Chapter 2
The Idea of a Semigroup

Problem 2 *Show that if f is a continuous function from R to R so that*

$$f(x) + f(y) = f(x + y) \text{ for all } x, y \in R,$$

then there is $c \in R$ so that

$$f(x) = cx, \ x \in R.$$

Problem 3 *Suppose that g is a continuous function from $[0, \infty)$ to R so that*

$$g(x)g(y) = g(x + y), \ x, y \in [0, \infty). \tag{2.1}$$

Show that either $g(x) = 0$ for all $x \geq 0$, or else there is $b \in R$ so that

$$g(x) = e^{bx}, \ x \geq 0.$$

Problem 4 *Contemplate the possibility of there being more solutions f in Problem 2 or more solutions g in Problem 3 if the word 'continuous' is deleted from the respective statements of these problems. Perhaps don't dwell on the present problem unless the term 'Hamel basis' is familiar.*

Problem 5 *In connection with Problem 4, find and read the second part of Hilbert's Fifth Problem (which is concerned with how algebraic and continuity conditions, taken together, may lead to sufficient differentiability to permit an analysis of a problem).*

Definition 1 *A semigroup on a set X is a function T with domain $[0, \infty)$ and range in the set of all functions from X to X so that*

$$T(0) = I \text{ and } T(t)T(s) = T(t + s), \ t, s \geq 0 \tag{2.2}$$

where $T(t)T(s)$ indicates the composition of the transformations $T(t)$ and $T(s)$. The identity transformation on X is I.

J.W. Neuberger, *A Sequence of Problems on Semigroups*,
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Problem 6 *Find a simple example of a semigroup T.*

Problem 7 *Find an example of a semigroup T that is not so simple.*

Problem 8 *Suppose X is a subset of a Banach space X_0 and F is a function $X \to X_0$ such that if $x \in X$ there is a unique $z : [0, \infty) \to X$ so that*

$$z(0) = x,\ z'(t) = F(z(t)),\ t \geq 0. \tag{2.3}$$

Denote by T the function with domain $[0, \infty)$ and range in the set of all transformations from $X \to X$ such that if $x \in X$ and $s \geq 0$, then

$$T(s)x = z(s)$$

where z satisfies (2.3). Show that T satisfies (2.2).

Note that for T a semigroup on X and $x \in X$, we write $T(t)x$ instead of the longer $(T(t))(x)$.

One can say, in this case, that F is a generator of T. The term *'generator'* will be used in at least four distinct senses in this collection of problems:

- (i) As the function F for which the solutions z of (2.3) serve to define the semigroup.
- (ii) For a semigroup T on X (if X is a Banach space),

  $$F \ = \ \{(x, y) \in X^2 : y = \lim_{t \to 0+} \frac{1}{t}(T(t)x - x)\},$$

  the derivative of T at zero. A function *is* a collection of ordered pairs. The above expression for F simply gives the set of all pairs comprising F.
- (iii) Given a semigroup T, the generator is a transformation F from which one can reconstruct T by means of an exponential formula (as we will see later in some problems).
- (iv) In Chapters 17 and 19 there is another notion of generator whose heritage goes back to Gauss and Riemann and then to Sophus Lie. In terms of this kind of generator, a complete theory of nonlinear semigroups was finally established (in about 1992). This will be the subject of a number of problems to follow.

It is often necessary to make understood the sense in which one is using the term 'generator' in a given discussion, but in many cases, a transformation F in items (i),(ii),(iii) is a generator in each of these senses. Often a generator in sense (iv) is related to such an F in a way that would have been familiar to Sophus Lie.

Definition 2 *If X is a topological space one says that a semigroup T on X is strongly continuous if for each $x \in X$ the function g such that*

$$g(t) \ = \ T(t)x,\ t \geq 0 \tag{2.4}$$

is continuous.

Definition 3 *If T is a semigroup on the set X, $x \in X$ and g satisfies (2.4), then g is called a trajectory of T.*

Problem 9 *Show that if a semigroup T arises from the setting of Problem 8, then T is strongly continuous.*

Some problems to follow give examples of semigroups, some of which are strongly continuous.

Problem 10 *Suppose $X = [0, 1]$ and T is the function with domain $[0, \infty)$ and range the collection of all functions from X to X such that*

$$T(t)x = \frac{x}{1 + tx},\ x \in X,\ t \geq 0.$$

Show that T is a semigroup on X.

Problem 11 *Does there exist a generator for T in Problem 10 in either sense (i) or sense (ii)?*

Problem 12 *Suppose $X = [0, 1]$ and T is the function with domain $[0, \infty)$ and range the collection of all functions from X to X so that if $t \geq 0$, then*

$$T(t)x = \begin{cases} 0 & \text{if } t \geq 0 \text{ and } x - t \leq 0, \\ x - t & \text{if } t \geq 0 \text{ and } x - t > 0. \end{cases}$$

Show that T is a semigroup on X.

Problem 13 *Does there exist a generator for T in sense (ii)?*

Definition 4 *If X is a Banach space, and T is a semigroup on X, then T is called linear provided that for each $t \geq 0$, $T(t) \in L(X, X)$, i.e., $T(t)$ is a continuous linear transformation from X to X. Otherwise, T is called nonlinear.*

Definition 5 *If T is a linear semigroup on a Banach space X, then T is said to be continuous if T is continuous as a function from $[0, \infty) \to L(X, X)$, using the operator norm, that is,*

$$\lim_{t \to s} |T(t) - T(s)| = 0 \text{ if } s \geq 0,$$

where if $K \in L(X, X)$, $|K|$ is the least number $M \geq 0$ so that

$$\|Kx\|_X \leq M\|x\|_X,\ x \in X.$$

Problem 14 *Suppose $X = C([-1, 1])$, the space of all continuous functions from $[-1, 1]$ to R with norm*

$$\|f\|_X = \sup_{t \in [-1,1]} |f(t)|.$$

Denote by F the function such that

$$F(x) = \begin{cases} x, & \text{if } x \leq 0, \\ 2x, & \text{if } x > 0. \end{cases}$$

Define T on $[0, \infty)$ so that if $t \geq 0$ and $f \in X$, then

$$(T(t)f)(x) \;=\; F(t + F^{-1}(f(x))), \quad x \in [-1, 1],\ t \geq 0. \tag{2.5}$$

Show that T is a semigroup on $C([-1, 1])$.

Problem 15 *Does T in Problem 14 have a generator in the second sense?*

This example is due to G. F. Webb ([73]) and was important in the development of the theory of nonlinear semigroups.

Problem 16 *Suppose $X = \ell_2$, the Hilbert space of all sequences $\{x_k\}_{k=1}^{\infty}$ such that $\Sigma_{k=1}^{\infty} x_k^2 < \infty$ with*

$$\|x\|_X = (\Sigma_{k=1}^{\infty} x_k^2)^{1/2},\ x \in \ell_2.$$

Define the semigroup T on X so that

$$T(t)(x_1, x_2, x_3, \dots) \;=\; (e^{-t}x_1, , e^{-2t}x_2, e^{-3t}x_3, \dots),\ t \geq 0.$$

Show that T is strongly continuous and think about the possibility of a generator in one of the first two senses.

Problem 17 *Show that T in Problem 16, if $t > 0$, then*

$$T(t)^{-1} \text{ exists}$$

but is only densely defined and is nowhere continuous.

Problem 18 *Suppose $X = \ell_2$, the complex Hilbert space of all sequences $\{x_k\}_{k=1}^{\infty}$ such that $\Sigma_{k=1}^{\infty} x_k^2 < \infty$ with*

$$\|x\|_X = (\Sigma_{k=1}^{\infty} |x_k^2|)^{1/2},\ x \in \ell_2.$$

Define the semigroup T on X so that

$$T(t)(x_1, x_2, x_3, \dots) \;=\; (e^{-it}x_1, e^{-2it}x_2, e^{-3it}x_3, \dots),\ t \geq 0.$$

Show that T is strongly continuous and think about the possibility of a generator in one of the first two senses.

Problem 19 *Show that T in Problem 18 has the property that*

$$T(t)^{-1} \text{ exists}$$

and is continuous with domain all of X*, for all* $t \in R$*. This semigroup* T *is actually a group. It is fundamental to the study of the Schrödinger equation of quantum mechanics.*

Problem 20 *Show that neither of the semigroups in Problems 16 and 18 is continuous.*

Problem 21 *Suppose* $X = [0, \infty)$*. Find solutions* u *to*

$$u(0) = x \in X,\ u' = u^2. \tag{2.6}$$

Show that if $x \in X, x > 0$*, then no solution* u *(solution has 'blowup') to* (2.6) *exists on all of* X *(one might solve* (2.6) *in this case in closed form in order to observe this).*

Problem 22 *Following Problem 21 consider trying to make a semigroup that arises from this problem in the manner of (looking ahead) Problem 315. Articulate any difficulties you encounter. Consider possible modifications of Definition 1 that might enable Equation 2.6 to be included in a broad study of semigroup theory. Eventually see Chapter 19.*

Problem 23 *Suppose* $X = [0, \infty)$*. Find solutions* u *to*

$$u(0) = x \geq 0 \in X,\ u' = -u^2. \tag{2.7}$$

Show that (2.7) *has a solution on all of* $[0, \infty)$*. Exhibit a semigroup generated by* (2.7).