Chapter 18
Measures and Linear Extension of Nonlinear Semigroups

Notation from Chapter 17 holds and the following is added:

Definition 29 *Denote by $M(X)$ the set of all Borel measures μ on X and by $B(X)$ the set of all Borel subsets of X.*

Problem 300 *Show that $M(X)$ in Definition 29 is a representation of the second dual of $C(X)$ in Chapter 17. See [67] for more information.*

Definition 30 *Suppose*

$$\mu \in M(X) \text{ and its range is a subset of } [0, \infty).$$

We say that μ is compact regular if

$$\mu(\Omega) = \sup\{\mu(\Omega') : \Omega' \subset \Omega \text{ and } \Omega' \text{ is compact}\}.$$

Definition 31 *More generally, we also say that $\mu \in M(X)$ is compact regular if μ is the difference of two compact regular measures as in Definition 30 which have range in $[0, \infty)$. In this case we say that $\mu \in MCR(X)$.*

Definition 32 *Suppose T is a jointly continuous semigroup on X. We say that U extends T if U is a function with domain $[0, \infty)$ and*

$$(U(t)\mu)(\Omega) = \mu\{T(t)^{-1}\Omega\},\ t \geq 0,$$

with $\Omega \in B(X), \mu \in MCR(X)$.

Problem 301 *Show that U in Definition 32 is a semigroup on $MCR(X)$.*

Problem 302 *Show that U in Definition 32 is a linear semigroup on $MCR(X)$.*

J.W. Neuberger, *A Sequence of Problems on Semigroups*,
Problem Books in Mathematics, DOI 10.1007/978-1-4614-0430-9_18,

Definition 33 *For U in Definition 32, define*

$$C = \{(\mu,\nu) \in MCR(X)^2 : \int_X f\, d\nu = \lim_{t\to 0+} \int_X \frac{1}{t}(f(T(t)) - f)\, d\mu,\ f \in C(X)\}.$$

(The integrals are Lebesgue integrals.) We say that C is the extended generator of T.

Definition 34 *Denote by $C(X)^{*\beta}$ the linear space of all β- continuous linear functions g,*

$$g : C(X) \to R,$$

such that g is continuous in the β- sense.

Problem 303 *Suppose $\mu \in MCR(X)$ and $p : C(X) \to R$ such that*

$$pf = \int_X f\, d\mu,\ f \in C(X).$$

Show that

$$p \in C(X)^{*\beta}.$$

Problem 304 *Suppose $p \in C(X)^{*\beta}$. Show that there is $\mu \in MCR(X)$ such that*

$$pf = \int_X f\, d\mu,\ f \in C(X).$$

(It might be helpful, for this problem, to find some lemmas in [67].)

Problem 305 *For C given in Definition 33, show that if $\lambda > 0$, then*

$$(I - \lambda C)^{-1} \ \text{exists}$$

with domain all of $MCR(X)$ and

$$|\int_X f\, d((I-\lambda C)^{-1}\mu)| \le |f|\,|\int_X |d\mu|,\ f \in C(X).$$

Problem 306 *For C as in Definition 33, show that if $\lambda \ge 0$, then*

$$\lim_{n\to\infty} \int_X f\, d((I-\lambda C)^{-n}\mu) = \int_X f\, d(U(\lambda)\mu),\ f \in CB(X).$$

Problem 307 *For C in Definition 33, show that there exists $A \in LG(X)$ such that*

$$\int_X f\, d(C\mu) = \int_X Af\, d\mu. \tag{18.1}$$

Problem 308 *Suppose* $A \in LG(X)$ *(see Definition 28) and* C *a linear transformation*

$$MCR(X) \to MCR(X)$$

so that (18.1) *is satisfied. Show that there exists a unique semigroup on* X *that has* C *as its extended generator.*

Problem 309 *Make a theorem which summarizes Problems 301–308.*

Problem 310 *Generalize results of the previous chapter to include the setting of the present chapter.*

Problem 311 *Refer to Problem 14 for a definition of a certain semigroup* T *on* $X = C([-1,1])$*. Review your work on these two problems. Try to conclude that the domain of the conventional generator of* T *contains only functions that are nonnegative or else are negative. Contemplate difficulties in using such a generator for purposes of recovering* T *in terms of it. Is* C *of Definition 33 a more promising choice of generator than that from Problem 14? See Notes to the present chapter to further understand the importance of this example.*

Problem 312 *For* T *as in Problem 14, calculate a generator* A *for* T *as in Chapter 17. In particular, determine* S *on* $C(X)$ *so that*

$$(S(t)f)(x) = f(T(t)x),\ t \geq 0,\ x \in X,$$

being as concrete as possible. Try to make an analysis of T *in terms of* A*.*

Problem 313 *Relate* C *of Problem 311 to* A *of Problem 312.*