Chapter 3
Translation Semigroups

Problem 24 *Suppose* $X = C([0,\infty))$, *the Banach space of all bounded continuous functions from* $[0,\infty)$ *to* R *with norm*

$$\|f\|_X = \sup_{t\geq 0} |f(t)|.$$

Define the semigroup T *on* X *by*

$$(T(t)f)(x) \ = f(x+t),\ x,t \geq 0,\ f \in C([0,\infty)). \tag{3.1}$$

Is T *strongly continuous?*

Problem 25 *Same as Problem 24 except that* X *is the set of all functions which are bounded and uniformly continuous from* $[0,\infty)$ *to* R. *Is the resulting semigroup* T *strongly continuous?*

Problem 26 *Is either of the semigroups in Problems 24 and 25 continuous (see Definition 5)?*

Problem 27 *Suppose* $f, g : [0,\infty) \to R$ *are continuous and if* $t \geq 0$, *then*

$$g(t) = \lim_{h\to 0+} \frac{1}{t}(f(t+h) - f(t)) \text{ for all } t \geq 0.$$

Show that f *is differentiable on* $[0,\infty)$.

Problem 28 *Denote by* X *the Banach space, under sup norm, of all bounded continuous functions from* $[0,\infty)$ *to* R. *Define* T *as in (3.1) and define*

$$A = \{(f,g) \in X :$$
$$g(x) = \lim_{t\to 0+} \frac{1}{t}((T(t)f)(x) - f(x)),\ f \in X = [0,1],\ x \geq 0\},$$

the generator for T *in the second sense. Show that*

$$Af = f' \in X \text{ if } f \in D(A), \text{ the domain of } A.$$

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Problem 29 *For A as in Problem 28, show that if $g \in X$ and $\lambda > 0$, then there is one and only one $f \in X$ such that f is in the domain of A and*

$$f - \lambda A f \;=\; g.$$

Problem 30 *For A, g as in Problem 29, show that f in that problem is given by*

$$f(x) \;=\; \frac{1}{\lambda}\int_0^\infty e^{-r/\lambda} g(r+x)\,dr,\ x \ge 0. \tag{3.2}$$

In a sense, f is a Laplace transform of g.

Problem 31 *Take A as in Problem 28. Show that if $\lambda \ge 0$, then*

$$(I - \lambda A)^{-1}$$

exists, is a member of $L(X,X)$ and

$$|(I - \lambda A)^{-1}| \le 1.$$

For Problems 32 through 36, suppose that A is as in Problem 28, $x \ge 0$, $\lambda > 0$ and $f \in X$ where X is the Banach space (under sup norm) of all bounded, real-valued uniformly continuous functions on $[0, \infty)$.

Problem 32 *Show that*

$$((I - (\lambda/2)A)^{-2} f)(x)$$
$$= (\frac{2}{\lambda})^2 \int_0^\infty \int_0^\infty \exp(-(2/\lambda)(s_1 + s_2)) f(s_1 + s_2 + x)\, ds_1\, ds_2. \tag{3.3}$$

Problem 33 *Convert (3.3) to a single integral (rotate coordinates 45 degrees) to obtain*

$$((I - (\lambda/2)A)^{-2} f)(x) = (\frac{1}{\lambda})^2 \int_0^\infty \exp(-(2s/\lambda)) s f(s+x)\, ds. \tag{3.4}$$

Problem 34 *Suppose n is a positive integer. Show that*

$$((I - (\lambda/n)A)^{-n} f)(x)$$
$$= (\frac{n}{\lambda})^n \int_0^\infty \exp(-(ns/\lambda))(s^{n-1}/(n-1)!) f(s+x)\, ds. \tag{3.5}$$

Problem 35 *Show that (3.5) may be rewritten, using a Stieltjes integral, as*

$$((I - (\lambda/n)A)^{-n} f)(x) = \int_0^\infty f(s+x)\, d\phi_{\lambda,n}(s) = \int_0^\infty f(s+x)\phi'_{\lambda,n}\, ds, \tag{3.6}$$

where

$$\phi_{\lambda,n}(s) = 1 - \sum_{k=0}^{n-1} \exp(-ns/\lambda)\frac{(ns/\lambda)^k}{k!}, \; s \geq 0. \tag{3.7}$$

Problem 36 *Show that if* $\lambda > 0, \; x \geq 0,$ *then*

$$\lim_{n\to\infty} ((I - (\lambda/n)A)^{-n} f)(x) = f(x + \lambda). \tag{3.8}$$

Some advice may be in order here. Equation 3.7 gives a well-known (at least to probabilists) Poisson distribution.

Problem 37 *Using the notation of Problem 35, show that*

$$\lim_{n\to\infty} \phi_{\lambda,n}(s) = \alpha_\lambda(s)$$

where α_λ *is defined by*

$$\alpha_\lambda(s) = \begin{cases} 0 & \text{if } 0 \leq s < \lambda, \\ 1/2 & \text{if } s = \lambda, \\ 1 & \text{if } s > \lambda. \end{cases}$$

Before working Problem 36, one might pause to review some probability theory concerning the law of large numbers (central limit theorem). The problem *can* be worked from the beginning by careful examination of the terms in (3.5) or (3.7), first noting which term is maximum and how the other terms 'fall off' as one proceeds from the mean in either direction.

The theory of semigroups has a rich connection with probability theory. It is rather widely known that semigroups have many applications to probability theory. What is not so well appreciated is that probability theory has a number of significant *applications* to semigroup theory (cf. [16]).