Chapter 4
Linear Continuous Semigroups

In this chapter we suppose that X is a Banach space with norm $\|\cdot\|$ and T is a semigroup on X such that

$$T(t) \in L(X,X), t \geq 0$$

where $L(X,X)$ represents the ring of all continuous linear transformations from X to X. Suppose also that T is continuous in the sense of Definition 5.

If $C \in L(X,X)$ denote by $|C|$ the norm of C, that is, the smallest non-negative number such that

$$\|Cx\| \leq |C|\|x\|, \; x \in X.$$

We will eventually see that the difference between a semigroup being continuous and its being merely strongly continuous is an important distinction. Continuous linear semigroups arise from what are essentially ordinary differential equations (even though the underlying space may be infinite dimensional). Strongly continuous semigroups that are not continuous can pertain to partial differential equations — a great difference! Note that 'strongly continuous' in Definition 2 is a notion weaker than the notion of 'continuous' in Definition 5. The terminology is an historical accident and is somewhat unfortunate. In Chapter 5 problems will deal with strongly continuous semigroups. Some of the problems in this chapter partially prepare one for problems in Chapter 5.

Problem 38 *If $t, s \geq 0$ and T is a continuous linear semigroup, show that*

$$(T(t) - I) \int_0^s T(r)\, dr = (T(s) - I) \int_0^t T(r)\, dr. \tag{4.1}$$

Problem 39 *If $s > 0$ and T is as in Problem 38, show that*

$$\lim_{t \to 0+} \frac{1}{t}(T(t) - I)\frac{1}{s}\int_0^s T(r)\, dr$$

exists in $L(X,X)$.

J.W. Neuberger, *A Sequence of Problems on Semigroups*,
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Problem 40 *Suppose that T is as in Problem 39. Show that*

$$\lim_{s\to 0+} \frac{1}{s}\int_0^s T(r)dr \;=\; I.$$

Problem 41 *Suppose that T is as in Problem 39. Show that there is $B \in L(X,X)$ such that*

$$\lim_{t\to 0+} \frac{1}{t}(T(t)-I) \;=\; B.$$

Note that B is a generator of T in the second sense.

Problem 42 *Suppose that T is as in Problem 39. Show that*

$$T'(t) \;=\; BT(t),\; t \geq 0$$

and

$$T(t) \;=\; I + \int_0^t BT(r)\,dr,\; t \geq 0.$$

Problem 43 *Suppose that $\lambda \in R$ is an eigenvalue of B as in Problem 42, i.e., there is $g \in X$ not equal to zero such that*

$$Bg = \lambda g.$$

Show that

$$T(t)g = \exp(t\lambda)g,\; t \geq 0.$$

Problem 44 *Suppose that T is as in Problem 39, n is a positive integer and that $X = R^n$. Suppose also that $\lambda_1, \lambda_2, \ldots, \lambda_n$ is a collection of distinct eigenvalues of B and that $x_1, x_2, \ldots, x_n$ is a corresponding sequence of eigenvectors with*

$$Bx_k = \lambda_k x,\; k = 1, 2, \ldots, n.$$

Show that if $x \in X$ and

$$x = c_1 x_1 + \cdots + c_n x_n,$$

then

$$T(t)x = c_1 \exp(\lambda_1 t)x_1 + \cdots + c_n \exp(\lambda_n t)x_n, t \in R.$$

Problem 45 *Give an appropriate generalization of Problem 44 to the case in which X is a finite-dimensional vector space over the complex numbers and some eigenvalue may have multiplicity greater than one (recall the Jordan normal form theorem).*

Some of the next problems may help to give an alternative for determining T from a generator.

Problem 46 *Suppose X is a Banach space, $B \in L(X,X)$, and $f_0, f_1, f_2, \ldots$ is a sequence of continuous functions from $[0,\infty) \to X$ such that*

$$f_n(t) = I + \int_0^t Bf_{n-1}(r)\,dr,\ t \geq 0, n = 1, 2, \ldots.$$

Show that

$$\|f_{n+1}(t) - f_n(t)\| \ \leq\ |B| \int_0^t \|f_n(r) - f_{n-1}(r)\|\,dr$$

for $t \geq 0, n = 1, 2, \ldots$.

Problem 47 *Using the notation of Problem 46, show that if $c > 0$, then there is $K > 0$ such that*

$$\|f_{n+1}(t) - f_n(t)\| \leq K\frac{|B|^n}{n!},\ t \in [0,c], n = 1, 2, \ldots.$$

Problem 48 *Show that $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy on $[0,c]$ for all $c > 0$. Denote by f the function with domain $[0,\infty)$ such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly in each $[0,c]$ for all $c > 0$. Show that*

$$f(t) = I + \int_0^t Bf(r)\,dr,\ t \geq 0.$$

(Method of successive approximations, Picard's method.)

Problem 49 *Show that*

$$T(t)\ =\ e^{tB} = \Sigma_{n=0}^{\infty}\frac{(tB)^n}{n!},\ t \geq 0$$

where the series converges in the norm of $L(X,X)$.

Problem 50 *Show that if $\lambda \geq 0$, then*

$$T(\lambda) = \lim_{n\to\infty}(I + \frac{\lambda}{n}B)^n. \tag{4.2}$$

Problem 51 *Suppose $C \in L(X,X)$. Show that if $\lambda \geq 0$, then*

$$\lim_{n\to\infty}(I + \frac{\lambda}{n}C)^n \tag{4.3}$$

exists and if S with domain $[0,\infty)$ is such that $S(\lambda)$ is this limit for all $\lambda \geq 0$, then S is a continuous semigroup.

Problem 52 *Suppose $C \in L(X,X)$. Show that if $t > 0$ and $t|C| < 1$, then*

$$(I - tC)^{-1}$$

exists and is in $L(X,X)$.

Problem 53 *For S as in Problem 51 show that*

$$S(\lambda) = \lim_{n\to\infty} (I - \frac{\lambda}{n} C)^{-n}.$$