Chapter 7
Stabilization of Fractional Order Unified Chaotic Systems via Linear State Feedback Controller

E.G. Razmjou, A. Ranjbar, Z. Rahmani, R. Ghaderi, and S. Momani

1 Introduction

The real world sometime possesses a fractional order dynamic [17]. Accordingly, fractional order controllers such as CRONE [13], TID [7], fractional PID controller [14], and lead-lag compensator [16] have been implemented to improve the performance and robustness of some closed loop control systems. An application of fractional algebra is the modeling of the fractional order chaotic systems. This kind of modeling provides more accuracy, less complexity as well as the possibility to increase the stability region [17].

Chaos, as an application of the fractional order modeling, is a very interesting nonlinear phenomenon. High sensitivity to initial conditions is a main characteristic of chaotic systems. Therefore, these systems are found to be difficult for synchronization or control [5]. Due to the complexity of these systems, control and stabilization task of chaotic nonlinear systems have been one of the arising interests in the control engineering area. In the past decade, great efforts have been devoted toward the chaos control, including stabilization of unstable equilibrium points, and more generally, unstable periodic solutions. Particularly, in case of chaos suppression of known chaotic systems, some useful methods have been developed. These include time delay feedback control [15], bang–bang control [19], optimal control [8], intelligent control [22], adaptive control [23], etc.

A unified chaotic system is a chaotic system that depends on a parameter, e.g., $\alpha \in [0,1]$. If $0 \leq \alpha < 0.8$, the unified chaotic system is reduced to the

E.G. Razmjou • A. Ranjbar (✉) • Z. Rahmani • R. Ghaderi
Intelligent System Research Group, Babol University of Technology,
P.O. Box 47135-484, Babol, Iran
e-mail: ehsan.razmju@gmail.com; a.ranjbar@nit.ac.ir; rahmaniz@gmail.com; r_ghaderi@nit.ac.ir

S. Momani
Department of Mathematics, The University of Jordan, Amman 11942, Jordan, Jordan

D. Baleanu et al. (eds.), *Fractional Dynamics and Control*,
DOI 10.1007/978-1-4614-0457-6_7, © Springer Science+Business Media, LLC 2012

generalized Lorenz chaotic system; the unified chaotic system is altered to the Lü chaotic system when $\alpha = 0.8$. For $0.8 < \alpha \leq 1$, the unified chaotic system is changed to the generalized Chen chaotic system.

Chen [3] considered that the parameter of the two unified chaotic systems is unknown. Hence, an adaptive controller was used to achieve synchronization based on Lyapunov stability theory. Chen [4] investigated the stabilization and synchronization of the unified chaotic system via an impulsive control method. Lu [10] used linear feedback and adaptive control to synchronize identical unified chaotic systems with only one controller. Ucar [18] used a nonlinear active controller to synchronize two coupled unified chaotic systems with three control inputs. Wang [20] proved that the unified chaotic system is equivalent to a passive system and asymptotically stabilized it at equilibrium points. Wang [21] studied the synchronization problem of two identical unified chaotic systems using three different methods. They used a linear feedback controller, a nonlinear feedback method, and an impulsive controller to synchronize the systems. In [24] based on the sliding mode theory, synchronization of two identical unified chaotic is discussed.

However, in this chapter, a linear state feedback controller stabilizes a fractional order unified chaotic system. An advantage of the proposed controller can be seen when it is used to stabilize a fractional order unified chaotic system, by means of increasing the stability region. In contrast, the application on the integer order system is shown to be failed.

The chapter is organized as follows: Sect. 2 includes the basic definition and preliminaries. A state feedback controller is proposed to stabilize the fractional order unified chaotic systems in Sect. 3. Results of numerical simulation are given in Sect. 4, to illustrate the effectiveness of the proposed controller. The chapter will be closed by a conclusion in Sect. 5.

2 Preliminary Definitions

2.1 Fractional Algebra

Among several definitions of fractional derivatives, the following Caputo-type definition [1] is more popular with respect the rest [17].

$$
{}_0D_t^q f(t) = \begin{cases} \frac{1}{\Gamma(m-q)} \int_0^t \frac{f^m(\tau)}{(t-\tau)^{q+1-m}} \mathrm{d}\tau, & m-1 < q \leq m \\ \frac{\mathrm{d}^m}{\mathrm{d}t^m} f(t) & q = m \end{cases} \tag{7.1}
$$

where m is the first integer number larger than q.

Definition 1. [6] A saddle point of index 2 is a saddle point with one stable eigenvalue and two unstable ones.

Definition 2. [2] Assume that a 3D fractional order chaotic system of $\dot{x} = f(x)$ displays a chaotic attractor. For every scroll existing in the chaotic attractor, this system has a saddle point of index 2 encircled by its respective scroll.

Theorem 1. *[11] Assume that a 3D chaotic system $\dot{x} = f(x)$ displays a chaotic attractor with n scrolls. Suppose Λ is a set of unstable eigenvalues of these n saddle points. A necessary condition for fractional system $D^q x = f(x)$ to exhibit an n-scroll chaotic attractor, similar to the chaotic attractor of system $\dot{x} = f(x)$, to keep the eigenvalues $\lambda \in \Lambda$ in the unstable region, satisfies:*

$$q > \frac{2}{\pi}\tan^{-1}\left(\frac{|\mathrm{Im}(\lambda)|}{\mathrm{Re}(\lambda)}\right), \qquad \forall\lambda \in \Lambda \tag{7.2}$$

Otherwise, at least one of these equilibriums becomes asymptotically stable and then attracts the nearby trajectories.

2.2 The Unified Chaotic System

[9] considered a kind of chaotic system which describes a class of unified form by:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = (25\alpha + 10)(y - x) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = (28 - 35\alpha)x - xz + (29\alpha - 1)y \\ \frac{\mathrm{d}z}{\mathrm{d}t} = xy - \frac{8+\alpha}{3}z \end{cases} \tag{7.3}$$

where x, y, z are the state variables and $\alpha \in [0,1]$ is a "homogeneity" parameter of the system. [9] calls (7.3) as unified chaotic system due to chaotic behavior for any $\alpha \in [0,1]$. When $0 \le \alpha < 0.8$, system (7.3) is called as the generalized Lorenz chaotic system. For $\alpha = 0.8$, it is called Lü chaotic system. Similarly, it is called generalized Chen chaotic system when $0.8 < \alpha \le 1$. However, let us introduce a fractional version of dynamic (7.3) as in (7.4). Standard derivatives of (7.3) are accordingly replaced by the following fractional derivatives:

$$\begin{cases} \frac{\mathrm{d}^q x}{\mathrm{d}t^q} = (25\alpha + 10)(y - x) \\ \frac{\mathrm{d}^q y}{\mathrm{d}t^q} = (28 - 35\alpha)x - xz + (29\alpha - 1)y \\ \frac{\mathrm{d}^q z}{\mathrm{d}t^q} = xy - \frac{8+\alpha}{3}z \end{cases} \tag{7.4}$$

where q with $0 < q \le 1$ is the fractional order. Chaos in the fractional order unified system of Chen, Lü, and Lorenz-Like for $q = 0.9, 0.95, 0.99$ are shown in [12].

From (7.4), a generalized scheme of the fractional order unified chaotic system can be given as follows:

$$\begin{cases} \frac{d^q x}{dt^q} = a(y-x) \\ \frac{d^q y}{dt^q} = bx - xz + cy \\ \frac{d^q z}{dt^q} = xy - dz \end{cases} \tag{7.5}$$

3 State Feedback Control

3.1 Design of the Controller for Fractional Order Chen System

The fractional order Chen system is given as follows [12]:

$$\begin{cases} \frac{d^q x}{dt^q} = a_1(y-x) \\ \frac{d^q y}{dt^q} = (c_1 - a_1)x - xz + c_1 y \\ \frac{d^q z}{dt^q} = xy - b_1 z \end{cases} \tag{7.6}$$

To obtain the Chen chaotic behavior, parameters in (7.6) are set to [12]:

$$a_1 = 40, b_1 = 3, c_1 = 28 \tag{7.7}$$

The equilibrium points of the Chen system are as follows:

$$\begin{aligned} O_1 &= (0,0,0) \\ O_2 &= (6.9282, 6.9282, 16) \\ O_3 &= (-6.9282, -6.9282, 16) \end{aligned} \tag{7.8}$$

From (7.6) the Jacobian matrix of the Chen system is achieved by:

$$J = \begin{bmatrix} -a_1 & a_1 & 0 \\ c_1 - a_1 - z & c_1 & -x \\ y & x & -b_1 \end{bmatrix} \tag{7.9}$$

Accordingly, the corresponding eigenvalues of the equilibrium (7.8) are obtained as:

$$\begin{aligned} &O_1 \to \lambda_1 = -3, \lambda_2 = 20, \lambda_3 - 32 \\ &O_{2,3} \to \lambda_1 = -20.2304, \lambda_{2,3} = 2.6152 \pm 13.5268j \end{aligned} \tag{7.10}$$

From definition 1, $O_{2,3}$ are of saddle point of index 2. Therefore, from theorem 1 the fractional order Chen system becomes chaotic when:

$$q > \frac{2}{\pi}\tan^{-1}\left(\frac{|\mathrm{Im}(\lambda_{2,3})|}{\mathrm{Re}(\lambda_{2,3})}\right) = 0.8784 \tag{7.11}$$

Otherwise the system is asymptotically stable. In order to stabilize the fractional order Chen system, a control input is added into the second state of the system, by the following:

$$\begin{cases} \frac{\mathrm{d}^q x}{\mathrm{d}t^q} = a_1(y-x) \\ \frac{\mathrm{d}^q y}{\mathrm{d}t^q} = (c_1-a_1)x - xz + c_1 y + u \\ \frac{\mathrm{d}^q z}{\mathrm{d}t^q} = xy - b_1 z \end{cases} \tag{7.12}$$

A linear state feedback controller is proposed to construct the input signal u as in the following form:

$$u = -(c_1-a_1)x - k_1 y \tag{7.13}$$

Where $k_1$is a constant gain by $k_1 = 12.7$.

Theorem 2. *The proposed state feedback controller in (7.13) increases the stability region of the fractional order Chen system and stabilizes the system at their stable equilibrium points.*

Proof. Using the state feedback controller changes the equilibrium points and the Jacobian matrix J to:

$$\begin{aligned} O'_1 &= (0,0,0) \\ O'_2 &= (6.7749, 6.7749, 15.3) \\ O'_3 &= (-6.7749, -6.7749, 15.3) \end{aligned} \tag{7.14}$$

$$J = \begin{bmatrix} -a_1 & a_1 & 0 \\ -z & c_1-k_1 & -x \\ y & x & -b_1 \end{bmatrix} \tag{7.15}$$

The corresponding eigenvalues of the equilibrium points in (7.14) are:

$$\begin{aligned} O'_1 &\to \lambda_1 = -3, \lambda_2 = 15.3, \lambda_3 = -40 \\ O'_{2,3} &\to \lambda_1 = -28.0829, \lambda_{2,3} = 0.1915 \pm 11.4331j \end{aligned} \tag{7.16}$$

Similarly, from the definition 1, $O'_{2,3}$ are of the saddle point of index 2. Hence, the fractional order Chen system becomes chaotic when:

$$q > \frac{2}{\pi}\tan^{-1}\left(\frac{|\mathrm{Im}(\lambda_{2,3})|}{\mathrm{Re}(\lambda_{2,3})}\right) = 0.9893 \tag{7.17}$$

Otherwise the system is asymptotically stable. This means that for $q < 0.9893$ the fractional order Chen system is asymptotically stable. ■

3.2 Design of the Controller for the Fractional Order Lü System

The fractional order Lü system is also given by [12]:

$$\begin{cases} \frac{d^q x}{dt^q} = a_1(y-x) \\ \frac{d^q y}{dt^q} = -xz + c_1 y \\ \frac{d^q z}{dt^q} = xy - b_1 z \end{cases} \tag{7.18}$$

The chaos in the Lü dynamic occurs when parameters in (7.18) are set to [12]:

$$a_1 = 35, b_1 = 3, c_1 = 30 \tag{7.19}$$

From (7.18) and (7.19), the equilibrium points and the Jacobian matrix of the Lü system are, respectively, as follows:

$$\begin{aligned} O_1 &= (0,0,0) \\ O_2 &= (9.4868, 9.4868, 30) \\ O_3 &= (-9.4868, -9.4868, 30) \end{aligned} \tag{7.20}$$

$$J = \begin{bmatrix} -a_1 & a_1 & 0 \\ -z & c_1 & -x \\ y & x & -b_1 \end{bmatrix} \tag{7.21}$$

Then the corresponding eigenvalues of the equilibrium points in (7.20) are:

$$\begin{aligned} &O_1 \to \lambda_1 = -3, \lambda_2 = 30, \lambda_3 = -35 \\ &O_{2,3} \to \lambda_1 = -19.3701, \lambda_{2,3} = 5.6851 \pm 17.1149j \end{aligned} \tag{7.22}$$

From definition 1, $O_{2,3}$ are of the saddle point of index 2. Thus, the fractional order Lü system becomes chaotic when:

$$q > \frac{2}{\pi}\tan^{-1}\left(\frac{|\mathrm{Im}(\lambda_{2,3})|}{\mathrm{Re}(\lambda_{2,3})}\right) = 0.7958 \tag{7.23}$$

Otherwise the system is asymptotically stable.

Similar to the previous section, to stabilize the fractional order Lü system, a controller is applied to the 2ns state, according to:

$$\begin{cases} \frac{d^q x}{dt^q} = a_1(y-x) \\ \frac{d^q y}{dt^q} = -xz + c_1 y + u \\ \frac{d^q z}{dt^q} = xy - b_1 z \end{cases} \tag{7.24}$$

The linear state feedback controller u in the following form stabilizes the chaotic dynamic:

$$u = -k_1 y \tag{7.25}$$

where k_1 as a constant gain is set to $k_1 = 16.5$.

Theorem 3. *The proposed state feedback controller in (7.25) stabilizes the system at their stable equilibrium points while increasing the stability region of the fractional order Lü system.*

Proof. The state feedback controller in the fractional order Lü system similar to (7.24) the equilibrium points and Jacobian matrix are, respectively, achieved by:

$$\begin{aligned} O'_1 &= (0,0,0) \\ O'_2 &= (6.3639, 6.3639, 13.5) \\ O'_3 &= (-6.3639, -6.3639, 13.5) \end{aligned} \tag{7.26}$$

$$J = \begin{bmatrix} -a_1 & a_1 & 0 \\ -z & c_1 - k_1 & -x \\ y & x & -b_1 \end{bmatrix} \tag{7.27}$$

Thus, the corresponding eigenvalues of the equilibrium points in (7.26) are:

$$\begin{aligned} &O'_1 \to \lambda_1 = -3, \lambda_2 = 13.5, \lambda_3 = -35 \\ &O'_{2,3} \to \lambda_1 = -24.863, \lambda_{2,3} = 0.1815 \pm 10.6767j \end{aligned} \tag{7.28}$$

Again from the definition 1, $O'_{2,3}$ are the saddle points of index 2. Therefore, the fractional order Lü system becomes chaotic when:

$$q > \frac{2}{\pi} \tan^{-1}\left(\frac{|\mathrm{Im}(\lambda_{2,3})|}{\mathrm{Re}(\lambda_{2,3})}\right) = 0.9892 \tag{7.29}$$

Otherwise the system is asymptotically stable. This means, for $q < 0.9892$ the fractional order Lü system is asymptotically stable. ■

4 Simulation

A simulation approach has been carried out using SIMULINK$^{\mathrm{TM}}$. Dormand–Prince solver is used to solve the system of differential equations during the simulation. Results of the unified chaotic Chen and Lü dynamics are shown for $q = 0.96$, $q = 0.98$, $q = 1$. Initial conditions of the states are selected as $(10, 15, 25)$. Simulation results show that the proposed state feedback controller stabilizes the fractional order unified chaotic systems while the behavior of the equivalent integer one still

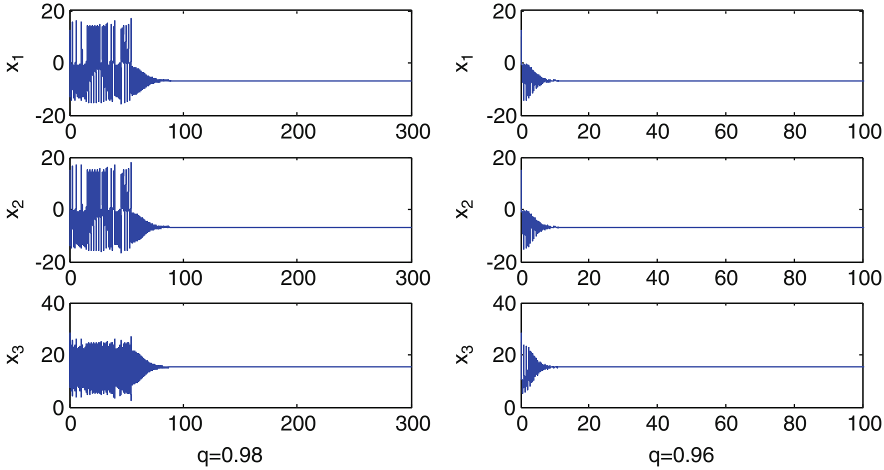

Fig. 7.1 Stabilization of the fractional order Chen system at their stable equilibrium points (O_2' and O_3'), via linear state feedback controller for $q = 0.96$ and $q = 0.98$

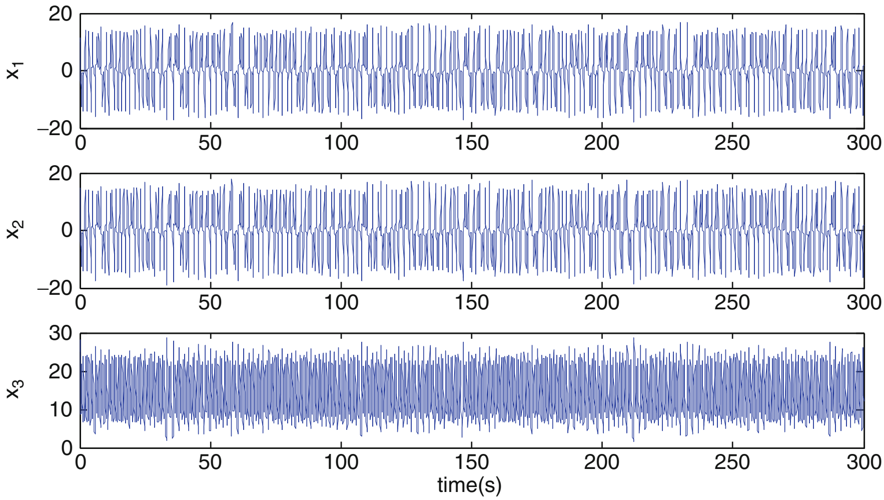

Fig. 7.2 Chaos behaviour in the integer order Chen system despite of using the state feedback controller

kept chaotic. Figure 7.1 shows that the fractional order Chen system is stabilized for $q = 0.96$ and $q = 0.98$ with state feedback controller in (7.13). Figure 7.2 shows the chaotic behavior of an integer order Chen system, despite of using the same state feedback controller in the system. Similar result is achieved in Fig. 7.3 when the fractional order Lü system is stabilized by the controller for $q = 0.96$ and $q = 0.98$. In the same way, Fig. 7.4 shows the chaotic behavior of integer order of the Lü system using the same state feedback controller.

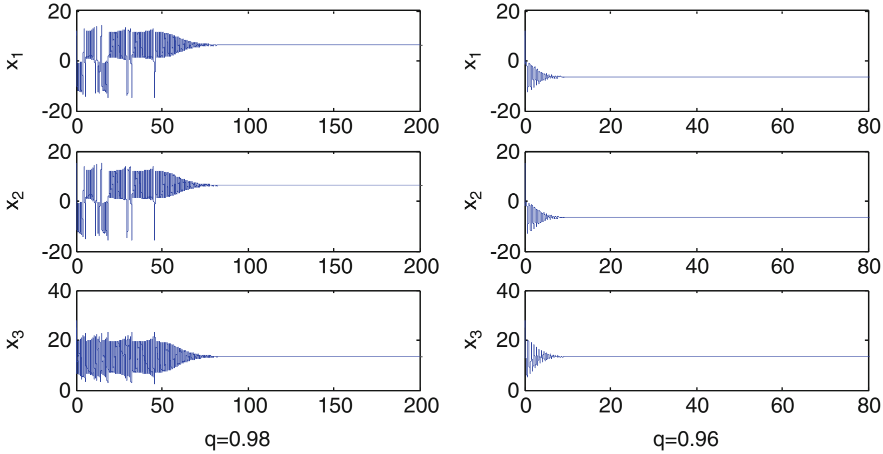

Fig. 7.3 Stabilization of the fractional order Lü system at their stable equilibrium points (O'_2 and O'_3), via linear state feedback controller for $q = 0.96$ and $q = 0.98$.

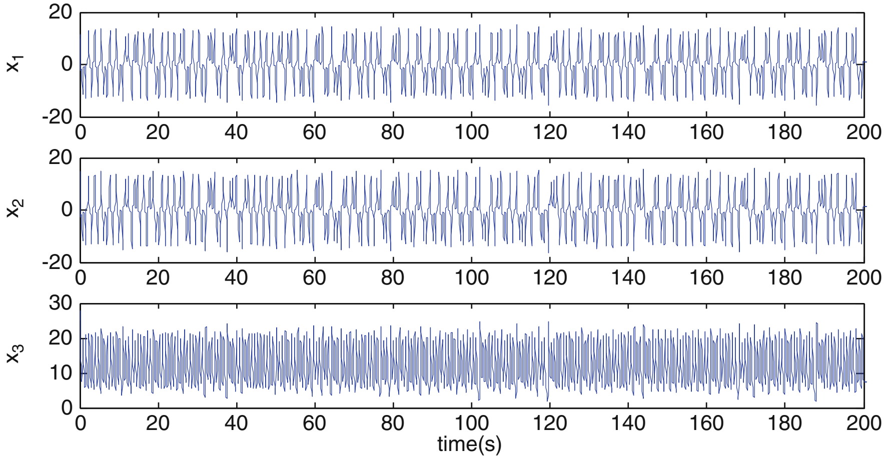

Fig. 7.4 Chaos behaviour in the integer order Lü system despite of using the state feedback controller

5 Conclusion

Three chaotic Lorenz, Chen, and Lü systems as unified systems will be separately shown unified by a same dynamic. These systems will separately be excited when a relevant parameter α is accordingly adjusted. A linear state feedback controller is gained to stabilize the unified chaotic systems at their stable equilibrium points. The controller also increases the stability region with respect to their integer order counterpart. Simulation approach is given to verify the outcome. The approach signifies the performance as well as the reliability of the proposed state feedback controller.

References

1. Caputo M (1967) Linear models of dissipation whose Q is almost frequency independent. Geophys J R Astron Soc 13:529–530
2. Cafagna D, Grassi G (2003) New 3-D-scroll attractors in hyperchaotic Chua's circuit forming a ring. Int J Bifurc Chaos 13:2889–2903
3. Chen SH, Lu JH (2002) Synchronization of an uncertain unified chaotic system via adaptive control. Chaos Solitons Fractals 14:643–647
4. Chen S, Yang Q, Wang C. (2004) Impulsive control and synchronization of unified chaotic systems. Chaos Solitons Fractals 20:751–758
5. Hosseinnia SH, Ghaderi R, Ranjbar A, Abdous F, Momani S (2010) Control of chaos via fractional-order state feedback controller. In: Baleanu D, Guvenc ZB, Machado JAT (eds) New trends in nanotechnology and fractional calculus applications. ISBN: 978-90-481-3292-8, DOI 10.1007/978-90-481-3293-5 46, pp 507–514
6. Khalil H (1992) Nonlinear systems. Macmillan, New York
7. Lurie BJ (1994) Tunable TID controller. US patent 5, 371, 670, December 6
8. Luce R, Kernevez JP (1991) Controllability of Lorenz equation. Int Ser Numer Math 97:257
9. Lü JH, Chen GR, Cheng DZ, Celikovsky S (2002) Bridge the gap between the Lorenz system and the Chen system. Int J Bifurcat Chaos 12:2917–2926
10. Lu J, Wu X, Han X, Lu J (2004) Adaptive feedback synchronization of a unified chaotic system. Phys Lett A 329:327–333
11. Matignon D (1996) Stability results for fractional differential equations with applications to control processing, in Computational Engineering in Systems Applications. IMACS, IEEE-SMC 2:963–968
12. Matouk AE (2009) Chaos synchronization between two different fractional systems of Lorenz family. Hindawi Publishing Corporation Mathematical Problems in Engineering, Article ID 572724, 11 p
13. Oustaloup A, Moreau X, Nouillant M (1996) The CRONE suspension. Control Eng Pract 4:1101–1108
14. Podlubny I (1999) Fractional-order systems and $PI^{\lambda}D^{\mu}$ controllers. IEEE Trans 44:208–214
15. Pyragas K, Tamasevieius A (1993) Experimental control of chaos by delayed self-controlling feedback. Phys Lett A 180:99–102
16. Raynaud HF, Zerga Inoh A (2000), State-space representation for fractional order controllers. Automatica 36:1017–1021
17. Tavazoei MS, Haeri M (2008) Chaotic attractors in incommensurate fractional order systems. Physica D 237:2628–2637
18. Ucar A, Lonngren K, Bai E (2006) Synchronization of the unified chaotic systems via active control. Chaos Solitons Fractals 27:1292–1297
19. Vincent TL, Yu J (1991) Control of a chaotic system. Dyn Contr 1:35–52
20. Wang F, Liu C (2007) Synchronization of unified chaotic system based on passive control. Phys D 225:55–60
21. Wang X, Song J (2008) Synchronization of the unified chaotic system, Nonlinear Anal 69:3409–3416
22. Yeap TH, Ahmed NU (1994) Feedback control of chaotic systems. Dyn Contr 4:97–114
23. Zeng Y, Singh SN (1997) Adaptive control of chaos in Lorenz system. Dyn Contr 7:143–154
24. Zribi M, Smaoui N, Salim H (2009) Synchronization of the unified chaotic systems using a sliding mode controller. Chaos Solitons Fractals 42:3197–3209