3
Interaction with fluids

3.1 Cattaneo theories

Within the field of fluid mechanics modifications of the Navier-Stokes equations to incorporate finite speed heat transport via a Cattaneo - like theory have not been as prevalent as they are in solid mechanics. The earliest approaches to doing this would appear to be those of (Müller, 1967b), of (Fox, 1969b) and of (Carrassi and Morro, 1972). Second sound in fluid mechanics has been known for a long time through heat waves in Helium II below the lambda point of about 2.2°K. (Peshkov, 1944) reports results of experiments on Helium II in which he detects a heat wave. (Peshkov, 1947) further analyses experimental results and relates these to Landau's theory. A review of the physics literature on this subject may be found in (Donnelly, 2009).

(Fox, 1969b) adopts a very general approach at the outset and writes the constitutive theory for the Helmholtz free energy function, ψ, stress tensor, t_{ij}, and entropy, η, as functions of the variables $F_{iA}, \theta, \theta_{,i}$ and q_i, these being the deformation gradient $F_{iA} = \partial x_i / \partial X_A$, temperature θ, and heat flux q_i. He proposes instead of a Fourier law for the heat flux $\mathbf{q}$, a general rate-type equation of form

$$h_k(\mathbf{F}, \theta, \theta_{,i}, q_i, \dot{\mathbf{F}}, \dot{\theta}, \dot{\theta}_{,i}, \dot{q}_i) = 0,$$

where the vector h_k is a linear function in each of the variables $\dot{\mathbf{F}}, \dot{\theta}, \dot{\theta}_{,i}$, and $\dot{q}_i$. In these expressions a superposed dot denotes the material derivative,

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e.g.

$$\dot{q}_i = \frac{\partial q_i}{\partial t} + v_j \frac{\partial q_i}{\partial x_j} \,.$$

He develops a general theory for what he calls a fluid phase, (Fox, 1969b), section 4. His full theory is totally nonlinear and involves a very general set of equations for a viscous fluid. However, he also develops a reduced theory for an inviscid fluid. (Fox, 1969b) stresses the use of an objective derivative rather than the material derivative $\dot{q}_i$ for the heat flux. His inviscid theory is based on the equations

$$\begin{aligned}
&\dot{\rho} + \rho d_{ii} = 0\,, \\
&\rho \dot{v}_i = \rho b_i - p_{,i}\,, \\
&\rho \theta \dot{\eta} + q_{i,i} = \rho r - 2\rho \frac{\partial \psi}{\partial \xi}\, q_i(\epsilon_1 \theta_{,i} + \epsilon_2 q_i), \\
&\dot{q}_i - \omega_{ij} q_j = \epsilon_1 \theta_{,i} + \epsilon_2 q_i\,,
\end{aligned} \tag{3.1}$$

where $\xi = q_i q_i$, $\omega_{ij} = (v_{i,j} - v_{j,i})/2$, and

$$p = \rho^2 \frac{\partial \psi}{\partial \rho}\,, \qquad \eta = -\frac{\partial \psi}{\partial \theta}\,.$$

The coefficients ϵ_1 and ϵ_2 are, in general, nonlinear functions of the scalar variables $\rho, \theta, \theta_{,i}\theta_{,i}, \xi$, and $\theta_{,i} q_i$. The derivative $\dot{q}_i - \omega_{ij} q_j$ is an objective (Jaumann) derivative. (Fox, 1969b) applies his theory to describe a fountain effect, and shows his theory is consistent with heat travelling as a wave.

(Müller, 1967b) adopts a different approach. He writes equations for $\dot{q}_i$, $\dot{t}_{ij}$ and couples these with the balances of mass, energy and momentum. This is effectively requiring the system of equations to form a hyperbolic system from the outset. The paper of (Müller, 1967b) has been very influential in that he developed the idea of an extended theory of thermodynamics. Theories of extended thermodynamics are described in detail in the books of (Müller and Ruggeri, 1998), (Jou et al., 2010a) and of (Lebon et al., 2008). We do not pursue this here, although the interested reader might wish to consult the article of (Muschik, 2007). For a gas, there is a connection with extended thermodynamics and the early work of (Grad, 1949), based on kinetic theory. We think it is worth drawing attention to the paper of (Truesdell, 1976) who writes, ... "*to claim that the kinetic theory can bear in any way upon the principle of material frame - indifference is presently ridiculous.*" (Truesdell, 1976) also writes, ... "*The kinetic theory of gases provides little support for continuum mechanics except in very special flows,*" and he writes, ..."*He who regards the kinetic theory as providing the one and only right approach to gas flows should discard all of continuum mechanics, not just one or another part of it.*" Whether one regards an equation like $(3.1)_4$ as a balance law or as a constitutive equation is a matter of some controversy in the literature. For the case of a balance law the material derivative, $\dot{q}_i$, is employed. When $(3.1)_4$ is regarded as a constitutive equation then an objective derivative is preferred for $\dot{q}_i$. Fourier's law, $q_i = -k\theta_{,i}$, is a constitutive equation and one viewpoint is to regard equation $(3.1)_4$ as a generalization of Fourier's law. Then, an objective derivative for $\dot{q}_i$ is natural. (Dauby et al., 2002) write, ..."*When the constitutive equations (like* $(3.1)_4$*) are used to describe heat transfer in a moving fluid as in the present work, it is important to recall that objective time derivatives (Jou et al., 2010a) must be introduced instead of the partial time derivatives.*" (The words in brackets have been added.) (Carrassi and Morro, 1972) also adopt a different approach. While they are interested in acoustic waves they do develop a general theory for a viscous fluid. They have the standard equations for balance of mass, momentum, and energy, namely

$$\begin{aligned} &\dot{\rho} + \rho d_{ii} = 0, \\ &\rho \dot{v}_i = -p_{,i} + t_{ji,j}\,, \\ &\rho \dot{\epsilon} = -p d_{ii} + t_{ij} d_{ij} - q_{i,i}\,. \end{aligned}$$

However, in addition to adopting a relaxation law for q_i they adopt a similar relation for the (extra) stress tensor t_{ij}. Thus, (Carrassi and Morro, 1972) suggest employing the evolution equations

$$\tau \frac{\partial q_i}{\partial t} + q_i = -k\theta_{,i}\,,$$

and

$$\tau_v \frac{\partial t_{ij}}{\partial t} + t_{ij} = 2\mu d_{ij} + \lambda \delta_{ij} d_{rr}\,.$$

The constant τ_v is a relaxation time for the stress. The paper of (Carrassi and Morro, 1972) then focusses on acoustic waves in some detail.

(Morro, 1980) is also interested in describing wave motion in a heat conducting viscous fluid. His is an inspiring paper which involves the use of hidden variables. (Morro, 1980) uses the balance equations

$$\begin{aligned} &\dot{\rho} + \rho d_{ii} = 0, \\ &\rho \dot{v}_i = t_{ji,j} + \rho b_i\,, \\ &\rho \dot{\epsilon} = \rho r + t_{ij} d_{ij} - q_{i,i}\,. \end{aligned} \tag{3.2}$$

However, he works with hidden variables, and these are the vector, α^1_i, and a tensor, α^2_{ij}, in component form; in direct notation the hidden variables are $\boldsymbol{\alpha}^1$ and $\boldsymbol{\alpha}^2$. (These, in certain cases approach the heat flux and stress tensor, respectively.) The governing equations for $\boldsymbol{\alpha}^1$ and $\boldsymbol{\alpha}^2$ have form

$$\begin{aligned} &\tau_1 \dot{\alpha}^1_i + \alpha^1_i = \theta_{,i} \\ &\tau_2 \dot{\alpha}^2_{ij} + \alpha^2_{ij} = d_{ij} \end{aligned}$$

for constants $\tau_1, \tau_2 > 0$. (Morro, 1980) shows that thermodynamics requires

$$\eta = -\frac{\partial \psi}{\partial \theta}, \qquad t_{ij} = -\rho^2 \psi_\rho \delta_{ij} + \frac{\rho}{\tau_2}\frac{\partial \psi}{\partial \alpha^2_{ij}}, \qquad q_i = -\frac{\rho\theta}{\tau_1}\frac{\partial \psi}{\partial \alpha^1_i},$$

and the free energy must have form

$$\psi = \Psi(\theta, \rho) + \frac{1}{\rho}\left[\frac{\kappa\tau_1}{2\theta}\,\alpha^1_i\alpha^1_i + \mu\tau_2\alpha^2_{ij}\alpha^2_{ij} + \frac{\lambda\tau_2}{2}(\alpha^2_{ii})^2\right].$$

The constitutive theory of (Morro, 1980) then becomes

$$\begin{aligned} \eta &= -\Psi_\theta + \frac{\kappa\tau_1}{2\rho\theta^2}\,\alpha^1_i\alpha^1_i\,, \\ t_{ij} &= -p\delta_{ij} + 2\mu\alpha^2_{ij} + \lambda\alpha^2_{rr}\delta_{ij}\,, \\ q_i &= -\kappa\alpha^1_i\,. \end{aligned}$$

(Morro, 1980) shows how one may develop an acceleration wave analysis in detail. It is important that he shows the free energy and the entropy depend on the variable α^1_i which is closely related to the heat flux q_i. (Morro, 1980) also considers objective derivatives for $\boldsymbol{\alpha}^1$ and $\boldsymbol{\alpha}^2$ which are generalizations of those of (Fox, 1969b).

3.1.1 Cattaneo-Fox theory

(Straughan and Franchi, 1984) adopted a specific form of incompressible thermoviscous fluid equations which uses a Boussinesq approximation in the buoyancy term in the momentum equation. They also employed the Jaumann derivative of (Fox, 1969b) for q_i in a Cattaneo model. Thus, the Cattaneo-Fox equations proposed by (Straughan and Franchi, 1984) have form

$$\begin{aligned} \dot{v}_i &= -\frac{1}{\rho}p_{,i} + k_i g\alpha\theta + \nu\Delta v_i\,, \\ v_{i,i} &= 0, \\ \dot{\theta} &= -q_{i,i}\,, \\ \tau(\dot{q}_i - \epsilon_{ijk}\omega_j q_k) &= -q_i - \kappa\theta_{,i}\,. \end{aligned} \tag{3.3}$$

Here $\mathbf{k} = (0, 0, 1)$ and $\boldsymbol{\omega} = \mathrm{curl}\,\mathbf{v}/2$. The quantities g, α, ν, τ and κ are, respectively, gravity, the thermal expansion coefficient of the fluid, kinematic viscosity, thermal relaxation time, and thermal conductivity of the fluid. In deriving equation $(3.3)_1$ one begins with the balance of momentum equation

$$\rho\dot{v}_i = t_{ji,j} + \rho f_i \tag{3.4}$$

where t_{ij} and f_i are the stress tensor and body force, respectively. For an incompressible, linear viscous fluid $t_{ij} = -p\delta_{ij} + 2\mu d_{ij}$, where μ is the dynamic viscosity and d_{ij} is the symmetric part of the velocity gradient, namely $d_{ij} = (v_{i,j} + v_{j,i})/2$. We note $\nu = \mu/\rho$ and suppose in the body force term $\mathbf{f} = -g\mathbf{k}$, and ρ is a linear function of temperature θ, i.e.

$$\rho = \rho_0\big(1 - \alpha(\theta - \theta_0)\big), \tag{3.5}$$

where ρ_0 is the value of ρ when $\theta = \theta_0$, and $\alpha(> 0)$ is the thermal expansion coefficient of the fluid. Then equation (3.4) becomes with ρ replaced by the constant ρ_0,

$$\rho_0 \dot{v}_i = -p_{,i} + 2\mu d_{ij,j} - \rho_0\big(1 - \alpha(\theta - \theta_0)\big)gk_i. \tag{3.6}$$

We note $2d_{ij,j} = \Delta v_i$ since $v_{j,j} = 0$ and we incorporate the constant terms $\rho_0[1 + \alpha\theta_0]g$ into p, i.e. redefine

$$p \to p + \rho_0 g[1 + \alpha\theta_0]z.$$

Then upon division by ρ_0 and replacing ρ_0 by a constant ρ, equation (3.6) yields equation $(3.3)_1$.

(Lebon and Cloot, 1984) suggested modifying the Jaumann derivative in (3.3) and studied a thermal convection problem incorporating the effect of surface tension.

3.1.2 Cattaneo-Christov theory

(Christov, 2009) is an inspiring piece of work and he has suggested another objective derivative be employed for q_i. He suggests the following Lie derivative which is based on very sound physical principles,

$$\dot{q}_i - q_j v_{i,j} + q_i d_{rr} \equiv \frac{\partial q_i}{\partial t} + v_j q_{i,j} - q_j v_{i,j} + v_{r,r} q_i \,. \tag{3.7}$$

When the fluid is incompressible $d_{rr} = 0$ and then instead of equations (3.3) one may pose the Cattaneo-Christov equations for thermoviscous fluid motions, namely

$$\begin{aligned} \dot{v}_i &= -\frac{1}{\rho}p_{,i} + k_i g\alpha\theta + \nu\Delta v_i \,, \\ v_{i,i} &= 0, \\ \dot{\theta} &= -q_{i,i} \,, \\ \tau(q_{i,t} + v_j q_{i,j} - q_j v_{i,j}) &= -q_i - \kappa\theta_{,i} \,. \end{aligned} \tag{3.8}$$

Uniqueness and structural stability questions for a general Cattaneo-Christov fluid are presented by (Ciarletta and Straughan, 2010). These writers allow compressibility but they restrict attention to the case where the velocity field is *a priori* known. A uniqueness result for the incompressible heat conducting Cattaneo-Christov model is given by (Tibullo and Zampoli, 2011).

A general non-isothermal thermodynamic theory for a compressible gas which is based on the Cattaneo-Christov equations is derived by

(Straughan, 2010a). He shows how an acceleration wave may propagate and derives an explicit formula for the wavespeeds. The Cattaneo-Christov theory has been placed on a sound thermodynamic footing by (Morro, 2010). He derives objective evolution equations for both the heat flux and the stress which allow the body to deform and are completely compatible with thermodynamics.

3.1.3 Guyer-Krumhansl model

(Franchi and Straughan, 1994b) suggested modifying equation $(3.3)_4$ by adding Guyer-Krumhansl terms for q_i. In this way one derives instead of (3.3) the system

$$\begin{aligned} \dot{v}_i &= -\frac{1}{\rho}p_{,i} + k_i g\alpha\theta + \nu\Delta v_i\,, \\ v_{i,i} &= 0, \\ \dot{\theta} &= -q_{i,i}\,, \\ \tau(\dot{q}_i - \epsilon_{ijk}\omega_j q_k) &= -q_i - \kappa\theta_{,i} + \hat{\tau}(\Delta q_i + 2q_{k,ki})\,, \end{aligned} \tag{3.9}$$

where the relaxation time $\hat{\tau}$ is discussed in section 1.3. (Franchi and Straughan, 1994b) study thermal convection on the basis of these equations.

(Dauby et al., 2002) propose a similar set of equations to (3.9) and investigate thermal convection also incorporating surface tension effects at a free surface.

3.1.4 Alternative Guyer-Krumhansl model

In view of the findings of (Straughan, 2010d; Straughan, 2010c) on thermal convection employing the Cattaneo-Christov equatons (3.8), it may be also worth considering a Guyer - Krumhansl invariant. Then, one would modify equations (3.8) to

$$\begin{aligned} \dot{v}_i &= -\frac{1}{\rho}p_{,i} + k_i g\alpha\theta + \nu\Delta v_i\,, \\ v_{i,i} &= 0, \\ \dot{\theta} &= -q_{i,i}\,, \\ \tau(q_{i,t} + v_j q_{i,j} - q_j v_{i,j}) &= -q_i - \kappa\theta_{,i} + \hat{\tau}(\Delta q_i + 2q_{k,ki})\,. \end{aligned} \tag{3.10}$$

3.1.5 Further Cattaneo type fluid models

(Puri and Kythe, 1997) worked with system (3.3) and solved a problem of a plate moving in a Maxwell-Cattaneo fluid. This allowed them to simplify the equations and seek a solution $\mathbf{v} = (0, 0, u(x, t))$ with a temperature field $\theta(x, t)$, x being the one-dimensional spatial variable. The reduced (linear) system of equations they worked with is

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + G\theta,\\ \lambda P\frac{\partial^2\theta}{\partial t^2} + P\frac{\partial\theta}{\partial t} &= \frac{\partial^2\theta}{\partial x^2}\,,\end{aligned} \tag{3.11}$$

where λ, P, G are positive constants.

(Puri and Kythe, 1998) analysed a similar class of problem but when the stress tensor is allowed to have a non-Newtonian form. In this case instead of equations (3.11) they derived the system

$$\begin{aligned}\frac{\partial u}{\partial t} - k\frac{\partial^3 u}{\partial t\partial x^2} &= \frac{\partial^2 u}{\partial x^2} + G\theta,\\ \lambda P\frac{\partial^2\theta}{\partial t^2} + P\frac{\partial\theta}{\partial t} &= \frac{\partial^2\theta}{\partial x^2}\,,\end{aligned} \tag{3.12}$$

where k is another positive constant, this term representing the viscoelastic effect.

(Puri and Jordan, 1999b) [see also (Puri and Jordan, 1999a)] analysed a problem of an oscillating vertical plate which is periodically heated. They adopted a Maxwell-Cattaneo fluid but also assumed the fluid was of dipolar type. This led them to study the system of equations

$$\begin{aligned}\frac{\partial u}{\partial t} - \ell_1^2\frac{\partial^3 u}{\partial t\partial x^2} &= \frac{\partial^2 u}{\partial x^2} - \ell_2^2\frac{\partial^4 u}{\partial x^4} + G\theta,\\ \lambda P\frac{\partial^2\theta}{\partial t^2} + P\frac{\partial\theta}{\partial t} &= \frac{\partial^2\theta}{\partial x^2}\,.\end{aligned} \tag{3.13}$$

The coefficient ℓ_2^2 is a positive dipolar constant.

If we analyse a problem like that of (Puri and Kythe, 1997) but instead of using equations (3.3) we employ the GMC system (3.9) then we may arrive at the system of partial differential equations

$$\begin{aligned}&\frac{\partial u}{\partial t} - \nu\Delta u = G\theta,\\ &c\frac{\partial\theta}{\partial t} = -\frac{\partial q_i}{\partial x_i}\,,\\ &\tau\,\frac{\partial^2 q_i}{\partial t\partial x_i} = -\frac{\partial q_i}{\partial x_i} - \kappa\Delta\theta + 3\hat{\tau}\,\Delta\frac{\partial q_i}{\partial x_i}\,.\end{aligned} \tag{3.14}$$

Upon elimination of $q_{i,i}$ we find

$$\begin{aligned}&\frac{\partial u}{\partial t} - \nu\Delta u = G\theta,\\ &\tau_1\frac{\partial^2\theta}{\partial t^2} + c\frac{\partial\theta}{\partial t} - \kappa\Delta\theta - \tau_2\,\Delta\frac{\partial\theta}{\partial t} = 0,\end{aligned} \tag{3.15}$$

where $\tau_1 = \tau c > 0$ and $\tau_2 = 3\hat{\tau}c > 0$.

Thus, equations (3.11), (3.12) and (3.15) represent interaction of a fluid with an MC or GMC thermodynamic law in a suitable linear sense.

The simplified systems (3.11), (3.12) or (3.15) are certainly much more amenable to analysis then the original systems (3.3) or (3.9).

3.2 Green-Laws theory

(Müller, 1971b) begins with the equations of balance of mass, balance of linear momentum, and balance of energy without body force and without heat supply, namely

$$\begin{aligned}
&\dot{\rho} + \rho v_{i,i} = 0, \\
&\rho \dot{v}_i - t_{ji,j} = 0, \\
&\rho \dot{\epsilon} + q_{i,i} = t_{ij} v_{i,j} \,,
\end{aligned}$$

where $\rho, v_i, t_{ij}, \epsilon, q_i$ are density, velocity, stress tensor, internal energy, and heat flux, respectively. He assumes constitutive theory of form

$$\begin{aligned}
t_{ij} &= t_{ij}(\rho, \theta, \theta_{,i}, \dot{\theta}, d_{rs}) \\
q_i &= q_i(\rho, \theta, \theta_{,i}, \dot{\theta}, d_{rs}) \\
\epsilon &= \epsilon(\rho, \theta, \theta_{,i}, \dot{\theta}, d_{rs})
\end{aligned} \tag{3.16}$$

where θ is the temperature, $d_{ij} = (v_{i,j} + v_{j,i})/2$. He exploits his entropy inequality

$$\rho \dot{\eta} + \Phi_{i,i} \geq 0,$$

for an entropy flux vector $\mathbf{\Phi}$ which like the entropy, η, depends on the constitutive list (3.16). (Müller, 1971b) derives equations for a viscous fluid and for an inviscid fluid. He also shows how one may include a body force and a heat supply and use the classical arguments of Coleman and Noll to reduce the constitutive theory.

In this section we describe the equations for an inviscid fluid derived using the thermodynamic arguments of (Green and Laws, 1972). The details may be found in (Lindsay and Straughan, 1978).

The equations presented by (Lindsay and Straughan, 1978) are conservation of mass, linear momentum, angular momentum, and energy and have form

$$\begin{aligned}
&\dot{\rho} + \rho v_{i,i} = 0, \\
&\rho \dot{v}_i = \rho f_i + t_{ji,j}, \\
&t_{ij} = t_{ji} \,, \\
&\rho \dot{\epsilon} = \rho r - q_{i,i} + t_{ij} d_{ij}
\end{aligned} \tag{3.17}$$

where f_i and r are the body force and externally supplied heat supply, respectively.

The entropy inequality employed is that of (Green and Laws, 1972),

$$\frac{d}{dt}\int_V \rho\eta\, dV - \int_V \frac{\rho r}{\phi}\, dV + \oint_{\partial V} \frac{q_i n_i}{\phi}\, dS \geq 0 \tag{3.18}$$

where V is any subbody in a continuous body B. The notation ∂V denotes the boundary of V, η is the entropy function and ϕ is a scalar function to be more precisely identified. In terms of the Helmholtz free energy function ψ,

$$\psi = \epsilon - \eta\phi \tag{3.19}$$

one may reduce (3.18) to a pointwise form and rewrite it with the aid of equation $(3.17)_4$ as

$$-\rho(\dot{\psi} + \eta\dot{\phi}) + t_{ij}d_{ij} - \frac{q_i\phi_{,i}}{\phi} \geq 0. \tag{3.20}$$

To describe an inviscid (perfect) fluid (Lindsay and Straughan, 1978) suppose that the functions

$$\psi, \phi, \eta, q_i, t_{ij} \tag{3.21}$$

depend on the independent variables

$$\rho, \theta, \dot{\theta}, \theta_{,i} \tag{3.22}$$

where θ is the temperature in the body. For the scalars ψ, ϕ, η (Lindsay and Straughan, 1978) show that the list (3.22) may be replaced by

$$\rho, \theta, \dot{\theta}, \lambda \tag{3.23}$$

where $\lambda = \theta_{,i}\theta_{,i}/2$. The forms (3.21) - (3.22) are now inserted into the entropy inequality (3.20) and (Lindsay and Straughan, 1978) deduce that

$$\phi = \phi(\theta, \dot{\theta}), \qquad \psi = \psi(\rho, \theta, \dot{\theta}, \lambda), \tag{3.24}$$

$$\eta = -\frac{\partial\psi/\partial\dot{\theta}}{\partial\phi/\partial\dot{\theta}} = \eta(\rho, \theta, \dot{\theta}, \lambda), \tag{3.25}$$

$$q_i = -K\theta_{,i}\,, \tag{3.26}$$

$$K = \frac{\rho\phi\partial\psi/\partial\lambda}{\partial\phi/\partial\dot{\theta}} = K(\rho, \theta, \dot{\theta}, \lambda), \tag{3.27}$$

$$t_{ij} = -p\delta_{ij} - \rho\frac{\partial\psi}{\partial\lambda}\,\theta_{,i}\theta_{,j}\,, \tag{3.28}$$

$$p = \rho^2\frac{\partial\psi}{\partial\rho} = p(\rho, \theta, \dot{\theta}, \lambda). \tag{3.29}$$

What remains of the entropy inequality (3.20) is

$$-\rho\Big(\frac{\partial\psi}{\partial\theta} + \eta\frac{\partial\phi}{\partial\theta}\Big)\dot{\theta} + 2K\frac{\partial\phi}{\partial\theta}\,\frac{\lambda}{\phi} \geq 0. \tag{3.30}$$

From this inequality (Lindsay and Straughan, 1978) deduce that in thermodynamic equilibrium (for which $\dot{\theta} = 0$, $\theta_{,i} = 0$ and is denoted by E) the following relations hold

$$\left(\frac{\partial\psi}{\partial\theta} + \eta\frac{\partial\phi}{\partial\theta}\right)\bigg|_E = 0, \tag{3.31}$$

$$\left(\frac{\partial\eta}{\partial\dot{\theta}}\frac{\partial\phi}{\partial\dot{\theta}}\right)\bigg|_E - \left(\frac{\partial\eta}{\partial\dot{\theta}}\right)\bigg|_E \geq 0, \tag{3.32}$$

$$K|_E \geq 0. \tag{3.33}$$

Thefunction $\phi(\theta, \dot{\theta})$ is usually called a generalized temperature. One may show that the system of equations (3.17) reduces to

$$\begin{aligned}
&\dot{\rho} + \rho v_{i,i} = 0,\\
&\rho\dot{v}_i = \rho f_i - p_{,i} - \left(\rho\frac{\partial\psi}{\partial\lambda}\,\theta_{,i}\theta_{,j}\right)_{,j},\\
&\rho\phi\eta_{\dot{\theta}}\ddot{\theta} + (\rho\psi_\theta + \eta\phi_\theta + \rho\phi\eta_\theta)\dot{\theta} + (\rho\psi_\lambda + \rho\phi\eta_\lambda - K_{\dot{\theta}})\dot{\lambda}\\
&\qquad - K_\lambda\theta_{,i}\theta_{,j}\theta_{,ij} - K\Delta\theta - K_\rho\rho_{,i}\theta_{,i} - 2\lambda K_\theta\\
&\qquad + \rho^2\phi\eta_\rho d_{ii} + (\rho\psi_\lambda - K_{\dot{\theta}})\theta_{,i}\theta_{,j}d_{ij} = \rho r.
\end{aligned} \tag{3.34}$$

Equations (3.34) represent the complete system of equations for thermodynamic motion in an inviscid fluid when one employs the thermodynamics of (Green and Laws, 1972).

(Lindsay and Straughan, 1978) develop a detailed analysis of acceleration wave behaviour for a solution to (3.34) including curved waves of arbitrary shape. Particular solutions are presented for a cylindrical shaped wave moving into a shear flow or for a spherical wave advancing into a radial flow.

3.3 Type II fluid

(Green and Naghdi, 1995a) used their thermal displacement variable α and their entropy balance equation to derive a new class of fluid theories. In this book we refer to their theories as being of a fluid of type II or type III. We believe that both of these theories may have application in the active area of research into heat transfer characteristics of nanofluids, cf. chapter 8. As we point out in chapter 8 nanofluids typically consist of a suspension of metals or their oxides, Cu, CuO, Al_2O_3, SiO_2, TiO_2, in water or a base fluid like ethylene glycol, see e.g. (Hwang et al., 2007), (Maiga et al., 2005), (Kim et al., 2007). An interesting article of (Vadasz et al., 2005) suggests that a mechanism for the increased heat transfer characteristics of a nanofluid may be through a hyperbolic equation for the temperature field. In view of the fact that the temperature displacement field essential to the type II theory of (Green and Naghdi, 1995a) satisfies what is effectively a hyperbolic equation it may be that the extension of the Green - Naghdi model developed by (Quintanilla and Straughan, 2008) which we now describe will be applicable to nanofluids.

(Quintanilla and Straughan, 2008) commence with the reduced energy balance equation

$$T_{ij}L_{ij} - p_i\gamma_i - \rho(\dot{\psi} + \eta\dot{\theta}) - \rho\theta\xi = 0, \tag{3.35}$$

written in the current configuration since we are dealing with a fluid. Here $T_{ij}, p_i, \rho, \psi, \eta, \theta$ and ξ are, respectively, the (symmetric) stress tensor, entropy flux vector, density, Helmholtz free energy function, entropy, absolute temperature, and the internal rate of production of entropy. Now, v_i denotes the velocity field, $L_{ij} = v_{i,j}$, $\gamma_i = (\dot{\alpha})_{,i}$, where $\alpha = \int_{t_0}^{t} \theta(\mathbf{X}, s)ds + \alpha_0$ is the thermal displacement field. We also require the (Green and Naghdi, 1995a) entropy balance law written in the current configuration

$$\rho\dot{\eta} = \rho s + \rho\xi - p_{i,i}, \tag{3.36}$$

where s is the external rate of supply of entropy per unit mass. Since we are now developing a fluid theory we also require the balance of mass,

$$\dot{\rho} + \rho v_{i,i} = 0, \tag{3.37}$$

and the balance of linear momentum,

$$\rho\dot{v}_i = T_{ji,j} + \rho b_i, \tag{3.38}$$

in which b_i is an externally supplied body force.

The development of (Quintanilla and Straughan, 2008) is different from that of (Green and Naghdi, 1995a). To understand this we observe that (Green and Naghdi, 1995a) commence with the assumption that ψ, η, T_{ij}, p_i and ξ depend on the variables $\rho, L_{ij}, \theta, \alpha_{,i}$ and γ_i. However, (Green and Naghdi, 1995a) p. 293 assume that p_i is linear in γ_i, T_{ij} is quadratic in d_{ij} $(d_{ij} = (v_{i,j} + v_{j,i})/2)$, ξ is quadratic in d_{ij} and γ_i, and ψ has the form

$$\psi = \frac{1}{2} m\delta_i\delta_i + f(\rho, \theta) \tag{3.39}$$

where $\delta_i = \alpha_{,i}$ and m is a constant. After this they analyse a class of dissipationless flows by assuming the Reynolds, Peclet and m numbers are suitably large and this leads to a restricted class of dissipationless flows. (Quintanilla and Straughan, 2008) develop what is a more general dissipationless theory from the outset. To do this they omit $\gamma_i = \theta_{,i} = (\dot{\alpha})_{,i}$ as a variable in the constitutive theory from the outset. (This corresponds to the way (Green and Naghdi, 1993) develop their theory of thermoelasticity without energy dissipation, discussed in section 2.3). (Quintanilla and Straughan, 2008) are thus able to obtain a more complete nonlinear constitutive theory in which a variable such as the entropy flux vector, p_i, is defined naturally in terms of the Helmholtz free energy rather than having a preimposed form.

The work of (Quintanilla and Straughan, 2008) begins with the assumption that

$$T_{ij}, \psi, \eta, p_i \text{ and } \xi \tag{3.40}$$

are functions of the independent variables

$$\rho, L_{ij}, \theta, \alpha_{,i}\,. \tag{3.41}$$

Next, write $L_{ij} = d_{ij} + \omega_{ij}$, $\omega_{ij} = (v_{i,j} - v_{j,i})/2$, and use (3.41) together with (3.40) in the energy balance law (3.35) to see that

$$\begin{aligned}
&\Big[T_{ij} + \delta_{ij}\rho^2\frac{\partial\psi}{\partial\rho} + \frac{\rho}{2}\Big(\frac{\partial\psi}{\partial\alpha_{,i}}\alpha_{,j} + \frac{\partial\psi}{\partial\alpha_{,j}}\alpha_{,i}\Big)\Big]d_{ij} + T_{ij}\omega_{ij}\\
&-\gamma_i\Big(p_i + \rho\frac{\partial\psi}{\partial\alpha_{,i}}\Big) - \rho\frac{\partial\psi}{\partial L_{ij}}\dot{\overline{L_{ij}}} - \dot\theta\rho\Big(\frac{\partial\psi}{\partial\theta} + \eta\Big)\\
&-\rho\theta\xi + \frac{\rho}{2}\omega_{ij}\Big(\frac{\partial\psi}{\partial\alpha_{,j}}\alpha_{,i} - \frac{\partial\psi}{\partial\alpha_{,i}}\alpha_{,j}\Big) = 0.
\end{aligned} \tag{3.42}$$

(Quintanilla and Straughan, 2008) deduce from (3.42) that p_i, η and ψ reduce to the forms

$$p_i = -\frac{\partial\psi}{\partial\alpha_{,i}}, \qquad \eta = -\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad \psi = \psi(\rho,\theta,\alpha_{,i}). \tag{3.43}$$

They then restrict attention to the situation where $\xi = \xi(\rho,\theta,\alpha_{,i})$ and equation (3.42) leaves

$$\begin{aligned}
&\Big[T_{ij} + \delta_{ij}\rho^2\frac{\partial\psi}{\partial\rho} + \frac{\rho}{2}\Big(\frac{\partial\psi}{\partial\alpha_{,i}}\alpha_{,j} + \frac{\partial\psi}{\partial\alpha_{,j}}\alpha_{,i}\Big)\Big]d_{ij} - \rho\theta\xi\\
&+\frac{\rho}{2}\omega_{ij}\Big(\frac{\partial\psi}{\partial\alpha_{,j}}\alpha_{,i} - \frac{\partial\psi}{\partial\alpha_{,i}}\alpha_{,j}\Big) = 0.
\end{aligned} \tag{3.44}$$

From (3.44) (Quintanilla and Straughan, 2008) show further that

$$\frac{\partial\psi}{\partial\alpha_{,i}}\alpha_{,j} = \frac{\partial\psi}{\partial\alpha_{,j}}\alpha_{,i}\,, \tag{3.45}$$

and

$$T_{ij} = -p\delta_{ij} - \frac{\rho}{2}\Big(\frac{\partial\psi}{\partial\alpha_{,i}}\alpha_{,j} + \frac{\partial\psi}{\partial\alpha_{,j}}\alpha_{,i}\Big) \tag{3.46}$$

where p is a pressure defined by $p = \rho^2\partial\psi/\partial\rho$. From the remainder of equation (3.44) it follows that $\xi = 0$, in agreement with (Green and Naghdi, 1995a).

In view of the above, the equations for a fluid of type II are given by the balance equations (3.36) - (3.38) with the constitutive theory (3.43), (3.45) and (3.46) together with $\xi = 0$. If we recollect these explicitly then the balances of mass, linear momentum, and entropy become

$$\begin{aligned} &\dot{\rho} + \rho v_{i,i} = 0, \\ &\rho \dot{v}_i = -p_{,i} - \frac{1}{2}\big[\rho(\psi_{\alpha_{,i}}\alpha_{,j} + \psi_{\alpha_{,j}}\alpha_{,i})\big]_{,j} + \rho b_i, \\ &-\rho\frac{d}{dt}\Big(\frac{\partial\psi}{\partial\theta}\Big) = \rho s + \frac{\partial}{\partial x_i}\Big(\frac{\partial\psi}{\partial\alpha_{,i}}\Big), \end{aligned} \tag{3.47}$$

where d/dt, like the superposed dot, denotes the material derivative.

3.4 Type III fluid

(Green and Naghdi, 1995a) develop a further theory for a thermoviscous fluid which utilizes their thermal displacement variable α,

$$\alpha(\mathbf{x}, t) = \int_{t_0}^{t} \theta(\mathbf{x}, s)ds + \alpha_0 \tag{3.48}$$

where θ is the temperature field and $\mathbf{x}$ refers to the current configuration. They begin with the equations of balance of mass, balance of linear momentum, and balance of entropy in the form

$$\rho_t + v_i\rho_{,i} + \rho v_{i,i} = 0, \tag{3.49}$$

$$\rho(v_{i,t} + v_j v_{i,j}) = T_{ji,j} + \rho b_i, \tag{3.50}$$

$$\rho(\eta_t + v_i\eta_{,i}) = -p_{i,i} + \rho s + \rho\xi\,. \tag{3.51}$$

In these equations ρ, v_i and η are the density, velocity and entropy. Additionally T_{ji} and p_i are the (Cauchy) stress tensor and entropy flux vector, while b_i, s are the externally supplied body force and entropy supply, respectively. The variable ξ is an internal rate of production of entropy per unit mass.

(Green and Naghdi, 1995a) also employ the reduced energy equation

$$-\rho(\dot{\psi} + \eta\dot{\theta}) - \rho\theta\xi - p_i\gamma_i + T_{ji}v_{i,j} = 0 \tag{3.52}$$

where a superposed dot denotes the material derivative and $\gamma_i = \theta_{,i} = \partial\dot{\alpha}/\partial x_i$. They also define the variable

$$\delta_i = \alpha_{,i} = \frac{\partial\alpha}{\partial x_i}\,. \tag{3.53}$$

They then define a thermoviscous fluid to be one for which the Helmholtz free energy function ψ, the entropy, stress tensor, entropy flux vector, and the internal rate of production of entropy depend on the independent constitutive variables

$$\rho, v_{i,j}, \theta, \delta_i, \gamma_i \tag{3.54}$$

i.e.

$$\begin{aligned}
\psi &= \psi(\rho, v_{i,j}, \theta, \delta_i, \gamma_i) \\
T_{ij} &= T_{ij}(\rho, v_{i,j}, \theta, \delta_i, \gamma_i) \\
\eta &= \eta(\rho, v_{i,j}, \theta, \delta_i, \gamma_i) \\
p_i &= p_i(\rho, v_{i,j}, \theta, \delta_i, \gamma_i) \\
\xi &= \xi(\rho, v_{i,j}, \theta, \delta_i, \gamma_i).
\end{aligned} \tag{3.55}$$

Unlike the theory of section 3.3 the constitutive list (3.54) contains the variable $\gamma_i = \dot{\alpha}_{,i}$ which is in addition to those of (3.41). For this reason we refer to this as a fluid of type III, by analogy with thermoelasticity of type III as defined in section 2.4.

By manipulating the energy equation (3.52) (Green and Naghdi, 1995a) are able to reduce the constitutive list and indeed, they demonstrate that ψ does not depend on γ_i and $v_{i,j}$, so

$$\psi = \psi(\rho, \theta, \alpha_{,i}), \tag{3.56}$$

and additionally

$$\eta = -\frac{\partial \psi}{\partial \theta}, \qquad \alpha_{,i} \frac{\partial \psi}{\partial \alpha_{,j}} = \alpha_{,j} \frac{\partial \psi}{\partial \alpha_{,i}}, \tag{3.57}$$

while the energy equation assumes the form

$$\Big(T_{ij} + p\delta_{ij} + \rho\alpha_{,j} \frac{\partial \psi}{\partial \alpha_{,i}}\Big) d_{ij} - \Big(p_i + \rho \frac{\partial \psi}{\partial \alpha_{,i}}\Big)\gamma_i - \rho\theta\xi_i = 0, \tag{3.58}$$

where $d_{ij} = (v_{i,j} + v_{j,i})/2$ and p is a pressure given by

$$p = \rho^2 \frac{\partial \psi}{\partial \rho}. \tag{3.59}$$

At this point (Green and Naghdi, 1995a) specialize to the situation in which

$$\psi = \frac{m}{2} \alpha_{,i}\alpha_{,i} + f(\rho, \theta) \tag{3.60}$$

for $m > 0$ a constant and

$$\begin{aligned}
p_i &= -\rho m \delta_i - \frac{\kappa}{\theta_0} \gamma_i \\
T_{ij} &= -p\delta_{ij} + \lambda d_{kk}\delta_{ij} + 2\mu d_{ij} - 2m\alpha_{,i}\alpha_{,j} \\
\rho\xi\theta &= \lambda d_{ii}^2 + 2\mu d_{ij} d_{ij} + \frac{\kappa}{\theta_0} \gamma_i\gamma_i,
\end{aligned} \tag{3.61}$$

where $\theta_0, \kappa, \lambda$ and μ are constants.

3.4.1 Type III viscous fluid

We do not in this work adopt equations (3.60) and (3.61). Instead we leave things more general. We dot not impose a form for ψ and select

$$p_i = -\rho \frac{\partial\psi}{\partial\alpha_{,i}} - \frac{\kappa}{\theta}\gamma_i \,. \tag{3.62}$$

This is different to equation $(3.61)_1$ for two reasons. One, the first term is more general. Secondly, we employ θ rather than a constant θ_0. This we believe leads to a more natural energy equation which reduces to the classical energy equation in appropriate circumstances. In (3.62) κ may depend on the variables in the constitutive list. Our viscous theory is completed by specifying

$$\begin{aligned} T_{ij} &= -p\delta_{ij} + \hat{T}_{ij} \,, \\ \hat{T}_{ij} &= -\rho\alpha_{,i}\frac{\partial\psi}{\partial\alpha_{,j}} + \lambda d_{kk}\delta_{ij} + 2\mu d_{ij} \,, \\ \rho\theta\xi &= \frac{\kappa}{\theta}\gamma_i\gamma_i \,. \end{aligned} \tag{3.63}$$

The governing equations of motion for a type III fluid are then obtained upon employment of (3.62) and (3.63) in the conservation laws (3.49) - (3.51).

3.4.2 Type III inviscid fluid

Since the theme of this book is heat waves it is appropriate to develop a theory for an inviscid type III fluid. To this end we effectively neglect the dependence on $v_{i,j}$ in the constitutive list and drop the d_{ij} terms. Thus, our constitutive theory for an inviscid fluid of type III is

$$p_i = -\rho\frac{\partial\psi}{\partial\alpha_{,i}} - \frac{\kappa}{\theta}\gamma_i \,, \tag{3.64}$$

together with

$$\begin{aligned} T_{ij} &= -p\delta_{ij} - \rho\alpha_{,i}\frac{\partial\psi}{\partial\alpha_{,j}} \,, \\ \rho\theta\xi &= \frac{\kappa}{\theta}\gamma_i\gamma_i \,, \end{aligned} \tag{3.65}$$

where in its fullest generality ψ has the functional form (3.56) and κ depends on the constitutive variables $\rho, \theta, \alpha_{,i}, \gamma_i$.

The governing equations for an inviscid fluid of type III then become upon utilizing (3.64) and (3.65) in the evolution equations (3.49) - (3.51)

$$\dot{\rho} + \rho v_{i,i} = 0, \tag{3.66}$$

$$\rho \dot{v}_i = -p_{,i} - (\rho \alpha_{,j} \psi_{\alpha_{,i}})_{,j} + \rho b_i \,, \tag{3.67}$$

$$\rho \dot{\eta} = (\rho \psi_{\alpha_{,i}})_{,i} + \left(\frac{\kappa}{\theta}\, \gamma_i\right)_{,i} + \rho s + \frac{\kappa}{\theta^2}\, \gamma_i \gamma_i \,, \tag{3.68}$$

$$= (\rho \psi_{\alpha_{,i}})_{,i} + \frac{1}{\theta} (\kappa \gamma_i)_{,i} + \rho s \,. \tag{3.69}$$

3.5 Green-Naghdi extended theory

(Green and Naghdi, 1996) continue their development for describing the behaviour of a continuous body which relies on an entropy balance law rather than an entropy inequality. Again they introduce a quantity T which is the "empirical" temperature and the "thermal displacement variable"

$$\alpha = \int_{t_0}^{t} T(\mathbf{x}, s) ds + \alpha_0 \,.$$

In fact, the full theory developed by (Green and Naghdi, 1996) is very general. They remark, (Green and Naghdi, 1996), p. 240, that ... "the theory ... leads to a set of differential equations ... which are rather unmanagable from the point of view of understanding turbulent or other flows." To produce a more tractable theory they restrict attention to a reduced version of their general theory which leaves only one equation as the mechanical differential equation. Precisely, the theory of (Green and Naghdi, 1996) develops a novel theory for fluids which involves vorticity and spin of vorticity. This introduces higher spatial gradients into the equations than those of Navier-Stokes theory and so is likely to be relevant where non-Newtonian fluid behaviour is expected. They work with two temperatures and are motivated by attempting to describe turbulence. In this respect, they are continuing the work of (Marshall and Naghdi, 1989a; Marshall and Naghdi, 1989b).

We simply describe the relevant differential equations for the model of (Green and Naghdi, 1996). Full details of the continuum thermodynamical development from the entropy balance law is given in (Green and Naghdi, 1996). The basic equations of (Green and Naghdi, 1996) are the balance of linear momentum, balance of mass, and balances of entropy for two temperatures θ_H and θ_T, which they regard as the usual temperature, and a turbulent temperature, respectively. However, other interpretations may be given to the different temperatures, see e.g. section 8.4 and (Straughan, 2010b). The balance of linear momentum, balance of mass, and balances of entropy as given by (Green and Naghdi, 1996) for their incompressible fluid may be written

$$\begin{aligned} &\rho\left(\dot{v}_i - \frac{\mu_1}{\mu}\frac{d}{dt}\Delta v_i\right) = \rho b_i - p_{,i} + \mu\Delta v_i - 2\mu_1\Delta^2 v_i,\\ &v_{i,i} = 0,\\ &\rho\dot{\eta}_H = \rho s_H + \rho\xi_H - p^H_{i,i}\,,\\ &\rho\dot{\eta}_T = \rho s_T + \rho\xi_T - p^T_{i,i}\,. \end{aligned} \tag{3.70}$$

Here a superposed dot denotes the material derivative $d/dt = \partial/\partial t + v_i\partial/\partial x_i$. The variables ρ, v_i, b_i, p are the density, velocity, body force and pressure. The coefficient μ is the kinematic viscosity of the fluid while μ_1 is another constant reflecting the geometry of the particles and the interaction with the fluid. The appendices to (Bleustein and Green, 1967) and (Green and Rivlin, 1964) derive an expression for the kinetic energy of a system of particles as a function of the velocity of the centroid and the derivative of this velocity. While neither of the articles of (Green and Rivlin, 1964) nor (Bleustein and Green, 1967) has a direct bearing on the fluid theory of (Green and Naghdi, 1996), their procedure leads to a kinetic energy which can be equated to the kinetic energy of the fluid currently being described. The quantities η_H, η_T are the entropies corresponding to the temperatures θ_H and θ_T. The terms s_H, s_T, are external supplies of entropy, ξ_H, ξ_T, are intrinsic supplies of entropy which depend on the variables of the theory, and p^H_i, p^T_i are entropy flux vectors. (Green and Naghdi, 1996) assume that the Helmholtz free energy function ψ has form

$$\psi = c_H(\theta_H - \theta_H\,\ln\theta_H) + c_T(\theta_T - \theta_T\,\ln\theta_T)\,, \tag{3.71}$$

with c_H, c_T positive constants, while the entropies and entropy fluxes assume the form

$$\eta_H = c_H\,\ln\theta_H, \qquad \eta_T = c_T\,\ln\theta_T, \tag{3.72}$$

and

$$p^H_i = -\frac{\kappa_H}{\theta_0}\frac{\partial\theta_H}{\partial x_i}\,, \qquad p^T_i = -\frac{\kappa_T}{\theta_0}\frac{\partial\theta_T}{\partial x_i}\,, \tag{3.73}$$

for positive constants $\kappa_H, \kappa_T, \theta_0$, with θ_0 being some reference temperature. The instrinsic entropy supply functions are given by

$$\rho\xi_H\theta_H = \frac{\kappa_H}{\theta_0}\frac{\partial\theta_H}{\partial x_i}\frac{\partial\theta_H}{\partial x_i} + 2\mu d_{ij}d_{ij} + \phi, \tag{3.74}$$

$$\rho\xi_T\theta_T = \frac{\kappa_T}{\theta_0}\frac{\partial\theta_T}{\partial x_i}\frac{\partial\theta_T}{\partial x_i} + 4\mu_1 d_{ij}P_{ij} + \frac{2\mu_1^2}{\mu}P_{ij}P_{ij} - \phi. \tag{3.75}$$

In these equations the variables d_{ij} and P_{ij} are defined by $d_{ij} = (v_{i,j} + v_{j,i})/2$, $P_{ij} = -\Delta v_{i,j}$ and ϕ is constant. It is very important to note, however, that (Green and Naghdi, 1996) observe that for some purposes ϕ could depend on temperatures, cf. section 8.4, and (Straughan, 2010b).

Thus, the complete system of equations for an incompressible viscous fluid in the (Green and Naghdi, 1996) extended theory are

$$\begin{aligned}
&\rho\frac{dv_i}{dt} - \frac{\rho\mu_1}{\mu}\frac{d}{dt}\Delta v_i = \rho b_i - p_{,i} + \mu\Delta v_i - 2\mu_1\Delta^2 v_i\,,\\
&v_{i,i} = 0,\\
&\rho c_H\frac{d\theta_H}{dt} = \rho s_H\theta_H + \frac{\kappa_H}{\theta_0}\frac{\partial\theta_H}{\partial x_i}\frac{\partial\theta_H}{\partial x_i} + 2\mu d_{ij}d_{ij} + \phi,\\
&\rho c_T\frac{d\theta_T}{dt} = \rho s_T\theta_T + \frac{\kappa_T}{\theta_0}\frac{\partial\theta_T}{\partial x_i}\frac{\partial\theta_T}{\partial x_i} + 4\mu d_{ij}P_{ij} + 2\frac{\mu_1^2}{\mu}P_{ij}P_{ij} - \phi,
\end{aligned} \tag{3.76}$$

where d/dt has been employed to denote the material derivative.

(Green and Naghdi, 1996) determine the basic solution to plane Poiseuille flow for their theory and show that it leads to a flattened profile rather than the parabolic one of classical Newtonian theory. They also address a similar basic solution to Poiseuille flow in a pipe. Additionally, (Green and Naghdi, 1996) address the problem of flow of a circular jet from a round hole. Finally, (Green and Naghdi, 1996) address two problems where the solution is time-dependent. All the problems addressed by (Green and Naghdi, 1996) are in an isothermal situation.