Chapter 5
Strongly Continuous Linear Semigroups

Suppose X is a Banach space. Here are some problems concerning the class of linear semigroups T which are strongly continuous and have the property that $|T(t)| \leq 1,\ t \geq 0$, that is, T is a strongly continuous semigroup of contractions (T is also called a nonexpansive semigroup). The contraction property makes our investigation a little easier but the general case of strongly continuous linear semigroups is actually an application of the contraction case. First we use a generator of T in the second sense:

$$A = \{(x, y) \in X^2 : y = \lim_{t \to 0+} \frac{1}{t}(T(t)x - x)\}. \tag{5.1}$$

For each $\lambda > 0$ denote by I_λ the transformation so that

$$I_\lambda x \ = \ \frac{1}{\lambda} \int_0^\infty e^{-r/\lambda} T(r)x\, dr,\ x \in X. \tag{5.2}$$

Problem 54 *Show that if* $\lambda > 0,$ *then* $|I_\lambda| \leq 1.$

Problem 55 *Show that if* $x \in X$, *then*

$$\lim_{\lambda \to 0+} I_\lambda x = x.$$

Problem 56 *Show that if* $x \in X$, *then* $I_\lambda x \in D(A)$, *the domain of* A.

Problem 57 *Show that if* $\lambda > 0$ *and* $x \in X$, *then*

$$(I - \lambda A) I_\lambda x = x,$$

that is, $I - \lambda A$ is a left inverse of I_λ.

Problem 58 *Show that if* $x \in D(A)$, *then*

$$I_\lambda (I - \lambda A)x = x,$$

that is, $(I - \lambda A)$ *is also a right inverse of* I_λ.

J.W. Neuberger, *A Sequence of Problems on Semigroups*,
Problem Books in Mathematics, DOI 10.1007/978-1-4614-0430-9_5,

Some help with this problem follows.

Problem 59 *Suppose* $x \in D(A)$ *and define* $h : [0, \infty) \to X$ *as*

$$h(t) = T(t)x, \; t \geq 0. \tag{5.3}$$

Show that the right derivative h^+ *of* h *exists in all of* $[0, \infty)$ *and* $h^+(t) = T(t)Ax, \; t \geq 0$. *Show also that* h^+ *is continuous.*

Problem 60 *Show that for* h *as in Problem 59,* h' *exists on* $[0, \infty)$.

Problem 61 *Show that* h *in Problem 59 satisfies*

$$h'(t) \; = \; T(t)Ax, \; t \geq 0,$$

and

$$h'(t) \; = \; Ah(t), \; t \geq 0 \tag{5.4}$$

provided that $x \in D(A)$, *the domain of* A.

Problem 62 *Suppose that* $x \in X$ *but* x *is not in* $D(A)$. *Show that there is a sequence* $\{x_n\}_{n=1}^{\infty}$ *of members of* $D(A)$, *converging to* x *so that if* $c > 0$, *then*

$$\{T(\cdot)x_n\}_{n=1}^{\infty}$$

converges uniformly to

$$T(\cdot)x \tag{5.5}$$

on $[0, c]$.

Definition 6 *The expression in* (5.5) *is called a generalized solution of* (5.4).

Definition 7 *Suppose* G *is a transformation from a subset of* X *into* X. *The statement that* G *is closed means that*

$$\{(x, Gx) : x \in D(G)\} \text{ is a closed subset of } X \times X.$$

Problem 63 *Show that if* $\lambda \geq 0$, *then* $(I - \lambda A)^{-1}$ *is closed and also that* A *is closed.*

Problem 64 *Show that* $A \in L(X, X)$ *if and only if* $D(A) = X$. *(Use the closed graph theorem.)*

Problem 65 *Suppose* $\lambda > 0, \; x \in X$ *and* m, n *are positive integers. Show that*

$$(I_{\lambda/n})^m x =$$

$$(\frac{n}{\lambda})^m \int_0^{\infty} \cdots \int_0^{\infty} e^{-(n/\lambda)(s_m + \cdots + s_1)} T(s_m + \cdots + s_1)x \; ds_m \cdots ds_1$$

$$= (\frac{n}{\lambda})^m \int_0^{\infty} e^{-sn/\lambda} \frac{s^{m-1}}{(m-1)!} T(s)x \; ds.$$

In particular,

$$(I_{\lambda/n})^n x = \int_0^\infty d\phi_{\lambda,n}\, T(\cdot)x$$

where

$$\phi_{\lambda,n}(s) \ = \ 1 - \Sigma_{k=0}^{n-1} e^{-ns/\lambda} \frac{(ns/\lambda)^k}{k!},\ s \geq 0. \tag{5.6}$$

This is the same distribution as in (3.7).

Problem 66 *Show that if $x \in X$ and $\lambda \geq 0$, then*

$$\lim_{n\to\infty} (I - \frac{\lambda}{n} A)^{-n} x \ = T(\lambda)x. \tag{5.7}$$

Problem 66 is, essentially, half of the famous theorem of Hille–Yosida for linear strongly continuous semigroups of contractions: Given one of these semigroups, define its generator (in the second sense) and reconstruct the semigroup from its generator by means of an exponential formula) (5.7).

The problems that follow in this sequence will make the other half of the Hille–Yosida theorem in the present case: Suppose A is a closed linear transformation with dense domain in X, with the property that $(I - \lambda A)^{-1}$ exists, with domain all of X and $|(I - \lambda A)^{-1}| \leq 1, \lambda \geq 0$.

Problem 67 *Show that*

$$A(I - \lambda A)^{-1} = \frac{1}{\lambda}((I - \lambda A)^{-1} - I). \tag{5.8}$$

Denote the expression in (5.8) by A_λ and call it the Yosida approximation to A at λ.

We want to construct a semigroup T which has A as its generator.

Problem 68 *Show that if $x \in D(A)$, then*

$$\lim_{\lambda\to 0+} A_\lambda x = Ax.$$

If $\lambda > 0$ denote by T_λ the semigroup with generator $A_\lambda,\ t \geq 0$, that is,

$$T_\lambda(t) = e^{tA_\lambda}.$$

Problem 69 *Show that*

$$|T_\lambda(t)| \leq 1,\ t \geq 0.$$

Problem 70 *Show that if $\lambda > 0$, then T_λ is a continuous linear semigroup.*

Problem 71 *Show that if $\alpha, \beta > 0$, then*

$$T_\alpha(t)(I - \beta A)^{-1} = (I - \beta A)^{-1} T_\alpha(t),\ t \geq 0.$$

Problem 72 *Show that if* $\alpha, \beta > 0$*, then*

$$T_\alpha(t)T_\beta(s) = T_\beta(s)T_\alpha(t),\ t, s \geq 0.$$

Problem 73 *Suppose that* $n \in Z^+$ *and each of*

$$\{C_k\}_{k=1}^n, \{D_k\}_{k=1}^n \in L(X, X),$$

$x \in X$ *and*

$$|C_k|, |D_k| \leq 1,\ k = 1, \ldots, n.$$

Find an inequality for

$$\|C_1C_2 \cdots C_n x - D_1D_2 \cdots D_n x\|.$$

Problem 74 *Suppose* $\alpha, \beta > 0$*. Show that*

$$\|T_\alpha(t)x - T_\beta(t)x\| \ \leq\ t\|A_\alpha x - A_\beta x\|,\ x \in X, t \geq 0.$$

Problem 75 *Show that there is a strongly continuous semigroup* T *of contractions such that*

$$T(t)x \ =\ \lim_{\lambda \to 0+} T_\lambda(t)x,\ x \in X,\ t \geq 0.$$

Problem 76 *Show that*

$$A \ =\ \{(x, y) \in X^2 : y = \lim_{t \to 0+} \frac{1}{t}(T(t)x - x)\} \tag{5.9}$$

where T *is as in Problem 75.*

Problem 77 *Show that* A *of the preceding problem is a generator of* T *in each of the first three senses.*

Now suppose that T is a linear strongly continuous semigroup which is not a semigroup of contractions.

Problem 78 *Show that there exists* $M > 0$ *so that*

$$|T(t)| \ \leq\ M,\ t \in [0, 1].$$

(Recall the theorem of uniform boundedness.)

Problem 79 *Show that for* M, T *as in Problem 78 and* $w = \ln(M)$*, it is true that*

$$|T(t)| \ \leq\ Me^{wt},\ t \geq 0.$$

Problem 80 *For* M, T, w *as in Problem 79, define* S *by*

$$S(t) \ =\ e^{-wt}T(t),\ t \geq 0.$$

Show that S is a strongly continuous semigroup and that

$$|S(t)| \ \le \ M, \ t \ge 0.$$

Problem 81 *For M, T, w, S as in Problem 80, define a norm $\|\cdot\|'$ by*

$$\|x\|' \ = \sup_{t \ge 0} \|S(t)x\|, \ x \in X.$$

Show that the norm $\|\cdot\|'$ is equivalent to $\|\cdot\|$ in the sense that there are k, $K > 0$ such that

$$k\|x\|' \ \le \|x\| \ \le K\|x\|', \ x \in X.$$

Show that S is a semigroup of contractions (i.e., nonexpansive) under the norm $\|\cdot\|'$.

Problem 82 *Make an analysis of the semigroup T by means of a study of S, using the fact that S, under the norm $\|\cdot\|'$, is a semigroup of the type studied in Problems 56–76.*

In (5.7) we have a generalization of formula

$$e^x \ = \ \lim_{n \to \infty} (1 - \frac{x}{n})^{-n}, \ x \in R.$$

Problem 83 *Discuss the possibility of using*

$$\lim_{n \to \infty} (I + \frac{\lambda}{n} A)^n x, \ x \in X$$

in place of one suggested by (5.7).

Problem 84 *Articulate why the second alternative below is more likely to be true than the first alternative:*

$$T(t)x = \lim_{n \to \infty} (I + \frac{t}{n} A)^n x \in X, t \ge 0,$$

$$T(t)x = \lim_{n \to \infty} (I - \frac{t}{n} A)^{-n}, x \in X, t \ge 0.$$