Chapter 7
Some Problems in Analysis

This chapter has some problems which are preliminary to problems in chapters to follow.

Definition 8 *Suppose each of X and Y is a Banach space and F is a function from a subset Ω of X into Y. The statement that F is Fréchet differentiable at $x \in X$ means that*

- *There is an open set G containing x so that $G \subset \Omega$.*
- *There is $M \in L(X,Y)$ so that if $\epsilon > 0$, there is $\delta > 0$ such that if $h \in H$ and $\|h\|_X < \delta$, then*

$$\|F(x+h) - F(x) - Mh\|_Y \ \leq \ \epsilon\|h\|_X, \ \textit{if } x+h \in D(F).$$

Problem 96 *Suppose that F is as in Definition 8. Show that the element M in the definition is unique.*

In this case, F' denotes the function whose domain is all $x \in H$ at which F is Fréchet differentiable. For each such $x \in X$, $F'(x)$ denotes the element M in Definition 8.

Definition 9 *A function F' as in Definition 8 is C^1 provided that F' is continuous as a function from $\Omega \to L(X,Y)$.*

Problem 97 *Suppose X is a Banach space $d_0, r > 0$, $(a,b) \in R \times X$ and f is a continuous function from*

$$\Omega = [a - d_0, a + d_0] \times B_r(b) \to X$$

such that for some $M > 0$ it is true that

$$\|f(t,x) - f(t,y)\| \leq M\|x-y\|, \ (t,x),(t,y) \in \Omega.$$

J.W. Neuberger, *A Sequence of Problems on Semigroups*,
Problem Books in Mathematics, DOI 10.1007/978-1-4614-0430-9_7,

Then there is $d \in (0, d_0)$ *and a unique function*

$$z : (a - d, a + d) \times X \to X$$

such that

$$z(a) = b,\ z'(t) = f(t, z(t)),\ t \in (a - d, a + d).$$

Consider using the method of successive approximations as in Problem 46.

Problem 98 *Suppose* $c > 0$ *and* h *is a function of class* C^1 *whose domain contains* $[0, c)$ *and whose range is a subset of the Banach space* X*. If there is* $M > 0$ *so that*

$$\int_0^t \|h'\|_X \ \le\ M,\ t \in [0, c),$$

then

$$\lim_{t \to c-} h(t) \text{ exists.}$$

Problem 99 *Suppose that* H *is a Hilbert space and* f *is a continuous linear function from* H *to* R*. Show that there is a unique* $y \in H$ *such that*

$$f(x) = \langle x, y \rangle_H,\ x \in H.$$

Problem 100 *Suppose that* H *is an infinite-dimensional separable Hilbert space and* $T \in L(X, X)$ *so that*

$$\langle Tx, y \rangle_H = \langle x, Ty \rangle_H,\ x, h \in H, \tag{7.1}$$

$$\langle Tx, x \rangle_H \ge 0,\ x \in H, \tag{7.2}$$

and if $\{x_k\}^\infty$ *is a bounded sequence in* H*, then the sequence* $\{Tx_k\}_{k=0}^\infty$ *has a convergent subsequence. Show that there is an orthonormal basis*

$$\{\phi_k\}_{k=1}^\infty$$

for H *and a nondecreasing sequence in* R*,*

$$\{\lambda_k\}_{k=1}^\infty \tag{7.3}$$

so that

$$T\phi_k = \lambda_k \phi_k,\ k \in Z^+.$$

Problem 101 *Show that the conclusion to Problem 100 still holds if (7.2) is removed and the word 'nondecreasing' is removed where it appears above in (7.3).*

Problem 102 *Suppose that H is a Hilbert space, $T \in L(X,X)$ and (7.1), (7.2) hold. Denote*

$$\sup_{x \in H, \|x\|=1} \langle Tx, x \rangle_H$$

by b.

Show that there is a function ϕ with domain $[0,b]$ and range in the set of orthogonal projections on H so that

- $\phi(0) = 0$.
- *If $0 \leq a < b \leq c < d$ and $x, y \in H$, then*

$$\langle [\phi(b) - \phi(a)]x, [\phi(d) - \phi(c)]x \rangle_H = 0$$

and

$$T = \int_0^b \lambda \, d\phi(\lambda).$$

(The function ϕ is called a spectral family for T - see notes to this chapter.)

Definition 10 *Suppose that T is a closed, densely defined linear transformation on the Hilbert space H into the Hilbert space K. Define T^t as follows: First let*

$$D(T^t) = \{y \in K : \text{ the transformation } W : x \in D(T) \to \langle Tx, y \rangle_K \tag{7.4}$$

is continuous$\}$.

For

$$y \in D(T^t),$$

define

$$T^t y = z$$

where z is the unique element of K such that

$$\langle Tx, y \rangle_K = \langle x, z \rangle_H, \; x \in D(T).$$

Problem 103 *Show that for T as in Definition 10*

$$\langle Tx, y \rangle_K = \langle x, T^t y \rangle, \; x \in D(T), y \in D(T^t),$$

Problem 104 *For T as in Definition 10, show that the range of*

$$(I + T^t T) \text{ is dense in } H.$$

Problem 105 *For T as in Definition 10, show that*

$$\|(I + T^t T)x\|_H \geq \|x\|_H, \; x \in D(T).$$

Problem 106 *Suppose that each of X, Y is a Hilbert space and T is a closed linear transformation on X into Y. Show that*

$$(I + T^t T)^{-1} \in L(X, X)$$

and

$$(I + TT^t)^{-1} \in L(Y, Y).$$

(See [66].)

Problem 107 *For T as in Definition 10, show that*

$$(I + T^t T)^{-1} T^t$$

is continuous and has continuous extension

$$T^t (I + TT^t)^{-1}.$$

Problem 108 *Suppose that each of H, K is a Hilbert space and $T \in L(H, K)$. Denote by T^* the member of $L(K, H)$ so that*

$$\langle Tx, y \rangle_K = \langle x, T^* y \rangle_H,\ x \in H, y \in K.$$

Show that the null space of T is the orthogonal complement of the range space of T^* and that the closure of the range of T is the orthogonal complement of the null space of T^*.

Problem 109 *Suppose that H is a Hilbert space and $T \in L(H, H)$ is self-adjoint, i.e.,*

$$\langle Tx, y \rangle_H = \langle x, Ty \rangle_H,\ x, y \in H$$

and that

$$\langle Tx, x \rangle_H \geq 0,\ x \in H,$$

i.e., T is nonnegative. Show that if $x \in H$, then

$$u = \lim_{t \to \infty} e^{-tT} x \text{ exists}$$

and is the orthogonal projection of x onto the null space of T.