# Chapter 3
# Group Representations

The goal of group representation theory is to study groups via their actions on vector spaces. Consideration of groups acting on sets leads to such important results as the Sylow theorems. By studying actions on vector spaces even more detailed information about a group can be obtained. This is the subject of representation theory. Our study of matrix representations of groups will lead us naturally to Fourier analysis and the study of complex-valued functions on a group. This in turn has applications to various disciplines like engineering, graph theory, and probability, just to name a few.

## 3.1 Basic Definitions and First Examples

The reader should recall from group theory that an *action* of a group $G$ on a set $X$ is by definition a homomorphism $\varphi\colon G \longrightarrow S_X$, where $S_X$ is the symmetric group on $X$. This motivates the following definition.

**Definition 3.1.1 (Representation).** A *representation* of a group $G$ is a homomorphism $\varphi\colon G \longrightarrow GL(V)$ for some (finite-dimensional) vector space $V$. The dimension of $V$ is called the *degree of* $\varphi$. We usually write $\varphi_g$ for $\varphi(g)$ and $\varphi_g(v)$, or simply $\varphi_g v$, for the action of $\varphi_g$ on $v \in V$.

*Remark 3.1.2.* We shall rarely have occasion to consider degree zero representations and so the reader can safely ignore them. That is, we shall tacitly assume in this text that representations are non-zero, although this is not formally part of the definition.

A particularly simple example of a representation is the trivial representation.

*Example 3.1.3 (Trivial representation).* The trivial representation of a group $G$ is the homomorphism $\varphi\colon G \longrightarrow \mathbb{C}^*$ given by $\varphi(g) = 1$ for all $g \in G$.

B. Steinberg, *Representation Theory of Finite Groups: An Introductory Approach*, Universitext, DOI 10.1007/978-1-4614-0776-8_3,

Let us consider some other examples of degree one representations.

*Example 3.1.4.* $\varphi\colon \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{C}^*$ given by $\varphi([m]) = (-1)^m$ is a representation.

*Example 3.1.5.* $\varphi\colon \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{C}^*$ given by $\varphi([m]) = i^m$ is a representation.

*Example 3.1.6.* More generally, $\varphi\colon \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{C}^*$ defined by $\varphi([m]) = e^{2\pi i m/n}$ is a representation.

Let $\varphi\colon G \longrightarrow GL(V)$ be a representation of degree $n$. To a basis $B$ for $V$, we can associate a vector space isomorphism $T\colon V \longrightarrow \mathbb{C}^n$ by taking coordinates. We can then define a representation $\psi\colon G \longrightarrow GL_n(\mathbb{C})$ by setting $\psi_g = T\varphi_g T^{-1}$ for $g \in G$. If $B'$ is another basis, we have another isomorphism $S\colon V \longrightarrow \mathbb{C}^n$, and hence a representation $\psi'\colon G \longrightarrow GL_n(\mathbb{C})$ given by $\psi'_g = S\varphi_g S^{-1}$. The representations $\psi$ and $\psi'$ are related via the formula $\psi'_g = ST^{-1}\psi_g TS^{-1} = (ST^{-1})\psi_g(ST^{-1})^{-1}$. We want to think of $\varphi$, $\psi$, and $\psi'$ as all being the same representation. This leads us to the important notion of equivalence.

**Definition 3.1.7 (Equivalence).** Two representations $\varphi\colon G \longrightarrow GL(V)$ and $\psi\colon G \longrightarrow GL(W)$ are said to be *equivalent* if there exists an isomorphism $T\colon V \longrightarrow W$ such that $\psi_g = T\varphi_g T^{-1}$ for all $g \in G$, i.e., $\psi_g T = T\varphi_g$ for all $g \in G$. In this case, we write $\varphi \sim \psi$. In pictures, we have that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi_g} & V \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ W & \xrightarrow[\psi_g]{} & W \end{array}$$

commutes, meaning that either of the two ways of going from the upper left to the lower right corner of the diagram give the same answer.

*Example 3.1.8.* Define $\varphi\colon \mathbb{Z}/n\mathbb{Z} \longrightarrow GL_2(\mathbb{C})$ by

$$\varphi_{[m]} = \begin{bmatrix} \cos\left(\dfrac{2\pi m}{n}\right) & -\sin\left(\dfrac{2\pi m}{n}\right) \\ \sin\left(\dfrac{2\pi m}{n}\right) & \cos\left(\dfrac{2\pi m}{n}\right) \end{bmatrix},$$

which is the matrix for rotation by $2\pi m/n$, and $\psi\colon \mathbb{Z}/n\mathbb{Z} \longrightarrow GL_2(\mathbb{C})$ by

$$\psi_{[m]} = \begin{bmatrix} e^{\frac{2\pi m i}{n}} & 0 \\ 0 & e^{\frac{-2\pi m i}{n}} \end{bmatrix}.$$

Then $\varphi \sim \psi$. To see this, let

$$A = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix},$$

and so

$$A^{-1} = \frac{1}{2i}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}.$$

Then direct computation shows

$$\begin{aligned} A^{-1}\varphi_{[m]}A &= \frac{1}{2i}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix}\begin{bmatrix} \cos\left(\frac{2\pi m}{n}\right) & -\sin\left(\frac{2\pi m}{n}\right) \\ \sin\left(\frac{2\pi m}{n}\right) & \cos\left(\frac{2\pi m}{n}\right) \end{bmatrix}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2i}\begin{bmatrix} e^{\frac{2\pi mi}{n}} & ie^{\frac{2\pi mi}{n}} \\ -e^{\frac{-2\pi mi}{n}} & ie^{\frac{-2\pi mi}{n}} \end{bmatrix}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2i}\begin{bmatrix} 2ie^{\frac{2\pi mi}{n}} & 0 \\ 0 & 2ie^{\frac{-2\pi mi}{n}} \end{bmatrix} \\ &= \psi_{[m]}. \end{aligned}$$

The following representation of the symmetric group is very important.

*Example 3.1.9 (Standard representation of $S_n$).* Define $\varphi\colon S_n \longrightarrow GL_n(\mathbb{C})$ on the standard basis by $\varphi_\sigma(e_i) = e_{\sigma(i)}$. One obtains the matrix for $\varphi_\sigma$ by permuting the rows of the identity matrix according to $\sigma$. So, for instance, when $n = 3$ we have

$$\varphi_{(1\ 2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \varphi_{(1\ 2\ 3)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Notice that in Example 3.1.9

$$\varphi_\sigma(e_1 + e_2 + \cdots + e_n) = e_{\sigma(1)} + e_{\sigma(2)} + \cdots + e_{\sigma(n)} = e_1 + e_2 + \cdots + e_n$$

where the last equality holds since $\sigma$ is a permutation and addition is commutative. Thus $\mathbb{C}(e_1 + \cdots + e_n)$ is invariant under all the $\varphi_\sigma$ with $\sigma \in S_n$. This leads to the following definition.

**Definition 3.1.10 ($G$-invariant subspace).** Let $\varphi\colon G \longrightarrow GL(V)$ be a representation. A subspace $W \leq V$ is *$G$-invariant* if, for all $g \in G$ and $w \in W$, one has $\varphi_g w \in W$.

For $\psi$ from Example 3.1.8, $\mathbb{C}e_1$ and $\mathbb{C}e_2$ are both $\mathbb{Z}/n\mathbb{Z}$-invariant and $\mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$. This is the kind of situation we would like to happen always.

**Definition 3.1.11 (Direct sum of representations).** Suppose that representations $\varphi^{(1)}\colon G \longrightarrow GL(V_1)$ and $\varphi^{(2)}\colon G \longrightarrow GL(V_2)$ are given. Then their (external) *direct sum*

$$\varphi^{(1)} \oplus \varphi^{(2)}\colon G \longrightarrow GL(V_1 \oplus V_2)$$

is given by

$$(\varphi^{(1)} \oplus \varphi^{(2)})_g(v_1, v_2) = (\varphi^{(1)}_g(v_1), \varphi^{(2)}_g(v_2)).$$

Let us try to understand direct sums in terms of matrices. Suppose that $\varphi^{(1)}\colon G \longrightarrow GL_m(\mathbb{C})$ and $\varphi^{(2)}\colon G \longrightarrow GL_n(\mathbb{C})$ are representations. Then

$$\varphi^{(1)} \oplus \varphi^{(2)}\colon G \longrightarrow GL_{m+n}(\mathbb{C})$$

has block matrix form

$$(\varphi^{(1)} \oplus \varphi^{(2)})_g = \begin{bmatrix} \varphi^{(1)}_g & 0 \\ 0 & \varphi^{(2)}_g \end{bmatrix}.$$

*Example 3.1.12.* Define representations $\varphi^{(1)}\colon \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{C}^*$ by $\varphi^{(1)}_{[m]} = e^{2\pi im/n}$, and $\varphi^{(2)}\colon \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{C}^*$ by $\varphi^{(2)}_{[m]} = e^{-2\pi im/n}$. Then

$$(\varphi^{(1)} \oplus \varphi^{(2)})_{[m]} = \begin{bmatrix} e^{\frac{2\pi im}{n}} & 0 \\ 0 & e^{\frac{-2\pi im}{n}} \end{bmatrix}.$$

*Remark 3.1.13.* If $n > 1$, then the representation $\rho\colon G \longrightarrow GL_n(\mathbb{C})$ given by $\rho_g = I$ all $g \in G$ is *not* equivalent to the trivial representation; rather, it is equivalent to the direct sum of $n$ copies of the trivial representation.

Since representations are a special kind of homomorphism, if a group $G$ is generated by a set $X$, then a representation $\varphi$ of $G$ is determined by its values on $X$; of course, not any assignment of matrices to the generators gives a valid representation!

*Example 3.1.14.* Let $\rho\colon S_3 \longrightarrow GL_2(\mathbb{C})$ be specified on the generators $(1\ 2)$ and $(1\ 2\ 3)$ by

$$\rho_{(1\ 2)} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix},\ \rho_{(1\ 2\ 3)} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

(check this is a representation!) and let $\psi\colon S_3 \longrightarrow \mathbb{C}^*$ be defined by $\psi_\sigma = 1$. Then

$$(\rho \oplus \psi)_{(12)} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ (\rho \oplus \psi)_{(123)} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We shall see later that $\rho \oplus \psi$ is equivalent to the representation of $S_3$ considered in Example 3.1.9.

Let $\varphi\colon G \longrightarrow GL(V)$ be a representation. If $W \leq V$ is a $G$-invariant subspace, we may restrict $\varphi$ to obtain a representation $\varphi|_W\colon G \longrightarrow GL(W)$ by setting $(\varphi|_W)_g(w) = \varphi_g(w)$ for $w \in W$. Precisely because $W$ is $G$-invariant, we have $\varphi_g(w) \in W$. Sometime one says $\varphi|_W$ is a *subrepresentation* of $\varphi$. If $V_1, V_2 \leq V$ are $G$-invariant and $V = V_1 \oplus V_2$, then one easily verifies $\varphi$ is equivalent to the

**Table 3.1** Analogies between groups, vector spaces, and representations

| Groups | Vector spaces | Representations |
|---|---|---|
| Subgroup | Subspace | $G$-invariant subspace |
| Simple group | One-dimensional subspace | Irreducible representation |
| Direct product | Direct sum | Direct sum |
| Isomorphism | Isomorphism | Equivalence |

(external) direct sum $\varphi|_{V_1} \oplus \varphi|_{V_2}$. It is instructive to see this in terms of matrices. Let $\varphi^{(i)} = \varphi|_{V_i}$ and choose bases $B_1$ and $B_2$ for $V_1$ and $V_2$, respectively. Then it follows from the definition of a direct sum that $B = B_1 \cup B_2$ is a basis for $V$. Since $V_i$ is $G$-invariant, we have $\varphi_g(B_i) \subseteq V_i = \mathbb{C}B_i$. Thus we have in matrix form

$$[\varphi_g]_B = \begin{bmatrix} [\varphi^{(1)}]_{B_1} & 0 \\ 0 & [\varphi^{(2)}]_{B_2} \end{bmatrix}$$

and so $\varphi \sim \varphi^{(1)} \oplus \varphi^{(2)}$.

In mathematics, it is often the case that one has some sort of unique factorization into primes, or irreducibles. This is the case for representation theory. The notion of "irreducible" in this context is modeled on the notion of a simple group.

**Definition 3.1.15 (Irreducible representation).** A non-zero representation $\varphi\colon G \longrightarrow GL(V)$ of a group $G$ is said to be *irreducible* if the only $G$-invariant subspaces of $V$ are $\{0\}$ and $V$.

*Example 3.1.16.* Any degree one representation $\varphi\colon G \longrightarrow \mathbb{C}^*$ is irreducible, since $\mathbb{C}$ has no proper non-zero subspaces.

Table 3.1 exhibits some analogies between the concepts we have seen so far with ones from Group Theory and Linear Algebra.

If $G = \{1\}$ is the trivial group and $\varphi\colon G \longrightarrow GL(V)$ is a representation, then necessarily $\varphi_1 = I$. So to give a representation of the trivial group is the same thing as to give a vector space. For the trivial group, a $G$-invariant subspace is nothing more than a subspace. A representation of the trivial group is irreducible if and only if it has degree one. So the middle column of Table 3.1 is a special case of the third column.

*Example 3.1.17.* The representations from Example 3.1.8 are not irreducible. For instance,

$$\mathbb{C}\begin{bmatrix} i \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbb{C}\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

are $\mathbb{Z}/n\mathbb{Z}$-invariant subspaces for $\varphi$, while the coordinate axes $\mathbb{C}e_1$ and $\mathbb{C}e_2$ are invariant subspaces for $\psi$.

Not surprisingly, after the one-dimensional representations, the next easiest class to analyze consists of the two-dimensional representations.

*Example 3.1.18.* The representation $\rho\colon S_3 \longrightarrow GL_2(\mathbb{C})$ from Example 3.1.14 is irreducible.

*Proof.* Since $\dim\ \mathbb{C}^2 = 2$, any non-zero proper $S_3$-invariant subspace $W$ is one-dimensional. Let $v$ be a non-zero vector in $W$; so $W = \mathbb{C}v$. Let $\sigma \in S_3$. Then $\rho_\sigma(v) = \lambda v$ for some $\lambda \in \mathbb{C}$, since by $S_3$-invariance of $W$ we have $\rho_\sigma(v) \in W = \mathbb{C}v$. It follows that $v$ must be an eigenvector for all the $\rho_\sigma$ with $\sigma \in S_3$.

*Claim.* $\rho_{(1\ 2)}$ and $\rho_{(1\ 2\ 3)}$ do not have a common eigenvector.

Indeed, direct computation reveals $\rho_{(1\ 2)}$ has eigenvalues $1$ and $-1$ with

$$V_{-1} = \mathbb{C}e_1 \text{ and } V_1 = \mathbb{C}\begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Clearly $e_1$ is not an eigenvector of $\rho_{(1\ 2\ 3)}$ as

$$\rho_{(1\ 2\ 3)}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Also,

$$\rho_{(1\ 2\ 3)}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix},$$

so $(-1, 2)$ is not an eigenvector of $\rho_{(1\ 2\ 3)}$. Thus $\rho_{(1\ 2)}$ and $\rho_{(1\ 2\ 3)}$ have no common eigenvector, which implies that $\rho$ is irreducible by the discussion above. □

Let us summarize as a proposition the idea underlying this example.

**Proposition 3.1.19.** *If $\varphi\colon G \longrightarrow GL(V)$ is a representation of degree 2 (i.e., $\dim V = 2$), then $\varphi$ is irreducible if and only if there is no common eigenvector $v$ to all $\varphi_g$ with $g \in G$.*

Notice that this trick of using eigenvectors only works for degree 2 and degree 3 representations (and the latter case requires finiteness of $G$).

*Example 3.1.20.* Let $r$ be rotation by $\pi/2$ and $s$ be reflection over the $x$-axis. These permutations generate the dihedral group $D_4$. Let the representation $\varphi\colon D_4 \longrightarrow GL_2(\mathbb{C})$ be defined by

$$\varphi(r^k) = \begin{bmatrix} i^k & 0 \\ 0 & (-i)^k \end{bmatrix},\ \varphi(sr^k) = \begin{bmatrix} 0 & (-i)^k \\ i^k & 0 \end{bmatrix}.$$

Then one can apply the previous proposition to check that $\varphi$ is an irreducible representation.

Our eventual goal is to show that each representation is equivalent to a direct sum of irreducible representations. Let us define some terminology to this effect.

**Definition 3.1.21 (Completely reducible).** Let $G$ be a group. A representation $\varphi\colon G \longrightarrow GL(V)$ is said to be *completely reducible* if $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ where the $V_i$ are $G$-invariant subspaces and $\varphi|_{V_i}$ is irreducible for all $i = 1, \ldots, n$.

Equivalently, $\varphi$ is completely reducible if $\varphi \sim \varphi^{(1)} \oplus \varphi^{(2)} \oplus \cdots \oplus \varphi^{(n)}$ where the $\varphi^{(i)}$ are irreducible representations.

**Definition 3.1.22 (Decomposable representation).** A non-zero representation $\varphi$ of a group $G$ is *decomposable* if $V = V_1 \oplus V_2$ with $V_1, V_2$ non-zero $G$-invariant subspaces. Otherwise, $V$ is called *indecomposable*.

Complete reducibility is the analog of diagonalizability in representation theory. Our aim is then to show that any representation of a finite group is completely reducible. To do this we show that any representation is either irreducible or decomposable, and then proceed by induction on the degree. First we must show that these notions depend only on the equivalence class of a representation.

**Lemma 3.1.23.** *Let $\varphi\colon G \longrightarrow GL(V)$ be equivalent to a decomposable representation. Then $\varphi$ is decomposable.*

*Proof.* Let $\psi\colon G \longrightarrow GL(W)$ be a decomposable representation with $\psi \sim \varphi$ and $T\colon V \longrightarrow W$ a vector space isomorphism with $\varphi_g = T^{-1}\psi_g T$. Suppose that $W_1$ and $W_2$ are non-zero invariant subspaces of $W$ with $W = W_1 \oplus W_2$. Since $T$ is an equivalence we have that

$$\begin{array}{ccc} V & \xrightarrow{\varphi_g} & V \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ W & \xrightarrow[\psi_g]{} & W \end{array}$$

commutes, i.e., $T\varphi_g = \psi_g T$ for all $g \in G$. Let $V_1 = T^{-1}(W_1)$ and $V_2 = T^{-1}(W_2)$. First we claim that $V = V_1 \oplus V_2$. Indeed, if $v \in V_1 \cap V_2$, then $Tv \in W_1 \cap W_2 = \{0\}$ and so $Tv = 0$. But $T$ is injective so this implies $v = 0$. Next, if $v \in V$, then $Tv = w_1 + w_2$ some $w_1 \in W_1$ and $w_2 \in W_2$. Then $v = T^{-1}w_1 + T^{-1}w_2 \in V_1 + V_2$. Thus $V = V_1 \oplus V_2$.

Finally, we show that $V_1, V_2$ are $G$-invariant. If $v \in V_i$, then $\varphi_g v = T^{-1}\psi_g T v$. But $Tv \in W_i$ implies $\psi_g T v \in W_i$ since $W_i$ is $G$-invariant. Therefore, we conclude that $\varphi_g v = T^{-1}\psi_g T v \in T^{-1}(W_i) = V_i$, as required. □

We have the analogous results for other types of representations, whose proofs we omit.

**Lemma 3.1.24.** *Let $\varphi\colon G \longrightarrow GL(V)$ be equivalent to an irreducible representation. Then $\varphi$ is irreducible.*

**Lemma 3.1.25.** *Let $\varphi\colon G \longrightarrow GL(V)$ be equivalent to a completely reducible representation. Then $\varphi$ is completely reducible.*

## 3.2 Maschke's Theorem and Complete Reducibility

In order to effect direct sum decompositions of representations, we take advantage of the tools of inner products and orthogonal decompositions.

**Definition 3.2.1 (Unitary representation).** Let $V$ be an inner product space. A representation $\varphi\colon G \longrightarrow GL(V)$ is said to be *unitary* if $\varphi_g$ is unitary for all $g \in G$, i.e.,

$$\langle \varphi_g(v), \varphi_g(w)\rangle = \langle v, w\rangle$$

for all $v, w \in W$. In other words, we may view $\varphi$ as a map $\varphi\colon G \longrightarrow U(V)$.

Identifying $GL_1(\mathbb{C})$ with $\mathbb{C}^*$, we see that a complex number $z$ is unitary (viewed as a matrix) if and only if $\overline{z} = z^{-1}$, that is $z\overline{z} = 1$. But this says exactly that $|z| = 1$, so $U_1(\mathbb{C})$ is exactly the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ in $\mathbb{C}$. Hence a one-dimensional unitary representation is a homomorphism $\varphi\colon G \longrightarrow \mathbb{T}$.

*Example 3.2.2.* Define $\varphi\colon \mathbb{R} \longrightarrow \mathbb{T}$ by $\varphi(t) = e^{2\pi i t}$. Then $\varphi$ is a unitary representation of the additive group of $\mathbb{R}$ since $\varphi(t+s) = e^{2\pi i(t+s)} = e^{2\pi i t}e^{2\pi i s} = \varphi(t)\varphi(s)$.

A crucial fact, which makes unitary representations so useful, is that every indecomposable unitary representation is irreducible as the following proposition shows.

**Proposition 3.2.3.** *Let $\varphi\colon G \longrightarrow GL(V)$ be a unitary representation of a group. Then $\varphi$ is either irreducible or decomposable.*

*Proof.* Suppose $\varphi$ is not irreducible. Then there is a non-zero proper $G$-invariant subspace $W$ of $U$. Its orthogonal complement $W^{\perp}$ is then also non-zero and $V = W \oplus W^{\perp}$. So it remains to prove that $W^{\perp}$ is $G$-invariant. If $v \in W^{\perp}$ and $w \in W$, then

$$\langle \varphi_g(v), w\rangle = \langle \varphi_{g^{-1}}\varphi_g(v), \varphi_{g^{-1}}(w)\rangle \tag{3.1}$$

$$= \langle v, \varphi_{g^{-1}}(w)\rangle \tag{3.2}$$

$$= 0 \tag{3.3}$$

where (3.1) follows because $\varphi$ is unitary, (3.2) follows because $\varphi_{g^{-1}}\varphi_g = \varphi_1 = I$ and (3.3) follows because $\varphi_{g^{-1}}w \in W$, as $W$ is $G$-invariant, and $v \in W^{\perp}$. We conclude $\varphi$ is decomposable. □

It turns out that for finite groups every representation is equivalent to a unitary one. This is not true for infinite groups, as we shall see momentarily.

**Proposition 3.2.4.** *Every representation of a finite group $G$ is equivalent to a unitary representation.*

*Proof.* Let $\varphi\colon G \longrightarrow GL(V)$ be a representation where $\dim V = n$. Choose a basis $B$ for $V$, and let $T\colon V \longrightarrow \mathbb{C}^n$ be the isomorphism taking coordinates with respect to $B$. Then setting $\rho_g = T\varphi_g T^{-1}$, for $g \in G$, yields a representation $\rho\colon G \longrightarrow GL_n(\mathbb{C})$ equivalent to $\varphi$. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{C}^n$. We define a new inner product $(\cdot, \cdot)$ on $\mathbb{C}^n$ using the crucial "averaging trick." It will be a frequent player throughout the text. Without further ado, define

$$(v, w) = \sum_{g \in G} \langle \rho_g v, \rho_g w \rangle.$$

This summation over $G$, of course, requires that $G$ is finite. It can be viewed as a "smoothing" process.

Let us check that this is indeed an inner product. First we check:

$$\begin{aligned}
(c_1 v_1 + c_2 v_2, w) &= \sum_{g \in G} \langle \rho_g (c_1 v_1 + c_2 v_2), \rho_g w \rangle \\
&= \sum_{g \in G} \left[ c_1 \langle \rho_g v_1, \rho_g w \rangle + c_2 \langle \rho_g v_2, \rho_g w \rangle \right] \\
&= c_1 \sum_{g \in G} \langle \rho_g v_1, \rho_g w \rangle + c_2 \sum_{g \in G} \langle \rho_g v_2, \rho_g w \rangle \\
&= c_1 (v_1, w) + c_2 (v_2, w).
\end{aligned}$$

Next we verify:

$$\begin{aligned}
(w, v) &= \sum_{g \in G} \langle \rho_g w, \rho_g v \rangle \\
&= \sum_{g \in G} \overline{\langle \rho_g v, \rho_g w \rangle} \\
&= \overline{(v, w)}.
\end{aligned}$$

Finally, observe that

$$(v, v) = \sum_{g \in G} \langle \rho_g v, \rho_g v \rangle \geq 0$$

because each term $\langle \rho_g v, \rho_g v \rangle \geq 0$. If $(v, v) = 0$, then

$$0 = \sum_{g \in G} \langle \rho_g v, \rho_g v \rangle$$

which implies $\langle \rho_g v, \rho_g v \rangle = 0$ for all $g \in G$ since we are adding non-negative numbers. Hence, $0 = \langle \rho_1 v, \rho_1 v \rangle = \langle v, v \rangle$, and so $v = 0$. We have now established that $(\cdot, \cdot)$ is an inner product.

To verify that the representation is unitary with respect to this inner product, we compute

$$(\rho_h v, \rho_h w) = \sum_{g \in G} \langle \rho_g \rho_h v, \rho_g \rho_h v \rangle = \sum_{g \in G} \langle \rho_{gh} v, \rho_{gh} w \rangle.$$

We now apply a change of variables by setting $x = gh$. As $g$ ranges over all $G$, $x$ ranges over all elements of $G$ since if $k \in G$, then when $g = kh^{-1}$, $x = k$. Therefore,

$$(\rho_h v, \rho_h w) = \sum_{x \in G} \langle \rho_x v, \rho_x w \rangle = (v, w).$$

This completes the proof. □

As a corollary we obtain that every indecomposable representation of a finite group is irreducible.

**Corollary 3.2.5.** *Let $\varphi \colon G \longrightarrow GL(V)$ be a non-zero representation of a finite group. Then $\varphi$ is either irreducible or decomposable.*

*Proof.* By Proposition 3.2.4, $\varphi$ is equivalent to a unitary representation $\rho$. Proposition 3.2.3 then implies that $\rho$ is either irreducible or decomposable. Lemmas 3.1.23 and 3.1.24 then yield that $\varphi$ is either irreducible or decomposable, as was desired. □

The following example shows that Corollary 3.2.5 fails for infinite groups and hence Proposition 3.2.4 must also fail for infinite groups.

*Example 3.2.6.* We provide an example of an indecomposable representation of $\mathbb{Z}$, which is not irreducible. Define $\varphi \colon \mathbb{Z} \longrightarrow GL_2(\mathbb{C})$ by

$$\varphi(n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

It is straightforward to verify that $\varphi$ is a homomorphism. The vector $e_1$ is an eigenvector of $\varphi(n)$ for all $n \in \mathbb{Z}$ and so $\mathbb{C}e_1$ is a $\mathbb{Z}$-invariant subspace. This shows that $\varphi$ is not irreducible. On the other hand, if $\varphi$ were decomposable, it would be equivalent to a direct sum of one-dimensional representations. Such a representation is diagonal. But we saw in Example 2.3.5 that $\varphi(1)$ is not diagonalizable. It follows that $\varphi$ is indecomposable.

*Remark 3.2.7.* Observe that any irreducible representation is indecomposable. The previous example shows that the converse fails.

The next theorem is the central result of this chapter. Its proof is quite analogous to the proof of the existence of a prime factorization of an integer or of a factorization of polynomials into irreducibles.

**Theorem 3.2.8 (Maschke).** *Every representation of a finite group is completely reducible.*

*Proof.* Let $\varphi\colon G \longrightarrow GL(V)$ be a representation of a finite group $G$. The proof proceeds by induction on the degree of $\varphi$, that is, $\dim V$. If $\dim V = 1$, then $\varphi$ is irreducible since $V$ has no non-zero proper subspaces. Assume the statement is true for $\dim V \leq n$. Let $\varphi\colon G \longrightarrow GL(V)$ be a representation with $\dim V = n + 1$. If $\varphi$ is irreducible, then we are done. Otherwise, $\varphi$ is decomposable by Corollary 3.2.5, and so $V = V_1 \oplus V_2$ where $0 \neq V_1, V_2$ are $G$-invariant subspaces. Since $\dim V_1, \dim V_2 < \dim V$, by induction, $\varphi|_{V_1}$ and $\varphi|_{V_2}$ are completely reducible. Therefore, $V_1 = U_1 \oplus \cdots \oplus U_s$ and $V_2 = W_1 \oplus \cdots \oplus W_r$ where the $U_i, W_j$ are $G$-invariant and the subrepresentations $\varphi|_{U_i}, \varphi|_{W_j}$ are irreducible for all $1 \leq i \leq s, 1 \leq j \leq r$. Then $V = U_1 \oplus \cdots U_s \oplus W_1 \oplus \cdots \oplus W_r$ and hence $\varphi$ is completely irreducible. □

*Remark 3.2.9.* If one follows the details of the proof carefully, one can verify that if $\varphi$ is a unitary matrix representation, then $\varphi$ is equivalent to a direct sum of irreducible unitary representations via an equivalence implemented by a unitary matrix $T$.

In conclusion, if $\varphi\colon G \longrightarrow GL_n(\mathbb{C})$ is any representation of a finite group, then

$$\varphi \sim \begin{bmatrix} \varphi^{(1)} & 0 & \cdots & 0 \\ 0 & \varphi^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \varphi^{(m)} \end{bmatrix}$$

where the $\varphi^{(i)}$ are irreducible for all $i$. This is analogous to the spectral theorem stating that all self-adjoint matrices are diagonalizable.

There still remains the question as to whether the decomposition into irreducible representations is unique. This will be resolved in the next chapter.

## Exercises

**Exercise 3.1.** Let $\varphi\colon D_4 \longrightarrow \mathrm{GL}_2(\mathbb{C})$ be the representation given by

$$\varphi(r^k) = \begin{bmatrix} i^k & 0 \\ 0 & (-i)^k \end{bmatrix}, \; \varphi(sr^k) = \begin{bmatrix} 0 & (-i)^k \\ i^k & 0 \end{bmatrix}$$

where $r$ is rotation counterclockwise by $\pi/2$ and $s$ is reflection over the $x$-axis. Prove that $\varphi$ is irreducible.

**Exercise 3.2.** Prove Lemma 3.1.24.

**Exercise 3.3.** Let $\varphi, \psi\colon G \longrightarrow \mathbb{C}^*$ be one-dimensional representations. Show that $\varphi$ is equivalent to $\psi$ if and only if $\varphi = \psi$.

**Exercise 3.4.** Let $\varphi\colon G \longrightarrow \mathbb{C}^*$ be a representation. Suppose that $g \in G$ has order $n$.

1. Show that $\varphi(g)$ is an $n\mathrm{th}$-root of unity (i.e., a solution to the equation $z^n = 1$).
2. Construct $n$ inequivalent one-dimensional representations $\mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{C}^*$.
3. Explain why your representations are the only possible one-dimensional representations.

**Exercise 3.5.** Let $\varphi\colon G \longrightarrow GL(V)$ be a representation of a finite group $G$. Define the *fixed subspace*

$$V^G = \{v \in V \mid \varphi_g v = v, \forall g \in G\}.$$

1. Show that $V^G$ is a $G$-invariant subspace.
2. Show that
$$\frac{1}{|G|}\sum_{h\in G}\varphi_h v \in V^G$$
for all $v \in V$.
3. Show that if $v \in V^G$, then
$$\frac{1}{|G|}\sum_{h\in G}\varphi_h v = v.$$
4. Conclude $\dim V^G$ is the rank of the operator
$$P = \frac{1}{|G|}\sum_{h\in G}\varphi_h.$$
5. Show that $P^2 = P$.
6. Conclude $\mathrm{Tr}(P)$ is the rank of $P$.
7. Conclude
$$\dim V^G = \frac{1}{|G|}\sum_{h\in G}\mathrm{Tr}(\varphi_h).$$

**Exercise 3.6.** Let $\varphi\colon G \longrightarrow GL_n(\mathbb{C})$ be a representation.

1. Show that setting $\psi_g = \overline{\varphi_g}$ provides a representation $\psi\colon G \longrightarrow GL_n(\mathbb{C})$. It is called the *conjugate representation*. Give an example showing that $\varphi$ and $\psi$ do not have to be equivalent.
2. Let $\chi\colon G \longrightarrow \mathbb{C}^*$ be a degree 1 representation of $G$. Define a map $\varphi^\chi\colon G \longrightarrow GL_n(\mathbb{C})$ by $\varphi^\chi_g = \chi(g)\varphi_g$. Show that $\varphi^\chi$ is a representation. Give an example showing that $\varphi$ and $\varphi^\chi$ do not have to be equivalent.

**Exercise 3.7.** Give a bijection between unitary, degree one representations of $\mathbb{Z}$ and elements of $\mathbb{T}$.

**Exercise 3.8.**

1. Let $\varphi\colon G \longrightarrow GL_3(\mathbb{C})$ be a representation of a finite group. Show that $\varphi$ is irreducible if and only if there is no common eigenvector for the matrices $\varphi_g$ with $g \in G$.
2. Given an example of a finite group $G$ and a decomposable representation $\varphi\colon G \longrightarrow GL_4(\mathbb{C})$ such that the $\varphi_g$ with $g \in G$ do not have a common eigenvector.