Chapter 25
Notes

Some General Comments

Semigroups, semiflows, semidynamical systems—all are the same. Once when I gave a talk at the University of Alaska in Fairbanks, I mentioned that I once heard that the Inuit had something like eighty different words for snow. One of my hosts countered that the Inuit had over a hundred words for snow. So much for a Texan making comments about snow to Alaskans! Anyway, there being three words—semigroups, semiflows, semidynamical systems—for the same thing is indicative of the importance of a certain idea—that of a one-parameter family of transformations T so that for some space X,

$$T(0) = I, \text{ the identity transformation on } X, \ T(t)T(s) = T(t+s), \ t, s \geq 0 \tag{25.1}$$

and

$$T(t) : X \to X, \ t \geq 0.$$

Now the origins of these three terms come from different mathematical cultures. The term 'semigroup', as used in this book, arose from abstraction of time-dependent autonomous partial differential equations. 'Semiflow' seems to me to have arisen from topology whereas 'semidynamical system' has a life of its own in the vast world of dynamical systems. The terms 'semiflow' and 'semidynamical system' are used infrequently, for objects in this list of problems, compared to the term 'semigroup'. For each of the three terms, dropping the 'semi' indicates that the members $T(t), \ t > 0$ all have inverses, commonly continuous ones, and that the semigroup law (25.1) extends to all of R. 'Semigroup', on the other hand, is more pervasive than 'group' in the context of one-parameter families of transformations. A reason for this is that time-dependent partial differential equations commonly are not reversible in time. Physically, knowing the heat distribution in a metal bar doesn't tell us much about how hot the bar was an hour ago. Actually, for T the heat equation semigroup $T(t)$ *is* invertible for $t > 0$, but although $T(t)^{-1}$ exists it is only densely defined and is discontinuous at each point at which it is defined. For plenty of examples of strongly continuous semigroups T, $T(t)$ is

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not invertible for some $t > 0$; for example the semigroup T on X, where X is the Banach space of bounded continuous functions on $[0, \infty)$ with sup norm and

$$(T(t)(f))(x) = f(t + x), \ t, x \geq 0.$$

Many things in life are not reversible, at least according to my own experience. Confining oneself to time-dependent partial differential equations which are reversible means one misses out on a lot. This said, however, there are many important time-dependent processes which *are* reversible, a most noteworthy one being the Schrödinger equation of quantum mechanics.

Something that conventionally separates things usually called 'semigroup, semiflow' or 'semidynamical system' is that generators of 'semigroups' are commonly densely defined everywhere discontinuous transformations. Such generators are not commonly found in papers dealing with semidynamical systems or semiflows, in my experience.

The recent incorporation of 'local semigroups' into existing semigroup theory (see Chapter 17) seems to be a significant extension but it is hardly new in that Sophus Lie considered them in his quest for integrating factors for systems of ordinary differential equations. My own introduction to Lie's work came in a graduate seminar that I was, many years ago, conducting at Emory University. We were trying to extend to non-locally compact spaces some of von Neumann's work, the part which was seminal to the eventual solution of Hilbert's Fifth Problem. I mentioned that one of us should find out what Sophus Lie did. It turns out that no one took me up on this, so I made the attempt myself, studying essentially work from [27]. Some five years later, in about 1970, I made use of my inspiration from Lie's work in [36], but it wasn't until my collaboration with Dorroh that the idea came to fruition as an alternative to the (stalled, in my opinion) theory of nonlinear semigroups (as a generalization of existing linear semigroup theory).

The present volume of these notes has its origins, [46], in notes I wrote in Spanish for the XIII Escuela Venezolana Matemàticas at the Universidad de los Andes in Merida, Venezuela, 6–15 September 2000. These notes comprised about 48 pages and contained about 111 problems. In the course the students (faculty, graduate students and a few undergraduates) vigorously attacked and settled many of the problems in these notes, leading to many intense and enlightening discussions. The present form of these notes started with my translation of the original notes into English. The number of problems is nearly quadruple that of the original notes. There are several chapters of problems dealing with subjects which didn't even exist in 2000.

The original notes benefited greatly, for both mathematics and Spanish, from the help of Alfonso and Miryam Castro, Mario Jimenez, Barbara Neuberger, Víctor Padrón, María Mera Rivas and María Cristina Trevisán.

I particularly thank the organizing hosts, Victor Padrón and Oswaldo Arajo, for their great hospitality and their great effort in arranging for this

Summer School. It was a long-cherished dream to be able to conduct a course using only Spanish, both in class and outside the classroom. This course represented a particular challenge since I was not lecturing but rather constantly interacting with the participants while they were presenting arguments. Explaining a flaw in someone's argument is a challenge even in one's native language. Everyone was very generous in putting up with my limited Spanish.

For some problems in semigroups, complex analysis enters in an essential way. Chapter 23 is one instance. The result of Beurling in Chapter 21 is another instance. The subject of holomorphic semigroups is generally complex-based. Earlier work (see in particular [21], for example) is based on Laplace transforms as in Chapter 17. Passage from semigroup to resolvent is frequently presented as a matter of inverting a Laplace transform—something often done using contour integrals but this is not done in the present volume.

Problem 437 *Find corresponding complex field results for the (majority) of problems in this book which are stated (usually implicitly) for the real field.*

25.1 Notes on Chapter 2

Problem 2 is the earliest functional equation of which I know. It is remarkable that it has a vast set of solution, if the hypothesis of continuity is omitted but only a simple family of solutions if continuity is assumed. Continuity, it turns out, implies differentiability. Problem 5 urges a reading of at least the second half of Hilbert's Fifth Problem. It was Hilbert who likely was the first to understand the profound power of combining algebraic and topological hypotheses in the presence of the possibility of analytic results. Almost all of the problems in this volume owe a debt to this legacy.

Problems 10, 12 are two examples considered at the start of the quest to generalize linear semigroups to nonlinear semigroups. These early examples led to [35] and attempts to incorporate the idea of resolvent into nonlinear study. See [9], [60], [29] and references contained therein as well as notes in Section 25.23 for more details.

The terms 'continuous' and 'strongly continuous' are somewhat misleading, but completely standard, when applied to semigroups. The term 'continuous' is a stronger notion than is 'strongly continuous'. To add to confusion, there is also 'weak continuity' of semigroups which refers to continuity with respect to a weak topology.

25.2 Notes on Chapter 3

Operator semigroup theory has a curious property that often results from a special case are applicable to more general cases. Many of the ideas developed in this chapter for translation semigroups have direct application to much more general cases. This holds true especially in Chapter 17 in which linear theory is *applied* to nonlinear theory.

For some decades, a thrust was to try to develop a nonlinear theory in *analogy* with linear theory. This led to many interesting developments but to this day has had a rather limited success. Generalized translation semigroups (see Chapter 17) ultimately gave a fairly satisfactory theory. For this reason alone, translation semigroups would be of considerable interest. Nonlinear semigroups in Chapter 17 give rise to linear semigroups which are essentially translation semigroups on a metric space.

In this chapter, some probability distributions arise in a natural way. Someone working their way through these problems has at least two choices. One is to find a source of information on Poisson distributions. Problem 36 is then essentially a consequence of an appropriate central limit theorem. The other choice is to closely study the distributions indicated in Problem 34 to see directly that as $n \to \infty$, then the sequence of distributions, for some $\lambda > 0$, converges to a stepfunction which is zero from $[0, \lambda)$ is $\frac{1}{2}$ at λ and is one on (λ, ∞). The same distribution appears in an essential way in Chapter 17, so effort spent on the Poisson distribution here will be rewarded later.

My own introduction to the application of probability to semigroup theory stems from my encountering Bernstein polynomials in what is outlined in this book as Problem 429. In 1958 while teaching my first graduate course. I rather idly was looking into how numerics worked out for certain simple partial differential equations. Much to my surprise, the Bernstein polynomials suddenly arose. I knew little of central limit theorems then and, before that time, Bernstein polynomials looked strange to me. My brute force approach in showing convergence of the numerical scheme in Problem 429 led me in a life-long affinity for how probability, semigroups and partial differential equations relate. This episode also led me to a study of quasianalyticity in terms of higher order differences, as indicated in Chapter 21. It is usually hard to put in a good word for ignorance, but in this case my lack of knowledge of central limit theorems led me to some nice things.

25.3 Notes on Chapter 4

Continuous semigroups are very special cases of semigroups of linear transformations. They are essentially based on ordinary differential equations in a Banach space. Continuous semigroups are essentially infinite-dimensional generalizations of constant coefficient systems of linear equations, but many of the problems in this chapters reveal things which help in the study of more general semigroups—those which do pertain to partial differential equations.

One result of working the problems in this chapter is to see in a rather direct way how differentiability may arise from algebraic semigroup properties taken together with continuity. The identity in Problem 38 is very useful in this regard. Problem 42 is an early chance for a reader to see a generator appearing for a semigroup.

A possible strategy in gaining understanding of the matters of this chapter is to follow through using the semigroup g from Problem 3.

Problem 46 indicates how Picard's method of successive approximations leads to existence of a semigroup with generator $B \in L(X, X)$, X a Banach space. Remaining problems in this chapter show how exponentials of such a transformation B represent the semigroup with generator B. Problems 49, 50 give a product expression and the equivalent series expression, respectively.

25.4 Notes on Chapter 5

As already indicated, linear semigroups which are strongly continuous but not continuous form the theoretical basis for autonomous time-dependent linear partial differential equations. This subject had its start with the work of Marshall Stone on the Schrödinger equation in the 1930s, showing that this equation gave rise to a group. The massive work of Hille and Phillips [21] in the 1950s developed the theory of strongly continuous linear semigroups into something close to its present form. The book [21] is an excellent reference, but one might still seek out Hille's original [20] book of the same title which has more concrete information about partial differential equations. A prized possession of mine is a copy of Hille's *Functional Analysis and Semigroups* given to me by Phillips.

The books [17], [19], [16] and [59] deal with linear semigroups and have a good deal of information on applications. In [66] there is an excellent chapter on strongly continuous linear semigroups.

To anyone reading any of these references it will be clear that problems in the present book contain just an introduction to the study of one-parameter semigroups.

25.5 Notes on Chapter 6

The heat equation gives the premier example of a semigroup that comes from a time-dependent PDE. In a sense it dates back to Fourier. One can solve the heat equations by Fourier's method, 'separation of variables', and then compare results with numerical solutions. Of particular interest here is how the relationship between 'implicit' and 'explicit' methods for solving the heat equation has its counterpart in the general theory of linear strongly continuous semigroups in that two plausible exponential formulae have strikingly different levels of viability (see Problem 84). Contemplate the First Law of Numerical Analysis, Chapter 6, in this regard.

25.6 Notes on Chapter 7

Definition 8 is for Fréchet derivative. Later problems give some properties. Problem 97 gives a local existence and uniqueness theorem for ordinary differential equations. It can be proved by the method of successive approximations, as in Problem 46. Problem 98 gives a limit theorem that is needed in Chapter 14. Problems 100, 102 give two versions of the spectral theorem that are useful in connection with Chapter 16. Problems 100, 101 give a generalization of the fact that in finite-dimensional Euclidean space, a symmetric linear transformation has a basis in terms of which the transformation has a diagonal matrix representation. A reference such as [66] might be consulted for lemmas and background.

The present chapter also contains some preliminaries to problems in Chapter 16 in which a single linear transformation T has two (related) adjoints. In this case, one adjoint of T is continuous and the other adjoint is not. This is characteristic of Sobolev gradients arising from problems in differential equations. It is helpful for a reader to reconcile this pair of adjoints. A reader might see later how gradients essentially based on one definition of adjoint lead to viable numerical methods whereas when based on ordinary gradients become a disaster (see Chapter 14 and [43] for problems on this issue).

For more background on Problems 100, 102, see, for example, [66] or other books that deal with spectral theory.

25.7 Notes on Chapter 8

Results on combining two (or more) continuous linear semigroups give some indication of what to expect for combining two strongly continuous linear semigroups and also for combining two nonlinear semigroups. A starting point is to note that for $n > 1$, a positive integer, and $A, B \in L(R^n, R^n)$, then it does not necessarily follow that

$$e^{tA}e^{tB} = e^{t(A+B)},\ t \geq 0,$$

but some aspects of this law of exponents can be regained by

$$e^{t(A+B)} = \lim_{k\to\infty} (e^{\frac{t}{k}A}e^{\frac{t}{k}B})^k.$$

25.8 Notes on Chapter 9

In regard to Problem 144, I seem to recall working this out sometime around the 1970s, but have been unable to recover my argument. A path to my original scratchings on this problem might be found by first doing a web search on ‘A Guide to the J. W. Neuberger Papers’. My actual papers up to 2002 (more to be added later) are in the

Dolph Briscoe Center for American History

at the University of Texas.

It might be easier to just work this out for yourself than to find it. I would not be surprised to find that others have found this result, but I cannot offer a reference.

What might be new to some is the introduction of a bit of near-ring theory, which seems rarely used in analysis problems.

The site indicated above contains many of my scratchings from the early 1960s to about 2001. I don’t particularly wish it upon someone to spend a lot of time searching these papers, but they are there if for some reason someone might want to attempt to reconstruct some of my (usually vague at best) mathematical thoughts.

In [62] there are results by Coke Reed on combining dynamical systems. Two continuous dynamical systems T and S on a Banach space X are said to combine provided that if $x \in X$ and $a, b \in R$, then there is $y \in X$ so that if $\epsilon > 0$ there is $\delta > 0$ so that if $t_0, t_1, \ldots, t_n$ is a partition from a to b of mesh less than δ, then

$$\| \left(\Pi_{k=1}^{n} \left(T\left(t_k - t_{k-1}\right) S\left(t_k - t_{k-1}\right)\right)\right)(x) - y\| < \epsilon. \tag{25.2}$$

If there is a continuous dynamical system U on X so that

$$K(b-a)x = y,$$

y as in (25.2), then it may be said that T, S combine to get U. The paper [62] contains a number of results on combining dynamical systems and surprisingly, contains counterexamples to some combination conjectures which struck me as plausible and probably true, but are not true.

Problem 438 *Obtain and study [62]. Ponder how results there show some limitations on combining semigroups as well as some promising directions of inquiry. Examine other papers of Coke Reed as found in MathSciNet.*

The paper [62] and others by Coke Reed show connections between the semigroup-dynamical system-flow cultures. (Disclaimer: I introduced Coke Reed to the study of dynamical systems in the 1960s. A reader might find it interesting to learn where his study of dynamical systems has led him.)

25.9 Notes on Chapter 10

This chapter contains a number of results which are an extension of Chapter 5 and are preliminary to Chapter 11. Perhaps the fundamental difficulty in dealing with strongly continuous linear semigroups that are not continuous, is that the generators of such semigroups are always closed densely defined linear transformations. Such transformations are always discontinuous at each point at which they are defined, but many calculations which are natural for continuous linear semigroups can be arrived at by rather convoluted reasoning (if they are true at all). Extensive use of resolvents of generators of strongly continuous linear semigroups is characteristic of many of the main developments in the theory. The present chapter gives some developments which will be used in Chapter 11. What is probably new to even seasoned researchers in the field is the use of probability measures as in Problem 155, again the Poisson distribution as introduced in Chapter 3.

25.10 Notes on Chapter 11

Results of the present chapter give an outline of some special cases of the celebrated Trotter–Kato development, which is of great interest in partial differential equations. Developments in this chapter follow the outline given in Chapter III, Section 5 of [17], but there are substantial differences in the present development, particularly in the use of probability distributions

$$\phi_{m,\lambda},\ m \in Z^+,\ \lambda > 0. \tag{25.3}$$

So far as I know, these distributions were first applied to semigroup theory in [14]. In the present volume, these distributions are used in Chapters 5 and 3, but their use there was taken from [14].

Feynman–Kac formulae, based on Trotter–Kato developments, are of interest in quantum mechanics. There is an account of this in [19] in which there is indicated a continuing mystery concerning these formulae. In Chapter 12 there are some problems involving applications of Trotter–Kato formulae to the numerical solution of time-dependent partial differential equations.

I consider Problems 179 to 181 rather speculative. My main reason for including these is the following: Consideration of results surrounding (25.3) might lead to a new and more comprehensive family of arguments for semigroups, those more based on ideas from probability, than have been usual in semigroup theory. Traditional Trotter–Kato results seem to be based on arguments such as those found in Chapter III, Section 5 of [17]. It just might be that some of Problems 179 to 181 can be proved true using these newer applications of probability to semigroup theory and consequently extend the theory.

In my opinion, to show that two semigroups T, S combine to give a third semigroup U is more important than the question of how possible generators of T, S can be used to gain a generator of U even though the second issue is certainly a significant one.

25.11 Notes on Chapter 12

Splitting methods are widely used in practice, often beyond the realm of established proofs indicating their validity. There is a considerable literature that may be consulted. The splitting method is a good way to deal with reaction–convection–diffusion equations. Each of the reaction, convection and diffusion equations has their own highly efficient method of solution. One main time step in the splitting method for such equations entails a sequence of three substeps: the first using only a diffusion method, the second a convection method and the third a reaction method.

25.12 Notes on Chapter 13

The idea of using a method of steepest descent to find zeros or critical points of real-valued functions goes back at least to Cauchy. This chapter gives some basic results concerning zeros of linear transformations between two Hilbert spaces. The two spaces may be Sobolev spaces, in which case results apply directly to systems of linear differential equations. In appropriate finite-dimensional spaces, results apply to finding numerical approximations of solutions to such equations, as illustrated in Section 25.13. A reference for problems in the present chapter is [43], Chapter 3.

25.13 Notes on Chapter 14

In [43] there is a fairly complete recent discussion of Sobolev gradients. There are applications to problems of transonic flow, minimal surfaces and superconductivity ([64], [65]). The gradient inequality (Definition 17) is a fairly strong hypothesis and I hope that future work will seek to replace it with weaker conditions.

A main issue relating semigroups and steepest descent is the following: Given a real-valued C^1 function ϕ on a Hilbert space X and a gradient $\nabla\phi$ for ϕ, that is, a function which satisfies

$$\phi'(x)h = \langle h, (\nabla\phi)(x)\rangle_X\,,\ x, h \in X,$$

determine conditions on ϕ so that if

$$z(0) = x,\ z'(t) = -(\nabla\phi)(z(t)),\ t \geq 0,$$

then

$$\lim_{t\to\infty} u = \lim_{t\to\infty} z(t) \text{ exists}$$

and

$$(\nabla\phi)(u) = 0.$$

There is the broad problem:

Problem 439 *Make an investigation of the gradient inequality and try to find weaker conditions which imply the conclusion of Problem 227.*

One can consult [66] for example, and Chapter 16, to obtain more information about the projections onto

$$\{\begin{pmatrix} x \\ Tx \end{pmatrix} : x \in D(T)\},$$

where H, K are Hilbert spaces and T is a closed densely defined linear transformation on H with range in K. The original formula was due to von Neumann [70]. These projections serve as a point of departure in the construction of Sobolev gradients. See also [1] as a general reference for Sobolev spaces.

This chapter gives only the barest introduction to Sobolev gradients, but [43] and references contained therein give a fairly comprehensive recent account and a bibliography. A reader might google 'Sobolev gradient' or 'Sobolev gradients' for a further impression of this subject.

Continuous steepest descent using Sobolev gradients leads naturally to nonlinear semigroups if the underlying problem is itself nonlinear. A dominant feature is that a properly formulated least squares or variational principle problem leads to a *continuous* nonlinear semigroup, not just a strongly continuous one. Hence the underlying steepest descent equation is essentially an *ordinary* differential equation. The contrast with more conventional formulations is illustrated by minimal surface problems: In [8], for instance, the process for finding a minimal surface is 'evolution by mean curvature' of a conventional time-dependent partial differential equation. A corresponding Sobolev gradient approach yields a steepest descent, with continually varying metric, which is an ordinary differential equation in function space (see [43], Chapters 11 and 16). This fact alone seems to justify interest in Chapter 9 which deals with continuous nonlinear semigroups.

This raises a question of both a numerical approach and a theoretical approach, using Sobolev gradients in both cases, to such problems as the Nash embedding problem and the Poincaré conjecture. These are, at present, descent or conventional Newton's method processes which can be embedded into continuous processes giving rise to effectively time-dependent partial differential equations.

Problem 440 *Try to formulate the two problems in the above paragraph in terms of Sobolev gradient descent in order that the process may be considered as a problem concerning ordinary differential equations in a function space.*

In a sense, such a corresponding pairing between time-dependent partial differential equations and Sobolev gradient ordinary differential equations in function space, is already illustrated in contrasting methods for Moser's inverse function theorem (Problem 402 of Chapter 22).

25.14 Notes on Chapter 15

A great deal of computational effort continues to be expended on numerical solution of time-dependent partial differential equations. Equations of Navier–Stokes, which govern a wide variety of fluid flows, are a notable example.

Semigroups as developed in these notes grew out of, and remain as, an abstraction of autonomous time-dependent partial differential equations. Even for problems, such as elliptic systems, which do not physically involve time, a semigroup is associated. For example, if X and Y are Hilbert spaces and $F : X \to Y$ is such that the problem of finding $u \in X$ such that

$$F(u) = 0 \tag{25.4}$$

represents a system of partial differential equations, then critical points of ϕ, defined by

$$\phi(x) = \frac{1}{2}\|F(x)\|_Y^2,\ x \in X,$$

can be turned into a semigroup problem by means of

$$z(0) = x,\ z'(t) = -(\nabla\phi)(z(t)),\ t \geq 0. \tag{25.5}$$

Numerical calculations such as those introduced here are at once a practical matter aiming to get concrete information about solutions, and also a means to gain insight into the theory of semigroups. Literature on numerical solution of time-dependent PDEs is truly vast, running from mathematics to physics, chemistry, various branches of engineering, biology and economics. No attempt is made here to do justice to this immense and important area. When (25.5) arises from (25.4), relevant gradients put the problem in somewhat different terms from conventional time-dependent problems, but semigroup issues remain for such problems.

In a practical sense, the theory of Sobolev gradients gives an organized way to determine and to compute preconditioners to be applied to ordinary gradients. Following the theory, preconditioners respect boundary and other supplementary conditions, even nonlinear conditions. Generally the ratio

$$\frac{\text{number of iterations needed using ordinary gradient}}{\text{number of iterations needed with Sobolev gradient}}$$

goes, for a given problem, to infinity as mesh size approaches zero. This is documented in [69], as well as a number of other papers authored or coauthored by Sultan Sial and in references contained in his work.

The above describes numerical symptoms when using Sobolev gradients as opposed to ordinary gradients in descent process for partial differential equations. Some issues may be illustrated by a simple example: Consider a least squares formulation for the simple problem of finding u on $[0,1]$ so that

$$u' - u = 0.$$

The least squares formulation is essentially (14.5):

$$\phi(u) = \frac{1}{2}\int_0^1 (u' - u)^2, \tag{25.6}$$

for u in what space? The space $L_2([0,1])$ is not good since then ϕ would be only densely defined and everywhere discontinuous where it is defined, giving a poor or nonexistent gradient. However, the ordinary gradient for a discrete version of (25.6) is in a sense trying to do just that. If, however, the Sobolev space $H^{1,2}([0,1])$ is used, then the resulting ϕ is continuous and differentiable—in fact it is a quadratic polynomial.

The fundamental idea of Sobolev gradients is this: For finite-dimensional emulations of least square (or energy functional) emulations, the finite-dimensional gradient should be taken with respect to a norm which emulates the theoretical norm which renders the functional in question at least a C^1 functional. This idea can be viewed as a consequence of the basic law of numerical analysis, given in Chapter 6, essentially saying that sensitivities in a functional should be matched by sensitivities in a gradient with respect to which steepest descent is being taken.

25.15 Notes on Chapter 16

References for this chapter are [25] and [43], Chapter 5, Section 5.4. Most of the material of this chapter comes directly from the first reference, as summarized in the second reference. The main object here is the systematic use of the embedding operator between two abstract Hilbert spaces H and H' where the points of H' form a dense subset of H and the norm in H' dominates the norm in H.

The material in this chapter had its origin in the seminal work [6] (reprinted in [5]), in which a 'kernel free' development of potential theory was presented. There are other interesting connections with semigroups in these references. The material of this chapter is essentially an abstract extension of [6] without specific reference to various measures used in potential theory.

The reference [25] contains a solution to an abstract symmetric version of the Kato conjecture, [2]. It is related to Problem 257 by observing that

$$H_1 = H'.$$

An application of the idea in Problem 261 may be found in [53].

This chapter concludes with problems leading up to a formula of von Neumann which applies the development in the first part of the chapter. This formula presents in a simple form the orthogonal projection onto

$$\begin{pmatrix} x \\ Tx \end{pmatrix}, \; x \in X,$$

where T is a closed densely defined linear transformation of X into Y.

Problem 441 *In* (16.2)*, describe why one might write*

$$T^t(I + TT^t)^{-1} \text{ instead of } (I + T^tT)^{-1}T^t.$$

Both of these expressions are linear and continuous. They agree on a dense subset of X. Are they precisely the same?

25.16 Notes on Chapter 17

Since the middle of 1950 some of us have sought a complete theory of nonlinear semigroups which has the power of the theory of strongly continuous linear semigroups. Perhaps the first paper in this direction was [35]. The book [9] has a good description of the case of strongly continuous semigroups of contractions on a convex subset of a Hilbert space. The books [60], [68], [29] deal with various extensions to spaces more general than Hilbert spaces. After 1971 or so there has been little substantial progress in the direction of a complete theory although many interesting results had been found. In this context, 'complete theory' means a theory in which a collection of semigroups SG, a collection of generators GEN, and a means of (a) for all elements of SG finding a member of GEN by means of differentiation at zero and (b) for all members of GEN, constructing by means of an exponential formula of a member of SG. The problems in this chapter, for the most part, are from [13], [14], [15].

In [71], von Neumann and Koopman consider Hamiltonian systems on a region Ω in a complex finite-dimensional space. Such systems are commonly a system of nonlinear ordinary equations. They take, using our present terms, a linear representation on complex $L_2(\Omega)$. This representation, using special features of Hamiltonian systems, turns out to be a strongly continuous group of unitary transformations, T. The generator of T turns out to be iA, where A is an unbounded, densely defined self-adjoint linear transformation on $L_2(\Omega)$. A spectral analysis using the spectral theorem, indicated in Chapter 7, is then related to dynamical properties of the Hamiltonian system. This work of von Neumann and Koopman gave encouragement to Dorroh and myself. M. G. Crandall indicated to me (private communication) that he and A. Pazy considered a study of nonlinear semigroups using linear representations, but indicated that they did not pursue this direction. I also had a private communication from G.-C. Rota that he once considered such a study. So, despite the work of Sophus Lie, [27], in the 1800s, work in [71] and the intense, but short, period of excitement briefly described in Chapter 24, it seems that using linear representation as a cornerstone to study nonlinear semigroups had to wait until [38] to seriously begin. After that paper, published in 1973, I occasionally returned to the subject over the next nearly 20 years, with not much in the way of results to report. It wasn't until [14] and its predecessor paper [13] that this direction of research finally made some significant progress in the early 1990s. I will briefly relate how that got started:

In 1992, there was a meeting on semigroups in Curaçao, organized by Jerry Goldstein. It was a small meeting with no parallel sessions. One afternoon there was scheduled a problem session. Several people presented some problems, but then there was silence. I volunteered to present work related to [38]. Bob Dorroh started asking some penetrating questions: 'Is my space X assumed to be locally compact?' for one (my answer was 'no'). This started a period of intense collaboration which eventually resulted in [14]. He knew some relevant things that I would have never 'Dreampt of in my philosophy' (adapted from William Shakespeare's Hamlet, Act I, Scene V). Work of Dennis Sentilles [67] was of crucial use for us. Sentilles was a Ph.D. student of Dorroh in the early 1970s; Dorroh and I have known each other since the 1960s. We could have had our conversations 20 years earlier!

Suppose that T is a jointly continuous semigroup on a subset X of a Banach space. Suppose also that T has a conventional generator B and also Lie generator A. A possible relationship between A and B is

$$(Af)(x) = \lim_{t \to 0+} \frac{f(T(t)x) - f(x)}{t}) = f'(x)Bx,\ x \in D(A), \tag{25.7}$$

assuming sufficient differentiability is available (otherwise the right side of (25.7) should be written as the directional derivative of f in the direction Bx, all evaluated at x). It is becoming clearer that zeros of B are important in an analysis of A, which in turn is crucial for an analysis of T from our present point of view. This will be made a little clearer in Chapter 19. However, using the material of Chapters 17, 18, 19 is still, for concrete applications, very much in its preliminary stages. I expect a good bit of progress will be led by numerical experiments, which in turn are in their very preliminary states.

In [57] G. E. Parker gives a way to recover semigroups from certian inverse limit sets. This initiated a still largely unexplored alternative way to associate a kind of generator with a nonlinear semigroup. This development merits additional attention. See also Parker's work in [56], [57], [58].

Problem 442 *Find and read [57], [56] to encounter a unique view of linear and nonlinear semigroup theory.*

Additional historical comments relevant to this chapter are in the Notes to Chapter 24.

25.17 Notes on Chapter 18

In a sense, for a nonlinear semigroup T on a Polish space X, the associated semigroup U on the indicated space of measures is more closely related to T than is the representation S of Chapter 17. These semigroups of measures remain, as of this writing, almost entirely unexplored.

If $T, X, B(X), MCR(X)$ are as in Chapter 18, define

$$(U(t)\mu)(\Omega) \;=\; \mu(T(t)^{-1}\Omega),\; t \geq 0, \Omega \in B(X),$$

where $\mu(T(t)^{-1}\Omega) = 0$ if there is not $y \in X$ such that $T(t)x \in \Omega$.

Note that if $x \in X$ and δ_x is the Dirac measure centered at x, then

$$(U(t)\delta_x) = \delta_{T(t)x},\; t \geq 0.$$

So, U restricted to the Dirac measures in $MCR(X)$ is essentially a copy of T itself. Since U is a linear semigroup, I have called U a linear extension of T.

The semigroups T, S, U of Problems 311, 312, 313 pose some questions of considerable importance. Webb's example (Problem 14), in my opinion, put a stop for some time to the search for a complete theory of nonlinear strongly continuous semigroups in terms of conventional generators. The conventional generator for Webb's example (see Chapter 24) had long seemed too sparsely defined to be useful in a generator-resolvent theory for nonlinear semigroups. The material of Chapters 17, 18 gave rise to possibilities of renewing the quest which was stopped for so long by Webb's example.

Problems 311, 312, 313 indicate a careful study of Webb's example in terms of these later developments. So far as I know, as of this writing, no one has attempted to use the suggestions to better understand Webb's example as a step toward a major advance in nonlinear semigroup theory.

Very few extensions or applications have been made of material from this chapter. I suspect that there are many interesting discoveries to be made in this regard.

25.18 Notes on Chapter 19

Problem 443 *Make a theory of local jointly continuous semigroups, on a complete separable metric space X, which is in analogy to the theory of jointly continuous semigroups in Chapter 17.*

A solution of this problem will make a good publication.

This problem appeared in 2000 notes (in Spanish) which were written for the XIII Escuela Venezolana Mathematicas, [46]. Since then, a substantial part of this problem has been solved by the developments described in Chapter 19 (see also [54]). However, a complete characterization of generators A for local semigroups is still lacking.

One of the Clay Millennium prizes, [18], is for establishing which of global and local existence holds for a Navier–Stokes equation in three dimensions. Developments in Chapter 19 yield a possible attack on this problem, since the *form* of a Lie generator A is clear from the form of Navier–Stokes (see (25.7)). In principle, one has only to decide whether there is a positive eigenvalue of the relevant generator A.

The 'dream' of a numerical attack on this problem is already the subject of serious work. The principle of a numerical attack seems clear, but it seems to be a very large computational problem. Even for a local or global nonlinear semigroup, a suitable discretization of the space of continuous functions whose domain is the Sobolev space $(H^{1,2}(R^2))^2$ needs to be made. Even for a rather rough discretization, the dimension of a corresponding problem is large. For a system of three ordinary differential equations the space that needs a discrete approximation is $(H^{1,2}(R^2))^3$, and so on. For a partial differential equation in time and one space dimension a suitable discretization might require a hundred ordinary differential equations in a 'method of lines' approximation. This would entail a numerical version of $(H^{1,2}(R^2))^{100}$. For Navier–Stokes in three space dimensions and time, the dimension of an appropriate approximation space would be vastly larger.

Problem 444 *Estimate the dimensionality of a reasonably close approximating space for time-dependent Navier–Stokes in three space dimensions, using the approach of the present chapter. Assess whether any present-day computer is up to the job of gaining meaningful evidence on the local–global time existence of time-dependent Navier–Stokes in three space dimensions. How long might we have to wait for an adequate computer?*

25.19 Notes on Chapter 20

For a system of partial differential equations, what condition on a solution is necessary and sufficient in order that there be one and only one function satisfying both the condition *and* the system. For many more-or-less standard systems, such additional conditions are well understood, but the more general question remains one of the outstanding unsettled questions in mathematics (I suspect that the question is considered so outrageous that no one has the nerve to give it a name). For some centuries, attention has been focused mainly, but not entirely, on specifying conditions on the boundary of a region on which a system is to be solved. I suspect that this frame of mind has arisen since so many systems arise as the Euler–Lagrange equations of an energy functional. For a given system on a bounded region to arise from an energy functional on some region, it is almost universal that one starts with a supposed critical point of the functional and then after an integration by parts, arrives at the fact that the Euler–Lagrange equations must be satisfied. When one integrates by parts, one is left with an integral around the boundary of the relevant region. What is to be done with these inevitable integrals around the boundary of the region? It is common that conditions are imposed on potential solutions so that functions satisfying these conditions are such that these boundary integrals become zero. Study of boundary conditions for known types of partial differential equations has led to many interesting results, of course, but a fixation on 'boundary conditions', to the exclusion of consideration of other supplementary conditions, seems not to be so productive. Examples of cases in which 'boundary conditions' may not be an appropriate way to pick out unique solutions include transonic flow problems in which nonlinearities determine type. There is usually no way to merely look at such an equation to pick out subregions of ellipticity or hyperbolicity, for example. The equation often has to be solved first in order to determine such regions. If one has only a method which requires boundary conditions to be known before a solution can be attempted, then one is often caught in an unhelpful circular path.

Now from Chapter 14, the method of Sobolev gradients gives a way to find critical points of some 'energy-like' functionals without having to deal with Euler–Lagrange equations. Using gradients derived from Sobolev metrics, both the start of a numerical theory and a theoretical one are indicated. Some reflection yields that this development gives a clue as to how systems of partial differential equations may be dealt with without first determining 'boundary conditions' appropriate to the system.

The main problem remains as to how one might classify the set of all solutions to a given system. The present chapter is a start in this direction. The main hypothesis is that continuous steepest, starting at any point of the underlying space, descent converges to a unique element. Two functions are called *equivalent* provided that when both are used as starting values for continuous steepest descent, then both descents converge to the same solution. In terms of an appropriate Lie generator, from Chapter 17, the relevant equivalence classes are characterized.

Problems in this chapter should be thought of as leading to a point of view on attacking the unsettled problem to which mention is given above. Finishing the project any time soon is unlikely of course, but interesting results can be expected from anyone seriously considering the problem. What is needed now are more examples.

Note the essential use of semigroup theory in this chapter. Note also that the development is essentially a linear one, even though the focus is on systems of nonlinear partial differential equations.

25.20 Notes on Chapter 21

Problem 368 deals with semigroups for denumerable Markov processes. David Kendall [26] was interested in the following question: If one knows one of the transition probability functions $p_{i,j}$ on some bounded subinterval $[a, b]$ of $[0, \infty)$, can one determine what that transition function is over all of $[0, \infty)$? If such a function were to be real analytic, then analytic continuation would determine it everywhere. However, such analyticity has not been established. About the best that is known in the general case is that these transitions functions are C^1. What Kendall, Kato, Beurling, this writer and others discovered is that the behavior of

$$|P(t) - I| \text{ as } t \to 0+,$$

has a striking effect on smoothness of these transition functions. Problem 366 gives that if

$$\limsup_{t\to 0+} |P(t) - I| < 2,$$

then all transition functions for P are analytic away from zero.

The gist of Problem 367 is effectively that if

$$\liminf_{t\to 0+} |P(t) - I| < 2, \tag{25.8}$$

then all transition functions for P lie in some quasianalytic collection. In [41] there is given a more-or-less constructive method, in case (25.8) is satisfied, of determining all of a trajectory from its values on any small interval, thus giving at least a partial solution of Kendall's problem. See [26], [40], [37] for a more in-depth discussion, references and history.

Problem 362 is Beurling's analyticity result which is used in Problem 366. See [4], [7] for an argument if you haven't yet figured out one for yourself. See [40] for some history of Beurling's result. It was this then-unpublished result which motivated me to seek the creation and publication of [4] and [5].

Problem 382 is from [11], which is concerned with analyticity 'away from zero' of strongly continuous linear semigroups. What is stated in Problem 382 is a weaker conclusion than is actually given in that reference. Results in this paper are a substantial generalization of Beurling's results in [7] (and in [5], in a paper of the same title as [7]). I feel that a great deal more can be learned about these 'analyticity away from zero' semigroups.

So far as I know, no progress has ever been made on Problem 392. In Problem 12 there is a nonlinear semigroup on $X = [0, 1]$ where all trajectories other than the zero trajectory start out at a positive number, then they hit and stay at zero. Even linear examples can have such a property:

Problem 445 *Find such a linear example.*

But many nonlinear semigroups have the property that the set of their trajectories themselves form a quasianalytic collection, hence Problem 392.

See [22] for another characterization of analyticity for a strongly continuous linear semigroup.

The first time I talked to David Kendall was in a transatlantic telephone call, a rarity for me in those days, in 1968. He had some questions about my paper [33], a complicated paper complete with some misprints that made it even harder. He had accepted an invitation to speak at a University of London analysis seminar. The only requirement on topic was that the 'details be sufficiently horrible'—no soft analysis here. He had chosen to present [33] for the occasion! I did get to spend the next summer visiting him in Cambridge. Paper [42], many years later, is a big improvement on [33], but it is quite involved too, sorry to say.

As noted in Chapter 21, Problem 363 is the hardest problem I have ever solved. Only Kendall and perhaps just a few others are known to me to have solved it (or its less general but even more complicated earlier version [33]). The result could use more sunlight shed upon it.

25.21 Notes on Chapter 22

The continuous Newton's method is a continuous generalization of the common Newton's method. The phenomenon that if

$$z(0) = x,\ z'(t) = -F'(z(t))^{-1}F(z(t)),\ t \geq 0,$$

then

$$F(z(t)) = \exp(-t)F(x), t \geq 0, \tag{25.9}$$

has no counterpart in discrete Newton's method. The chaotic domains of attraction, for roots of polynomials (Chapter 23), that come with the discrete Newton's method, do not appear in the continuous case, but rather are artifacts of the discretization. Note that a discretization of the continuous

Newton's method yields the ordinary Newton's method when the discretization parameter is one—the damped Newton's method when the discretization parameter falls in $(0, 1)$.

I once saw a sign on someone's door at Oak Ridge National Laboratory:

'One Man's Error is another Man's Data'

In a way, this sign could be a parody of some step-by-step numerical calculations.

Problem 398 is a continuous form of a Nash–Moser inverse function theorem, [30]. See [43], Chapter 8, as well as [48], [49], [51] for more results related to Problem 398. Moser's epic result uses a scale of Banach spaces and smoothing operators. It is a triumph of intricate analysis. Results related to Problem 398 give similar results in a vastly simpler fashion. The key to this simplicity is that in a continuous version of Nash–Moser type results there is generally no 'loss of smoothness'. I will try to make this clearer:

Suppose F is a function from a Sobolev space H to a Sobolev space K. Assume that the problem of finding $u \in H$ such that

$$F(u) = 0 \tag{25.10}$$

represents a system of nonlinear partial differential equations, say of order m. A step, starting with $u \in H$, toward finding a zero of F by the conventional Newton's method generally involves finding $h \in H$ such that

$$F'(u)h = -F(u). \tag{25.11}$$

Now u usually needs (when applied to problems in differential equations) to have some smoothness in order to get a smooth enough solution h so that when u is updated to $u + h$, the new u has sufficient smoothness. If the system is of order m, then calculating $F(u)$ from u involves taking m derivatives of u. If u has $k > m$ derivatives, then $F(u)$ generally would have only $k - m$ derivatives and one would not expect to find h with more derivatives than this. As the iteration (25.11) progresses, the situation deteriorates to the point where it can't be continued. This 'loss of derivatives' is countered by Moser, inspired by Nash's work, [31]. At each step the current value u is replaced by an approximation to u which has more derivatives. This adds an inner loop to the process. An eventual proof of convergences is an intricate process requiring a list of additional assumptions on the function F. A close examination of Problem 401 shows that this iteration, derived from the continuous Newton's method, does not generally suffer this loss. The key to avoiding this loss is found in (22.2): The right-hand side of this equation can be fixed at $-F(x)$ (due to (25.9)) throughout the iteration; all of the linear systems, essentially

$$F'(y)h = -F(x)$$

for a succession of elements y have the same right-hand side. See [48], [49], [51], [43] (Chapter 8) for further explanation.

Alfonso Castro, some years ago pointed out to me a relationship between Moser's, [30], inequalities for his inverse function theorem and some of my thoughts on gradients inequalities in Chapter 14. This suggestion resulted in [10] and eventually to the Nash-Moser type results in Chapter 22.

Problem 338 concerns the continuous Newton's method and might well belong to the present chapter.

25.22 Notes on Chapter 23

Before Chapter 23 all semigroups considered were deterministic in the sense that they had the forward uniqueness property, i.e., for a given semigroup T on a space X, if one knows $T(t)x$ for some $t \geq 0,\ x \in X$, then $T(s)x$ is completely determined for all $s > t$ (a slight modification of this statement might be made for local semigroups). In Chapter 23 this condition is relaxed. For p a nonconstant complex polynomial, we interpret the continuous Newton's method as that of finding a continuous function $z : R \to C$ so that

$$z(t_0) = x \in C,\ p(z)'(t) = -p(z(t)),\ t \in R. \tag{25.12}$$

For some $x \in C$, this problem has multiple solutions. This occurs when for some $y \in C,\ s \in R$,

$$p'(z(s)) = 0, z'(s) \text{ doesn't exist and} \lim_{t \to s} z(t) = y. \tag{25.13}$$

It turns out then there are at least two ways to continue $z(t)$ continuously for $t > s$. Such a thing could never happen if (25.12) had a unique solution. Nevertheless, (25.12) does lead to a generalized kind of semigroup, closely following axioms given in [3].

Problem 446 *Do you agree that it is reasonable to call* (25.12) *an equation for the continuous Newton's method?*

The Mathematica code given in Chapter 23 is intended for use in plotting vector fields generated by (25.12). Such plots are in marked contrast to plots obtained with the conventional Newton's method. There are famous plots for complex polynomials arising from the conventional Newton's method. For example, if

$$p(w) = w^3 - 1,\ w \in C,$$

and one colors the complex plane using red, green, blue and black according to the following:

- Color a point red if starting with it, the conventional Newton's method converges to the first root of p.
- Color a point green if starting with it, the conventional Newton's method converges to the second root of p.

- Color a point blue if starting with it, the conventional Newton's method converges to the third root of p.
- Color a point black if starting with it, the conventional Newton's method does not converge.

Similar plots for a Möbius transformation and for other rational functions are of interest. One gets a great fractal mixture of red, green and blue (black ones exist but are not seen in a plot). Such pictures are great for coffee table books, but they are an analyst's nightmare. I consider the 'chaos' represented by the fractal nature of the plot to be an artifact of the discretization of (25.12). Essentially the chaos seen comes from truncation error and is in a sense not natural to the problem.

Problem 447 *References in [44] point to an extension of the above results to polynomials on higher dimensional spaces. Read these references and contemplate an extension of developments in the present chapter.*

Vector fields coming from the Riemann Zeta function suggest a more qualitative approach to the Riemann hypothesis. An idea is this:

In an examination of vector field plots, using the Mathematica code in this chapter but with the polynomial definition replaced by 'Zeta', one notes a quite regular arrangement of arrows in the vector plot. For this I suggest a window, say $\{x, -2, 12\}$,$\{y, 0, 100\}$, maybe done in pieces which are patched together. As one progresses upward from the real axis, staying fairly close to the critical line, the vector field patterns become a bit more complicated, forming discernible groups, but still quite regular. Particular attention might be paid to the roots of the derivative of the Zeta function. These are points where it appears, in the corresponding vector field plot, that two constant argument lines collide and two leave. Take your choice of which path to follow after such a collision. Now if the Riemann hypothesis is not true, it seems likely that vector field patterns would be severely altered around such a pair of exceptional roots of Zeta. Such a possibility might allow a more topological, global approach to the Riemann hypothesis problem, trying to show that such exceptions can't occur.

25.23 Notes on Chapter 24

As a graduate student in the mid-1950s, in a seminar course under H. S. Wall (all of his courses were like that), we were studying linear evolution equations, essentially problems such as finding

$$u : [a, b] \to X$$

so that

$$u'(t) = A(t)u(t) \; t \in [a, b], \tag{25.14}$$

where X is a Banach space and

$$[a,b] \subset R, A : [a,b] \to L(X,X) \text{ is continuous.}$$

Commonly, for each $t \in [a,b]$, $A(t)$ is a generator of a strongly continuous semigroup and results in semigroup theory are applied to evolution equations.

I asked myself, 'why does everything have to be linear?' A result of this query was [32], my thesis, which dealt with problems of finding

$$Y : [a,b] \to X$$

such that

$$Y(t) = c + \int_a^t dF\, Y,\ t \in [a,b], \tag{25.15}$$

where F is a given function with domain $[a,b]$ and range a set of *nonlinear* functions from X to X. I don't mean that just F itself is nonlinear (this could easily happen in (25.14)) but rather that $F(t)$ itself is a possibly nonlinear transformation for each t in the domain of F. It was at this point that I realized that poor notation from calculus had a bad effect on the development of nonlinear functional analysis. Someone stuck with saying things like

$$f = f(x)$$

was rather frozen out of the subject. It could easily be that f is a function, $x \in D(f)$ and $f(x)$ is also a function, but decidedly $f \neq f(x)$. I had the good fortune of having a rare calculus class, in 1952-53 under R. L. Moore, which maintained good functional notation. This was crucial for me when I began to think about nonlinear evolution equations.

In (25.15), I thought of F as a kind of nonlinear measure-valued function and the integral in (25.15) emulating a Stieltjes integral.

I went from [32] in 1958, to [34] in 1965 (work done much earlier but was slow to be published), in which I started with a collection of functions and derived a function F as in (25.15). Then I wrote [35], published in 1966, in which I found how to approximate resolvents of nonlinear transformations.

Essentially for a strongly continuous nonexpansive semigroup T on a Hilbert space X and for $\lambda, \delta > 0$, I defined

$$A_\delta = \frac{1}{\delta}(T(\delta) - I)$$

and then thought about resolvents:

$$(I - \lambda A_\delta)^{-1},$$

seeking to find when

$$\lim_{\delta \to 0+} (I - \lambda A_\delta)^{-1} x$$

might exist for $x \in X$.

In my 1966 paper, [35], there is a rudimentary exponential formula for a nonlinear semigroup which was under, according to Brezis in 1973, [9], 'hypothèses très restrictives'. I had assumed some differentiability which I wanted to be able to prove (and was later proved by others—see [9], [60],[29]). I agree with Brezis' judgment, but point out that such criticism appeared only after a number of years of work had given improved understanding. At the time of [35] some told me that 'nonlinear functional analysis', was a contradiction in terms. Ten years later it was considered 'mainstream' (whatever that is supposed to mean).

The problems of this chapter use freely ideas from [60] for the development of a form of the Crandall–Liggett Theorem [12] (which in its general form holds in any Banach space). It also uses [9]. These references together with references in [68], [29] contain a great deal for additional study in the direction of this chapter. Other references on monotone operators are [74] and [28] for some very important early ideas on the subject.

After [35], for about 8 years, there was a great deal of activity on nonlinear semigroups, attempting to develop a theory that generalized existing linear theory. A paradigm of that era was to attempt to analyze nonlinear semigroups in a manner *analogous* to the linear theory. Starting in [38], there was an attempt to develop a different idea of generator, one suggested by Sophus Lie's work in which a semigroup was not directly differentiated at zero (to get a conventional generator), but rather to analyze the effect that a semigroup has on real-valued functions when *composed* with a trajectory of a semigroup. This activity, after about 20 years, yielded [14], in which a rather satisfactory theory was obtained linking the set of all jointly continuous semigroups on a Polish space with a clearly defined set of generators in the Sophus Lie sense (Chapters 17, 19).

A quote from Lie's work [27], near the start of his Chapter 4:

"··· es wird sich nämlich später immer zeigen, dass alle auf die eingliedrige Gruppe bezüglichen Probleme durch Benutzung der infinitesimalen Transformation derselben allein gelöst werden können."

A translation coming from an anonymous translator at Oak Ridge National Laboratory reads:

"··· it will be clear later that all problems related to the one-parameter group may be solved by use of the infinitesimal transformation."

In [55], there is a discussion of the ideas that led Gauss and Riemann to their analysis of finite-dimensional surfaces that were *not* specified as a subset of a Euclidean space. This led to Riemannian geometry, where tangent spaces to a manifold are defined as a collection of tangent vectors, each of which is essentially defined as a means of differentiating real-valued functions on the manifold. Our present Lie's generators were adapted from this idea and Lie's ideas underlie the theory in Chapter 17. The work in Chapters 17 and 19 uses only metric spaces without a differential structure; it is essentially a melding of linear semigroup theory (in which generators are only closed and densely defined) with fundamental ideas of Gauss–Riemann–Lie.

Problem 448 *In Chapter 17, if the metric space X is required to be a Riemannian manifold that is not necessarily locally compact (Hilbert manifold), can the theory presented there be enhanced by using a differential structure somewhat similar to what is used for a Riemannian manifold?*