(\sigma) = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|^2 = \langle \chi_{\widetilde{\sigma}}, \chi_{\widetilde{\sigma}}\rangle$$

hold.

Proof. Since $\mathrm{rank}(\sigma)$ is the number of orbits of σ^2 on $X \times X$ and the number of fixed points of g on $X \times X$ is $|\mathrm{Fix}(g)|^2$ (Proposition 7.1.11), the first equality is a consequence of Burnside's lemma. For the second, we compute

$$\langle \chi_{\widetilde{\sigma}}, \chi_{\widetilde{\sigma}}\rangle = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|\overline{|\mathrm{Fix}(g)|} = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|^2$$

completing the proof. □

Assume now that $\sigma\colon G \longrightarrow S_X$ is a transitive action. Let $v_0 = \sum_{x\in X} x$. Then $\mathbb{C}X^G = \mathbb{C}v_0$ by Proposition 7.2.8. Since $\widetilde{\sigma}$ is a unitary representation, $V_0 = \mathbb{C}v_0^{\perp}$ is a G-invariant subspace (cf. the proof of Proposition 3.2.3). Usually, $\mathbb{C}v_0$ is called the *trace* of σ and V_0 the *augmentation* of σ. Let $\widetilde{\sigma}'$ be the restriction of $\widetilde{\sigma}$ to V_0; we call it the *augmentation representation* associated to σ. As $\mathbb{C}X = V_0 \oplus \mathbb{C}v_0$, it follows that $\chi_{\widetilde{\sigma}} = \chi_{\widetilde{\sigma}'} + \chi_1$ where χ_1 is the trivial character. We now characterize when the augmentation representation $\widetilde{\sigma}'$ is irreducible.

Theorem 7.2.11. *Let $\sigma\colon G \longrightarrow S_X$ be a transitive group action. Then the augmentation representation $\widetilde{\sigma}'$ is irreducible if and only if G is 2-transitive on X.*

Proof. This is a simple calculation using Corollary 7.2.10 and the fact that G is 2-transitive on X if and only if $\mathrm{rank}(\sigma) = 2$ (Proposition 7.1.10). Indeed, if χ_1 is the trivial character of G, then

$$\begin{aligned}\langle \chi_{\widetilde{\sigma}'}, \chi_{\widetilde{\sigma}'}\rangle &= \langle \chi_{\widetilde{\sigma}} - \chi_1, \chi_{\widetilde{\sigma}} - \chi_1\rangle \\ &= \langle \chi_{\widetilde{\sigma}}, \chi_{\widetilde{\sigma}}\rangle - \langle \chi_{\widetilde{\sigma}}, \chi_1\rangle - \langle \chi_1, \chi_{\widetilde{\sigma}}\rangle + \langle \chi_1, \chi_1\rangle. \end{aligned} \tag{7.2}$$

Now by Proposition 7.2.8 $\langle \chi_{\widetilde{\sigma}}, \chi_1\rangle = 1$ because G is transitive. Therefore, $\langle \chi_1, \chi_{\widetilde{\sigma}}\rangle = 1$. Also, $\langle \chi_1, \chi_1\rangle = 1$. Thus (7.2) becomes, in light of Corollary 7.2.10,

$$\langle \chi_{\widetilde{\sigma}'}, \chi_{\widetilde{\sigma}'}\rangle = \mathrm{rank}(\sigma) - 1$$

and so $\chi_{\widetilde{\sigma}'}$ is an irreducible character if and only if $\mathrm{rank}(\sigma) = 2$, that is, if and only if G is 2-transitive on X. □

Remark 7.2.12. The decomposition of the standard representation of S_3 in Example 4.3.17 corresponds precisely to the decomposition into the direct sum of the augmentation and the trace.

With Theorem 7.2.11 in hand, we may now compute the character table of S_4.

Example 7.2.13 (Character table of S_4). First of all S_4 has five conjugacy classes, represented by $Id, (1\ 2), (1\ 2\ 3), (1\ 2\ 3\ 4), (1\ 2)(3\ 4)$. Let χ_1 be the trivial character and χ_2 the character of the sign homomorphism. As S_4 acts 2-transitively on $\{1, \ldots, 4\}$, Theorem 7.1.10 implies that the augmentation representation is irreducible. Let χ_4 be the character of this representation; it is the character of the standard representation minus the trivial character so $\chi_4(g) = |\mathrm{Fix}(g)| - 1$. Let $\chi_5 = \chi_2 \cdot \chi_4$. That is, if τ is the representation associated to χ_4, then we can define a new representation $\tau^{\chi_2}\colon S_4 \longrightarrow GL_3(\mathbb{C})$ by $\tau_g^{\chi_2} = \chi_2(g)\tau_g$. It is easily verified that $\chi_{\tau^{\chi_2}}(g) = \chi_2(g)\chi_4(g)$ and τ^{χ_2} is irreducible. This gives us four of the five irreducible representations. How do we get the fifth? Let d be the degree of the missing representation. Then

$$24 = |S_4| = 1^2 + 1^2 + d^2 + 3^2 + 3^2 = 20 + d^2$$

and so $d = 2$. Let χ_3 be the character of the missing irreducible representation and let L be the regular representation of S_4. Then

$$\chi_L = \chi_1 + \chi_2 + 2\chi_3 + 3\chi_4 + 3\chi_5$$

so for $Id \neq g \in S_4$, we have

$$\chi_3(g) = \frac{1}{2}\left(-\chi_1(g) - \chi_2(g) - 3\chi_4(g) - 3\chi_5(g)\right).$$

In this way we are able to produce the character table of S_4 in Table 7.1.

Table 7.1 Character table of S_4

	Id	(1 2)	(1 2 3)	(1 2 3 4)	(1 2)(3 4)
χ_1	1	1	1	1	1
χ_2	1	−1	1	−1	1
χ_3	2	0	−1	0	2
χ_4	3	1	0	−1	−1
χ_5	3	−1	0	1	−1

The reader should try to produce a representation with character χ_3. As a hint, observe that $K = \{Id, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ is a normal subgroup of S_4 and that $S_4/K \cong S_3$. Construct an irreducible representation by composing the surjective map $S_4 \longrightarrow S_3$ with the degree 2 irreducible representation of S_3 coming from the augmentation representation for S_3.

7.3 The Centralizer Algebra and Gelfand Pairs

Let $\sigma\colon G \longrightarrow S_X$ be a transitive group action. Our goal in this section is to study the ring $\mathrm{Hom}_G(\widetilde{\sigma}, \widetilde{\sigma})$. We only scratch the surface of this topic in this section. Much more information, as well as applications to probability and statistics, can be found in [3].

Let us assume that $X = \{x_1, \ldots, x_n\}$. Define a matrix representation $\varphi\colon G \longrightarrow GL_n(\mathbb{C})$ by $\varphi_g = [\widetilde{\sigma}_g]_X$. Then $\varphi \sim \widetilde{\sigma}$ and so $\mathrm{Hom}_G(\widetilde{\sigma}, \widetilde{\sigma}) \cong \mathrm{Hom}_G(\varphi_g, \varphi_g)$. Next observe that

$$\begin{aligned}\mathrm{Hom}_G(\varphi, \varphi) &= \{A \in M_n(\mathbb{C}) \mid A\varphi_g = \varphi_g A, \forall g \in G\}\\ &= \{A \in M_n(\mathbb{C}) \mid \varphi_g A \varphi_g^{-1} = A, \forall g \in G\}.\end{aligned}$$

From now on we denote $\mathrm{Hom}_G(\varphi, \varphi)$ by $C(\sigma)$ and call it the *centralizer algebra* of σ.

Proposition 7.3.1. *$C(\sigma)$ is a unital subring of $M_n(\mathbb{C})$.*

Proof. Trivially, $\varphi_g I_n \varphi_g^{-1} = I_n$ for all $g \in G$. If $A, B \in C(\sigma)$, then

$$\varphi_g(A + B)\varphi_g^{-1} = \varphi_g A \varphi_g^{-1} + \varphi_g B \varphi_g^{-1} = A + B$$

for all $g \in G$, and similarly $\varphi_g(AB)\varphi_g^{-1} = \varphi_g A \varphi_g^{-1} \varphi_g B \varphi_g^{-1} = AB$. Thus $C(\sigma)$ is indeed a unital subring of $M_n(\mathbb{C})$. □

We aim to show that $\dim C(\sigma) = \mathrm{rank}(\sigma)$ and exhibit an explicit basis. Let $V = M_n(\mathbb{C})$ and define a representation $\tau\colon G \longrightarrow GL(V)$ by $\tau_g(A) = \varphi_g A \varphi_g^{-1}$. The reader should perform the routine verification that τ is indeed a representation. Notice that

$$V^G = \{A \in M_n(\mathbb{C}) \mid \varphi_g A \varphi_g^{-1} = A, \forall g \in G\} = C(\sigma).$$

Let $\sigma^2\colon G \longrightarrow S_{X\times X}$ be as per Definition 7.1.8. We exhibit an explicit equivalence between τ and $\widetilde{\sigma^2}$. We can then use Proposition 7.2.8 to obtain a basis for $C(\sigma)$.

Proposition 7.3.2. *Define a mapping* $T\colon M_n(\mathbb{C}) \longrightarrow \mathbb{C}(X\times X)$ *by*

$$T(a_{ij}) = \sum_{i,j=1}^{n} a_{ij}(x_i, x_j)$$

where we have retained the above notation. Then T *is an equivalence between* τ *and* $\widetilde{\sigma^2}$*.*

Proof. The map T is evidently bijective and linear with inverse

$$\sum_{i,j=1}^{n} a_{ij}(x_i, x_j) \longmapsto (a_{ij}).$$

Let us check that it is an equivalence. Let $g \in G$ and let $A = (a_{ij}) \in M_n(\mathbb{C})$. Put $B = \tau_g A$; say $B = (b_{ij})$. Define an action $\gamma\colon G \longrightarrow S_n$ by $\sigma_g(x_i) = x_{\gamma_g(i)}$ for $g \in G$. Then

$$b_{ij} = \sum_{k=1,\ell=1}^{n} \varphi(g)_{ik} a_{k\ell} \varphi(g^{-1})_{\ell j} = a_{\gamma_g^{-1}(i),\gamma_g^{-1}(j)}$$

because

$$\varphi(g)_{ik} = \begin{cases} 1 & x_i = \sigma_g(x_k) \\ 0 & \text{else} \end{cases} \quad \text{and} \quad \varphi(g^{-1})_{\ell j} = \begin{cases} 1 & x_\ell = \sigma_g^{-1}(x_j) \\ 0 & \text{else.} \end{cases}$$

Therefore, we have

$$\begin{aligned} T\tau_g A &= \sum_{i,j=1}^{n} b_{ij}(x_i, x_j) = \sum_{i,j=1}^{n} a_{\gamma_g^{-1}(i),\gamma_g^{-1}(j)}(x_i, x_j) \\ &= \sum_{i,j=1}^{n} a_{ij}(\sigma_g(x_i), \sigma_g(x_j)) = \sum_{i,j=1}^{n} a_{ij}\sigma_g^2(x_i, x_j) = \widetilde{\sigma^2}_g TA \end{aligned}$$

and so T is an equivalence, as required. □

We can now provide a basis for $C(\sigma)$. If Ω is an orbital of σ, define a matrix $A(\Omega) \in M_n(\mathbb{C})$ by

$$A(\Omega)_{ij} = \begin{cases} 1 & (x_i, x_j) \in \Omega \\ 0 & \text{else.} \end{cases}$$

Corollary 7.3.3. *Let* $\sigma\colon G \longrightarrow S_X$ *be a transitive group action. We retain the above notation. Let* $\Omega_1, \ldots, \Omega_r$ *be the orbitals of* σ *where* $r = \operatorname{rank}(\sigma)$*. Then the set* $\{A(\Omega_1), \ldots, A(\Omega_r)\}$ *is a basis for* $C(\sigma)$ *and consequently* $\dim C(\sigma) = \operatorname{rank}(\sigma)$.

Proof. Proposition 7.2.8 implies that a basis for $\mathbb{C}(X \times X)^G$ is given by the elements $v_1, \ldots, v_r$ where

$$v_k = \sum_{(x_i,x_j)\in\Omega_k} (x_i, x_j).$$

Clearly, $A(\Omega_k) = T^{-1}v_k$. As T restricts to an equivalence of $C(\sigma) = M_n(\mathbb{C})^G$ and $\mathbb{C}(X \times X)^G$ (cf. Exercise 7.7), it follows that $\{A(\Omega_1), \ldots, A(\Omega_r)\}$ is a basis for $C(\sigma)$, as required. □

An important notion in applications is that of a Gelfand pair; the reader is referred to [3, 4.7] and [7, 3.F] where a Fourier transform is defined in this context and applied to probability theory.

Definition 7.3.4 (Gelfand pair). Let G be a group and H a subgroup. Let $\sigma\colon G \longrightarrow S_{G/H}$ be the coset action. Then (G, H) is said to be a *Gelfand pair* if the centralizer algebra $C(\sigma)$ is commutative.

Example 7.3.5. Let G be a group and let $H = \{1\}$. The coset action of G on G/H is none other than the regular action $\lambda\colon G \longrightarrow S_G$ and so $\widetilde{\lambda}$ is none other than the regular representation L. We claim that $C(\lambda) \cong L(G)$. For this argument, we identify the centralizer algebra with the ring $\operatorname{Hom}_G(L, L)$.

Let $T \in C(\lambda)$ and define $f_T\colon G \longrightarrow \mathbb{C}$ by

$$T1 = \sum_{x\in G} f_T(x^{-1})x.$$

We claim that the mapping $T \mapsto f_T$ is an isomorphism $\psi\colon C(\lambda) \longrightarrow L(G)$. First note that, for $g \in G$, one has

$$Tg = TL_g1 = L_gT1 = L_g\sum_{x\in G} f_T(x^{-1})x.$$

Thus T is determined by f_T and hence ψ is injective. It is also surjective because if $f\colon G \longrightarrow \mathbb{C}$ is any function, then we can define $T \in \operatorname{End}(\mathbb{C}G)$ on the basis by

$$Tg = L_g\sum_{x\in G} f(x^{-1})x.$$

First note that T belongs to the centralizer algebra because if $g, y \in G$, then

$$TL_yg = Tyg = L_{yg}\sum_{x\in G} f(x^{-1})x = L_yTg.$$

Also, we have

$$T1 = \sum_{x \in G} f(x^{-1})x$$

and so $f_T = f$. Thus ψ is surjective. Finally, we compute, for $T_1, T_2 \in C(\lambda)$,

$$\begin{aligned} T_1T_21 &= T_1 \sum_{x \in G} f_{T_2}(x^{-1})x = \sum_{x \in G} f_{T_2}(x^{-1})T_1L_x1 \\ &= \sum_{x \in G} f_{T_2}(x^{-1})L_x \sum_{y \in G} f_{T_1}(y^{-1})y = \sum_{x,y \in G} f_{T_1}(y^{-1})f_{T_2}(x^{-1})xy. \end{aligned}$$

Setting $g = xy$, $u = x^{-1}$ (and hence $y^{-1} = g^{-1}u^{-1}$) yields

$$T_1T_21 = \sum_{g \in G} \sum_{u \in G} f_{T_1}(g^{-1}u^{-1})f_{T_2}(u)g = \sum_{g \in G} f_{T_1} * f_{T_2}(g^{-1})g.$$

Thus $f_{T_1T_2} = f_{T_1} * f_{T_2}$ and so ψ is a ring homomorphism. We conclude that ψ is an isomorphism.

Consequently, $(G, \{1\})$ is a Gelfand pair if and only if G is abelian because $L(G)$ is commutative if and only if $L(G) = Z(L(G))$. But $\dim Z(L(G)) = |Cl(G)|$ and $\dim L(G) = |G|$, and so $Z(L(G)) = L(G)$ if and only if G is abelian.

It is known that (G, H) is a Gelfand pair if and only if $\widetilde{\sigma}$ is *multiplicity-free*, meaning that each irreducible constituent of $\widetilde{\sigma}$ has multiplicity one [3, Theorem 4.4.2]. We content ourselves here with the special case of so-called symmetric Gelfand pairs.

If $\sigma\colon G \longrightarrow S_X$ is a transitive group action, then to each orbital Ω of σ, we can associate its transpose

$$\Omega^T = \{(x_1, x_2) \in X \times X \mid (x_2, x_1) \in \Omega\}.$$

It is easy to see that Ω^T is indeed an orbital. Let us say that Ω is *symmetric* if $\Omega = \Omega^T$. For instance, the diagonal orbital Δ is symmetric. Notice that $A(\Omega^T) = A(\Omega)^T$ and hence Ω is symmetric if and only if the matrix $A(\Omega)$ is symmetric (and hence self-adjoint, as it has real entries).

Definition 7.3.6 (Symmetric Gelfand pair). Let G be a group and H a subgroup with corresponding group action $\sigma\colon G \longrightarrow S_{G/H}$. Then (G, H) is called a *symmetric Gelfand pair* if each orbital of σ is symmetric.

Of course, we must show that a symmetric Gelfand pair is indeed a Gelfand pair! First we provide some examples.

Example 7.3.7. Let $H \leq G$ and suppose that the action of G on G/H is 2-transitive. Then the orbitals are Δ and $(G/H \times G/H) \setminus \Delta$. Clearly, each of these is symmetric. Thus (G, H) is a symmetric Gelfand pair.

Example 7.3.8. Let $n \geq 2$ and let $[n]^2$ be the set of all two-element subsets of $\{1, \ldots, n\}$. Then S_n acts on $[n]^2$ as follows. Define $\tau\colon S_n \longrightarrow S_{[n]^2}$ by $\tau_\sigma(\{i, j\}) = \{\sigma(i), \sigma(j)\}$. This action is clearly transitive since S_n is 2-transitive on $\{1, \ldots, n\}$. Let H be the stabilizer in S_n of $\{n-1, n\}$. Notice that H is the internal direct product of S_{n-2} and $S_{\{n-1,n\}}$ and so $H \cong S_{n-2} \times S_2$. The action of S_n on $[n]^2$ can be identified with the action of S_n on S_n/H.

If Ω is a non-trivial orbital, then a typical element of Ω is of the form $(\{i, j\}, \{k, \ell\})$ where these two subsets are different. There are essentially two cases. If i, j, k, and ℓ are all distinct, then $(i\ k)(j\ \ell)$ takes the above element to $(\{k, \ell\}, \{i, j\})$ and so Ω is symmetric. Otherwise, the two subsets have an element in common, say $i = k$. Then $(j\ \ell)$ takes $(\{i, j\}, \{i, \ell\})$ to $(\{i, \ell\}, \{i, j\})$. Thus Ω is symmetric in this case, as well. We conclude (S_n, H) is a symmetric Gelfand pair.

The proof that a symmetric Gelfand pair is in fact a Gelfand pair relies on the following simple observation on rings of symmetric matrices.

Lemma 7.3.9. *Let R be a subring of $M_n(\mathbb{C})$ consisting of symmetric matrices. Then R is commutative.*

Proof. If $A, B \in R$, then $AB = (AB)^T = B^T A^T = BA$ since A, B, and AB are assumed symmetric. □

And now we turn to the proof that symmetric Gelfand pairs are Gelfand.

Theorem 7.3.10. *Let (G, H) be a symmetric Gelfand pair. Then (G, H) is a Gelfand pair.*

Proof. As usual, let $\sigma\colon G \longrightarrow S_{G/H}$ be the action map. Denote by $\Omega_1, \ldots, \Omega_r$ the orbitals of σ. Then because each Ω_i is symmetric, it follows that each matrix $A(\Omega_i)$ is symmetric for $i = 1, \ldots r$. Since the symmetric matrices form a vector subspace of $M_n(\mathbb{C})$ and $\{A(\Omega_1), \ldots, A(\Omega_r)\}$ is a basis for $C(\sigma)$ by Corollary 7.3.3, it follows that $C(\sigma)$ consists of symmetric matrices. Thus $C(\sigma)$ is commutative by Lemma 7.3.9 and so (G, H) is a Gelfand pair. □

Exercises

Exercise 7.1. Show that if $\sigma\colon G \longrightarrow S_X$ is a group action, then the orbits of G on X form a partition X.

Exercise 7.2. Let $\sigma\colon G \longrightarrow S_X$ be a transitive group action with $|X| \geq 2$. If $x \in X$, let

$$G_x = \{g \in G \mid \sigma_g(x) = x\}. \tag{7.3}$$

G_x is a subgroup of G called the *stabilizer* of x. Prove that the following are equivalent:

1. G_x is transitive on $X \setminus \{x\}$ for *some* $x \in X$;

2. G_x is transitive on $X \setminus \{x\}$ for *all* $x \in X$;
3. G acts 2-transitively on X.

Exercise 7.3. Compute the character table of A_4. (Hints:

1. Let $K = \{Id, (12)(34), (13)(24), (14)(23)\}$. Then K is a normal subgroup of A_4 and $A_4/K \cong \mathbb{Z}/3\mathbb{Z}$. Use this to construct 3 degree one representations of A_4.
2. Show that A_4 acts 2-transitively on $\{1, 2, 3, 4\}$.
3. Conclude that A_4 has four conjugacy classes and find them.
4. Produce the character table.)

Exercise 7.4. Two group actions $\sigma \colon G \longrightarrow S_X$ and $\tau \colon G \longrightarrow S_Y$ are isomorphic if there is a bijection $\psi \colon X \longrightarrow Y$ such that $\psi\sigma_g = \tau_g\psi$ for all $g \in G$.

1. Show that if $\tau \colon G \longrightarrow S_X$ is a transitive group action, $x \in X$ and G_x is the stabilizer of x (cf. (7.3)), then τ is isomorphic to the coset action $\sigma \colon G \longrightarrow S_{G/G_x}$.
2. Show that if σ and τ are isomorphic group actions, then the corresponding permutation representations are equivalent.

Exercise 7.5. Let p be a prime. Let G be the group of all permutations $\mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$ of the form $x \mapsto ax + b$ with $a \in \mathbb{Z}/p\mathbb{Z}^*$ and $b \in \mathbb{Z}/p\mathbb{Z}$. Prove that the action of G on $\mathbb{Z}/p\mathbb{Z}$ is 2-transitive.

Exercise 7.6. Let G be a finite group.

1. Suppose that G acts transitively on a finite set X with $|X| \geq 2$. Show that there is an element $g \in G$ with no fixed points on X. (Hint: Assume that the statement is false. Use that the identity e has $|X|$ fixed points to contradict Burnside's lemma.)
2. Let H be a proper subgroup of G. Prove that

$$G \neq \bigcup_{x \in G} xHx^{-1}.$$

(Hint: Use the previous part.)

Exercise 7.7. Let $\varphi \colon G \longrightarrow GL(V)$ and $\rho \colon G \longrightarrow GL(W)$ be representations and suppose that $T \colon V \longrightarrow W$ is an equivalence. Show that $T(V^G) = W^G$ and the restriction $T \colon V^G \longrightarrow W^G$ is an equivalence.

Exercise 7.8. Show that if Ω is an orbital of a transitive group action $\sigma \colon G \longrightarrow S_X$, then the transpose Ω^T is an orbital of σ.

Exercise 7.9. Suppose that G is a finite group of order n with s conjugacy classes. Suppose that one chooses a pair $(g, h) \in G \times G$ uniformly at random. Prove that the probability g and h commute is s/n. (Hint: Apply Burnside's lemma to the action of G on itself by conjugation.)

Exercise 7.10. Give a direct combinatorial proof of Burnside's lemma, avoiding character theory.

Exercise 7.11. Let G be a group and define $\Lambda\colon G \longrightarrow GL(L(G))$ by putting $\Lambda_g(f)(h) = f(g^{-1}h)$.

1. Verify that Λ is a representation.
2. Prove that Λ is equivalent to the regular representation L.
3. Let K be a subgroup of G. Let $L(G/K)$ be the subspace of $L(G)$ consisting of functions $f\colon G \longrightarrow \mathbb{C}$ that are *right K-invariant*, that is, $f(gk) = f(g)$ for all $k \in K$. Show that $L(G/K)$ is a G-invariant subspace of $L(G)$ and that the restriction of Λ to $L(G/K)$ is equivalent to the permutation representation $\mathbb{C}(G/K)$. (Hint: show that $L(G/K)$ has a basis consisting of functions that are constant on left cosets of K and compute the character.)