2
Interaction with elasticity

2.1 Cattaneo theories

2.1.1 Cattaneo-Lord-Shulman theory

In sections 1.2 - 1.12 we have seen various ways of modifying the classical diffusion equation in order to allow heat to be transported with a finite wavespeed. The assumption was that the body would remain rigid. However, in many cases this is too strong since the body itself deforms or vibrates. Thus, in this chapter we wish to look at ways of coupling heat propagation in the case where the body is an elastic solid. This is the domain of thermoelasticity and, in particular, we shall review theories of thermoelasticity which allow temperature to travel as a wave with finite speed.

It would appear that the first attempts to couple elasticity with a way in which temperature can travel with a finite wavespeed are due to (Lord and Shulman, 1967) and to (Popov, 1967), as is observed in the short but very informative review by (Jordan and Puri, 2001). (Jordan and Puri, 2001) also derive a very useful comparison of the classical theory of thermoelasticity with two theories capable of allowing temperature to travel with a finite wavespeed. Extensive reviews of the early literature on thermoelasticity with temperature waves are by (Chandrasekharaiah, 1986), (Chandrasekharaiah, 1998) and by (Hetnarski and Ignaczak, 1999), and the recent book by (Ignaczak and Ostoja-Starzewski, 2009) concentrates on thermoelasticity with temperature waves, although the overlap with the current book is minimal.

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To understand the situation we commence, as do (Lord and Shulman, 1967), with the classical theory of *linear* thermoelasticity. (Lord and Shulman, 1967) consider the isotropic case, but it is no more difficult to begin with the anisotropic situation and this we do now. In terms of the elastic displacement, u_i, and the temperature field, θ, the equations of classical linear thermoelasticity for an anisotropic and inhomogeneous body may be written,

$$\begin{aligned} \rho \ddot{u}_i &= (c_{ijkh} u_{k,h})_{,j} - (a_{ij}\theta)_{,j} + \rho f_i \,, \\ c\dot{\theta} &= -a_{ij}\dot{u}_{i,j} + (k_{ik}\theta_{,k})_{,i} + \rho r, \end{aligned} \tag{2.1}$$

where $\dot{\theta} = \theta_{,t}$ and standard indicial notation is used. Here ρ, c, f_i and r are, respectively, the density, density multiplied by the specific heat, externally supplied body force, and external supply of heat. The coefficients $c_{ijkh}(\mathbf{x}, t)$ are the elastic coefficients, or elasticities, $k_{ij}(\mathbf{x}, t)$ denote the components of the thermal conductivity tensor, and $a_{ij}(\mathbf{x}, t)$ are the components of a coupling tensor connecting the equations of elasticity to those for heat transport in the solid. We observe immediately that if we set $a_{ij} \equiv 0$, $f_i = 0$ and $r = 0$ then system (2.1) decouples into the two linear equations

$$\rho \ddot{u}_i = (c_{ijkh} u_{k,h})_{,j} \tag{2.2}$$

and

$$c\dot{\theta} = (k_{ik}\theta_{,k})_{,i} \,. \tag{2.3}$$

Equation (2.2) represents the standard equations of linear elasticity which under appropriate conditions on the elasticities c_{ijkh} allow elastic wave propagation and define a hyperbolic system, cf. (Knops and Payne, 1971b), (Knops and Wilkes, 1973). On the other hand, equation (2.3) for $c > 0$ and k_{ik} a positive-definite tensor, is the classical parabolic equation for the diffusion of θ. Thus, θ effectively has an infinite wavespeed as we saw in section 1.2. Thus, for the combined system (2.1) we expect a coupled hyperbolic - parabolic system of partial differential equations with the temperature field diffusing with infinite wavespeed.

(Lord and Shulman, 1967) proposed combining the Cattaneo approach (Maxwell-Cattaneo theory of section 1.2) together with the standard development of elasticity to derive a Cattaneo - type theory of thermoelasticity as we now describe. The approach of (Lord and Shulman, 1967) begins with the full nonlinear equations but they are mainly interested in developing a linear theory since they begin with ... "*small strains and small temperature changes*". With $\epsilon, \eta, t_{ij}, q_i$ and $e_{ij} = (u_{i,j} + u_{j,i})/2$ being the internal energy, entropy, stress tensor, heat flux and strain tensor for the elastic body, respectively, (Lord and Shulman, 1967) write the energy balance law as

$$\rho\theta\dot{\eta} = -q_{i,i} \,, \tag{2.4}$$

where η and ϵ are connected by the equation

$$\rho\dot{\epsilon} = \rho\theta\dot{\eta} + t_{ij}\dot{e}_{ij}\,, \tag{2.5}$$

superposed dot being the partial time derivative, $\partial/\partial t$. (Lord and Shulman, 1967) propose the general anisotropic equation for q_i which generalizes Cattaneo's equation $(1.45)_2$, namely,

$$A_{ij}\dot{q}_j + a\dot{q}_i + q_i = b\theta_{,i} + B_{ij}\theta_{,j}\,, \tag{2.6}$$

where the coefficients A_{ij}, a, b and B_{ij} depend on the material comprising the elastic body. They are principally interested in deriving an isotropic version of their theory and so note that in the isotropic case equation (2.6) may be replaced by

$$\tau\dot{q}_i + q_i = -k\theta_{,i}\,. \tag{2.7}$$

(Lord and Shulman, 1967) call τ the relaxation time, and they say it "*represents the time-lag needed to establish steady - state heat conduction in an element of volume when a temperature gradient is suddenly imposed on that element*".

(Lord and Shulman, 1967) proceed to introduce the Helmholtz free energy function $\psi = \psi(e_{ij}, \theta) = \epsilon - \eta\theta$ and then note

$$\frac{\partial\psi}{\partial t} = \dot{\psi} = \frac{\partial\psi}{\partial e_{ij}}\,\dot{e}_{ij} + \frac{\partial\psi}{\partial\theta}\,\dot{\theta} \tag{2.8}$$

and

$$\dot{\psi} = \dot{\epsilon} - \eta\dot{\theta} - \dot{\eta}\theta\,. \tag{2.9}$$

Equations (2.8) and (2.9) are employed in equation (2.5) to see that

$$\begin{aligned} t_{ij}\dot{e}_{ij} =& \rho(\dot{\epsilon} - \theta\dot{\eta}) \\ =& \rho(\dot{\psi} + \eta\dot{\theta}) \\ =& \rho\Big(\frac{\partial\psi}{\partial e_{ij}}\,\dot{e}_{ij} + \frac{\partial\psi}{\partial\theta}\,\dot{\theta} + \eta\dot{\theta}\Big)\,. \end{aligned} \tag{2.10}$$

From equation (2.10) (Lord and Shulman, 1967) infer that the stress tensor and entropy have the forms

$$\eta = -\frac{\partial\psi}{\partial\theta}\,, \qquad t_{ij} = \rho\frac{\partial\psi}{\partial e_{ij}}\,. \tag{2.11}$$

(Lord and Shulman, 1967) then employ the relation $(2.11)_1$ in the energy balance law (2.4) to derive the equation

$$\rho\theta\Big(\frac{\partial^2\psi}{\partial e_{ij}\partial\theta}\,\dot{e}_{ij} + \frac{\partial^2\psi}{\partial\theta^2}\,\dot{\theta}\Big) = q_{i,i}\,. \tag{2.12}$$

Let us observe that equations (2.7) and (2.12) (with replacement of appropriate time derivatives) could form the basis for a nonlinear Cattaneo -

Lord - Shulman theory. (Lord and Shulman, 1967) do not pursue this line and proceed to combine equations (2.7) and (2.12) linearizing in the process. In this way they derive the *linearized* energy balance law

$$-\rho\theta\psi_{\theta\theta}(\dot{\theta}+\tau\ddot{\theta})-\rho\theta\,\frac{\partial^2\psi}{\partial e_{ij}\partial\theta}\,(\dot{e}_{ij}+\tau\ddot{e}_{ij})=k\Delta\theta. \tag{2.13}$$

(Lord and Shulman, 1967) then proceed to develop their theory in the isotropic case and expand about a constant temperature θ_0 and expand in terms of the strain invariants of elasticity theory. In this way they produce their famous system of equations for isotropic thermoelasticity. Their equation for the displacement u_i is the isotropic equivalent of equation $(2.1)_1$ and coupled to the isotropic equation which arises from (2.13) the Lord-Shulman system of equations is

$$\begin{aligned}&\rho\ddot{u}_i=(\lambda+\mu)u_{j,ij}+\mu\Delta u_i-(3\lambda+2\mu)\alpha\theta_{,i}\,,\\&\rho c(\tau\ddot{\theta}+\dot{\theta})+(3\lambda+2\mu)\alpha\theta_0(\tau\ddot{e}_{rr}+\dot{e}_{rr})=k\Delta\theta\,.\end{aligned} \tag{2.14}$$

In equations (2.14), c is the specific heat and λ,μ are the coefficients which arise in isotropic elasticity, the Lamé moduli, the connection with the elastic coefficients c_{ijkh} being

$$c_{ijkh}=\lambda\delta_{ij}\delta_{kh}+\mu(\delta_{ik}\delta_{jh}+\delta_{ih}\delta_{jk})\,.$$

(Lord and Shulman, 1967) write their equations in non-dimensional form and then solve a one-dimensional problem which corresponds to zero initial conditions in a half space with the velocity $\partial u/\partial t$ experiencing a discontinuous input at time $t=0$ along the half space boundary, i.e. a displacement shock problem.

2.1.2 Cattaneo-Fox theory

The first development of a fully nonlinear thermoelastic theory which employs a Cattaneo equation for the heat flux is that of (Fox, 1969a). Fox begins with the momentum and continuity equations written in the *current* configuration as

$$\begin{aligned}&\rho\dot{v}_i=t_{ji,j}+\rho f_i\,,\\&\dot{\rho}+\rho d_{rr}=0,\end{aligned} \tag{2.15}$$

where t_{ij} is the symmetric Cauchy stress tensor, a superposed dot denotes the *material* derivative, e.g. $\dot{\rho}=\partial\rho/\partial t+v_i\partial\rho/\partial x_i$, f_i is the externally supplied body force, and $d_{ij}=(v_{i,j}+v_{j,i})/2$, $v_i(\mathbf{x},t)$ being the velocity in the current reference frame. (Fox, 1969a) begins with a balance of energy equation and an entropy inequality postulated for arbitrary sub-bodies of an elastic body, and reduces these to pointwise form. In terms of the internal energy ϵ, entropy η, heat flux q_i, and temperature θ these are

$$\begin{aligned} \rho\dot{\epsilon} - \rho r + q_{i,i} - t_{ij}d_{ij} &= 0, \\ \rho\theta\dot{\eta} - \rho r + q_{i,i} - \frac{q_i\theta_{,i}}{\theta} &\geq 0, \end{aligned} \tag{2.16}$$

where r is the externally supplied source of heat. The entropy inequality $(2.16)_2$ is rewritten in terms of the Helmholtz free energy function $\psi = \epsilon - \eta\theta$ as

$$\rho\dot{\psi} + \rho\eta\dot{\theta} - t_{ij}d_{ij} + \frac{q_i\theta_{,i}}{\theta} \leq 0. \tag{2.17}$$

The constitutive theory of (Fox, 1969a) requires that

$$\psi, \eta, t_{ij}$$

depend on the independent variables

$$F_{iA} = \frac{\partial x_i}{\partial X_A} = x_{i,A} \qquad \text{and} \qquad \theta,$$

where $x_i = x_i(\mathbf{X}, t)$ is the mapping of points in the reference configuration to equivalent points in the current configuration. Upon introducing the right Cauchy - Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ (i.e. $C_{AB} = x_{i,A}x_{i,B}$) (Fox, 1969a) notes

$$\dot{C}_{AB} = 2d_{ij}x_{i,A}x_{j,B}$$

and rewrites inequality (2.17) in the form

$$-\rho\Big(\frac{\partial\psi}{\partial\theta} + \eta\Big)\dot{\theta} + \Big(t_{ij} - 2\rho\frac{\partial\psi}{\partial C_{AB}}\,x_{i,A}x_{j,B}\Big)d_{ij} - \frac{q_i\theta_{,i}}{\theta} \geq 0. \tag{2.18}$$

Using the fact that r and f_i may be selected at will it is now deduced from (2.18) that

$$\eta = -\frac{\partial\psi}{\partial\theta} \qquad \text{and} \qquad t_{ij} = 2\rho x_{i,A}x_{j,B}\,\frac{\partial\psi}{\partial C_{AB}}\,. \tag{2.19}$$

The residual of the entropy inequality (2.18) is

$$-q_i\theta_{,i} \geq 0, \tag{2.20}$$

and the energy balance law becomes

$$\theta\dot{\eta} = -q_{i,i} + \rho r. \tag{2.21}$$

(Fox, 1969a) uses superposed rigid body arguments and requests that the nonlinear time derivative of q_i in a Cattaneo law should be an objective derivative. This leads him to propose the general equation generalizing Cattaneo's one,

$$\dot{q}_i - \omega_{ij}q_j = \alpha q_i + \beta\theta_{,i}\,, \tag{2.22}$$

where $\omega_{ij} = (v_{i,j} - v_{j,i})/2$, $\dot{q}_i = q_{i,t} + v_k q_{i,k}$, and α, β depend on θ and the scalar invariants $q_i q_i, q_i \theta_{,i}$ and $\theta_{,i}\theta_{,i}$. (Fox, 1969a) specializes to the case where α and β are constants and introduces constants τ and κ by $\tau = -1/\alpha$, $\kappa = \beta/\alpha$ so that his equation (2.22) becomes

$$\tau(q_{i,t} + v_j q_{i,j} - \omega_{ij} q_j) = -q_i - \kappa\theta_{,i} \,. \tag{2.23}$$

Thus, the full nonlinear system of equations derived by (Fox, 1969a) to describe motion in a thermoelastic body generalizing the (Lord and Shulman, 1967) approach comprise equations (2.15), (2.21) and (2.23).

For easy reference we collect these here recalling the forms for η and t_{ij} given in equations (2.19),

$$\begin{aligned} &\dot{\rho} + \rho d_{ii} = 0, \\ &\rho\dot{v}_i = 2\frac{\partial}{\partial x_j}\Big(\rho x_{i,A} x_{j,B}\,\frac{\partial \psi}{\partial C_{AB}}\Big) + \rho f_i \,, \\ -&\theta\frac{d}{dt}\Big(\frac{\partial\psi}{\partial\theta}\Big) = -\frac{\partial q_i}{\partial x_i} + \rho r, \\ &\tau(\dot{q}_i - \omega_{ij} q_j) = -q_i - \kappa\theta_{,i} \,, \end{aligned} \tag{2.24}$$

where d/dt denotes the material derivative.

I am not aware of further use of the nonlinear system (2.24) apart from the solutions derived by (Fox, 1969a) himself. However, (Fox, 1969a) deserves full credit for producing a nonlinear invariant system of thermoelastic equations using a Cattaneo theory. The solutions given by (Fox, 1969a) involve a static deformation where he shows the heat flux decays exponentially in time, and one where the deformation is

$$x = 2ktY, \;\; y = Y, \;\; z = Z.$$

For this definition he solves his equation for q_i, $(2.24)_4$, exactly.

2.1.3 Hidden variables

(Caviglia et al., 1992) begin with the idea of introducing an internal vector variable ξ_i; an internal variable is sometimes also referred to as a hidden variable, and an extensive description of such variables may be found in (Maugin, 1990), (Maugin and Muschik, 1994a; Maugin and Muschik, 1994b). The vector $\boldsymbol{\xi}$ refers to a current configuration $\mathcal{R}$ which has deformed from a reference configuration $\mathcal{R}_0$ by a mapping $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ or $x_i = x_i(X_A, t)$. They define the Cauchy stress tensor t_{ij}, the second Piola-Kirchoff stress tensor Y_{AB} and the first Piola-Kirchoff stress tensor S_{Ai}, where $\mathbf{Y} = J\mathbf{F}^{-1}\mathbf{t}(\mathbf{F}^{-1})^T$, $\mathbf{F}$ being the deformation gradient defined by $F_{iA} = \partial x_i/\partial X_A$ and $J = \det(F_{iA})$. They also introduce the heat flux q_i, the Helmholtz free energy ψ, the temperature θ, and temperature gradients $g_i = \theta_{,i}$ and $G_A = \theta_{,A}$ where $\theta_{,i} \equiv \partial\theta/\partial x_i$ whereas $\theta_{,A} \equiv \partial\theta/\partial X_A$. In terms of the displacement $u_i = x_i - X_i$, (Caviglia et al., 1992) have the balance of momentum equation

$$\rho_0 \ddot{u}_i = \frac{\partial}{\partial X_A} S_{Ai} + \rho_0 b_i \tag{2.25}$$

where ρ_0 is the density in $\mathcal{R}_0$, b_i is the body force and a superposed dot denotes $\partial/\partial t$ holding $\mathbf{X}$ fixed.

The thermodynamic procedure of (Caviglia et al., 1992) introduces the Cauchy-Green right tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and requires the equivalent of the internal variable $\boldsymbol{\xi}$ referred to the reference configuration, namely $\boldsymbol{\Xi} = \mathbf{F}^T\boldsymbol{\xi}$. Then, (Caviglia et al., 1992) define their thermoelastic body to be one for which

$$\begin{aligned} \mathbf{t} &= \mathbf{F}\tilde{\mathbf{t}}(\mathbf{C}, \theta, \mathbf{G}, \boldsymbol{\Xi})\mathbf{F}^T\,, \\ \mathbf{q} &= \mathbf{F}\tilde{\mathbf{q}}(\mathbf{C}, \theta, \mathbf{G}, \boldsymbol{\Xi})\,, \\ \psi &= \tilde{\psi}(\mathbf{C}, \theta, \mathbf{G}, \boldsymbol{\Xi})\,, \end{aligned} \tag{2.26}$$

where $\tilde{\mathbf{t}}, \tilde{\mathbf{q}}, \tilde{\psi}$ are the functional forms of the indicated variables. The entropy inequality

$$-\rho_0(\dot{\psi} + \eta\dot{\theta}) + \frac{1}{2}\mathbf{Y}\cdot\dot{\mathbf{C}} - \frac{1}{\theta}\,\mathbf{Q}\cdot\mathbf{G} \,\geq\, 0 \tag{2.27}$$

is posed where η is the entropy. (Caviglia et al., 1992) show that inequality (2.27) leads to the deductions

$$\frac{\partial\tilde{\psi}}{\partial\mathbf{G}} = 0\,, \qquad \eta = -\frac{\partial\tilde{\psi}}{\partial\theta}\,, \qquad \mathbf{Y} = 2\rho_0\,\frac{\partial\tilde{\psi}}{\partial\mathbf{C}}\,, \tag{2.28}$$

and the residual entropy inequality is

$$\rho_0\theta\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\Xi}}\cdot\dot{\boldsymbol{\Xi}} + \mathbf{Q}\cdot\mathbf{G} \,\leq\, 0. \tag{2.29}$$

Then, from the first of (2.28), the Helmholtz free energy function reduces to the form $\psi = \psi(\mathbf{C}, \theta, \boldsymbol{\Xi})$.

For the internal variable $\boldsymbol{\xi}$, (Caviglia et al., 1992) propose that $\boldsymbol{\Xi}$ satisfies an evolution equation of form

$$\dot{\boldsymbol{\Xi}} = -m\mathbf{G} - n\boldsymbol{\Xi} \tag{2.30}$$

where m, n are functions of the variables θ and $\mathbf{C}$ with $n > 0$. Upon employing $\dot{\boldsymbol{\Xi}}$ as given by (2.30) in the inequality (2.29) they deduce that

$$\mathbf{Q} = \rho_0 m\theta\,\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\Xi}}\,. \tag{2.31}$$

Further, from (2.29), there remains the restriction

$$\boldsymbol{\Xi}\cdot\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\Xi}} \,\geq\, 0\,. \tag{2.32}$$

To progress further (Caviglia et al., 1992) require that under stationary conditions $\mathbf{Q}$ satisfies Fourier's law so that

$$\mathbf{Q} = -\mathbf{K}\mathbf{G}$$

for $\mathbf{K}$ a positive-definite tensor which depends on θ and $\mathbf{C}$. Under stationary conditions equation (2.30) yields the connection

$$\mathbf{G} = -\frac{n}{m}\boldsymbol{\Xi}$$

and the last two relations lead (Caviglia et al., 1992) to propose the relationship

$$\mathbf{Q} = \frac{n}{m}\mathbf{K}\boldsymbol{\Xi}\,. \tag{2.33}$$

Then, from (2.31) they deduce that

$$\frac{\partial\tilde{\psi}}{\partial\boldsymbol{\Xi}} = \frac{n}{\rho_0\theta m^2}\,\mathbf{K}\boldsymbol{\Xi}$$

whence

$$\psi = \hat{\psi}(\theta,\mathbf{C}) + \frac{n}{2\rho_0\theta m^2}\,\Xi_A K_{AB}\Xi_B\,, \tag{2.34}$$

where $\hat{\psi}$ denotes a functional relationship of the indicated variables. Upon introducing the internal energy $\epsilon = \psi + \theta\eta$ one then sees that

$$\epsilon = \hat{\psi} - \theta\frac{\partial\hat{\psi}}{\partial\theta} + \frac{1}{2}\left(\frac{nK_{AB}}{\rho_0\theta m^2} - \theta\frac{\partial}{\partial\theta}\left[\frac{nK_{AB}}{\rho_0\theta m^2}\right]\right)\Xi_A\Xi_B\,. \tag{2.35}$$

(Caviglia et al., 1992) then require that ϵ be independent of $\boldsymbol{\Xi}$ and hence of $\mathbf{Q}$ and so

$$\frac{n}{\rho_o\theta m^2}\mathbf{K} = n\hat{\mathbf{K}}(\mathbf{C})$$

where $\hat{\mathbf{K}}$ denotes the functional form, $\hat{\mathbf{K}}$ also being a positive-definite tensor.

The constitutive theory of (Caviglia et al., 1992) may be summarized as

$$\begin{aligned}
\psi &= \hat{\psi}(\theta,\mathbf{C}) + \frac{\theta}{2}\,\hat{K}_{AB}\Xi_A\Xi_B\,,\\
Y_{AB} &= 2\rho_0\,\frac{\partial\tilde{\psi}}{\partial C_{AB}} + \frac{\theta}{2}\,\frac{\partial\hat{K}_{RS}}{\partial C_{AB}}\,\Xi_R\Xi_S\,,\\
\eta &= -\frac{\partial\hat{\psi}}{\partial\theta} - \frac{1}{2}\hat{K}_{RS}\Xi_R\Xi_S\,.
\end{aligned} \tag{2.36}$$

The (fully nonlinear) evolution equations for the model then follow from (2.25), the energy balance equation, equation (2.30), and may then be written as,

$$\rho_0 \ddot{u}_i = \frac{\partial}{\partial X_A} S_{Ai} + \rho_0 b_i \,,$$
$$-\rho_0 \theta \Big(\frac{\partial^2 \tilde{\psi}}{\partial \theta^2} \, \dot{\theta} + \frac{\partial^2 \tilde{\psi}}{\partial \theta \partial C_{AB}} \, \dot{C}_{AB} \Big) = -\frac{\partial}{\partial X_A} \big(\rho_0 m \theta^2 \hat{K}_{AB} \Xi_B \big), \tag{2.37}$$
$$\dot{\Xi}_A = -m\theta_{,A} - n\Xi_A \,,$$

where

$$S_{Ai} = \rho_0 \, \frac{\partial \psi}{\partial F_{iA}} = \rho_0 \, \frac{\partial \psi}{\partial C_{RS}} \, \frac{\partial C_{RS}}{\partial F_{iA}} = Y_{RS} \, \frac{\partial C_{RS}}{\partial F_{iA}}$$

and so

$$S_{Ai} = (\delta_{AR} F_{iS} + \delta_{SA} F_{iR}) Y_{RS} = F_{iS} Y_{AS} + F_{iR} Y_{RA} \,.$$

(Caviglia et al., 1992) then develop a linearized version of their theory. It is, however, important to note that they do this by considering a potentially large deformation from $\mathcal{R}_0$ to $\mathcal{R}$ followed by a "small" deformation to a new current configuration $\mathcal{R}^*$. In this way they are not simply developing a linear theory by suitably restricting $\hat{\psi}$ and $\hat{\mathbf{K}}$, they are producing a linearized theory which allows for linearization about a (nonlinear) state of pre-stress and possibly non-uniform temperature.

Let $\mathbf{X}$ denote the position of a particle in the reference configuration $\mathcal{R}_0$, let $\mathbf{x}$ be its position in $\mathcal{R}$, and let $\mathbf{x}^*$ be the corresponding position in $\mathcal{R}^*$. (Caviglia et al., 1992) assume that in $\mathcal{R}$ the temperature θ is constant so that $\mathbf{G} = \mathbf{0}$ and $\boldsymbol{\Xi} = \mathbf{0}$. The values of these variables in $\mathcal{R}^*$ are denoted by θ^*, $\mathbf{G}^*$ and $\boldsymbol{\Xi}^*$, with $\mathbf{C}$ and $\mathbf{C}^*$ denoting the values of the Cauchy-Green right tensor in $\mathcal{R}$ and $\mathcal{R}^*$. The perturbations to $\mathbf{x}, \theta$ and $\boldsymbol{\Xi}$ in $\mathcal{R}$ are written as $\mathbf{u}, \phi$ and $\boldsymbol{\Lambda}$, i.e.

$$x_i^* = x_i + u_i \,, \qquad \theta^* = \theta + \phi, \qquad \Xi_i^* = \Xi_i + \Lambda_i = \Lambda_i \,.$$

Then, equations (2.37) are linearized keeping only quantities linear in u_i, ϕ, Λ_i and their derivatives, in the equations which result. Full details are given in (Caviglia et al., 1992), we here only record the equations. However, we point out that (Caviglia et al., 1992) take $\mathbf{F} = \mathbf{I}$ so that in $\mathcal{R}$ the right Cauchy-Green tensor satisfies $\mathbf{C} = \mathbf{I}$, where θ is uniform. The pre-stress in $\mathcal{R}$ is maintained through the body force b_i and in equilibrium equation (2.37) is

$$\frac{\partial}{\partial X_A} (F_{iB} Y_{BA}) + \rho_0 b_i = 0. \tag{2.38}$$

The linearization of equations (2.37) relies on the fact that this procedure is performed about the solution of (2.38). It is important to note that the steady state deformation given by (2.38) is, in general, not homogeneous and represents a true nonlinear deformation before linearization.

The linearized equations of (Caviglia et al., 1992) are

$$\begin{aligned} \rho_0 \ddot{u}_i &= \big[(\delta_{ij} t_{kh} + A_{ihjk}) u_{j,k} - B_{ih}\phi\big]_{,h} \,, \\ c\dot{\phi} + B_{ij}\dot{u}_{i,j} &= -\frac{1}{\theta}\, q_{i,i} \,, \\ \tau a_{ij}\dot{q}_i + a_{ij} q_j &= -\phi_{,i} \,, \end{aligned} \tag{2.39}$$

where the coefficients involve quantities evaluated in the configuration $\mathcal{R}$ in which θ is uniform and $\mathbf{C} = \mathbf{I}$. The quantity t_{ij} is the Cauchy pre-stress tensor, and

$$\begin{aligned} A_{ijkh} &= 2\rho_0 \frac{\partial^2 \hat{\psi}}{\partial C_{ij} \partial C_{kh}} \,, \qquad B_{ij} = -2\rho_0 \frac{\partial^2 \hat{\psi}}{\partial \theta \partial C_{ij}} \,, \\ c &= -\rho_0 \frac{\partial^2 \hat{\psi}}{\partial \theta^2} \,, \qquad \tau = \frac{1}{n} \,, \qquad a_{ij} = \frac{n}{\rho_0 m^2 \theta^2} (\hat{\mathbf{K}}^{-1})_{ij} \,. \end{aligned} \tag{2.40}$$

We point out that the coefficients in (2.40) are all evaluated in $\mathcal{R}$.

When the body is isotropic, the coefficients become

$$\begin{aligned} A_{ijkh} &= \lambda \delta_{ij}\delta_{kh} + \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}), \\ t_{ij} &= \alpha \delta_{ij} \,, \qquad B_{ij} = \beta \delta_{ij} \,, \qquad a_{ij} = \frac{1}{\kappa}\, \delta_{ij} \end{aligned}$$

where κ is a constant and then equations (2.39) become

$$\begin{aligned} \rho_0 \ddot{u}_i &= (\alpha u_{i,j})_{,j} + (\lambda u_{j,j})_{,i} + (\mu u_{i,j})_{,j} + (\mu u_{j,i})_{,j} - (\beta\phi)_{,i} \,, \\ c\dot{\phi} + \beta \dot{u}_{i,i} &= -\frac{1}{\theta}\, q_{i,i} \,, \\ \tau \dot{q}_i + q_i &= -\kappa \phi_{,i} \,. \end{aligned} \tag{2.41}$$

In equations (2.39) and (2.41) the pre-stress is provided by the body force in equation (2.38). We could follow the procedure of (Iesan, 1980; Iesan, 1988) and allow a deformation from $\mathcal{R}_0$ to $\mathcal{R}$ which is induced by non-homogeneous boundary conditions in both x_i and θ. This would lead to coefficients which have pre-stress present due to the deformation but also due to non-uniform temperature in $\mathcal{R}$. The linearized equations which then arise contain extra terms to those in (2.39) and (2.41).

(Chandrasekharaiah, 1998), p. 722, remarks that the linearized theory of (Caviglia et al., 1992) closely resembles the Lord - Shulman theory. We point out that there is a resemblance, but equations (2.39) and (2.41) are different from those of Lord - Shulman. Firstly, in (2.39) the equations are for the anisotropic case. However, importantly both sets of equations (2.39) and (2.41) contain the effects of pre-stress. This is evident in (2.39) via the t_{kh} term but also in the equation for q_i through the coefficient a_{ij} which contains the pre-stress via $\hat{\mathbf{K}}$, see (2.40). In particular, due to the presence of the Cauchy pre-stress t_{ij} it is not true that, in general, the elastic coefficients $c_{ijkh} = \delta_{ij} t_{kh} + A_{ijkh}$ would be sign-definite.

2.2 Green-Lindsay theory

When one develops the classical theory of nonlinear thermoelasticity it is usual to begin with a constitutive assumption which is equivalent to requiring

$$\psi, \eta, q_i \quad \text{and} \quad S_{Ai} \tag{2.42}$$

to depend on the variables

$$X_A, \rho_0, \theta, \theta_{,A} \quad \text{and} \quad e_{AB}\,. \tag{2.43}$$

Here ψ, η are the Helmholtz free energy function and the entropy function, q_i is the heat flux vector and S_{Ai} is the Piola-Kirchoff stress tensor. The independent variables are $\mathbf{X}$, the coordinates of a point in the reference configuration, ρ_0 the density in the reference configuration, the temperature $\theta(\mathbf{X}, t)$, the temperature gradient $\theta_{,A} = \partial\theta/\partial X_A$, and the strain tensor, $e_{AB} = (x_{i,A}x_{i,B} - \delta_{AB})/2$, acting at time t but referred to the reference configuration. The function $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ denotes the map defining the deformation (motion) of the elastic body.

The above prescription leads to a coupled set of nonlinear partial differential equations for the displacement $u_i = x_i - X_i$ and the temperature field θ. The balance of momentum equation which results may be thought of as yielding a hyperbolic equation but the corresponding balance of energy equation contains $\partial\theta/\partial t$ as the highest time derivative of θ and is effectively a parabolic equation. Thus, the system may be thought of as one of coupled parabolic - hyperbolic type. This has the undesirable feature that the temperature field essentially travels with infinite speed, cf. section 1.2. (This argument generalizes the analogous one from linear thermoelasticity as explained in section 2.1.1.) An appealing way to overcome this was suggested by (Müller, 1971a). His idea is to include $\dot{\theta}$ in the list of independent constitutive variables in (2.43). He develops a complete theory of thermoelasticity beginning with the balance laws for conservation of mass, momentum, and energy. In the balances of momentum and energy (Müller, 1971a) does not include a body force or external supply of heat. The thermodynamics of (Müller, 1971a) is based on his entropy inequality, (Müller, 1967a),

$$\rho\dot{\eta} + \frac{\partial \Phi_i}{\partial x_i} \geq 0 \tag{2.44}$$

where Φ_i is his entropy flux vector, see (Müller, 1967a). (Müller, 1971a) expands inequality (2.44) using the extended constitutive list, and he then argues that the balance equations which arise must hold in such a way that he is able to deduce relations between the functions $\psi, \eta, \Phi_i, S_{Ai}$, and q_i. In this way (Müller, 1971a) develops a fully nonlinear theory for thermoelasticity which, unlike the classical theory, allows θ to travel with a finite wavespeed. (Müller, 1971a) develops complete expressions for the stress, heat flux, and his entropy flux for an isotropic solid and deduces restrictions in equilibrium. He also shows the heat conduction tensor must be symmetric.

We here describe a theory due to (Green and Lindsay, 1972) which also employs $\dot{\theta}$ in the constitutive list. (Green and Lindsay, 1972) commence with the balance laws of mass, momentum, angular momentum, and energy, which are

$$\begin{aligned} &\rho_0 = \rho \det(x_{i,A}), \\ &\rho_0 \dot{v}_i = \rho_0 F_i + S_{Ai,A}\,, \\ &Y_{AB} = Y_{BA} \qquad \text{where} \qquad S_{Ai} = F_{iR} Y_{RA}\,, \\ &\rho_0 \dot{\epsilon} = \rho_0 r + Y_{AB}\dot{e}_{AB} - Q_{A,A}\,. \end{aligned} \tag{2.45}$$

Here ρ and ρ_0 denote the density in the current and reference configurations, v_i is the velocity, S_{Ai} is the Piola-Kirchoff stress tensor, ϵ the internal energy, Q_A the heat flux vector per unit area in the X_K frame but acting over the corresponding surface at time t, and e_{AB} (defined by (Green and Lindsay, 1972) as $e_{AB} = (x_{i,A}x_{i,B} - \delta_{AB})$) is the strain tensor referred to the reference configuration. The quantities $\mathbf{F}$ and r are an external body force and an external supply of heat, respectively. The Cauchy stress tensor, t_{ij}, (in the current frame) and the equivalent heat flux vector, q_i, are given in terms of Y_{AB} and Q_A as

$$\begin{aligned} (\det x_{r,K})\, t_{ij} &= x_{i,A}x_{j,B}Y_{AB} \\ (\det x_{r,K})\, q_i &= x_{i,A}Q_A\,. \end{aligned}$$

(Green and Lindsay, 1972) employ a general entropy inequality over any sub-body, this being based on the entropy inequality of (Green and Laws, 1972). However, they effectively reduce this to the following pointwise entropy inequality

$$\rho_0\dot{\eta} - \frac{\rho_0 r}{\phi} + \left(\frac{Q_A}{\phi}\right)_{,A} \geq 0. \tag{2.46}$$

This inequality resembles the Clausius-Duhem inequality but the function ϕ is a generalized temperature which will be specified by constitutive theory. If one introduces the Helmholtz free energy function in terms of the generalized temperature ϕ, i.e.

$$\psi = \epsilon - \eta\phi \tag{2.47}$$

then inequality (2.46) may be rearranged with the aid of the energy conservation equation $(2.45)_4$, noting $\phi > 0$, as

$$-\rho_0(\dot{\psi} + \eta\dot{\phi}) + Y_{AB}\dot{e}_{AB} - \frac{Q_A\phi_{,A}}{\phi} \geq 0. \tag{2.48}$$

(Green and Lindsay, 1972) essentially use as constitutive theory the assertion that

$$\psi, \eta, \phi, Q_A \quad \text{and} \quad Y_{AB} \tag{2.49}$$

depend on the independent variables

$$X_A, \rho_0, \theta, \theta_{,A}, \dot{\theta} \quad \text{and} \quad e_{AB}\,. \tag{2.50}$$

Upon using (2.49) and (2.50) in inequality (2.48) (Green and Lindsay, 1972) deduce that

$$\begin{aligned} &-\rho_0(\psi_\theta + \eta\phi_\theta)\dot{\theta} - \rho_0(\psi_{\dot{\theta}} + \eta\phi_{\dot{\theta}})\ddot{\theta} - \rho_0(\psi_{\theta_{,A}} + \eta\phi_{\theta_{,A}})\dot{\theta}_{,A} \\ &\quad + \Big[Y_{AB} - \frac{\rho_0}{2}\Big(\frac{\partial\psi}{\partial e_{AB}} + \frac{\partial\psi}{\partial e_{BA}}\Big) - \frac{\rho_0}{2}\eta\Big(\frac{\partial\phi}{\partial e_{AB}} + \frac{\partial\phi}{\partial e_{BA}}\Big)\Big]\dot{e}_{AB} \\ &\quad - \frac{Q_A}{\phi}\Big[\phi_\theta\theta_{,A} + \frac{\partial\phi}{\partial\theta_{,B}}\,\theta_{,BA} + \phi_{\dot{\theta}}\dot{\theta}_{,A} \\ &\quad + \frac{1}{2}\Big(\frac{\partial\phi}{\partial e_{RS}} + \frac{\partial\phi}{\partial e_{SR}}\Big)e_{RS,A} + \frac{\partial\phi}{\partial\rho_0}\,\rho_{,A} + \frac{\partial\phi}{\partial X_A}\Big] \,\geq\, 0. \end{aligned} \tag{2.51}$$

(Green and Lindsay, 1972) then argue that $\ddot{\theta}, \dot{\theta}_{,A}\,, \dot{e}_{AB}\,, e_{RS,A}\,, \theta_{,AB}\,, \rho_{0,A}\,,$ may be selected independently in inequality (2.51) balancing the momentum and energy equations $(2.45)_2$ and $(2.45)_4$ by a suitable choice of F_i and r. In this manner they deduce the relations

$$\begin{aligned} &\eta = -\frac{\partial\psi/\partial\dot{\theta}}{\partial\phi/\partial\dot{\theta}}\,, \\ &Y_{AB} = \frac{\rho_0}{2}\Big(\frac{\partial\psi}{\partial e_{AB}} + \frac{\partial\psi}{\partial e_{BA}}\Big) + \frac{\rho_0}{2}\eta\Big(\frac{\partial\phi}{\partial e_{AB}} + \frac{\partial\phi}{\partial e_{BA}}\Big)\,, \\ &\rho_0\Big(\frac{\partial\psi}{\partial\theta_{,A}} + \eta\frac{\partial\phi}{\partial\theta_{,A}}\Big) + \frac{Q_A}{\phi}\,\phi_{\dot{\theta}} \;= 0\,, \\ &Q_A\,\frac{\partial\phi}{\partial\theta_{,B}} + Q_B\,\frac{\partial\phi}{\partial\theta_{,A}} = 0\,, \\ &Q_A\,\frac{\partial\phi}{\partial\rho_0} = 0, \qquad Q_A\Big(\frac{\partial\phi}{\partial e_{RS}} + \frac{\partial\phi}{\partial e_{SR}}\Big) = 0\,. \end{aligned} \tag{2.52}$$

The residual entropy inequality follows from (2.51). However, (Green and Lindsay, 1972) then restrict attention to the case where the reference body is homogeneous (i.e. does not depend on $\mathbf{X}$) and then upon use of $(2.52)_{4,5,6}$ one finds

$$\phi = \phi(\theta, \dot{\theta}). \tag{2.53}$$

The residual entropy inequality then has form

$$-\rho_0\big(\psi_\theta + \eta\phi_\theta\big)\dot{\theta} - \frac{Q_A}{\phi}\,\phi_\theta\theta_{,A} \,\geq\, 0. \tag{2.54}$$

Since ρ_0 is non constant one employs (2.53) in equations $(2.52)_{2,3}$ to derive the forms for the stress tensor and heat flux, namely

$$\begin{aligned} Y_{AB} &= \frac{\rho_0}{2}\Big(\frac{\partial\psi}{\partial e_{AB}} + \frac{\partial\psi}{\partial e_{BA}}\Big) \\ Q_A &= -\rho_0\phi\,\frac{\dfrac{\partial\psi}{\partial\theta_{,A}}}{\dfrac{\partial\phi}{\partial\dot\theta}}\,. \end{aligned} \tag{2.55}$$

(Green and Lindsay, 1972) further reduce the energy equation $(2.45)_4$. One may then show that the full nonlinear system of equations for thermoelasticity of (Green and Lindsay, 1972) are given by

$$\begin{aligned} \rho_0\ddot x_i &= \rho_0 F_i + \frac{\partial S_{Ai}}{\partial X_A}\,, \\ \rho_0\phi\dot\eta &= \rho_0 r - \frac{\partial Q_A}{\partial X_A} - \rho_0\big(\psi_\theta + \eta\phi_\theta\big)\dot\theta - \rho_0\psi_{\theta,A}\,\dot\theta_{,A}\,, \end{aligned} \tag{2.56}$$

where S_{Ai} and Q_A are given by equations (2.55) with $S_{Ai} = F_{iR}Y_{RA}$.

A detailed analysis of acceleration waves, including curved waves, for system (2.56) is given by (Lindsay and Straughan, 1979).

(Green and Lindsay, 1972) write down expressions for ψ and ϕ which are quadratic in the variables $\theta, \dot\theta, \theta_{,i}, e_{ij}$ to develop a linearized theory of thermoelasticity from (2.56). They linearize about an initial body with zero stress and heat flux. The complete system of equations for linearized thermoelasticity derived by (Green and Lindsay, 1972) for an anisotropic thermoelastic body then have form

$$\begin{aligned} \rho\ddot u_i &= \rho F_i + (c_{ijkh}u_{k,h})_{,j} + \big[a_{ij}(\theta + \alpha\dot\theta)\big]_{,j}\,, \\ \rho(h\ddot\theta + d\dot\theta - a_{ij}\dot u_{i,j} - b_i\dot\theta_{,i}) &= \frac{\rho r}{\theta_0} + (b_i\dot\theta + k_{ij}\theta_{,j})_{,i}\,. \end{aligned} \tag{2.57}$$

Here u_i is the displacement about a reference state with positions denoted by X_i, ρ is the density, $h, d, b_i, c_{ijkh}, a_{ij}, k_{ij}$ are coefficients which have the symmetries

$$c_{ijkh} = c_{khij} = c_{jikh}\,, \;\; a_{ij} = a_{ji}\,, \;\; k_{ij} = k_{ji}\,. \tag{2.58}$$

(Green, 1972) has shown that the boundary-initial value problem for (2.57) is unique requiring only symmetry of the elastic coefficients c_{ijkh}. His proof employs a Lagrange identity technique. Uniqueness and continuous dependence on the initial data for a solution to the boundary-initial value problem for (2.57) requiring only symmetry of the elastic coefficients c_{ijkh} was established by (Straughan, 1974). His proof introduced a natural logarithmic convexity functional into thermoelasticity.

A very interesting study comparing the solutions to the equations of classical thermoelasticity, Cattaneo-Lord-Shulman theory, cf. section 2.1.1, and the (Green and Lindsay, 1972) theory is provided by (Jordan and Puri, 2001). These writers investigate the propagation of a thermal pulse in a thermoelastic shell employing each of the linearized equations for the three thermoelastic theories, classical, Lord-Shulman, and Green-Lindsay. Their numerical results are very revealing. They typically demonstrate that the classical theory leads to a smooth pulse while that of Lord-Shulman is less smooth showing discontinuities in derivatives. The theory of (Green and Lindsay, 1972) leads to strong pulse behaviour displaying distinct jumps. For the applications they have in mind, such as to the behaviour of stainless steel run tanks which hold cryogenic liquids for rocket fuel at NASA's John C. Stennis Space Center, the strong pulse solution is definitely of interest.

2.3 Green-Naghdi type II theory

(Green and Naghdi, 1993) adopt a different approach to thermoelasticity to other writers, this approach being based on an extension of the type II theory of heat propagation in a rigid solid developed by (Green and Naghdi, 1991), see section 1.10. The idea is to define a temperature θ, an empirical temperature T, and a thermal displacement α, such that θ depends on T and the properties of the material with $\theta > 0, \partial\theta/\partial T > 0$, and

$$\alpha(\mathbf{X}, t) = \int_{t_0}^{t} T(\mathbf{X}, s)ds + \alpha_0 \,. \tag{2.59}$$

Here t_0 is a "start time" at our disposal and α_0 is the value of α at $t = t_0$. (Although (Green and Naghdi, 1993) define T and θ in this way at the outset they later show that there is no loss in generality if one identifies T with θ.)

As usual, $x_i = x_i(X_A, t)$ denotes the motion of a body with positions $\mathbf{X}$ in the reference configuration, $\mathbf{x}$ being their counterparts in the current configuration. (Green and Naghdi, 1993) observe that

$$\dot{\alpha} = T \tag{2.60}$$

and they introduce the variables β_A and γ_i as

$$\beta_A = \frac{\partial \alpha}{\partial X_A}\,, \qquad \gamma_i = \frac{\partial T}{\partial x_i} = \dot{\alpha}_{,i}\,. \tag{2.61}$$

The variables $\dot{\boldsymbol{\beta}}$ and $\boldsymbol{\gamma}$ are connected by the equation

$$\dot{\beta}_A = F_{Ai}\gamma_i\,.$$

(Green and Naghdi, 1993) define t_{ij} to be the Cauchy stress tensor, $p_i = q_i/\theta$ to be the entropy flux vector, ψ, η, to be the Helmholtz free energy and entropy, respectively. Their momentum equation is

$$\rho\dot{v}_i = \rho b_i + t_{ji,j} \tag{2.62}$$

where ρ, v_i, b_i are the density, velocity, and body force. They work with an entropy balance *equation* rather than an entropy inequality and this requires them to introduce an intrinsic supply of entropy ξ in order to postulate their entropy balance equation as

$$\rho\dot{\eta} = \rho s + \rho\xi - p_{i,i}\,. \tag{2.63}$$

Here s is the external supply of entropy given by $s = r/\theta$, where r is the external supply of heat. The balance of energy equation employed by (Green and Naghdi, 1993) has form

$$t_{ij}d_{ij} - p_i\theta_{,i} - \rho(\dot{\psi} + \eta\dot{\theta}) - \rho\theta\xi = 0, \tag{2.64}$$

where $d_{ij} = (v_{i,j} + v_{j,i})/2$.

(Green and Naghdi, 1993) define a classical thermoelastic body to be one for which

$$t_{ij}, p_i, \psi, \eta, \theta \quad \text{and} \quad \xi$$

depend on the variables

$$T, \gamma_i = T_{,i} = \dot{\alpha}_{,i}\,, \quad \text{and} \quad F_{iA} = x_{i,A}\,. \tag{2.65}$$

This leads to the usual "hyperbolic-parabolic" system of nonlinear equations of thermoelasticity. The goal of (Green and Naghdi, 1993) is to introduce a new class of thermoelasticity equations by requiring

$$t_{ij}, p_i, \psi, \eta, \theta \quad \text{and} \quad \xi \tag{2.66}$$

to depend on

$$T, \alpha_{,A} \quad \text{and} \quad F_{iA}\,. \tag{2.67}$$

(Green and Naghdi, 1993) call this type of thermoelasticity, thermoelasticity of type II. They remark that ... "*it involves no dissipation of energy*" ... "*is perhaps a more natural candidate for its identification as thermoelasticity than the usual theory*".

(Green and Naghdi, 1993) employ relations (2.66) together with (2.67) in equation (2.64). They show that one may deduce from this the relations

$$\frac{\partial\theta}{\partial\beta_A} = 0, \qquad \frac{\partial\theta}{\partial F_{iA}} = 0 \tag{2.68}$$

whence

$$\theta = \theta(T).$$

They then argue that they may write $T = \theta - \theta_0$ and henceforth replace T by θ in the ensuing development. Thus,

$$\psi = \psi(\theta, \beta_A, F_{iA}) = \psi(\theta, \alpha_{,A}, F_{iA}).$$

They further show that the expanded energy equation leads to the results

$$\begin{aligned} \eta &= -\frac{\partial\psi}{\partial\theta}\,, \qquad t_{ij} = \rho\frac{\partial\psi}{\partial F_{iA}}\,F_{Aj}\,,\\ p_i &= -\rho F_{iA}\,\frac{\partial\psi}{\partial\alpha_{,A}} \qquad \text{and} \qquad \xi = 0. \end{aligned} \tag{2.69}$$

(An equivalent reduction employing the Piola-Kirchoff stress tensor S_{Ai} is given in section 4.4 where the forms more suitable for an acceleration wave analysis are derived.) (Green and Naghdi, 1993) then replace F_{iA} by the right Cauchy-Green tensor $C_{AB} = F_{Ai}F_{iB}$ to deduce

$$t_{ij} = \rho F_{iA}F_{Bj}\left(\frac{\partial\psi}{\partial C_{AB}} + \frac{\partial\psi}{\partial C_{BA}}\right). \tag{2.70}$$

The complete nonlinear equations of thermoelasticity of type II are then given in the current frame by equations (2.62) and (2.63) with η, p_i, ξ and t_{ij} given by (2.69) and (2.70). For ease of reference these are collected here as

$$\begin{aligned} \rho\ddot{x}_i &= \rho b_i + \frac{\partial}{\partial x_j}\left\{\rho F_{iA}F_{Bj}\left(\frac{\partial\psi}{\partial C_{AB}} + \frac{\partial\psi}{\partial C_{BA}}\right)\right\},\\ -\rho\frac{d}{dt}\left(\frac{\partial\psi}{\partial\theta}\right) &= \rho s + \frac{\partial}{\partial x_i}\left(\rho F_{iA}\frac{\partial\psi}{\partial\alpha_{,A}}\right), \end{aligned} \tag{2.71}$$

where b_i and s are externally supplied and d/dt denotes the material derivative. Once a prescription of the functional form of $\psi = \psi(\theta, \alpha_{,A}, F_{iA})$ is known, equations (2.71) yield a nonlinear system of partial differential equations for x_i and θ.

Linearized forms of the equations for type II thermoelasticity are given in the isotropic case by (Green and Naghdi, 1993) and in the anisotropic case by (Quintanilla, 1999; Quintanilla, 2002b). In terms of the displacement u_i and temperature perturbation θ these may be written for the isotropic case as

$$\begin{aligned} \rho_0\ddot{u}_i &= \rho_0 b_i - E_1\theta_{,i} + \mu\Delta u_i + (\lambda+\mu)u_{j,ij}\,,\\ c\ddot{\theta} &= \rho_0 r + \kappa\Delta\theta + \theta_0 E_1\ddot{u}_{i,i}\,, \end{aligned} \tag{2.72}$$

where $\rho_0, E_1, \kappa, \theta_0$ are constants and μ, λ are the Lamé coefficients. In the anisotropic case for a body with a centre of symmetry the respective linear equations are

$$\begin{aligned} \rho\ddot{u}_i &= (c_{ijkh}u_{k,h})_{,j} - (a_{ij}\theta)_{,j} + \rho f_i,\\ c\ddot{\theta} &= -a_{ij}\ddot{u}_{i,j} + (k_{ij}\theta_{,j})_{,i} + \rho r, \end{aligned} \tag{2.73}$$

where f_i, r are the externally supplied body force and heat supply, ρ, c are positive constants, c_{ijkh} are the elastic coefficients, a_{ij} define a coupling tensor, and k_{ij} defines the thermal conductivity tensor.

A general uniqueness theorem for a solution to equations (2.73) requiring no definiteness of the elastic coefficients c_{ijkh} is given by (Quintanilla and

Straughan, 2000). Their proof relies on a logarithmic convexity argument. These writers also derive a variety of growth estimates for the solution depending on the elastic coefficients and the initial energy, see sections 6.2 and 6.3 of this book. Reciprocal theorems and variational principles for type II linear thermoelasticity are given by (Chirita and Ciarletta, 2010a).

As we mentioned in the introduction, section 1.1, the paper of (Green and Naghdi, 1991), and their companion papers (Green and Naghdi, 1992; Green and Naghdi, 1993) on type II and type III thermoelasticity (discussed in the next section), brought a new way of thinking to the area of heat wave propagation and their articles have influenced many subsequent developments. In fact, work since 1991 in this area has definitely increased as may be witnessed for example from the papers, and the references therein, of (Abd-Alla and Abo-Dahab, 2009), (Alvarez-Ramirez et al., 2006; Alvarez-Ramirez et al., 2008), (Anile and Romano, 2001), (Bargmann et al., 2008b), (Bargmann et al., 2008a), (Brusov et al., 2003), (Buishvili et al., 2002), (Caviglia and Morro, 2005), (Chandrasekharaiah, 1998), (Cai et al., 2006), (Christov and Jordan, 2005), (Christov, 2008), (Cimmelli and Frischmuth, 2007), (Ciancio and Quintanilla, 2007), (De Cicco and Diaco, 2002), (Duhamel, 2001), (Fabrizio et al., 1998), (Fabrizio et al., 2008), (Fichera, 1992), (Green and Naghdi, 1991; Green and Naghdi, 1992; Green and Naghdi, 1993; Green and Naghdi, 1995b; Green and Naghdi, 1995a; Green and Naghdi, 1996), (Han et al., 2006), (Hetnarski and Ignaczak, 1999), (Horgan and Quintanilla, 2005), (Iesan, 2002; Iesan, 2004; Iesan, 2008), (Iesan and Nappa, 2005), (Jaisaardsuetrong and Straughan, 2007), (Johnson et al., 1994), (Jordan and Puri, 2001), (Jou and Criado-Sancho, 1998), (Kalpakides and Maugin, 2004), (Lin and Payne, 2004a), (Linton-Johnson et al., 1994), (Loh et al., 2007), (Messaoudi and Said-Houari, 2008), (Metzler and Compte, 1999), (Meyer, 2006), (Mitra et al., 1995), (Morro, 2006), (Payne and Song, 2002; Payne and Song, 2004b), (Puri and Jordan, 1999b; Puri and Jordan, 1999a; Puri and Jordan, 2004; Puri and Jordan, 2006), (Puri and Kythe, 1997; Puri and Kythe, 1998), (Quintanilla, 2001b; Quintanilla, 2002a; Quintanilla, 2007b), (Quintanilla and Racke, 2003; Quintanilla and Racke, 2006a; Quintanilla and Racke, 2007; Quintanilla and Racke, 2008), (Quintanilla and Straughan, 2000; Quintanilla and Straughan, 2002; Quintanilla and Straughan, 2004; Quintanilla and Straughan, 2005b; Quintanilla and Straughan, 2005a; Quintanilla and Straughan, 2008), (Roy et al., 2009), (Ruggeri, 2001), (Saleh and Al-Nimr, 2008), (Sanderson et al., 1995), (Serdyukov, 2001), (Serdyukov et al., 2003), (Shnaid, 2003), (Straughan, 2004; Straughan, 2008), (Su et al., 2005), (Su and Dai, 2006), (Tzou, 1995b; Tzou, 1995a), (Vadasz, 2005), (Vadasz et al., 2005), (Vedavarz et al., 1992), (Zhang and Zuazua, 2003).

2.4 Green-Naghdi type III theory

The theory of type III thermoelasticity was formulated by (Green and Naghdi, 1992). The development starts very much like that for type II in section 2.3. Hence, the governing equations are (2.62) and (2.63) with the energy balance law (2.64) being used to reduce the constitutive theory. Again, the temperatures θ and T are introduced as is the thermal displacement α. The difference between type II and type III is in the constitutive list (2.67). The theory of type III adds the variable $\dot{\alpha}_{,i} = T_{,i}$ to the list (2.67). Thus, a thermoelastic material of type III is defined as one for which

$$t_{ij}, p_i, \psi, \eta, \theta \quad \text{and} \quad \xi \tag{2.74}$$

depend on

$$T, \alpha_{,A}\,, \dot{\alpha}_{,i} \quad \text{and} \quad F_{iA}\,. \tag{2.75}$$

(In a sense, type III combines the classical theory with that of type II as the list (2.75) is the union of the lists (2.65) and (2.67).)

(Green and Naghdi, 1992) employ (2.74) and (2.75) in the energy balance equation (2.64). After expanding the derivatives $\dot{\psi}$ and $\dot{\theta}$ in terms of the variables (2.75) the expanded energy equation is reduced. (Green and Naghdi, 1992) deduce that

$$\frac{\partial \theta}{\partial \dot{\alpha}_{,i}} = 0, \qquad \frac{\partial \theta}{\partial \alpha_{,A}} = 0, \qquad \frac{\partial \theta}{\partial F_{iA}} = 0 \qquad \text{and} \qquad \frac{\partial \psi}{\partial \dot{\alpha}_{,i}} = 0\,. \tag{2.76}$$

Thus,

$$\theta = \theta(T)$$

and (Green and Naghdi, 1992) show that T may be replaced by θ. Then, $(2.76)_4$ yields

$$\psi = \psi(\theta, \theta_{,A}, F_{iA}). \tag{2.77}$$

Further, (Green and Naghdi, 1992) show that

$$\eta = -\frac{\partial \psi}{\partial \theta}\,, \qquad t_{ij} = \rho\,\frac{\partial \psi}{\partial F_{iA}}\,F_{Aj} \tag{2.78}$$

but, unlike (2.69) for a type II material they cannot deduce an explicit expression for p_i, nor is ξ zero. Instead, the residual of the energy balance equation yields

$$p_i \dot{\alpha}_{,i} + \rho \frac{\partial \psi}{\partial \alpha_{,A}} F_{Ai} \dot{\alpha}_{,i} + \rho \theta \xi = 0. \tag{2.79}$$

We might think of equation (2.79) as defining the variable ξ.

To complete the theory of a type III thermoelastic material one needs, therefore, to specify the functional form of

$$p_i = p_i(\theta, \alpha_{,A}, \dot{\alpha}_{,i}, F_{iA}), \tag{2.80}$$

or equivalently, one needs to specify the heat flux $q_i = \theta p_i$. Clearly, one can write a general expression for p_i as a function of the vector terms which arise from combinations of $\alpha_{,A}, \dot{\alpha}_{,i}$ and F_{iA}. I am not aware of where this has been done, although (Quintanilla and Straughan, 2004) do study acceleration waves in the complete nonlinear theory.

The nonlinear theory for a thermoelastic body of type III then consists of equations (2.62) and (2.63) combined with (2.78), (2.79) and an explicit representation for p_i from (2.80). The general equations have form

$$\begin{aligned} \rho \ddot{x}_i &= \rho b_i + \frac{\partial}{\partial x_j}\Big(\rho \frac{\partial \psi}{\partial F_{iA}} F_{Aj}\Big)\,, \\ -\rho \frac{d}{dt}\Big(\frac{\partial \psi}{\partial \theta}\Big) &= \rho s - \frac{\partial p_i}{\partial x_i} - \frac{1}{\theta} p_i \dot{\alpha}_{,i} - \frac{\rho}{\theta}\frac{\partial \psi}{\partial \alpha_{,A}} F_{Ai}\dot{\alpha}_{,i}\,. \end{aligned} \tag{2.81}$$

Linearized forms of the equations for type III thermoelasticity are given by (Green and Naghdi, 1992) in the isotropic case and by (Quintanilla, 2001c) in the anisotropic case. In the isotropic case they are

$$\begin{aligned} \rho_0 \ddot{u}_i &= \rho_0 b_i - E_1 \theta_{,i} + \mu \Delta u_i + (\lambda + \mu) u_{j,ij}\,, \\ \rho_0 c \ddot{\theta} + E_1 \theta_0 \ddot{u}_i &= \rho_0 \dot{r} + \kappa \Delta \dot{\theta} + \kappa^* \Delta \theta\,, \end{aligned} \tag{2.82}$$

where $\rho_0, E_1, c, \kappa, \kappa^*$ are constants, μ, λ are the Lamé constants, and b_i, r are the externally supplied body force and heat supply. In the anisotropic case when the body has a centre of symmetry the relevant equations are

$$\begin{aligned} \rho \ddot{u}_i &= (c_{ijkh} u_{k,h})_{,j} - (a_{ij}\theta)_{,j} + \rho f_i\,, \\ c\ddot{\theta} &= -a_{ij}\ddot{u}_{i,j} + (k_{ij}\theta_{,j})_{,i} + (b_{ij}\dot{\theta}_{,j})_{,i} + \rho r, \end{aligned} \tag{2.83}$$

where ρ, c are positive functions which may depend on $\mathbf{x}$, c_{ijkh} are the elastic coefficients, a_{ij} are coupling coefficients, and k_{ij}, b_{ij} represent the coefficients of thermal tensors. The terms f_i and r represent the body force and heat supply.

A general uniqueness theorem for a solution to (2.83) requiring only symmetry of the elastic coefficients c_{ijkh} is provided by (Quintanilla and Straughan, 2000). Their proof employs a Lagrange identity method, see section 6.4. Non-standard problems for thermoelasticity of type II or type III are considered by (Quintanilla and Straughan, 2005b), see also section 6.6.

2.5 Thermoelasticity with Voids

A class of theories which may be thought of as describing certain properties of porous media were derived by (Nunziato and Cowin, 1979). The key idea is to suppose there is an elastic body which has a distribution of voids throughout. The voids are gaps full of air, water, or some other fluid. This theory provides equations for the displacement of the elastic matrix of the porous medium and the void fraction occupied by the fluid. We believe the voids theory has a large potential, especially in wave propagation problems.

The theory of an elastic body containing voids essentially generalizes the classical theory of nonlinear elasticity by adding a function $\nu(\mathbf{X}, t)$ to describe the void fraction within the body. Here $\mathbf{X}$ denotes a point in the reference configuration of the body. Thus, in addition to the momentum equation for the motion $x_i = x_i(\mathbf{X}, t)$ as time evolves, one needs to prescribe an evolution equation for the void fraction ν. For a non-isothermal situation one also needs an energy balance law which effectively serves to determine the temperature field $T(\mathbf{X}, t)$. The original theory is due to (Nunziato and Cowin, 1979) and the temperature field development was largely due to D. Iesan, see details in chapter 1 of (Iesan, 2004). This theory has much in common with the continuum theory for granular materials, cf. (Massoudi, 2005; Massoudi, 2006a; Massoudi, 2006b).

In this chapter we wish to examine theories of thermoelastic materials containing voids. Such theories are particularly useful to describe nonlinear wave motion and account well for the elastic behaviour of the matrix, being a generalisation of nonlinear elasticity theory. Interestingly, while there are many studies involving the linearised theory of elastic materials with voids, see e.g. (Ciarletta and Iesan, 1993) or (Iesan, 2004), analysis of the fully nonlinear equations is only beginning, see e.g. (Iesan, 2005; Iesan, 2006).

The basic idea of including voids in a continuous body is due to (Goodman and Cowin, 1972), although they developed constitutive theory appropriate to a fluid. This they claim is more appropriate to flow of a granular medium. Acceleration waves in the Goodman-Cowin theory of granular media were studied by (Nunziato and Walsh, 1977; Nunziato and Walsh, 1978). For a reader interested in the theory of voids I would suggest first reading the article of (Goodman and Cowin, 1972), and then progressing to the theory of elastic materials with voids as given by (Nunziato and Cowin, 1979). General descriptions of the theory of elastic materials with voids and various applications are given in the books of (Ciarletta and Iesan, 1993) and (Iesan, 2004). Continuous dependence on the coupling coefficients of the voids theory (a structural stability problem) is studied by (Chirita et al., 2006).

The potential application area for the theory of elastic materials with voids is huge. In particular, wave motion in elastic materials with voids has many applications. (Ciarletta et al., 2007) mention four application areas of immediate interest. To appreciate the potential uses we briefly describe these areas. (Ouellette, 2004) is a beautiful and inspiring article which deals with many applications of acoustic microscopy. We are all aware of optical microscopy, but the potential uses of acoustic microscopy are enormous. (Ouellette, 2004) points out that the presence of voids presents a serious problem for acoustic microscopy, and a study of wave motion in an elastic material with voids is likely to be very helpful here. She observes that, "acoustic microscopy remains a niche technology and is especially sensitive to variations in the elastic properties of semiconductor materials, such as air gaps, known as delaminations or voids ..." In particular, (Ouellette, 2004) draws attention to several novel applications of acoustic microscopy in diagnostic medicine. She notes that one may, "apply a special ultrasound scanner to deliver pathological assessments of skin tumours or lesions, non-invasively," and especially there is, "no need to kill the specimen as is usually needed in optical microscopy." (Diebold, 2005) further emphasizes these and other applications.

Wave motion is important in the production of ceramics, or certainly in ceramic behaviour. (Saggio-Woyansky et al., 1992) observe that porous ceramics are either reticulate or foam and are made up of a porous network which has relatively low mass, low thermal conductivity, and low density, and (Raiser et al., 1994) report experimental results where microcracking along grain boundaries in ceramics is caused by compressive waves. Since reticulate porous ceramics are used for molten metal filters, diesel engine exhaust filters, as catalyst supports, and industrial hot-gas filters, and both reticulate and foam porous ceramics are used as light-structure plates, in gas combustion burners, and in fire - protection and thermal insulation materials, a study of wave motion in such materials is clearly useful.

A further important application area for elastic materials with voids is in the production of building materials such as bricks. Modern buildings are usually made with lighter, thinner bricks, often with many voids in the building materials. In seismic areas lighter materials are necessary and much applied research activity is taking place. However, the use of lighter materials, especially those with voids is creating an environmental problem because noise transmission through such objects is considerably greater. Consequently, there is much applied research ongoing in the area of acoustic materials with voids, cf. (Garai and Pompoli, 2005), (Maysenhölder et al., 2004), (Wilson, 1997), and any theoretical model for acoustic wave propagation in an elastic material with voids which yields useful results is desirable.

2.5.1 Basic theory of elastic materials with voids

To present ideas clearly we begin with the classical theory of thermoelasticity with voids, where the energy balance equation is essentially parabolic, so temperature is not transported as a wave. The balance equations for a continuous body containing voids are given by (Goodman and Cowin, 1972). We use the equations as given by (Nunziato and Cowin, 1979) since these are appropriate for an elastic body.

The key thing is to assume that there is a distribution of voids throughout the body B. If $\gamma(\mathbf{X}, t)$ denotes the density of the elastic matrix, then the mass density $\rho(\mathbf{X}, t)$ of B has form

$$\rho = \nu\gamma \tag{2.84}$$

where $0 < \nu \leq 1$ is a volume distribution function with $\nu = \nu(\mathbf{X}, t)$. Since the density or void distribution in the reference configuration can be different we also have

$$\rho_0 = \nu_0\gamma_0$$

where ρ_0, γ_0, ν_0 are the equivalent functions to ρ, γ, ν, but in the reference configuration.

The first balance law is the balance of mass

$$\rho|\det \mathbf{F}| = \rho_0 \,.$$

With π_{Ai} being the Piola-Kirchoff stress tensor and $F_{iA} = x_{i,A}$ as before, the balance of angular momentum states

$$\boldsymbol{\pi}\mathbf{F}^T = \mathbf{F}\boldsymbol{\pi}^T.$$

The balance of linear momentum has form

$$\rho_0\ddot{x}_i = \pi_{Ai,A} + \rho_0 f_i, \tag{2.85}$$

f_i being an external body force. The balance law for the voids distribution is

$$\rho_0 k\ddot{\nu} = h_{A,A} + g + \rho_0\ell, \tag{2.86}$$

where k is an inertia coefficient, h_A is a stress vector, g is an intrinsic body force (giving rise to void creation/extinction inside the body), and ℓ is an external void body force. Actually, (Nunziato and Cowin, 1979) allow the inertia coefficient k to depend on $\mathbf{X}$ and/or t, but, for simplicity, we follow (Goodman and Cowin, 1972) and assume it to be constant.

The energy balance in the body may be expressed as

$$\rho_0\dot{\epsilon} = \pi_{Ai}\dot{F}_{iA} + h_A\dot{\nu}_{,A} - g\dot{\nu} - q_{A,A} + \rho_0 r, \tag{2.87}$$

where ϵ, q_A and r are, respectively, the internal energy function, the heat flux vector, and the externally supplied heat supply function. To understand equation (2.87) we may integrate it over a fixed body B, integrate by parts, and use the divergence theorem to see that

$$\frac{d}{dt}\int_B \rho_0\epsilon dV + \int_B (g\dot{\nu} + h_{A,A}\dot{\nu})dV = \int_B \pi_{Ai}\dot{F}_{iA}dV - \oint_{\partial B} q_A N_A dS + \int_B \rho_0 r dV,$$

where ∂B is the boundary of B. Employing (2.86) with $\ell = 0$ we may rewrite the above as

$$\frac{d}{dt}\int_B (\rho_0\epsilon + \frac{\rho_0 k}{2}\,\dot{\nu}^2)dV = \int_B \pi_{Ai}\dot{F}_{iA}dV - \oint_{\partial B} q_A N_A dS + \int_B \rho_0 r dV.$$

In this form we recognise the equation as an energy balance equation with a term added due to the kinetic energy of the voids. In fact, (Iesan, 2004), pp. 3–5, shows how one may begin with a conservation of energy law for an arbitrary sub-body of a continuous medium with voids, and then derive equations (2.85), (2.86) and (2.87) from the initial energy balance equation.

It is usual in continuum thermodynamics to also introduce an entropy inequality. We use the Clausius-Duhem inequality

$$\rho_0 \dot{\eta} \geq -\left(\frac{q_A}{\theta}\right)_{,A} + \frac{\rho_0 r}{\theta}, \tag{2.88}$$

where η is the specific entropy function. Observe that the sign of the first term on the right of (2.88) is different from that of (Nunziato and Cowin, 1979). (One could use a more sophisticated entropy inequality where q_A/θ is replaced by a general entropy flux $\mathbf{k}$, as in (Goodman and Cowin, 1972), but the above is sufficient for our purpose.)

2.5.2 Thermodynamic restrictions

We consider an elastic body containing voids to be one which has as constitutive variables the set

$$\Sigma = \{\nu_0, \nu, F_{iA}, \theta, \theta_{,A}, \nu_{,A}\} \tag{2.89}$$

supplemented with $\dot{\nu}$. Thus, the constitutive theory assumes

$$\begin{aligned} &\epsilon = \epsilon(\Sigma, \dot{\nu}), \qquad \pi_{Ai} = \pi_{Ai}(\Sigma, \dot{\nu}), \qquad q_A = q_A(\Sigma, \dot{\nu}), \\ &\eta = \eta(\Sigma, \dot{\nu}), \qquad h_A = h_A(\Sigma, \dot{\nu}), \qquad g = g(\Sigma, \dot{\nu}). \end{aligned} \tag{2.90}$$

This is different from (Nunziato and Cowin, 1979) who regard η as the independent variable rather than θ and they also assume $q_A = 0$.

To proceed we introduce the Helmholtz free energy function ψ in the manner

$$\epsilon = \psi + \eta\theta. \tag{2.91}$$

Next, (2.87) is employed to remove the terms $-q_{A,A} + \rho_0 r$ from inequality (2.88) and then utilize (2.91) to rewrite (2.88) as

$$-\rho_0(\dot{\psi} + \eta\dot{\theta}) - \frac{q_A \theta_{,A}}{\theta} + \pi_{Ai}\dot{F}_{iA} + h_A \dot{\nu}_{,A} - g\dot{\nu} \geq 0. \tag{2.92}$$

The chain rule is used together with (2.90) to expand $\dot{\psi}$ and then (2.92) may be written as

$$\begin{aligned} &-\left(\rho_0 \frac{\partial \psi}{\partial \nu} + g\right)\dot{\nu} - \frac{q_A \theta_{,A}}{\theta} - \left(\rho_0 \frac{\partial \psi}{\partial F_{iA}} - \pi_{Ai}\right)\dot{F}_{iA} \\ &-\left(\rho_0 \frac{\partial \psi}{\partial \theta} + \rho_0 \eta\right)\dot{\theta} - \left(\rho_0 \frac{\partial \psi}{\partial \nu_{,A}} - h_A\right)\dot{\nu}_{,A} \\ &-\rho_0 \frac{\partial \psi}{\partial \theta_{,A}} \dot{\theta}_{,A} - \rho_0 \frac{\partial \psi}{\partial \dot{\nu}} \ddot{\nu} \geq 0. \end{aligned} \tag{2.93}$$

The next step is to observe that $\dot{F}_{iA}, \dot{\theta}, \dot{\theta}_{,A}, \dot{\nu}_{,A}$ and $\ddot{\nu}$ appear linearly in inequality (2.93). We may then follow the procedure of (Coleman and Noll, 1963) and assign an arbitrary value to each of these quantities in turn, balancing equations (2.85), (2.86) and (2.87) by a suitable choice of the externally supplied functions f_i, ℓ and r. We may in this manner violate inequality (2.93) unless the coefficients of $\dot{F}_{iA}, \dot{\theta}, \dot{\theta}_{,A}, \dot{\nu}_{,A}$ and $\ddot{\nu}$ are each identically zero. Hence, we deduce that

$$\psi \neq \psi(\dot{\nu}, \theta_{,A}),$$

$$h_A = \rho_0 \frac{\partial \psi}{\partial \nu_{,A}} \quad \Rightarrow \quad h_A \neq h_A(\dot{\nu}, \theta_{,A}), \tag{2.94}$$

$$\pi_{Ai} = \rho_0 \frac{\partial \psi}{\partial F_{iA}} \quad \Rightarrow \quad \pi_{Ai} \neq \pi_{Ai}(\dot{\nu}, \theta_{,A}), \tag{2.95}$$

$$\eta = -\frac{\partial \psi}{\partial \theta} \quad \Rightarrow \quad \eta \neq \eta(\dot{\nu}, \theta_{,A}),$$

and further

$$\epsilon \neq \epsilon(\dot{\nu}, \theta_{,A}).$$

The residual entropy inequality, left over from (2.93), which must hold for all motions is

$$-\left(\rho_0 \frac{\partial \psi}{\partial \nu} + g\right)\dot{\nu} - \frac{q_A \theta_{,A}}{\theta} \geq 0.$$

Thus, to specify a material for an elastic body containing voids we have to postulate a suitable functional form for $\psi = \psi(\nu_0, \nu, F_{iA}, \theta, \nu_{,A})$. Such a form is usually constructed with the aid of experiments. The functions g and q_A still involve $\dot{\nu}$ and this can lead to behaviour almost viscoelastic-like, see (Nunziato and Cowin, 1979). Other writers, e.g. (Iesan, 2004), (Ciarletta and Iesan, 1993), omit $\dot{\nu}$ from the constitutive list at the outset. In this manner one deduces that g may be given as a derivative of the Helmholtz free energy, (Iesan, 2004), p. 7, although some of the possibly desirable features of viscoelasticity are lost. The wavespeeds of acceleration waves in this case are derived in (Iesan, 2004), (Ciarletta and Iesan, 1993).

2.5.3 Voids and Green - Lindsay thermoelasticity

In this section we consider a theory of voids as developed by (Nunziato and Cowin, 1979) but we allow for the possibility of propagation of a temperature wave, by generalizing the voids theory in the thermodynamic framework of (Green and Laws, 1972). In addition to allowing us to explicitly examine the important effects of temperature this allows us to study the propagation of a temperature wave in a porous material. In this section we concentrate on the theory of (Green and Laws, 1972) where a generalized temperature $\phi(\theta, \dot{\theta})$, θ being absolute temperature, is introduced. The theory was originally developed by (Ciarletta and Scarpetta, 1989).

The current literature increasingly recognises the importance thermal waves have in the theory of porous media. A very clever way to dry a saturated porous material via second sound is due to (Meyer, 2006) and (Johnson et al., 1994) show how second sound may be employed to calculate physical properties of water saturated porous media. Both of these cover highly important and useful topics. (Kaminski, 1990) reports experimental results for materials with non-homogeneous inner structures which indicate relaxation times of order $11 - 54$ seconds rather than order picoseconds as was previously thought. There is evidence that second sound may be a key mechanism for heat transfer in some biological tissues as the experiments of (Mitra et al., 1995) and the work of (Vedavarz et al., 1992) indicate. Thus, we believe a theory of elastic materials with voids coupled to a suitable thermodynamic theory capable of admitting second sound has a place in modern engineering. One has to be careful how the theory of voids is married to the thermodynamics, however. The incorporation of time derivatives does present a serious problem. The thermodynamics of Green and his co-workers were specifically developed to incorporate into other areas of continuum mechanics and thus we believe these are natural approaches to use.

In this section we describe a thermo-poroacoustic theory which allows for nonlinear elastic effects and for the presence of voids, by using the thermodynamics of (Green and Laws, 1972). This thermodynamics utilises a generalized temperature $\phi(\theta, \dot{\theta})$ rather than just the standard absolute temperature θ.

The starting point is to commence with the standard balance equations for an elastic material containing voids, cf. (Nunziato and Cowin, 1979), or equations (2.85), (2.86), (2.87), and we follow the approach of (Ciarletta and Scarpetta, 1989), see also (Ciarletta and Straughan, 2007b),

$$\rho \ddot{x}_i = \pi_{Ai,A} + \rho F_i, \tag{2.96}$$

$$\rho k \ddot{\nu} = h_{A,A} + g + \rho \ell, \tag{2.97}$$

$$\rho \dot{\epsilon} = -q_{A,A} + \pi_{Ai} \dot{x}_{i,A} + h_A \dot{\nu}_{,A} - g \dot{\nu} + \rho r. \tag{2.98}$$

Here X_A denote reference coordinates, x_i denote spatial coordinates, a superposed dot denotes material time differentiation holding $\mathbf{X}$ fixed, and $_{,A}$ signifies $\partial/\partial X_A$. The variable ρ is the reference density, and we use ρ rather than ρ_0 henceforth, for simplicity. Furthermore, ν is the void fraction, ϵ is the specific internal energy, k is the inertia coefficient, F_i, ℓ and r are externally supplied body force, extrinsic equilibrated body force, and externally supplied heat. The tensor π_{Ai} is the stress per unit area of the X_A-plane in the reference configuration acting over corresponding surfaces at time t (the Piola-Kirchoff stress tensor), q_A is the heat flux vector, and h_A and g are a vector and a scalar function arising in the conservation law for void evolution. (Nunziato and Cowin, 1979) refer to h_A as the equilibrated stress and they call g the intrinsic equilibrated body force.

The thermodynamic development commences with the entropy inequality of (Green and Laws, 1972), and this is

$$\rho\dot{\eta} - \frac{\rho r}{\phi} + \left(\frac{q_A}{\phi}\right)_{,A} \geq 0. \tag{2.99}$$

In this inequality η is the specific entropy and $\phi(>0)$ is a generalised temperature function which reduces to θ in the equilibrium state. Next, introduce the Helmholtz free energy function ψ by $\psi = \epsilon - \eta\phi$ and rewrite inequality (2.99) using the energy equation (2.98) to obtain

$$-\rho\dot{\psi} - \rho\dot{\phi}\eta + \pi_{Ai}\dot{x}_{i,A} - \frac{q_A\phi_{,A}}{\phi} - g\dot{\nu} + h_A\dot{\nu}_{,A} \geq 0. \tag{2.100}$$

Now, we assume that the constitutive functions

$$\psi, \phi, \eta, \pi_{Ai}, q_A, h_A, g \tag{2.101}$$

depend on the variables

$$x_{i,A}, \nu, \nu_{,A}, \theta, \dot{\theta}, \theta_{,A}. \tag{2.102}$$

Note that we do not include $\dot{\nu}$ in the constitutive list and are so effectively following the voids approach of (Iesan, 2004), (Ciarletta and Iesan, 1993). One then expands $\dot{\psi}$ and $\dot{\phi}$ in (2.100) to reduce the constitutive equations. Inequality (2.100) expanded is

$$\begin{aligned}
&\dot{x}_{i,A}\Big(\pi_{Ai} - \rho\,\frac{\partial\psi}{\partial x_{i,A}} - \rho\eta\,\frac{\partial\phi}{\partial x_{i,A}}\Big) - \dot{\nu}\Big(\rho\,\frac{\partial\psi}{\partial\nu} + g + \rho\eta\,\frac{\partial\phi}{\partial\nu}\Big)\\
&- \dot{\theta}\Big(\rho\,\frac{\partial\psi}{\partial\theta} + \rho\eta\,\frac{\partial\phi}{\partial\theta}\Big) - \ddot{\theta}\Big(\rho\,\frac{\partial\psi}{\partial\dot{\theta}} + \rho\eta\,\frac{\partial\phi}{\partial\dot{\theta}}\Big)\\
&- \dot{\theta}_{,A}\Big(\rho\,\frac{\partial\psi}{\partial\theta_{,A}} + \rho\eta\,\frac{\partial\phi}{\partial\theta_{,A}} + \frac{q_A}{\phi}\,\frac{\partial\phi}{\partial\dot{\theta}}\Big) - \dot{\nu}_{,A}\Big(\rho\eta\,\frac{\partial\phi}{\partial\nu_{,A}} + \rho\,\frac{\partial\psi}{\partial\nu_{,A}} - h_A\Big)\\
&- \frac{q_A}{\phi}\,x_{i,AB}\,\frac{\partial\phi}{\partial x_{i,AB}} - \frac{q_A}{\phi}\,\frac{\partial\phi}{\partial\nu_{,J}}\,\nu_{,JA} - \frac{q_A}{\phi}\,\frac{\partial\phi}{\partial\theta_{,J}}\,\theta_{,JA}\\
&- \frac{q_A}{\phi}\Big(\frac{\partial\phi}{\partial\nu}\,\nu_{,A} + \frac{\partial\phi}{\partial\theta}\,\theta_{,A}\Big) \geq 0.
\end{aligned} \tag{2.103}$$

The terms in $x_{i,AB}, \nu_{,JA}$ and $\theta_{,JA}$ appear linearly and so using the fact that ℓ, r and F_i may be selected as we like to balance (2.96) – (2.98), we find

$$\frac{\partial\phi}{\partial x_{i,A}} = 0, \qquad \frac{\partial\phi}{\partial\nu_{,A}} = 0, \qquad \frac{\partial\phi}{\partial\theta_{,A}} = 0. \tag{2.104}$$

Thus

$$\phi = \phi(\theta, \dot{\theta}, \nu). \tag{2.105}$$

It is important to observe that the generalized temperature depends on ν in addition to θ and $\dot{\theta}$. Hence, the void fraction ν directly influences ϕ.

Furthermore, the linearity of $\dot{x}_{i,A}, \dot{\nu}, \ddot{\theta}, \dot{\theta}_{,A}$ and $\dot{\nu}_{,A}$ in (2.103) then allows us to deduce that

$$\begin{aligned} \pi_{Ai} &= \rho \frac{\partial \psi}{\partial x_{i,A}}, \qquad q_A = -\rho \frac{\partial \psi}{\partial \theta_{,A}} \bigg/ \frac{1}{\phi} \frac{\partial \phi}{\partial \dot{\theta}}, \\ h_A &= \rho \frac{\partial \psi}{\partial \nu_{,A}}, \qquad g = -\rho \left(\frac{\partial \psi}{\partial \nu} + \eta \frac{\partial \phi}{\partial \nu} \right), \end{aligned} \tag{2.106}$$

and

$$\eta = -\frac{\partial \psi}{\partial \dot{\theta}} \bigg/ \frac{\partial \phi}{\partial \dot{\theta}} . \tag{2.107}$$

The residual entropy inequality which remains from (2.103) after this procedure, has form

$$-\dot{\theta} \left(\rho \frac{\partial \psi}{\partial \theta} + \rho \eta \frac{\partial \phi}{\partial \theta} \right) - \frac{q_A}{\phi} \left(\frac{\partial \phi}{\partial \nu} \nu_{,A} + \frac{\partial \phi}{\partial \theta} \theta_{,A} \right) \geq 0. \tag{2.108}$$

This inequality places a further restriction on all constitutive equations and motions.

Thus, the complete nonlinear theory of Green - Lindsay thermoelasticity with voids as derived by (Ciarletta and Scarpetta, 1989) consists of equations (2.96) - (2.98) together with the constitutive equations (2.105) - (2.107). One needs functional forms for ψ and ϕ and then $\pi_{Ai}, h_A, g, \epsilon$ and q_A follow and the balance equations (2.96) - (2.98) are, in principle, determinate.

2.5.4 Voids and type II thermoelasticity

In this section we describe the theory of (De Cicco and Diaco, 2002). These writers generalize the thermodynamic procedure of (Green and Naghdi, 1993) and use a thermal displacement variable

$$\alpha = \int_{t_0}^{t} \theta(\mathbf{X}, s) ds + \alpha_0, \tag{2.109}$$

where $\mathbf{X}$ is the spatial coordinate in the reference configuration of the body with θ being the absolute temperature. A general procedure for deriving the equations for a continuous body from a single balance of energy equation is developed by (Green and Naghdi, 1995b). These writers derive the conservation equations for balance of mass, momentum, and entropy. The work of (De Cicco and Diaco, 2002), like that of (Green and Naghdi, 1993) starts with an entropy balance equation. (De Cicco and Diaco, 2002) extend the (Green and Naghdi, 1993) thermoelasticity theory to include voids in the manner of (Nunziato and Cowin, 1979). The full nonlinear equations are derived by (De Cicco and Diaco, 2002), although they only utilize a linearized version. We follow (Ciarletta et al., 2007) and rederive the (De Cicco and Diaco, 2002) theory referring to a reference configuration and employing a first Piola-Kirchoff stress tensor, as opposed to the symmetric stress tensor formulation of (De Cicco and Diaco, 2002).

It is worth observing that (Green and Naghdi, 1993) write, ... "*This type of theory, ... thermoelasticity type II, since it involves no dissipation of energy is perhaps a more natural candidate for its identification as thermoelasticity than the usual theory.*" Moreover, (Green and Naghdi, 1993) observe that, ... "*This suggests that a full thermoelasticity theory - along with the usual mechanical aspects - should more logically include the present type of heat flow (type II) instead of the heat flow by conduction (classical theory, type I).*" (The words in brackets have been added for clarity.) We would argue that it is beneficial to develop a fully nonlinear acceleration wave analysis for a Green - Naghdi type II thermoelastic theory of voids.

The starting point in the development of the theory is to consider the momentum and balance of voids equations for an elastic material containing voids, see (2.85), (2.86),

$$\rho \ddot{x}_i = \pi_{Ai,A} + \rho F_i, \tag{2.110}$$

$$\rho k \ddot{\nu} = h_{A,A} + g + \rho \ell. \tag{2.111}$$

One needs a balance of energy and from (De Cicco and Diaco, 2002) this is

$$\rho \dot{\epsilon} = \pi_{Ai} \dot{x}_{i,A} + h_A \dot{\nu}_{,A} - g \dot{\nu} + \rho s \theta + (\theta \Phi_A)_{,A}. \tag{2.112}$$

In these equations X_A denote reference coordinates, x_i denote spatial coordinates, a superposed dot denotes material time differentiation and $_{,A}$ stands for $\partial/\partial X_A$. The variables ρ, ν, ϵ, k, are the reference density, the void fraction, the specific internal energy, and the inertia coefficient. The terms F_i, ℓ and s denote externally supplied body force, extrinsic equilibrated body force, and externally supplied heat. The tensor π_{Ai} is the stress per unit area of the X_A-plane in the reference configuration acting over corresponding surfaces at time t (the Piola-Kirchoff stress tensor), Φ_A is the entropy flux vector, and h_A and g are a vector and a scalar function arising in the conservation law for void evolution. These are referred to by (Nunziato and Cowin, 1979) as the equilibrated stress and the intrinsic equilibrated body force, respectively.

The next step is to use the entropy balance equation, see (Green and Naghdi, 1993), (De Cicco and Diaco, 2002),

$$\rho \theta \dot{\eta} = \rho \theta s + \rho \theta \xi + (\theta \Phi_A)_{,A} - \Phi_A \theta_{,A} \tag{2.113}$$

where ξ is the internal rate of production of entropy per unit mass, and η, θ are the specific entropy and the absolute temperature. Introduce the Helmholtz free energy function $\psi = \epsilon - \eta\theta$ and then equation (2.112) is rewritten with the aid of (2.113) as

$$\rho \dot{\psi} + \rho \eta \dot{\theta} = \pi_{Ai} \dot{x}_{i,A} + h_A \dot{\nu}_{,A} - g \dot{\nu} + \Phi_A \theta_{,A} - \rho \theta \xi. \tag{2.114}$$

The constitutive theory of (De Cicco and Diaco, 2002) writes the functions

$$\psi, \eta, \pi_{Ai}, \Phi_A, h_A, g, \xi, \tag{2.115}$$

as depending on

$$x_{i,A}, \nu, \nu_{,A}, \dot{\alpha}, \alpha_{,A}. \tag{2.116}$$

The function $\dot{\psi}$ is expanded using the chain rule, and rearranging terms, recollecting $\dot{\alpha} = \theta$, equation (2.114) may be written as

$$\begin{aligned} &\dot{x}_{i,A}\Big(\rho\frac{\partial\psi}{\partial x_{i,A}} - \pi_{Ai}\Big) + \dot{\nu}_{,A}\Big(\rho\frac{\partial\psi}{\partial\nu_{,A}} - h_{Ai}\Big) + \dot{\alpha}_{,A}\Big(\rho\frac{\partial\psi}{\partial\alpha_{,A}} - \Phi_A\Big) \\ &+ \rho\ddot{\alpha}\Big(\frac{\partial\psi}{\partial\dot{\alpha}} - \eta\Big) + \dot{\nu}\Big(\rho\frac{\partial\psi}{\partial\nu} + g\Big) + \rho\theta\xi = 0. \end{aligned} \tag{2.117}$$

We now use the fact that $\dot{x}_{i,A}, \dot{\nu}_{,A}, \dot{\alpha}_{,A}, \ddot{\alpha}$ and $\dot{\nu}$ appear linearly in (2.117) and so one derives the forms, cf. (De Cicco and Diaco, 2002), equations (19),

$$\begin{aligned} &\pi_{Ai} = \rho\frac{\partial\psi}{\partial x_{i,A}}, \qquad \Phi_A = \rho\frac{\partial\psi}{\partial\alpha_{,A}}, \qquad h_A = \rho\frac{\partial\psi}{\partial\nu_{,A}}, \\ &g = -\rho\frac{\partial\psi}{\partial\nu}, \qquad \eta = -\frac{\partial\psi}{\partial\theta} = -\frac{\partial\psi}{\partial\dot{\alpha}}, \qquad \xi = 0. \end{aligned} \tag{2.118}$$

A theory of type II thermoelasticity containing voids is then given by equations (2.110) - (2.112) with the constitutive theory prescribed by equations (2.118).

2.5.5 Voids and type III thermoelasticity

As we have seen in section 2.5.4, (De Cicco and Diaco, 2002) have developed a theory of thermoelasticity with voids which is a generalization of the dissipationless theory of thermoelasticity of (Green and Naghdi, 1993). The latter writers refer to this as thermoelasticity of type II, type I being the classical theory where the equation governing the temperature field is effectively parabolic as opposed to hyperbolic in type II theory. The theory of a thermoelastic body with voids corresponding to type I thermoelasticity was developed by D. Iesan, see e.g. (Iesan, 2004). However, as shown in section 2.4 (Green and Naghdi, 1992) have developed a further theory of thermoelasticity which employs the thermal displacement variable α and the thermodynamics of (Green and Naghdi, 1991; Green and Naghdi, 1995b). This theory leads to what is essentially a second order in time equation for the thermal displacement field, but differently from the type II theory of (Green and Naghdi, 1993) the theory of (Green and Naghdi, 1992) does have damping and hence dissipation. (Green and Naghdi, 1991; Green and Naghdi, 1992) refer to this theory as being of type III, cf. section 2.4.

The goal of this section is to develop a type III theory of thermoelasticity, but allowing for the accommodation of a distribution of voids throughout the body. The essential difference between type II and type III thermoelasticity is that the variable $\dot{\alpha}_{,A}$ is added to the constitutive list (2.116), whereas it is absent in section 2.5.4, cf. section 2.4. The presentation follows (Straughan, 2008), chapter 7.

We commence with the balance laws for a thermoelastic body with voids, equations (2.85), (2.86) and (2.87). With ρ denoting the density in the reference configuration and referring everything to this configuration, we have the equation of momentum balance

$$\rho \ddot{x}_i = \pi_{Ai,A} + \rho f_i. \tag{2.119}$$

The equation of voids distribution is

$$\rho k \ddot{\nu} = h_{A,A} + g + \rho \ell. \tag{2.120}$$

The equation of energy balance is

$$\rho \dot{\epsilon} = \pi_{Ai} \dot{x}_{i,A} + h_A \dot{\nu}_{,A} - g\dot{\nu} + \rho s \theta - (\theta p_A)_{,A}. \tag{2.121}$$

We let s be the heat supply and $p_A = q_A/\theta$ is the entropy flux vector. We choose this representation to keep in line with (Green and Naghdi, 1991; Green and Naghdi, 1992), and observe that $p_A = -\Phi_A$ where Φ_A is the entropy flux vector of (De Cicco and Diaco, 2002). We follow (Green and Naghdi, 1992) and postulate an entropy balance equation

$$\rho \dot{\eta} = \rho s + \rho \xi - p_{A,A}, \tag{2.122}$$

where ξ is the internal rate of production of entropy per unit mass. The variable θ is the absolute temperature and $\alpha(\mathbf{X}, t)$ is the thermal displacement.

We next introduce the Helmholtz free energy function ψ in terms of the internal energy ϵ, entropy η and temperature θ, by $\psi = \epsilon - \eta\theta$. Then, from (2.121) and (2.122) it is a straightforward matter to derive the reduced energy equation, cf. (Green and Naghdi, 1992), equation (2.5),

$$\rho \dot{\psi} + \rho \eta \dot{\theta} = \pi_{Ai} \dot{x}_{i,A} + h_A \dot{\nu}_{,A} - g\dot{\nu} - \rho \xi \theta - \theta_{,A} p_A. \tag{2.123}$$

A thermoelastic body of type III which contains a distribution of voids is defined to be one for which the functions

$$\psi, \eta, \pi_{Ai}, p_A, h_A, g \text{ and } \xi \tag{2.124}$$

depend on the independent variables

$$F_{iA} = x_{i,A}, \nu, \nu_{,A}, \dot{\alpha}, \alpha_{,A}, \dot{\alpha}_{,A}\,. \tag{2.125}$$

We do not consider the inhomogeneous situation which would also require inclusion of X_A in the list (2.125), cf. (Iesan, 2004). Observe that we do not include $\dot{\nu}$ in the list (2.125). This follows (Iesan, 2004) and allows us to determine g from ψ.

The procedure now is to expand ψ in terms of the variables in the list (2.125), and recalling $\dot{\alpha} = \theta$, we obtain from (2.123),

$$\begin{aligned}&(\rho\psi_{F_{iA}} - \pi_{Ai})\dot{F}_{iA} + \dot{\nu}(\rho\psi_\nu + g) + \dot{\nu}_{,A}(\rho\psi_{\nu,A} - h_A)\\&+ \ddot{\alpha}(\rho\psi_{\dot{\alpha}} + \rho\eta) + \rho\psi_{\dot{\alpha}_{,A}}\ddot{\alpha}_{,A} + \dot{\alpha}_{,A}(p_A + \rho\psi_{\alpha,A}) + \rho\xi\dot{\alpha} = 0.\end{aligned} \tag{2.126}$$

We observe that $\dot{F}_{iA}, \dot{\nu}_{,A}, \ddot{\alpha}, \ddot{\alpha}_{,A}, \dot{\nu}$, appear linearly in (2.126). Thus, we may deduce that the coefficients of these terms in (2.126) must be zero. The process is akin to that described in Appendix A of (Green and Naghdi, 1992). Thus, we find that

$$\begin{aligned}&\pi_{Ai} = \rho\psi_{F_{iA}}\,, \qquad g = -\rho\psi_\nu\,, \qquad h_A = \rho\psi_{\nu,A},\\&\eta = -\psi_{\dot{\alpha}}, \qquad \psi \neq \psi(\dot{\alpha}_{,A}).\end{aligned} \tag{2.127}$$

Hence, once we prescribe a functional form for the Helmholtz free energy function ψ we also know the stress tensor, entropy, and the voids functions h_A and g. What remains from (2.126) is

$$\rho\xi\dot{\alpha} + \dot{\alpha}_{,A}(\rho\psi_{\alpha_{,A}} + p_A) = 0. \tag{2.128}$$

This leads to further restrictions on constitutive functions. We now also have that

$$\begin{aligned}&\psi = \psi(x_{i,A}, \nu, \nu_{,A}, \dot{\alpha}, \alpha_{,A}),\\&p_A = p_A(x_{i,A}, \nu, \nu_{,A}, \dot{\alpha}, \alpha_{,A}, \dot{\alpha}_{,A}),\\&\xi = \xi(x_{i,A}, \nu, \nu_{,A}, \dot{\alpha}, \alpha_{,A}, \dot{\alpha}_{,A}).\end{aligned} \tag{2.129}$$

Thus, once we have a form for the functional dependence of ψ on its variables, and a form for p_A, equations (2.119) - (2.121) yield the complete nonlinear theory of type III thermoelasticity with voids, the function ξ being determined by equation (2.128).

2.5.6 Linear voids type III thermoelasticity

One may study acceleration waves in the nonlinear theory of section 2.5.5. The acceleration waves in this case do not have a separately propagating temperature wave. The reason is that in some sense type III thermoelasticity behaves more like type I thermoelasticity. For acceleration wave motion in thermoelasticity without voids this is explained in detail by (Quintanilla and Straughan, 2004), and a similar explanation holds here. Nevertheless, the extra damping present in the current theory may be useful in practical problems and with this in mind we now develop the equations for a linear theory. Let the body have a centre of symmetry although we allow it to be anisotropic. We denote the displacement in this section as u_i. We then write ψ as a quadratic function of the variables in the list (2.129). Thus,

$$\begin{aligned} \rho\psi =& \frac{1}{2} a_{iAjB} u_{i,A} u_{j,B} - \frac{a_1}{2}\theta^2 - \frac{a_2}{2}\nu^2 + A_{iA}\theta u_{i,A} + B_{iA}\nu u_{i,A} \\ &+ \frac{R_{AB}}{2}\nu_{,A}\nu_{,B} + S_{AB}\nu_{,A}\alpha_{,B} + \frac{T_{AB}}{2}\alpha_{,A}\alpha_{,B}\,, \end{aligned} \tag{2.130}$$

where a_{iAjB}, R_{AB}, T_{AB} have the following symmetries,

$$a_{iAjB} = a_{jBiA}, \qquad R_{AB} = R_{BA}, \qquad T_{AB} = T_{BA}.$$

From (2.127) we now see that

$$\begin{aligned} &\pi_{Ai} = a_{iAjB} u_{j,B} + A_{iA}\theta + B_{iA}\nu, \qquad h_A = R_{AB}\nu_{,B} + S_{AB}\alpha_{,B}, \\ &\rho\eta = a_1\theta - A_{iA}u_{i,A}\,, \qquad g = a_2\nu - B_{iA}u_{i,A}\,. \end{aligned} \tag{2.131}$$

We also write

$$\begin{aligned} \rho\xi &= \phi_1\nu + \phi_2\dot{\alpha}, \\ p_A &= -K_{AB}\nu_{,B} - L_{AB}\alpha_{,B} - M_{AB}\dot{\alpha}_{,B}\,. \end{aligned}$$

From (2.128) one may use the cyclic thermomechanical process argument of (Green and Naghdi, 1991), section 9, to infer that L_{AB}, M_{AB}, R_{AB} are non-negative tensor forms, $\phi_2 \leq 0$, $\phi_1 = 0$, and $S_{AB} = K_{AB}$, $T_{AB} = L_{AB}$.

In this manner, equations (2.119), (2.120) and (2.122) lead to the linear equations

$$\begin{aligned} &\rho\ddot{u}_i = (a_{iAjB}u_{j,B})_{,A} + (A_{iA}\theta)_{,A} + (B_{iA}\nu)_{,A}, \\ &\rho k\ddot{\nu} = (R_{AB}\nu_{,B})_{,A} + (K_{AB}\alpha_{,B})_{,A} + a_2\nu - B_{iA}u_{i,A}\,, \\ &a_1\ddot{\alpha} = A_{iA}\dot{u}_{i,A} + \phi_2\dot{\alpha} + (K_{AB}\nu_{,B})_{,A} + (T_{AB}\alpha_{,B})_{,A} + (M_{AB}\dot{\alpha}_{,B})_{,A}\,. \end{aligned} \tag{2.132}$$

One may study the boundary - initial value problem for (2.132). For example, uniqueness and stability are easily investigated either by using an energy method, or if definiteness of the elastic coefficients a_{iAjB} is not imposed, by a logarithmic convexity argument. For the latter one will be better employing a time integrated version of α as done by (Ames and Straughan, 1992; Ames and Straughan, 1997) and (Quintanilla and Straughan, 2000), these articles following the introduction of this method for the (Green and Laws, 1972), (Green, 1972), version of thermoelasticity in (Straughan, 1974). One may also study one-dimensional waves as in (Green and Naghdi, 1992) and then (2.132) essentially reduce to

$$\begin{aligned} \rho u_{tt} &= a u_{xx} + A\theta_x + B\nu_x, \\ \rho k\nu_{tt} &= R\nu_{xx} + K\alpha_{xx} + a_2\nu - Bu_x, \\ a_1\alpha_{tt} &= Au_{tx} + \phi_2\alpha_t + K\nu_{xx} + T\alpha_{xx} + M\alpha_{txx}. \end{aligned} \tag{2.133}$$

The damped character of the temperature wave is evident from (2.133) as is observed in the non voids case by (Green and Naghdi, 1992), page 262. If the displacement and voids effects are absent from $(2.133)_3$, then we see that α satisfies the equation

$$a_1 \frac{\partial^2 \alpha}{\partial t^2} - M \frac{\partial^3 \alpha}{\partial t \partial x^2} = \phi_2 \frac{\partial \alpha}{\partial t} + T \frac{\partial^2 \alpha}{\partial x^2}.$$

This equation clearly does not permit the possibility of undamped thermal waves, unless $M = \phi_2 = 0$. The damping evident in equations (2.133) may be useful for description of some practical situations.

(Eringen, 1990; Eringen, 2004) develops a voids theory which has a richer structure than the (Nunziato and Cowin, 1979) model. This is achieved by incorporating an equation for the spin at each point of the body. Again, this theory is likely to have rich application in wave propagation problems. (Straughan, 2008) describes this theory in connection with nonlinear wave motion in section 7.6. A general study of singular surface propagation in a continuous body formed of a thermo-microstretch material which has memory is given by (Iesan and Scalia, 2006).

The theory developed by (Eringen, 1990) includes temperature effects while (Eringen, 2004) also includes electromagnetic effects which could be important in wave motion in ceramics, for example. However, we here ignore electromagnetic effects. The basic variables of the theory of (Eringen, 1990; Eringen, 2004) are the displacement u_i, microstretch φ, and the microrotation vector ϕ_i. The microstretch theory of (Eringen, 1990; Eringen, 2004) is based on balance laws for these quantities. These are balance of momentum,

$$\rho_0 \ddot{u}_i = \pi_{Ai,A} + \rho_0 f_i \tag{2.134}$$

and balance of microstretch

$$\rho_0 \frac{j_0}{2} \ddot{\varphi} = m_{A,A} + T + \rho_0 \ell, \tag{2.135}$$

in which we measure quantities in the current configuration but refer back to the reference configuration. Thus, π_{Ai} is a Piola-Kirchoff stress tensor, f_i is a prescribed body force, j_0 is the microinertia, m_A is a microstretch couple, ℓ is a prescribed microstretch source term and T (denoted by $t - s$ in (Eringen, 2004)) is the microstretch stress. Here $,A$ denotes $\partial/\partial X_A$. In addition to equations (2.134) and (2.135), the Eringen theory has a balance of spins equation of form

$$\rho_0 J \ddot{\phi}_i = m_{Ai,A} + \epsilon_{iAj} \pi_{Aj} + \rho_0 \ell_i, \tag{2.136}$$

where ℓ_i is an applied body couple density, m_{Ai} is the couple stress tensor, and we have taken the microinertia tensor $J_{ik} = J\delta_{ik}$ for simplicity. The constitutive theory assumes that

$$\pi_{Ai}, m_A, T \text{ and } m_{Ai} \tag{2.137}$$

are functions of the variables

$$F_{iA} = u_{i,A},\ \phi_i,\ \phi_{i,A},\ \varphi \text{ and } \varphi_{,A}. \tag{2.138}$$

In fact, (Eringen, 2004) combines $u_{i,A}$ and ϕ_i into a single strain measure $e_{iA} = u_{i,A} + \epsilon_{Ami}\phi_m$.

(Straughan, 2008) addresses some new questions regarding singular surfaces for the (Eringen, 1990) theory.

A detailed account of many properties of elastic bodies containing voids may also be found in the book by (Iesan, 2004), chapters 1 to 3.

2.6 Generalized thermoelasticity with microstructure

2.6.1 Hetnarski-Ignaczak theory

(Ignaczak, 1990) and (Hetnarski and Ignaczak, 1996; Hetnarski and Ignaczak, 1997; Hetnarski and Ignaczak, 1999) present an interesting thermoelastic theory which is capable of describing soliton - like thermoelastic waves. The wave aspect is further analysed in (Hetnarski and Ignaczak, 2000) where a comparison is made with wave propagation in other thermoelastic models. The model described by (Hetnarski and Ignaczak, 1999) consists of equations for the displacement u_i, temperature θ, and an elastic heat flow field b_i. In the isotropic case these equations are given by (Hetnarski and Ignaczak, 1999) as

$$\begin{aligned}
\zeta^2 \ddot{u}_i &= f_i - \epsilon\theta_{,i} + \frac{1}{2(1-\nu)}\, u_{j,ij} + \kappa\Delta u_i \,, \\
\dot{\theta} &= r - \theta\dot{u}_{i,i} + \Delta\theta + \frac{b_i\theta_{,i}}{\theta} - b_{i,i} \,, \\
\omega\dot{b}_i &= -\frac{\theta_{,i}}{\theta} \,,
\end{aligned} \tag{2.139}$$

where θ is the absolute temperature, ζ, ϵ are constants, f_i and r are body force and heat supply, ν is Poisson's ratio and $\kappa = (1-2\nu)/(2-2\nu)$. The constant ω is much less than 1 although positive. (Hetnarski and Ignaczak, 1999) show how equations (2.139) lead to soliton - like thermoelastic waves which move with different wavespeeds.

2.6.2 Micropolar, dipolar, affine microstructure

A type II thermoelastic theory incorporating micropolar effects was developed by (Ciarletta, 1999). He concentrates on producing a linear theory. In addition to the type II thermoelasticity theory of section 2.3 (Ciarletta, 1999) introduces a microrotation vector ϕ_i which represents spin at a point.

His basic equations, in the current frame are

$$\begin{aligned} \rho_0 \ddot{u}_i &= t_{ji,j} + \rho_0 f_i \,, \\ \rho_0 \dot{\eta} &= \rho_0 s + \Phi_{i,i} \,, \\ I_{ij} \ddot{\phi}_j &= m_{ji,j} + \epsilon_{ijk} t_{jk} + \rho_0 g_i \,. \end{aligned} \tag{2.140}$$

Equation $(2.140)_1$ is the balance of linear momentum, ρ_0 being density, u_i displacement, t_{ij} Cauchy stress, and f_i body force. Equation $(2.140)_2$ is the balance of entropy equation, η being entropy, s entropy supply, Φ_i entropy flux, and we observe the intrinsic entropy supply ξ is shown by (Ciarletta, 1999) to be zero. In the equation $(2.140)_3$ I_{ij} represents the coeffcients of inertia, m_{ij} is the couple stress tensor, and g_i is the body couple density. (Ciarletta, 1999) introduces the variables e_{ij} and κ_{ij} by

$$e_{ij} = u_{j,i} + \epsilon_{jik}\phi_k \,, \qquad \kappa_{ij} = \phi_{j,i} \,, \tag{2.141}$$

and he shows the energy balance law may be written as

$$\rho_0 \dot{\psi} - t_{ij}\dot{e}_{ij} - m_{ij}\dot{\kappa}_{ij} + \rho_0 \eta \dot{\theta} - \Phi_i \theta_{,i} = 0, \tag{2.142}$$

where ψ is the Helmholtz free energy and θ is the temperature.

(Ciarletta, 1999) linearizes about a reference state in which $\theta = T_0, \alpha = \alpha_0$, T_0 and α_0 being constants, where α is the thermal displacement. By introducing a free energy ψ which is quadratic in e_{ij}, κ_{ij}, T and $\tau_{,i}$, where $T = \theta - T_0$, $\tau = \int_{t_0}^t T ds$, he shows the constitutive equations are

$$\begin{aligned} t_{ij} &= A_{ijrs}e_{rs} + B_{ijrs}\kappa_{rs} - D_{ij}T + G_{ijr}\tau_{,r} \,, \\ m_{ij} &= B_{rsij}e_{rs} + C_{ijrs}\kappa_{rs} - E_{ij}T + H_{ijr}\tau_{,r} \,, \\ \rho_0 \eta &= D_{ij}e_{ij} + E_{ij}\kappa_{ij} + aT + b_i\tau_{,i} \,, \\ \Phi_i &= G_{rsi}e_{rs} + H_{rsi}\kappa_{rs} - b_i T + K_{ij}\tau_{,j} \,. \end{aligned}$$

(Ciarletta, 1999) principally works with the isotropic theory for a body with a centre of symmetry. For this case he shows the governing evolutionary equations become

$$\begin{aligned} &\rho_0 \ddot{u}_i = (\mu + \kappa)\Delta u_i + (\lambda + \mu)u_{j,ji} + \kappa\epsilon_{irs}\phi_{s,r} - mT_{,i} + \rho_0 f_i \,, \\ &I\ddot{\phi}_i = \gamma\Delta\phi_i + (\alpha + \beta)\phi_{j,ji} + \kappa\epsilon_{irs}u_{s,r} - 2\kappa\phi_i + \rho_0 g_i \,, \\ &aT_0\ddot{T} = k\Delta T - mT_0\ddot{u}_{i,i} + \rho_0\dot{s}. \end{aligned} \tag{2.143}$$

(Ciarletta, 1999) solves a problem of a concentrated heat source and proves a continuous dependence result. (Passarella and Zampoli, 2011) derive reciprocal and variational principles.

(Quintanilla, 2002c) develops a theory for thermoelasticity of type II for a body which includes an affine microstructure term x_{iK}. He writes that this determines the homogeneous deformation of the particle with centre of mass at **X**. He uses the equation of balance of linear momentum,

$$\rho \ddot{x}_i = t_{Ki,K} + \rho f_i \,, \tag{2.144}$$

where t_{Ki} is here the Piola-Kirchoff stress tensor. His balance of entropy is

$$\rho\dot{\eta} = \rho S + \rho\xi + \Phi_{A,A} \,. \tag{2.145}$$

He also needs an equation for micromotion,

$$\rho J_{KL}\ddot{x}_{iL} = S_{LiK,L} - S_{iK} + \rho f_{iK} \,, \tag{2.146}$$

where J_{KL} is an inertia tensor, S_{LiK} is the dipolar stress tensor, S_{iK} is a second order tensor defined below, and f_{iK} is a source term for the micromotion. The energy balance equation is

$$\rho(\dot{\psi} + \dot{\theta}\eta) - t_{Ki}\dot{x}_{i,K} - S_{LiK}\dot{x}_{iK,L} - S_{iK}\dot{x}_{iK} + \rho\theta\xi - \Phi_A\theta_{,A} = 0. \tag{2.147}$$

(Quintanilla, 2002c) postulates constitutive theory that

$$\psi, t_{Kj}, S_{LiK}, S_{iK}, \eta, \Phi_A \quad \text{and} \quad \xi$$

depend on the variables

$$x_{i,K}, x_{iK}, x_{iK,L}, \theta \quad \text{and} \quad \alpha_{,K} \,,$$

α being the thermal displacement. He shows that this leads to

$$\begin{aligned} t_{Kj} &= \rho\,\frac{\partial\psi}{\partial x_{j,K}}\,, \quad S_{Kj} = \rho\,\frac{\partial\psi}{\partial x_{Kj}}\,, \quad S_{KiJ} = \rho\,\frac{\partial\psi}{\partial x_{iJ,K}}\,, \\ \Phi_A &= \rho\,\frac{\partial\psi}{\partial\alpha_{,A}}\,, \quad \eta = -\frac{\partial\psi}{\partial\theta} \qquad \text{and} \qquad \xi = 0. \end{aligned} \tag{2.148}$$

Then, a nonlinear theory for thermoelasticity of type II including affine microstructure consists of the differential equations (2.144) - (2.146) together with the constitutive equations (2.148).

(Quintanilla, 2002c) linearizes about a state in which $\alpha = \alpha_0$ and $\theta = T_0$. He puts $T = \theta - T_0$, $u_i = x_i - X_i$, $u_{iA} = x_{iA} - X_{iA}$, and postulates a Helmholtz free energy function ψ which is quadratic. In this way he derives the governing evolution equations

$$\begin{aligned} \rho\ddot{u}_i &= (A_{iJRs}u_{s,R} + B_{iJrS}u_{r,S} - \beta_{Ji}T)_{,J} + \rho f_i \,, \\ \rho J_{KL}\ddot{u}_{iL} &= (E_{KiLSjR}u_{jR,S} + M_{KiLR}\tau_{,R})_{,L} \\ &\qquad - (B_{rSiK}u_{r,S} + C_{SriK}u_{r,S} - \chi_{iK}T) + \rho f_{iK} \,, \\ a\ddot{\tau} &= -\beta_{Ki}\dot{u}_{i,K} - \chi_{iK}\dot{u}_{iK} + M_{LjKI}\,u_{jL,KI} + K_{IJ}\,\tau_{,IJ} + \frac{\rho}{T_0}\,R \,, \end{aligned} \tag{2.149}$$

where $\tau = \int_{t_0}^t T ds$ is a thermal displacement. (Quintanilla, 2002c) introduces an interesting functional to establish uniqueness via logarithmic convexity without assuming definiteness of the elastic coefficients. He also establishes an existence theorem using a semigroup approach.

Thermoelasticity theories based on Green-Naghdi type II and type III thermodynamics are also investigated with internal variables in the interesting article of (Ciancio and Quintanilla, 2007).

2.6.3 Piezoelectricity and thermoelasticity

Piezoelectricity is an interesting phenomenon. It is basically the ability of some materials to generate an electric field or an electric potential when a mechanical stress is applied. Some crystals and especially certain ceramics exhibit piezoelectric behaviour. In this section we briefly describe some work which has developed and employed theories for piezoelectricity in a thermoelastic body when the temperature wave behaviour arises from a Lord-Shulman, Green-Lindsay, or Green-Naghdi type II approach.

Since ceramics are porous materials it makes sense to develop a piezoelectric theory for thermoelasticity which also incorporates porosity. This is what (Ciarletta and Scalia, 1993) did. They derive a thermoeleastic theory which allows the body to have a distribution of voids. Their thermodynamics is based on the (Green and Laws, 1972) and (Green and Lindsay, 1972) θ and $\dot{\theta}$ theory. Let u_i denote the displacement and ν the void fraction. Then (Ciarletta and Scalia, 1993) begin with the balance of linear momentum and balance equation for the voids, i.e.

$$\begin{aligned} \rho_0 \ddot{u}_i &= t_{ji,j} + f_i, \\ \rho_0 \chi \ddot{\nu} &= H_{i,i} + g + \ell\,, \end{aligned} \tag{2.150}$$

where t_{ij}, H_i are the Cauchy stress tensor and the equilibrated stress vector, f_i and ℓ are externally supplied body forces, χ is an inertia coefficient, and g is an intrinsic equilibrated body force. They adopt Maxwell's equations in the form

$$D_{i,i} = f, \qquad E_i = -\phi_{,i}\,, \tag{2.151}$$

where $\mathbf{D}, \mathbf{E}$ are the electric displacement field and the electric field, f is the charge density and ϕ is the electric potential. Their equation of energy balance is

$$\rho_0 \dot{\epsilon} = t_{ij}\dot{e}_{ij} + H_i \dot{\nu}_{,i} - g\dot{\nu} - q_{i,i} + E_i \dot{D}_i + \rho_0 r, \tag{2.152}$$

in which ϵ is the internal energy, $e_{ij} = (u_{i,j} + u_{j,i})/2$, q_i is the heat flux and r is the heat supply.

(Ciarletta and Scalia, 1993) employ the entropy inequality of (Green and Laws, 1972)

$$\rho_0 \dot{\eta} \geq \frac{\rho_0 r}{\phi} - \left(\frac{q_i}{\phi}\right)_{,i},$$

with η being entropy and ϕ a function depending on the constitutive variables. They assume there is a constant temperature T_0 in the reference state and ν_0 is the distribution of ν in that state. They then put $\theta = T - T_0, \zeta = \nu - \nu_0$, and define a generalized Helmholtz free energy of form

$$G = \epsilon - \phi\eta - \frac{1}{\rho_0} D_i E_i\,.$$

(Ciarletta and Scalia, 1993) define a piezoelectric material to be one for which

$$G, t_{ij}, H_i, q_i, g, \eta, D_i \quad \text{and} \quad \phi$$

depend on the variables

$$e_{ij}, \theta, \dot{\theta}, \theta_{,i}\,, E_i, \zeta \quad \text{and} \quad \zeta_{,i}\,.$$

They then exploit the entropy inequality to show that

$$\begin{aligned} & t_{ij} = \rho_0 \frac{\partial G}{\partial e_{ij}}\,, \qquad D_i = -\rho_0 \frac{\partial G}{\partial E_i}\,, \qquad H_i = \rho_0 \frac{\partial G}{\partial \zeta_{,i}}\,, \\ & q_i = \rho_0 \phi \frac{\partial G}{\partial \theta_{,i}} \Big/ \frac{\partial \phi}{\partial \dot{\theta}}\,, \qquad \eta = -\frac{\partial G}{\partial \dot{\theta}} \Big/ \frac{\partial \phi}{\partial \dot{\theta}}\,, \\ & g = -\rho_0 \Big(\frac{\partial G}{\partial \zeta} + \eta \frac{\partial \phi}{\partial \zeta} \Big) \\ & \text{and} \qquad \phi = \phi(\zeta, \theta, \dot{\theta}). \end{aligned} \tag{2.153}$$

They assume further that in thermodynamic equilibrium ϕ becomes $T_0 + \theta$, i.e. $\phi(\zeta, \theta, 0) = T_0 + \phi$.

Thus, the full system of nonlinear equations for piezoelectric behaviour in a thermoelastic body as derived by (Ciarletta and Scalia, 1993) are equations (2.150), (2.151), and (2.152) together with (2.153).

(Ciarletta and Scalia, 1993) futher develop a linear version of their theory and establish reciprocity relations and a uniqueness theorem.

The paper of (Iesan, 2008) proceeds along the lines of Green-Naghdi type II thermoelasticity to develop a theory of piezoelectricity in a microstretch continuous body. The idea of microstretch was introduced in section 2.5.6. As (Iesan, 2008) usefully points out a microstretch continuum is a dipolar one with a dipolar displacement u_{ij} where $u_{ij} = \phi\delta_{ij} + \epsilon_{ijk}\phi_k$. Here ϕ is a microstretch function (i.e. a porosity function) while ϕ_i is a microrotation vector. He remarks that ϕ may be thought of as a breathing motion whereas ϕ_i represents a rigid microrotation. He also notes that when ϕ is zero one obtains a Cosserat continuum.

The lucid paper of (Iesan, 2008) employs balance equations for entropy, linear momentum, moment of momentum, energy, microstretch, and Maxwell's equations. The full thermodynamic development is given in (Iesan, 2008). We simply present the relevant equations and constitutive theory. The form of Maxwell's equations are

$$D_{i,i} = f, \qquad E_i = -\psi_{,i}\,, \tag{2.154}$$

where D_i, E_i are the electric displacement field and the electric field, f is the charge density and ψ is the electric potential. The balance of entropy equation is

$$\rho_0 \dot{\eta} = \rho_0 s + \rho_0 \xi + \Phi_{i,i} \tag{2.155}$$

where ρ_0 is density, η entropy, Φ_i entropy flux, s is the external supply of entropy, and ξ is the internal rate of production of entropy. The balance of linear momentum is

$$\rho_0 \ddot{u}_i = \rho_0 f_i + t_{ji,j} \tag{2.156}$$

where u_i is the elastic displacement, f_i is the prescribed body force, and t_{ij} is the Cauchy stress tensor. The balance of moment of momentum equation is

$$I_{ij}\ddot{\phi}_j = \rho_0 g_i + \epsilon_{ijk} t_{jk} + m_{ji,j} \tag{2.157}$$

where I_{ij} is an inertia tensor, g_i is the external body couple, and m_{ij} is the couple stress tensor. Finally the equation for microstretch balance is

$$j_0 \ddot{\phi} = \pi_{i,i} + \rho_0 \ell - \sigma. \tag{2.158}$$

Here j_0 is a coefficient, π_i is the microstretch stress vector, ℓ is an extrenally supplied microstretch body force and σ is a function defined in terms of the electric enthalpy, see below.

(Iesan, 2008) introduces the electric enthalpy function A by

$$A = \epsilon - \eta\theta - \frac{1}{\rho_0} D_i E_i \tag{2.159}$$

where ϵ is the internal energy. His constitutive theory for a piezoelectric thermoelastic body requires that

$$A, t_{ij}, m_{ij}, \pi_i, \sigma, \eta, \Phi, \xi \qquad \text{and} \qquad D_i$$

depend on the variables

$$e_{ij}, \phi_{j,i}, \phi_{,i}, \phi, \theta \qquad \text{and} \qquad \alpha_{,i}$$

where

$$e_{ij} = u_{j,i} + \epsilon_{jik}\phi_k \qquad \text{and} \qquad \dot{\alpha} = \theta,$$

θ being the temperature. (Iesan, 2008) shows that

$$\begin{aligned} m_{ij} &= \rho_0 \frac{\partial A}{\partial \phi_{j,i}}, \qquad t_{ij} = \rho_0 \frac{\partial A}{\partial e_{ij}}, \qquad \Phi_i = \rho_0 \frac{\partial A}{\partial \alpha_{,i}}, \\ \eta &= -\frac{\partial A}{\partial \theta}, \qquad D_i = -\rho_0 \frac{\partial A}{\partial E_i}, \qquad \pi_i = \rho_0 \frac{\partial A}{\partial \phi_{,i}}, \\ \sigma &= \rho_0 \frac{\partial A}{\partial \phi}, \qquad \text{and} \qquad \xi = 0. \end{aligned} \tag{2.160}$$

The fully nonlinear theory of (Iesan, 2008) then consists of equations (2.154) - (2.158) with the forms (2.160). Once a form for functional dependence of A is prescribed this yields a complete set of equations.

(Iesan, 2008) further develops a linear theory. He linearizes about a reference state in which $\theta = T_0$ and $\alpha = \alpha_0$, T_0 and α_0 being constants. He defines $T = \theta - T_0$ and $\tau = \int_{t_0}^t T ds$ and then proposes a quadratic form for A. The complete form for the functions $t_{ij}, m_{ij}, \pi_i, \sigma, \Phi_i, \eta$ and D_i is then given in the general anisotropic case by (Iesan, 2008) in his equations (2.25) For an isotropic and homogeneous body (Iesan, 2008) develops the linear equations as

$$\begin{aligned}
\rho_0 \ddot{u}_i &= (\mu+\kappa)\Delta u_i + (\lambda+\mu)u_{j,ji} + \kappa\epsilon_{ijk}\phi_{k,j} \\
&\qquad + \lambda_0\phi_{,i} - \beta_0\dot{\tau}_{,i} + \rho_0 f_i\,, \\
I\ddot{\phi}_i &= \gamma\Delta\phi_i + (\alpha+\beta)\phi_{j,ji} + \kappa\epsilon_{ijk}u_{k,j} - 2\kappa\phi_{,i} + \rho_0 g_i\,, \\
j_0\ddot{\phi} &= (a_0\Delta - \xi_0)\phi - \lambda_2\Delta\psi + \nu_1\Delta\tau \\
&\qquad - \lambda_0 u_{j,j} + c_0\dot{\tau} + \rho_0\ell\,, \\
a\ddot{\tau} &= k\Delta\tau + \nu_1\Delta\phi - \nu_3\Delta\psi - \beta_0\dot{u}_{i,i} - c_0\dot{\phi} + \frac{\rho_0}{T_0}\,S\,, \\
\lambda_2\Delta\phi &+ \chi\Delta\psi + \nu_3\Delta\tau = -f,
\end{aligned} \tag{2.161}$$

where f_i, g_i, ℓ, S are external supplies. (Iesan, 2008) pointedly remarks that equation $(2.161)_5$ generalizes the classical equation $\chi\Delta\psi = -f$ for the electric potential. Here, the λ_2 term represents a porosity effect on the electric potential while the ν_3 term represents a thermal effect.

(Iesan, 2008) establishes a general uniqueness theorem and a continuous dependence result for his linear theory. He also obtains the solution for the problem of a concentrated heat source and for an impulsive body force. He also derives the solution for the problem of a thick-walled spherical shell where the shell surfaces are subject to different but constant pressures.

(Walia et al., 2009) study the propagation of Lamb waves in a transversely isotropic thermoelastic piezoelectric plate which is rotating about an axis orthogonal to the plate. They allow for finite speed thermal wave propagation by using both a Lord-Shulman type theory and a Green-Lindsay one, with the appropriate modifications to account for piezoelectric effects. Many numerical results are presented and their theory is applied specifically to a plate made of PZT-5A piezoelectric thermoelastic material. Other relevant references are provided by (Walia et al., 2009), see also (Ciarletta and Scarpetta, 1996).

2.6.4 Other theories

There are several other theories of thermoelasticity which cater for second sound effects which have been proposed and analysed in the literature. We briefly mention some.

(Iesan and Quintanilla, 2009) develop a type II thermoelasticity theory which includes microstretch effects and also allows for microtemperatures. Within the linearized theory they study uniqueness, existence, and instability of solutions. (Green and Naghdi, 1995c) present a general development of their entropy balance thermodynamics to Cosserat continua, Cosserat surfaces and to Cosserat curves. In (Green and Naghdi, 1995d) they present a similar development for the theory of mixtures of interacting continua. (Caviglia and Morro, 2005) present a general theory for a class of linear thermoviscoelastic materials and study this in detail when there is variation in a particular direction, the z-direction say. They also investigate the energy flux, and problems of reflection and transmission of waves.

Functionally graded elastic bodies are man made and have the property that elastic coefficients or other coefficients are not constant but change continuously throughout in a way that the material is designed for a specific purpose. Within second sound theory functionally graded thermoelastic bodies have been studied by (Ghosh and Kanoria, 2009) and (Mallik and Kanoria, 2007). The work of (Ghosh and Kanoria, 2009) is based on a Green-Lindsay type of thermoelasticity whereas that of (Mallik and Kanoria, 2007) proposes equations based on type II thermoelasticity. The effect of a magnetic field on the response of a thermoelastic body in the context of second sound theories has also been studied. (Aouadi, 2008) studies magnetic field effects within Green-Lindsay thermoelasticity. (Abd-Alla and Abo-Dahab, 2009) investigate a time-dependent problem with a magnetic field in type II thermoelasticity theory. (Sharma and Thakar, 2006) analyse the effect of rotation and a magnetic field for both Lord-Shulman and Green-Lindsay theories of thermoelasticity.

A thermoelasticity theory based on the two temperature approach, see section 1.7, was developed by (Chen et al., 1969). A variety of shock wave problems within the context of this theory were tackled by (Warren and Chen, 1973). (Puri and Jordan, 2006) also present an in-depth study of harmonic waves in the two-temperature thermoelastic theory. They investigate particularly the low and high frequency regimes and present detailed numerical results for both the elastic and temperature waves. Another study of wave propagation in the two temperature thermoelasticity theory is due to (Kumar and Mukhopadhyay, 2010). We also mention the study of (Othman and Singh, 2007) who study a rotating micropolar thermoelastic body. They present solutions for harmonic waves and compare the results within theories of classical thermoelasticity, Lord-Shulman theory, Green-Lindsay theory, type II theory, and a dual phase lag theory.

Analytical results for the solution to thermoelasticity of type III for a beam are given by (Zelati et al., 2010), while (Liu and Quintanilla, 2010a) establish analyticity results for a type III plate. Energy decay in a mixed thermoelastic system of type II and type III is studied by (Liu and Quintanilla, 2010b).

A novel result for a Timoshenko beam system is established by (Sare and Racke, 2009), who show that exponential decay of the solution is to be expected for a Timoshenko system with Fourier's law, but incorporation of a Cattaneo - like heat flux law does not lead to exponential decay.

2.7 Exercises

Exercise 2.7.1 *Consider the boundary - initial value problem, $\mathcal{P}$, for equations (2.132) with u_i, ν and α prescribed on the boundary Γ, of a bounded domain $\Omega \subset \mathbb{R}^3$. Let (u_i^1, ν_1, α_1) and (u_i^2, ν_2, α_2) be solutions to $\mathcal{P}$ for the same boundary and initial data. Write out the boundary initial value problem for the difference solution $u_i = u_i^1 - u_i^2, \nu = \nu_1 - \nu_2, \alpha = \alpha_1 - \alpha_2$ to $\mathcal{P}$. For appropriate symmetry conditions on the coefficients derive the energy equation*

$$\begin{aligned} \frac{d}{dt}\Big[\frac{1}{2}\int_\Omega \rho \dot{u}_i \dot{u}_i \, dx + \frac{1}{2}\int_\Omega a_{iAjB} u_{j,B} u_{i,A} \, dx + \frac{1}{2}\int_\Omega \rho k \dot{\nu}^2 dx \\ + \frac{1}{2}\int_\Omega R_{AB}\nu_{,A}\nu_{,B} dx + \frac{a_1}{2}\|\theta\|^2 + \frac{1}{2}\int_\Omega T_{AB}\alpha_{,A}\alpha_{,B} dx \\ + \int_\Omega K_{AB}\alpha_{,A}\nu_{,B} dx + \int_\Omega B_{iA} u_{i,A}\nu \, dx\Big] \\ + \int_\Omega M_{AB}\theta_{,A}\theta_{,B} dx - \phi_2\|\theta\|^2 = 0, \end{aligned} \tag{2.162}$$

where $\|\cdot\|$ is the norm on $L^2(\Omega)$. Use this equation to deduce uniqueness for appropriate signs on and relations between coefficients.

Exercise 2.7.2 *For the Hetnarsky - Ignazcak equations (2.139) with $f_i = 0$ and $r = 0$, show that*

$$\omega b_i \dot{b}_i = \frac{\partial}{\partial t}\frac{\omega}{2}|\mathbf{b}|^2 = -\frac{b_i \theta_{,i}}{\theta}\,.$$

Then show that

$$\dot{\theta} + \frac{\partial}{\partial t}\frac{\omega}{2}|\mathbf{b}|^2 = -\theta \dot{u}_{i,i} + \Delta\theta - b_{i,i}\,.$$

Show further that if Ω is a bounded domain in $\mathbb{R}^3$ with boundary Γ,

$$-\oint_\Gamma n_i \theta_{,i} dS = \omega \oint_\Gamma \theta \dot{b}_i n_i \, dS$$

and so $\partial\theta/\partial n = 0$ on Γ is consistent with $b_i n_i = 0$ on Γ.

Deduce also that with $u_i = 0$ on Γ,

$$\frac{d}{dt}\frac{\zeta^2}{2}\|\dot{\mathbf{u}}\|^2 = -\epsilon\int_\Omega \theta_{,i}\dot{u}_i dx - \frac{d}{dt}\frac{A}{2}\int_\Omega (u_{i,i})^2 dx - \frac{d}{dt}\frac{\kappa}{2}\|\nabla \mathbf{u}\|^2\,,$$

where $\|\cdot\|$ is the norm on $L^2(\Omega)$. Hence, conclude that with $u_i = 0, b_i n_i = 0$, and $\partial\theta/\partial n = 0$ on Γ,

$$F(t) = \frac{\zeta^2}{2\epsilon}\|\dot{\mathbf{u}}\|^2 + \frac{A}{2\epsilon}\|u_{i,i}\|^2 + \frac{\kappa}{2\epsilon}\|\nabla\mathbf{u}\|^2 + \frac{\omega}{2}\|\mathbf{b}\|^2 + \int_\Omega \theta \, dx$$

satisfies

$$F(t) = F(0) \qquad \text{for all } t > 0.$$

Exercise 2.7.3 *Prove that a solution to the boundary initial value problem* $\mathcal{P}$ *for* (2.143) *is unique.*
Hint. Let (2.143) *be defined on a bounded spatial domain* $\Omega \subset \mathbb{R}^3$*, for* $t > 0$*. Let* Γ *be the boundary of* Ω*. On* Γ *suppose* u_i, ϕ_i *and* T *are given. Also, initial values are given for* $u_i, u_{i,t}, \phi_i, \phi_{i,t}, T$ *and* T_t*. Let* (u_i^1, ϕ_i^1, T^1) *and* (u_i^2, ϕ_i^2, T^2) *be solutions which satisfy* $\mathcal{P}$ *for the same boundary and initial data. Define the difference solution* $u_i = u_i^1 - u_i^2, \phi_i = \phi_i^1 - \phi_i^2, T = T_1 - T_2$*. Integrate in time the equation which arises for* T *and set* $\tau = \int_{t_0}^t T\, ds$*. Show that one may find*

$$\begin{aligned}\frac{d}{dt}\bigg[\int_\Omega \frac{\rho_0}{2}\dot{u}_i\dot{u}_i dx + \Big(\frac{\mu+\kappa}{2}\Big)\|\nabla\mathbf{u}\|^2 + \Big(\frac{\mu+\lambda}{2}\Big)\|u_{i,i}\|^2 \\ - \kappa\epsilon_{irs}\int_\Omega u_i\phi_{s,r}dx + \frac{I}{2}\int_\Omega \dot{\phi}_i\dot{\phi}_i dx + \frac{\gamma}{2}\|\nabla\boldsymbol{\phi}\|^2 + \Big(\frac{\alpha+\beta}{2}\Big)\|\phi_{i,i}\|^2 \\ + \frac{a}{2}\|T\|^2 + \frac{k}{T_0}\|\nabla\tau\|^2\bigg] = -2\kappa\|\boldsymbol{\phi}\|^2.\end{aligned}$$

(Note $\dot{\tau} = T$*.) Hence, deduce uniqueness when* κ *is suitably restricted (a restriction which does follow from thermodynamics).*