Chapter 19
Local Semigroups and Lie Generators

Some nonlinear differential equations do not generate a semigroup such as those that are the subject of previous problems. For example there is:

Problem 314 *Investigate solutions u to*

$$u(0) = x \geq 0,\ u' = u^2 \tag{19.1}$$

where for each $x \geq 0$, the interval of existence contains zero, is a subset of $[0, \infty)$ and is as long as possible.

Problem 315 *For the setting of Problem 314, define T so that*

- *If $t \geq 0$, then $T(t)$ is a function from a subset of $[0, \infty)$ into $[0, \infty)$.*
- *If $x \geq 0$, and x is in the domain of $T(t)$ if and only if there is a solution u to (19.1) so that t is in the domain of u satisfying (19.1).*
- *If $x \geq 0$, then $T(t)x = u(t)$ where u satisfies (19.1).*

Articulate as many properties of T as you can.

Definition 35 *Suppose that X is a complete separable metric space. The statement that T is a local semigroup on X means that*

- *There is ω so that $0 < \omega \leq \infty$.*
- *If $t < \omega$, then $T(t)$ is a function from a subset of X into X.*
- *$T(0)x = x,\ x \in X$.*
- *There is a function $m : X \to (0, \infty]$ such that $1/m$ is continuous and so that $x \in D(T(t)) \iff t \in [0, m(x))$.*
- *$m(x) < \infty$ for some $x \in X$.*
- *If $t, s \geq 0$ and $x \in X$, then*

$$T(t)T(s)x = T(t + s)x \iff t + s < m(x).$$

- *T is jointly continuous.*
- *T is maximal, i.e., if $s \in [0, \infty)$ and $\lim_{t \to s-} T(t)x$ exists, then $s < m(x)$.*

J.W. Neuberger, *A Sequence of Problems on Semigroups*,
Problem Books in Mathematics, DOI 10.1007/978-1-4614-0430-9_19,

Problem 316 *Show that T from Problem 315 is a local semigroup.*

Definition 36 *Suppose that T is a local semigroup on X. Then the Lie generator of T is*

$$A = \{(f, g) \in C(X)^2 : g(x) = \lim_{t \to 0+} \frac{1}{t}(f(T(t)x) - f(x)),\ x \in X.$$

Problem 317 *Calculate the Lie generator for T in Problem 315.*

For Problems 318 to 323, suppose that T is a local semigroup on X and A is its Lie generator.

Problem 318 *Show that A is a linear derivation, i.e., if $f, g \in D(A)$, then $fg \in D(A)$ and*

$$A(fg) = f(Ag) + (Af)g.$$

Problem 319 *Show that $D(A)$ is β-dense in $C(X)$.*

Problem 320 *Suppose that $\lambda > 0$. Define I_λ on $C(X)$ so that if $f \in C(X)$, then*

$$(I_\lambda f)(x) = \frac{1}{\lambda} \int_0^{m(x)} \exp(-j/\lambda) f(T(j)x),\ x \in X.$$

Show that

- *I_λ is a linear transformation from $C(X)$ to $C(X)$.*
- *$|I_\lambda| \leq 1$.*
- *$(I - \lambda A) I_\lambda f = f,\ f \in C(X)$.*

Problem 321 *Use assumptions and notation of Problem 320. Show that if $f \in D(A)$, then*

$$I_\lambda (I - \lambda A) f = f - g$$

for $g \in C(X)$ for

$$g(x) = \lim_{t \to m(x)-} \exp(-t/\lambda) f(T(t)x),\ x \in X.$$

Problem 322 *Suppose that g, λ, T, A, f are as in Problem 321. Show that*

$$(I - \lambda A)g = 0.$$

Show that if $f(y) > 0, y \in X$, then the resulting member $g \in C(X)$ is an eigenvector of A, if g is not zero.

Problem 323 *Using assumptions and notation of Problem 320, show that*

$$\lim_{n \to \infty} (I^n_{\lambda/n} f)(x) = f(T(\lambda)x),\ f \in C(X),\ x \in X.$$

Problem 324 *For T as in Problem 315, verify all the conclusions to Problems 318–323.*

Problem 325 *For T as in Problem 315 calculate the corresponding element g as in Problem 321 for various $f \in C(X)$.*

Problem 326 *Suppose T is a local semigroup. Define m as in Definition 35 and that*

$$f(x) = \exp^{-m(x)},\ x \in X.$$

Show that

$$Af = f.$$

Problem 327 *For f as in Problem 326, show that*

$$A(f^2) = 2f.$$

Problem 328 *Generalize the result of Problem 327.*

Problem 329 *Suppose T is a local semigroup and $\lambda > 0$. Define m as in Definition 35 and*

$$f(x) = \exp^{-\lambda m(x)},\ x \in X.$$

Then

$$Af = \lambda f.$$

Problem 330 *Show that the following is true: Suppose that T is either a local semigroup or a global semigroup and A is its Lie generator. Then T is global if and only if A has no positive eigenvalue.*

Problem 331 *Pick various systems of time-dependent autonomous differential, ordinary or partial differential equations that are known to generate either a local or global semigroup. For such systems, calculate a Lie generator. Use some of the results of the problems in this chapter to try to decide if corresponding semigroups are local or global. The problem of three-dimensional Navier–Stokes is such a problem, surely not an easy one.*

Problem 332 *Dream of a numerical attack corresponding to Problems 330 and 331. For various choices of X and T, make a discrete version of X, then a discrete version of the Lie generator A of T on a discrete version of $C(X)$. Then use numerics in order to test whether this discrete version of A might have a positive eigenvalue. Consider first testing for situations for which the answer is known.*

The following problems are late additions to the problem set. The 'dream' mentioned in Problem 332 has come a little closer to being realized. Within the past year, I have written a code in which X is one-dimensional, i.e., to test a single ordinary ODE for global or local existence in time. J. W. Swift and John M. Neuberger have written a code for cases in which X is two-dimensional. Preliminary results are encouraging.

Problem 333 *Write a MatLab code to test various one-dimensional examples for global versus local existence. Examples might include*

- $u' - u^2 = 0,\ X = [0, b),\ 0 < b \leq \infty.$
- $u' - u(1-u) = 0,\ X = [0, b),\ 0 < b \leq \infty.$
- $u' + u^2 = 0,\ X = [0, b),\ 0 < b \leq \infty.$

The eigenvalue–eigenvector routines from MatLab may be used.

Problem 334 *Consider the use of finite element approximations, first in one-, then two-dimensional problems in connection with Problem 333. Realize that in the present context, 'two-dimensional' problems refer to a system of two ordinary differential equations.*

Problem 335 *Consider the local–global existence problem for the partial differential equation which seeks, for some $w > 0$, a function u with domain $[0, w] \times [0, 1]$ so that*

$$u_1(t, x) = mu_{2,2}(t, x) + \epsilon|u(t, x)|^p \tag{19.2}$$

for some $p > 1$ and some $m > 0$. For boundary conditions, take

$$u(t, 0) = 0 = u(t, 1),\ t \in [0, w],$$

and

$$u(0, x) = f(x),\ x \in [0, 1],$$

for some given function f. Set up a corresponding finite-dimensional version and investigate the existence or nonexistence of eigenvectors of the relevant Lie generator in order to decide which of local or global describes the underlying semigroup.

Problem 336 *Realize that for Equation 19.2, for $\epsilon = 0$ this is just the heat equation, which has existence on $[0, w]$ for any $w > 0$ for reasonable f. Realize also that for $m = 0$, Equation 19.2 is essentially an uncountable system of ordinary differential equations. Consider that in a proper discretization of this problem the number of grid points will be very large. Estimate how many grid points it would take for such a proper discretization if results of this chapter are to be applied.*

Problem 337 *Now back to dreaming. Write down a time-dependent Navier–Stokes system in three space dimension. Make an estimate of the number of grid points needed for a reasonable finite-dimensional version of a transformation A for which a positive eigenvalue indicates only local existence and for which the lack of such an eigenvalue is indicative of global existence. Try to form an opinion as to whether any present-day computer is sufficient for this task. If your opinion is negative on this matter, is that sufficient reason for not working on this problem? Do you believe that the power of computers will increase in the future somewhat as they have in the past?*

Another instance where ‘local’ versus ‘global’ solutions is of interest is the following:

Problem 338 *Suppose that* n *is a positive integer,* F *is a polynomial function:* $C^n \to C^n$ *so that for each* $x \in C^n$*, the derivative* $F'(x)$ *has an inverse. Is it true that for each* $x \in C^n$ *there is* $z : [0,1] \to C^n$ *so that*

$$z(0) = x, \; z'(t) = -(F'(z(t))^{-1} F(x), \; t \in [0,1]?$$

Problem 339 *Show that an affirmative answer, for each such function* F *in Problem 338, implies the truth of:*

Jacobian Conjecture: Every *F* as in Problem 338 is a bijection.

See [72] and [50] for some background on the Jacobian Conjecture. A Web search yields an extensive literature on this problem. The problem is unsolved by anyone, if $n > 1$, so far as I know.