Chapter 6
An Application to the Heat Equation

Denote by H the subspace of $L_2([0,1])$ consisting of all $g \in L_2([0,1])$ for which there is $f \in L_2([0,1])$ such that for some $c \in R$,

$$g(t) = c + \int_0^t f,\ t \in [0,1]. \tag{6.1}$$

In this case f is denoted as g', and is considered to be a generalized derivative of g. Denote by H the vector space of all functions g as in (6.1) with

$$\|g\|_H^2 = \|g\|_{L_2([0,1])}^2 + \|g'\|_{L_2([0,1])}^2.$$

Problem 85 *Show that H is a Hilbert space.*

H will also be denoted by $H^{1,2}([0,1])$ and is called a Sobolev space. An alternate, but equivalent definition will be given later in this problem sequence.

Problem 86 *Suppose that*

$$f(x) = \begin{cases} 0 & \textit{if } 0 \le x < 1/2, \\ 1/2 & \textit{if } x = 1/2, \\ 1 & \textit{if } 1/2 < x \le 1, \end{cases}$$

and g is as in (6.1). Critique the assertion that $g' = f$. Show that all members of H are continuous.

Problem 87 *Suppose that H is as above and H_0 is the subspace of H so that*

$$H_0 = \{f \in H : f(0) = 0 = f(1)\}$$

and that

$$A = \{(f, f'') : f \in H_0, f' \in H\}. \tag{6.2}$$

Show that A is the generator of a strongly continuous semigroup T on H_0.

J.W. Neuberger, *A Sequence of Problems on Semigroups*,
Problem Books in Mathematics, DOI 10.1007/978-1-4614-0430-9_6,

Problem 88 *For the setting in Problem 87, show that if* $u : [0, \infty) \times [0, 1] \to R$ *is defined by*

$$u(t, x) = (T(t)f)(x),\ t \geq 0,\ x \in [0, 1],$$

then

$$u(0, x) = f(x), u_1(t, x) = u_{2,2}(t, x), u(t, 0) = 0 = u(t, 1), x \in [0, 1], t \geq 0 \tag{6.3}$$

where $u_1(t, x)$ *is the partial derivative of* u *of the first order with respect to the first argument of* u *at the point* (t, x) *and* $u_{2,2}(t, x)$ *is the partial derivative of* u *of the second order in the second argument of* u*. The partial derivatives are taken in the generalized sense above. This is the famous heat equation.*

The next two problems in this chapter deal with numerical problems in approximating semigroups. There are two reasons for these problems. The first is to introduce some useful numerical ideas and the second is to illustrate the First Law of Numerical Analysis:

'Numerical difficulties and analytical difficulties always come in pairs.'

This is illustrated with the heat equation in Problem 88.

Problem 89 *Suppose* $n, N \in Z^+$ *and*

$$u^{0,k} \ = \ f(k/n),\ k = 1, \ldots, n-1,\ u^{0,0} = 0 = u^{0,n}, \tag{6.4}$$

where f *is the function given as initial data in* (6.3)*, being in this case continuous.*

Given $w > 0$ *and an integer* N*, write a computer program to calculate*

$$u^{j,k}\ , k = 1, \ldots, n-1,\ j = 1, 2, \ldots, N \tag{6.5}$$

such that

$$u^{j,0},\ u^{j,n} \ = \ 0,\ j = 1, \ldots, N \tag{6.6}$$

and

$$\frac{u^{j,k} - u^{j-1,k}}{\delta} \ = \ \frac{u^{j-1,k+1} - 2u^{j-1,k} + u^{j-1,k-1}}{h^2}, \tag{6.7}$$

$$k = 1, \ldots, n-1,\ j = 1, 2, \ldots, N,$$

where

$$\delta = 1/n, \quad \text{and} \ \ h = w/N.$$

(This method is called explicit.)

Problem 90 *Suppose that u, n, f, w, n, N are as in Problem 89 such that (6.4),(6.6) hold. Write a computer program for calculating the quantities in (6.7) using, in place of (6.7), the following scheme (it is called implicit):*

$$\frac{u^{j,k} - u^{j-1,k}}{\delta} = \frac{u^{j,k+1} - 2u^{j,k} + u^{j,k-1}}{h^2}, \tag{6.8}$$

$$k = 1, \ldots, n-1, \; j = 1, 2, \ldots, N.$$

Problem 91 *Observe that it is necessary to solve the system (6.8) for the quantities in (6.5). This is a triangular system. One can use Gaussian elimination, the method of Gauss–Seidel or other methods to solve this system. One can use Mathematica, MatLab, C, Fortran or almost any other computer language. Compare several methods for solving the system in Problem 90.*

Problem 92 *After your codes for Problems 89 and 90 work, make a comparison between the numerical phenomena in Problems 89 and 90 and the analytical phenomena suggested by Problem 84. Think about the first law of numerical analysis in this connection. In particular think about the fact that for $\lambda > 0$, $(I + \lambda A)$ is not continuous but $(I - \lambda A)^{-1}$ is continuous, where A is as in (6.2).*

Problem 93 *Carry out the classical 'separation of variables' method on the heat equation. First determine all solutions $u : [-\pi, \pi] \times [0, \infty)$ so that*

$$u(t, x) = f(t)g(x), \; t \geq 0, x \in [-\pi, \pi],$$
$$\text{with } u(t, -\pi) = 0 = u(t, \pi), \; t \geq 0.$$

Determine that any such nonzero pair must satisfy

$$\frac{f'(t)}{f(t)} = \frac{g''(x)}{g(x)} = \lambda$$

for some $\lambda \in R$, $0 < x < 1$ and $t \geq 0$. Determine all such numbers λ. Show that there is only a countable collection

$$\{\lambda_n\}_{n=0}^{\infty}$$

of such numbers λ. Consider all linear combinations of these 'separated' solutions

$$\sum_{n=1}^{\infty} c_n f_n(t) g_n(x), \; x \in [-\pi, \pi], \; t \geq 0$$

so that

$$\sum_{n=1}^{\infty} c_n^2 \text{ converges. (Why this?)} \tag{6.9}$$

Make a semigroup from this setting.

Problem 94 *Denote by* A *the transformation with domain all* $x = (x_1, x_2, \dots) \in \ell_2$ *so that*

$$Ax = (-x_1, -2x_2, -3x_3, -4x_4, \dots) \in \ell_2.$$

Find a semigroup T *on* ℓ_2 *which has generator* A.

Problem 95 *Compare semigroups in Problems 93 and 94.*