Chapter 13
On the Hadamard Type Fractional Differential System

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1 Introduction

In recent decades, the fractional differential equations has been paid more and more attention, which mostly involve the Riemann–Liouville fractional calculus or the Caputo one [1–6]. The Hadamard calculus (differentiation and integration) has not been mentioned so much as other kinds of fractional derivative, even if it has been presented many years before [7].

In the following, the definitions of the Hadamard derivative and integral are introduced [8].

Definition 13.1. The Hadamard fractional integral of order $\alpha \in R^+$ of function $f(x)$, $\forall x > 1$, is defined by:

$$ {}_H D_{1,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_1^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t) \frac{dt}{t}, \tag{13.1}$$

where $\Gamma(\cdot)$ is the Euler Gamma function.

Definition 13.2. The Hadamard derivative of order $\alpha \in [n-1, n)$, $n \in Z^+$, of function $f(x)$ is given as follows:

$$ {}_H D_{1,x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(x \frac{d}{dx}\right)^n \int_1^x \left(\ln \frac{x}{t}\right)^{n-\alpha-1} f(t) \frac{dt}{t}. \tag{13.2}$$

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From the above definitions, the differences between Hadamard fractional derivative and the Riemann–Liouville fractional derivative are obvious, which include two aspects: firstly, no matter what the definitions of integral or derivative, the kernel in the Hadamard integral has the form of $\ln\frac{x}{t}$ instead of the form of $(x-t)$ which is involved in the Riemann–Liouville integral; secondly, the Hadamard derivative has the operator $\left(x\frac{d}{dx}\right)^n$, whose construction is well suited to the case of the half-axis and is invariant relative to dilation [9], while the Riemann–Liouville derivative has the operator $\left(\frac{d}{dx}\right)^n$.

Next, some of propositions with the Hadamard calculus are formed as follows.

Proposition 13.1. *If* $0<\alpha<1$*, the following relations hold*

(i) ${}_HD_{1,x}^{-\alpha}(\ln x)^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\ln x)^{\beta+\alpha-1}$;

(ii) ${}_HD_{1,x}^{\alpha}(\ln x)^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\ln x)^{\beta-\alpha-1}$.

Proof. Here we only prove (ii), (i) can be proved similar to (ii). Direct calculations yield

$$
\begin{aligned}
HD_{1,x}^{\alpha}(\ln x)^{\beta-1} &= \left(x\frac{d}{dx}\right)\cdot\frac{1}{\Gamma(1-\alpha)}\int_1^x\left(\ln\frac{x}{t}\right)^{-\alpha}(\ln t)^{\beta-1}\frac{dt}{t}\\
&= \left(x\frac{d}{dx}\right)\cdot\frac{(\ln x)^{\beta-\alpha}}{\Gamma(1-\alpha)}\int_1^x\left(1-\frac{\ln t}{\ln x}\right)^{-\alpha}\left(\frac{\ln t}{\ln x}\right)^{\beta-1}d\frac{\ln t}{\ln x}\\
&= \left(x\frac{d}{dx}\right)\cdot\frac{(\ln x)^{\beta-\alpha}}{\Gamma(1-\alpha)}B(1-\alpha,\beta)\\
&= \left(x\frac{d}{dx}\right)\cdot\frac{(\ln x)^{\beta-\alpha}}{\Gamma(1-\alpha)}\frac{\Gamma(1-\alpha)\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\\
&= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}\cdot x\cdot\frac{d\left((\ln x)^{(\beta-\alpha)}\right)}{dx}\\
&= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\ln x)^{(\beta-\alpha-1)}.
\end{aligned}
$$

This completes the proof. □

The following results are available in [8].

Proposition 13.2. *If* $\alpha\geq 0$ *and* $\beta=1$*, for any* $j=[\alpha]+1$*, the following relations hold*

(i) $({}_HD_{1,t}^{\alpha}1)(x) = \dfrac{1}{\Gamma(1-\alpha)}(\ln x)^{-\alpha}$;

(ii) $\left({}_HD_{1,t}^{\alpha}(\ln t)^{\alpha-j}\right)(x) = 0$.

Next, we will introduce the weighted space $C_{\gamma,\ln}[a,b]$, $C^n_{\delta,\gamma}[a,b]$ of the function f on the finite interval $[a,b]$, if $\gamma \in C(0 \le Re(\gamma) < 1)$, $n-1 < \alpha \le n$, then

$$C_{\gamma,\ln}[a,b] := \left\{ f(x) : \ln(\tfrac{x}{a})^{\gamma} f(x) \in C[a,b], ||f||_{C_\gamma} = ||(\ln \tfrac{x}{a})^{\gamma} f(x)||_C \right\},$$
$$C_{0,\ln}[a,b] = C[a,b],$$

and

$$C^n_{\delta,\gamma}[a,b] := \Big\{ g(x) : (\delta^n g)(x) \in C_{\gamma,\ln}[a,b],$$
$$||g||_{C_{\gamma,\ln}} = \textstyle\sum_{k=0}^{n-1} ||\delta^k g||_C + ||\delta^n g||_{C_{\gamma,\ln}} \Big\},$$
$$\delta = x\frac{d}{dx}.$$

Theorem 13.1. *Let $\alpha > 0$, $n = -[-\alpha]$ and $0 \le \gamma < 1$. Let G be an open set in R and let $f : (a,b] \times G \longrightarrow R$ be a function such that: $f(x,y) \in \mathrm{C}_{\gamma,\ln}[a,b]$ for any $y \in G$, then the following problem*

$$_H D^{\alpha}_{a,t}(x) = f(x,y(x)), (\alpha > 0), \tag{13.3}$$

$$_H D^{\alpha-k}_{a,t}(a+) = b_k, b_k \in R, (k = 1,\ldots,n, n = -[-\alpha]), \tag{13.4}$$

satisfies the following Volterra integral equation:

$$y(x) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)} \left(\ln \frac{t}{a}\right)^{\alpha-j} + \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f[t,y(t)]\frac{dt}{t}, (x > a > 0), \tag{13.5}$$

i.e., $y(x) \in \mathrm{C}_{n-\alpha,\ln}[a,b]$ satisfies the relations (13.3)–(13.4) if and only if it satisfies the Volterra integral equation (13.5).

In particular, if $0 < \alpha \le 1$, the problem (13.3)–(13.4) is equivalent to the following equation:

$$y(x) = \frac{b}{\Gamma(\alpha)} \left(\ln \frac{t}{a}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f[t,y(t)]\frac{dt}{t}, \ (x > a > 0). \tag{13.6}$$

2 The Generalized Gronwall Inequality

The Gronwall inequality, which plays a very important role in classical differential systems, has been generalized by Ye and Gao [10] which is used to fractional differential equations with Riemann–Liouville derivative. In this paper we further generalize the inequality. We firstly recall the classical Gronwall inequality which can be found in [11].

Theorem 13.2. *If*

$$x(t) \leq h(t) + \int_{t_0}^{t} k(s)x(s)ds,\, t \in [t_0, T),$$

where all the functions involved are continuous on $[t_0, T)$, $T \leq \infty$, *and* $k(t) \geq 0$, *then* $x(t)$ *satisfies*

$$x(t) \leq h(t) + \int_{t_0}^{t} h(s)k(s)exp\left[\int_{s}^{t} k(u)du\right] ds,\, t \in [t_0, T).$$

If, in addition, $h(t)$ *is nondecreasing, then*

$$x(t) \leq h(t)exp\left(\int_{t_0}^{t} k(s)ds\right),\, t \in [t_0, T).$$

The generalized Gronwall inequality corresponding to the Riemann–Liouville type fractional differential system is introduced as follows which is presented in Ye and Gao [10].

Theorem 13.3. *Suppose* $\alpha > 0$, $a(t)$ *is a nonnegative function and locally integrable on* $0 \leq t < T$ *(some* $T \leq +\infty$*) and* $g(t)$ *is a nonnegative, nondecreasing, continuous function defined on* $0 \leq t < T$, $g(t) \leq M$ *(constant), and suppose* $u(t)$ *is nonnegative and locally integrable on* $0 \leq t < T$ *with*

$$u(t) \leq a(t) + g(t)\int_{0}^{t} (t-s)^{\alpha-1}u(s)ds,$$

on the interval. Then

$$u(t) \leq a(t) + \int_{0}^{t}\left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}a(s)\right] ds,\; 0 \leq t < T.$$

This inequality can be used to estimate the bound of the Lyapunov exponents for both the Riemann–Liouville fractional differential systems and the Caputo ones [5]. In the following, we derive another inequality which can be regarded as a modification of Theorem 3.

Theorem 13.4. *Suppose* $\alpha > 0$, $a(t)$ *and* $u(t)$ *are nonnegative functions and locally integrable on* $1 \leq t < T$ $(\leq +\infty)$, *and* $g(t)$ *is a nonnegative, nondecreasing, continuous function defined on* $1 \leq t < T$, $g(t) \leq M$ *(constant). If the following inequality*

$$u(t) \leq a(t) + g(t)\int_{1}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1} u(s)\frac{ds}{s},\; 1 \leq t < T, \tag{13.7}$$

holds. Then

$$u(t) \le a(t) + \int_1^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\ln \frac{t}{s}\right)^{n\alpha-1} a(s) \right] \frac{ds}{s}, \ 1 \le t < T. \tag{13.8}$$

Proof. Let

$$B\phi(t) = g(t) \int_1^t \left(\ln \frac{t}{s}\right)^{n\alpha-1} \phi(s) \frac{ds}{s}.$$

Then

$$u(t) \le a(t) + Bu(t).$$

Iterating the inequality, one has

$$u(t) \le \sum_{k=0}^{n-1} B^k a(t) + B^n u(t).$$

In the following, we prove

$$B^n u(t) \le \int_1^t \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\ln \frac{t}{s}\right)^{n\alpha-1} u(s) \frac{ds}{s}, \tag{13.9}$$

and $B^n u(t) \to +\infty$ for each $t \in (1,T)$.

Obviously, (13.9) holds when $n = 1$. Suppose it holds for $n = k$. Let $n = k+1$, then one has

$$B^{k+1}u(t) = B\left(B^k u(t)\right) \le g(t) \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \left[\int_1^s \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\ln \frac{s}{\tau}\right)^{k\alpha-1} u(\tau) \frac{d\tau}{\tau}\right] \frac{ds}{s}.$$

Under the condition that $g(t)$ is nondecreasing, one obtains

$$B^{k+1}u(t) \le (g(t))^{k+1} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \left[\int_1^s \frac{(\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\ln \frac{s}{\tau}\right)^{k\alpha-1} u(\tau) \frac{d\tau}{\tau}\right] \frac{ds}{s}.$$

By interchanging the order of integration, we get

$$\begin{aligned} B^{k+1}u(t) &\le (g(t))^{k+1} \int_1^t \left[\int_\tau^t \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\ln \frac{t}{s}\right)^{\alpha-1} \left(\ln \frac{s}{\tau}\right)^{k\alpha-1} \frac{ds}{s}\right] u(\tau) \frac{d\tau}{\tau} \\ &= \int_1^t \frac{(g(t)\Gamma(\alpha))^{k+1}}{\Gamma((k+1)\alpha)} \left(\ln \frac{t}{s}\right)^{(k+1)\alpha-1} u(s) \frac{ds}{s}, \end{aligned}$$

where the integral

$$\int_\tau^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\left(\ln\frac{s}{\tau}\right)^{k\alpha-1}\frac{ds}{s} = \left(\ln\frac{t}{\tau}\right)^{k\alpha+\alpha-1}\int_0^1 (1-z)^{\alpha-1}z^{k\alpha-1}dz$$
$$= \left(\ln\frac{t}{\tau}\right)^{(k+1)\alpha-1} B(k\alpha,\alpha)$$
$$= \frac{\Gamma(\alpha)\Gamma(k\alpha)}{\Gamma((k+1)\alpha)}\left(\ln\frac{t}{\tau}\right)^{(k+1)\alpha-1},$$

is obtained, where $\ln s = \ln\tau + z\ln\frac{t}{\tau}$ is used.

Therefore, (13.9) is true.

Moreover, since

$$B^n u(t) \le \int_1^t \frac{(M\Gamma(\alpha))^n}{\Gamma(n\alpha)}\left(\ln\frac{t}{s}\right)^{n\alpha-1} u(s)\frac{ds}{s} \to 0,$$

as $n \to +\infty$, for $t \in [1,T)$.

Hence this completes the proof. □

Corollary 13.1. *Let $g(t) = b > 0$ in (13.7). The inequality (13.7) turns into the following form*

$$u(t) \le a(t) + b\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} u(s)\frac{ds}{s}.$$

Furthermore

$$u(t) \le a(t) + \int_1^t \left[\sum_{n=1}^{\infty}\frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)}\left(\ln\frac{t}{s}\right)^{n\alpha-1} a(s)\right]\frac{ds}{s},\ (1 \le t < T).$$

Corollary 13.2. *Under the assumption of Theorem 4, suppose that $a(t)$ is a nondecreasing function on $[1,T)$. Then*

$$u(t) \le a(t)E_{\alpha,1}(g(t)\Gamma(\alpha)(\ln t)^{\alpha}),$$

where $E_{\alpha,1}$ is the Mittag-leffler function defined by

$$E_{\alpha,1} = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}.$$

Proof. The assumptions imply

$$
\begin{aligned}
u(t) &\le a(t)\left[1+\int_1^t \sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\left(\ln\frac{t}{s}\right)^{n\alpha-1}\frac{ds}{s}\right] \\
&= a(t)\sum_{n=0}^{\infty}\frac{(g(t)\Gamma(\alpha)\ln t)^n}{\Gamma(n\alpha+1)} \\
&= a(t)E_\alpha(g(t)\Gamma(\alpha)(\ln t)^\alpha).
\end{aligned}
$$

This ends the proof. □

3 The Dependence of Solution on Parameters

As far as we are concerned, there have been some papers dedicated to study the dependence of the solution on the order and the initial condition to the fractional differential equation with Riemann–Liouville type and Caputo type derivative, while quite few papers are contributed to the Hadamard type fractional differential system. In this section, we study the dependence of the solution on the order and the initial condition of the fractional differential equation with Hadamard fractional derivative.

Now we consider the following fractional system:

$$ {}_H D_{1,t}^{\alpha} y(t) = f(t, y(t)), \tag{13.10} $$

$$ {}_H D_{1,t}^{\alpha-1} y(t)|_{t=1} = \eta, \tag{13.11} $$

where $0<\alpha<1, 1\le t<T\le+\infty, f:[1,T)\times R\to R$.

The existence and uniqueness of the initial value problem (13.10)–(13.11) have been studied in [8], in which the dependence of a solution on initial conditions has also been considered. Here, we investigate the dependence on both the initial value conditions and the derivative order.

Obviously, the problem (13.10)–(13.11) can be changed into the Volterra integral equation.

$$ y(t) = \frac{\eta}{\Gamma(\alpha)}(\ln t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t (\ln t)^{\alpha-1} f(\tau, y(\tau))\frac{d\tau}{\tau}. \tag{13.12} $$

In effect, the Volterra equation (13.12) is equivalent to the initial value problem (13.10)–(13.11).

Theorem 13.5. *Let $\alpha>0$ and $\delta>0$ such that $0<\alpha-\delta<\alpha\le 1$. Also let the function f be continuous and satisfy the Lipschitz condition with respect to the second variable:*

$$ |f(t,y)-f(t,z)| \le L|y-z|, $$

for a constant L independent of t, y, z *in R. For* $1 \le t \le h < T$*, assume that y and z are the solutions of the initial value problems (13.10)–(13.11) and*

$$ {}_{H}D_{1,t}^{\alpha-\delta} z(t) = f(t, z(t)), \tag{13.13} $$

$$ {}_{H}D_{1,t}^{\alpha-\delta-1} z(t)|_{t=1} = \bar{\eta}, \tag{13.14} $$

respectively. Then, the following relation holds for $1 < t \le h$*:*

$$ |z(t) - y(t)| \le A(t) + \int_1^t \left[\sum_{n=1}^{\infty} \left(\frac{L}{\Gamma(\alpha)} \Gamma(\alpha - \delta) \right)^n \frac{(\ln \frac{t}{s})^{n(\alpha-\delta)-1}}{\Gamma(n(\alpha - \delta))} A(s) \right] \frac{ds}{s}, $$

where

$$ A(t) = \left| \frac{\bar{\eta}}{\Gamma(\alpha-\delta)} (\ln t)^{\alpha-\delta-1} - \frac{\eta}{\Gamma(\alpha)} (\ln t)^{\alpha-1} \right| + \left| \frac{(\ln t)^{\alpha-\delta}}{(\alpha-\delta)\Gamma(\alpha)} - \frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)} \right| \cdot \|f\| $$
$$ + \left| \frac{(\ln t)^{\alpha-\delta}}{\alpha - \delta} \left[\frac{1}{\Gamma(\alpha - \delta)} - \frac{1}{\Gamma(\alpha)} \right] \right| \cdot \|f\|, $$

and

$$ \|f\| = \max_{1 \le t \le h} |f(t, y)|. $$

Proof. The solutions of the initial value problem (13.10)–(13.11) and (13.13)–(13.14) are as follows:

$$ y(t) = \frac{\eta}{\Gamma(\alpha)} (\ln t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t (\ln t)^{\alpha-1} f(\tau, y(\tau)) \frac{d\tau}{\tau}, $$

and

$$ z(t) = \frac{\bar{\eta}}{\Gamma(\alpha - \delta)} (\ln t)^{\alpha-\delta-1} + \frac{1}{\Gamma(\alpha - \delta)} \int_1^t (\ln t)^{\alpha-\delta-1} f(\tau, z(\tau)) \frac{d\tau}{\tau}. $$

So we have

$$ \begin{aligned} |z(t) - y(t)| \le& \left| \frac{\bar{\eta}}{\Gamma(\alpha - \delta)} (\ln t)^{\alpha-\delta-1} - \frac{\eta}{\Gamma(\alpha)} (\ln t)^{\alpha-1} \right| \\ &+ \left| \frac{1}{\Gamma(\alpha-\delta)} \int_1^t (\ln t)^{\alpha-\delta-1} f(\tau, z(\tau)) \frac{d\tau}{\tau} - \frac{1}{\Gamma(\alpha)} \int_1^t (\ln t)^{\alpha-\delta-1} f(\tau, z(\tau)) \frac{d\tau}{\tau} \right| \\ &+ \left| \frac{1}{\Gamma(\alpha)} \int_1^t (\ln t)^{\alpha-\delta-1} f(\tau, z(\tau)) \frac{d\tau}{\tau} - \frac{1}{\Gamma(\alpha)} \int_1^t (\ln t)^{\alpha-\delta-1} f(\tau, y(\tau)) \frac{d\tau}{\tau} \right| \\ &+ \left| \frac{1}{\Gamma(\alpha)} \int_1^t (\ln t)^{\alpha-\delta-1} f(\tau, y(\tau)) \frac{d\tau}{\tau} - \frac{1}{\Gamma(\alpha)} \int_1^t (\ln t)^{\alpha-1} f(\tau, y(\tau)) \frac{d\tau}{\tau} \right| \\ \le& A(t) + \frac{1}{\Gamma(\alpha)} \int_1^t (\ln t)^{\alpha-\delta-1} L |z(\tau) - y(\tau)| \frac{d\tau}{\tau}, \end{aligned} $$

where

$$A(t)=\left|\frac{\bar{\eta}}{\Gamma(\alpha-\delta)}(\ln t)^{\alpha-\delta-1}-\frac{\eta}{\Gamma(\alpha)}(\ln t)^{\alpha-1}\right|+\left|\frac{(\ln t)^{\alpha-\delta}}{(\alpha-\delta)\Gamma(\alpha)}-\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}\right|\cdot\|f\|$$

$$+\left|\frac{(\ln t)^{\alpha-\delta}}{\alpha-\delta}\left[\frac{1}{\Gamma(\alpha-\delta)}-\frac{1}{\Gamma(\alpha)}\right]\right|\cdot\|f\|.$$

Applying Theorem 1 to the above inequality yields:

$$|z(t)-y(t)|\leq A(t)+\int_1^t\left[\sum_{n=1}^{\infty}\left(\frac{L}{\Gamma(\alpha)}\Gamma(\alpha-\delta)\right)^n\frac{(\ln\frac{t}{s})^{n(\alpha-\delta)-1}}{\Gamma(n(\alpha-\delta))}A(s)\right]\frac{ds}{s}.$$

The proof is finished. □

Next, we give an example to discuss the approximate solution of the Hadamard fractional differential equation.

$$_HD_{1,t}^{1-\delta}x(t)=x(t), \tag{13.15}$$

$$_HD_{1,t}^{-\delta}x(t)|_{t=1}=1, \tag{13.16}$$

where $1\leq t<T\leq+\infty$, $\delta\in R^+$ is small enough.

For the above question, we need not get its asymptotic solution. We can find its approximate solution quickly in the other way. Now we consider the simple problem as follows:

$$_HD_{1,t}^{1}y(t)=y(t), \tag{13.17}$$

$$_HD_{1,t}^{0}y(t)|_{t=1}=1. \tag{13.18}$$

Combining the corresponding evaluation and Theorem 5, one has

$$A(t)=\left|\frac{1}{\Gamma(1-\delta)}(\ln t)^{-\delta}-1\right|+\left|\frac{(\ln t)^{1-\delta}}{1-\delta}-\ln t\right|\cdot\|x\|+\left|\frac{(\ln t)^{1-\delta}}{1-\delta}\left[\frac{1}{\Gamma(1-\delta)}-1\right]\right|\cdot\|y\|.$$

When $\delta\longrightarrow 0$ and $t\in[1,T)$, we get $A(t)\longrightarrow 0$.

Actually, $\delta\longrightarrow 0$ and $t\in[1,T)$, one has

$$|x(t)-y(t)|=|e^{\ln t}-(\ln t)^{\delta}e^{(\ln t)^{1-\delta}}|\longrightarrow 0.$$

The example shows that the Hadamard differential equation is dependent on both the initial value conditions and the order of derivative.

4 Estimation of the Bound of the Lyapunov Exponents

Recently, Li, Chen and Li, Xia have obtained the bound of the Lyapunov exponents of the discrete-time system, the ordinary differential system respectively. For details, see [12, 13]. Also, Li, et al. firstly introduced the Lyapunov exponents for the fractional differential systems with Riemann–Liouville derivative and Caputo derivative, and determined the bounds of their Lyapunov exponents [5]. In this paper, we use the modified Gronwall inequality to derive the bound of the Lyapunov exponents of the fractional differential system with Hadamard derivative.

Theorem 13.6. *The following fractional differential system with Hadamard derivative*

$$\begin{cases} {}_HD^{\alpha}_{t_0,t}x(t) = f(x,t), \\ (x,t) \in \Omega \times (t_0,+\infty) \subset R^n \times (t_0,+\infty),\ \alpha \in (0,1), t_0 > 0, \\ {}_HD^{\alpha-1}_{t_0,t}x(t)|_{t=t_0} = x_0, \end{cases} \tag{13.19}$$

has its first variation equation

$$\begin{cases} {}_HD^{\alpha}_{t_0,t}\Phi(t) = f_x(x,t)\Phi(t), \\ (x,t) \in \Omega \times (t_0,+\infty) \subset R^n \times (t_0,+\infty),\ \alpha \in (0,1), t_0 > 0, \\ \Phi(t_0) = I, \end{cases} \tag{13.20}$$

where I is an identity matrix and

$$\Phi(t) = \frac{\partial}{\partial s}\phi(t;x_0 + s\Phi(t))|_{s=0} = D_x\phi(t_0;x_0),$$

$\phi(t_0;x_0)$ *is the fundamental solution to the system.*

Proof. The proof is similar to that in [5], we omit the details here. □

Definition 13.3. Let $u_k(t),\ k = 1,2,\ldots,n$ be the eigenvalues of $\Phi(t)$ of system (13.20), which satisfy $|u_1(t)| \le |u_2(t)| \le \cdots \le |u_n(t)|$. Then the Lyapunov exponents l_k of the trajectory $x(t)$ solving (13.20) are defined by:

$$l_k = \lim_{t\to\infty} \sup \frac{1}{t} \ln |u_k(t)|,\ \ k = 1,2,\ldots,n.$$

These exponents $l_k, k = 1,2,\ldots,n$, are real numbers. The existence of the limit for the classical differential system was established [14]. For the fractional differential system, it still holds. Obviously, Φ is not invertible when $u_1(t) = 0$, which implies $l_1 = -\infty$. But this case does not happen in general. Hence, we always assume that $u_1(t)$ is not (identically) equal to zero. Therefore, Φ is always supposed to be invertible.

Next, we estimate the bound of the Lyapunov exponents for the fractional differential systems with Hadamard derivative. But firstly, let's take a look at the following lemma [15].

Lemma 13.1. *If* $0 < \alpha < 2$, β *is an arbitrary complex number,* u *is an arbitrary real number such that* $\frac{\pi\alpha}{2} < u < \min\{\pi, \pi\alpha\}$, *then for an arbitrary integer* $p \geq 1$ *the following expansion holds*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - k\alpha)} + O(|z|^{-1-p}), \ \ |z| \to \infty, \ |\arg(z)| \leq u.$$

By Lemma 1, we can directly obtain the asymptotic expansion of the Mittag-Leffler function

$$E_{\alpha,\alpha}(K(\ln t)^{\alpha}) \approx \frac{e^{K^{\frac{1}{\alpha}}}}{\alpha} K^{\frac{1}{\alpha}-1} (\ln t)^{1-\alpha} t, \ \ t \to +\infty,$$

where K is a positive constant.

Integrating system (13.19) gives

$$\Phi(t) = \frac{\left(\ln \frac{t}{t_0}\right)^{\alpha-1}}{\Gamma(\alpha)} I + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\ln \frac{t}{\tau}\right)^{\alpha-1} f_x(x,\tau) \Phi(\tau) \frac{d\tau}{\tau}.$$

Taking the matrix norm of both sides of the above equation leads to

$$\|\Phi(t)\| \leq \frac{\left(\ln \frac{t}{t_0}\right)^{\alpha-1}}{\Gamma(\alpha)} + \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \|\Phi(\tau)\| \frac{d\tau}{\tau},$$

where the constant M is assumed the bound of $\|f_x(x,t)\|$.

Applying Corollary 2 to the above integral inequality brings about

$$\|\Phi(t)\| \leq \left(\ln \frac{t}{t_0}\right)^{\alpha-1} E_{\alpha,\alpha}\left(M \left(\ln \frac{t}{t_0}\right)^{\alpha}\right).$$

By the fact that the spectral radius of a given matrix is not bigger than its norm, we have

$$|u_n(t)| \leq \|\Phi(t)\| \leq \left(\ln \frac{t}{t_0}\right)^{\alpha-1} E_{\alpha,\alpha}\left(M \left(\ln \frac{t}{t_0}\right)^{\alpha}\right).$$

Using the definition of the Lyapunov exponents and applying Lemma 1, one gets

$$
\begin{aligned}
l_n &= \lim_{t\to+\infty}\sup\frac{1}{t}\ln|u_n(t)| \le \lim_{t\to+\infty}\sup\frac{1}{t}\ln\|\Phi(t)\| \\
&\le \lim_{t\to+\infty}\sup\frac{1}{t}\ln\left(\left(\ln\frac{t}{t_0}\right)^{\alpha-1}E_{\alpha,\alpha}\left(M\left(\ln\frac{t}{t_0}\right)^{\alpha}\right)\right) \\
&= \lim_{t\to+\infty}\sup\frac{1}{t}\ln\left(\frac{e^{K^{\frac{1}{\alpha}}}}{\alpha}K^{\frac{1}{\alpha}-1}\left(\ln\frac{t}{t_0}\right)^{1-\alpha}\frac{t}{t_0}\right) \\
&= 0.
\end{aligned}
$$

So the Lyapunov exponents of systems (13.19) satisfy

$$
-\infty < l_1 \le \cdots \le l_n \le 0.
$$

Therefore we eventually derive the upper bound of the Lyapunov exponents for the fractional differential systems with Hadamard derivative and the upper bound is zero, which means that generally the fractional differential system with Hadamard derivative has no chaotic attractor in the sense of the definition 3. We do not know whether or not such a system is chaotic in the other sense. Such a problem is still open. We hope the studies in this respect will appear elsewhere.

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