Chapter 17
Nonlinear Semigroups Studied by Linear Methods

In this chapter, denote by X a separable complete metric space, that is to say, a Polish space.

Definition 22 *Suppose T is a semigroup on X. The statement that T is jointly continuous means that if*

$$g : [0, \infty) \times X \to X \tag{17.1}$$

such that

$$g(t, x) \ = \ T(t)x, \ t \in [0, \infty), \ \ x \in X,$$

then g is continuous.

Definition 23 *$C(X)$ denotes the Banach space of all bounded and continuous functions $f : X \to R$. The norm in $C(X)$ is given by*

$$\|f\|_{C(X)} \ = \ \sup_{x \in X} |f(x)|.$$

Problem 272 *Suppose T is a jointly continuous semigroup on X and*

$$S : [0, \infty) \to \ \textit{set of transformations on } C(X)$$

such that

$$(S(t)f)(x) = f(T(t)x), \ t \geq 0, x \in X, f \in C(X).$$

Show that if $t \geq 0$, then

$$S(t) \in L(C(X), C(X))$$

and

$$|S(t)| \leq 1.$$

S is called a representation of T.

Problem 273 *Show that S is a linear semigroup on $C(X)$.*

J.W. Neuberger, *A Sequence of Problems on Semigroups*,
Problem Books in Mathematics, DOI 10.1007/978-1-4614-0430-9_17,

Problem 274 *Find an example of a semigroup S as in Problem 273 such that S is not strongly continuous under the sup norm of $C(X)$.*

Here is a second topology for $C(X)$. It is presented by means of a notion of convergence.

Definition 24 *We say that the sequence $f_1, f_2, \ldots$ in $C(X)$ β-converges to $f \in C(X)$ if*

$$\text{there exists } M > 0 \text{ such that } \|f_n\|_{C(X)} \le M, \; n = 1, 2, \ldots,$$

and also $f_1, f_2, \ldots$ converges uniformly to f on each compact subset of X. In this case we write

$$\beta - \lim_{n \to \infty} f_n = f.$$

Problem 275 *Show that S in Problem 273 is β-strongly continuous in the sense that if*

$$f \in C(X) \text{ and } \{t_n\}_{n=1}^{\infty} \text{ in } (0, \infty), \text{ converges to } 0,$$

then

$$\{S(t_n)f\}_{n=1}^{\infty} \;\beta\text{-converges to } f.$$

Problem 276 *Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence in $C(X)$, β-convergent to $f \in C(X)$, S as in Problem 273 and $t \ge 0$. Show that*

$$\{S(t)f_n\}_{n=1}^{\infty} \text{ is } \beta\text{-convergent to } S(t)f.$$

Definition 25 *For T a jointly continuous semigroup on X, the Lie generator of T is*

$$A = \{(f, g) \in C(X)^2 : g(x) = \lim_{t \to 0+} \frac{1}{t}(f(T(t)x) - f(x)), \; x \in X\}.$$

Problem 277 *Suppose that T is the semigroup on $[0, \infty)$ so that*

$$T(t)x = \frac{x}{1 + tx}, \; t \ge 0, x \ge 0.$$

Calculate the Lie generator A of T.

Problem 278 *Take additional examples of jointly continuous semigroups on a Banach space and calculate their Lie generators.*

Problem 279 *Show that the generator A in Definition 25 is a derivation in the sense that if $f, g \in D(A)$, then $fg \in D(A)$ and*

$$A(fg) = f(Ag) + (Af)g.$$

Definition 26 *For $\lambda > 0$ and T as in Problem 273, define a transformation I_λ with domain $C(X)$ such that*

$$(I_\lambda f)(x) \ = \ \frac{1}{\lambda}\int_0^\infty e^{-s/\lambda} f(T(s)x)\, ds, \ f \in C(X), \ x \in X.$$

Problem 280 *Show that if $\lambda > 0$, then $(I - \lambda A)I_\lambda = I$.*

Problem 281 *Show that if $\lambda > 0$ and f is in the domain of A, then*

$$I_\lambda(I - \lambda A)f = f.$$

Problem 282 *Show that if $\lambda > 0$,*

$$(I - \lambda A)^{-1}$$

exists and

$$(I - \lambda A)^{-1} \ = \ I_\lambda.$$

Problem 283 *Show that if λ and A are as in Problem 282, then*

$$\|(I - \lambda A)^{-1} f\|_{C(X)} \ \le \|f\|_{C(X)}, \ f \in C(X).$$

Problem 284 *With A as in Problem 283 and $\{t_n\}_{n=1}^\infty$ a sequence in $(0, \infty)$ convergent to 0, show that if $n \in Z^+$, $\lambda \ge 0$ and $x \in X$, then*

$$((I - t_n A)^{-n} f)(x) = \int_0^\infty f(T(\cdot)x)\, d\phi_{\lambda,n}$$

where $\phi_{\lambda,n}$ is as in Problem 65.

Problem 285 *With A as in Problem 283 and $\{t_n\}_{n=1}^\infty$ a sequence in $(0, \infty)$ convergent to 0, show that*

$$\beta - \lim_{n\to\infty} (I - t_n A)^{-1} f \ = \ f,$$

and consequently, A has dense domain in the β-sense.

Problem 286 *With A and T given in this chapter and $\lambda > 0$, show that*

$$\beta - \lim_{n\to\infty} (I - \frac{\lambda}{n} A)^{-n} f \ = \ f(T(\lambda)), \ f \in C(X).$$

Definition 27 *A set Q of transformation from $C(X)$ to $C(X)$ is β-equicontinuous if for each sequence $\{f_n\}_{n=1}^\infty \in Q$, β-convergent to $f \in C(X)$, there is $M > 0$ such that*

$$\|W f_n\|_{C(X)} \le M, \ W \in Q, \ n = 1, 2, \ldots,$$

and if Ω is a compact subset of X and $\epsilon > 0$, there is N such that if $n > N$,

$$|(Wf_n)(x) - (Wf)(x)| < \epsilon,\ x \in \Omega,\ W \in Q.$$

Problem 287 *Show that if $\eta > 0$, then*

$$\{(I - \lambda/nA)^{-n},\ 0 \leq \lambda \leq \eta,\ n = 1, 2, \ldots\}$$

is β-equi-continuous.

Definition 28 *Denote by $LG(X)$ the set of all linear transformations A with domain and range in $C(X)$ which have the properties given in Problems 279, 283, 285, 287, that is,*

- *A is a derivation.*
- *The domain of A is dense in $C(X)$ in the β-sense.*
- *If $\lambda \geq 0$, $(I - \lambda A)^{-1}$ exists and is a contraction in $L(C(X), C(X))$.*
- *If $\eta \geq 0$, $\{(I - (\lambda/n)A)^{-n},\ 0 \leq \lambda \leq \eta, n = 1, 2, \ldots\}$ is β-equi-continuous.*

Denote by $LG(X)$ the collection of all such transformations A satisfying the above items.

Problem 288 *Make a theorem from Problems 273–287.*

For Problems 289–292 suppose that $A \in LG(X)$. Define

$$I_\lambda = (I - \lambda A)^{-1} \text{ for } \lambda > 0.$$

Problem 289 *Show that there is a linear semigroup $\overline{V}$, which is strongly continuous (in norm of $C(X)$ but on $\overline{D(A)}$), such that*

$$\overline{V}(\lambda)f = \lim_{n\to\infty} (I_{\lambda/n})^{-n} f,\ \lambda > 0, f \in D(A).$$

Problem 290 *Show that there is a unique β-continuous extension S of $\overline{V}$, to all of $C(X)$.*

Problem 291 *Show that S in Problem 290 is a β-continuous linear semigroup on all of $C(X)$.*

Problem 292 *Show that if S as in Problem 290, then*

$$A = \{(f, g) \in C(X) : g = \beta - \lim_{t\to\infty} \frac{1}{t}(S(t)f - f)\}.$$

Problem 293 *Suppose that $z \in X$ and $\mu : C(X) \to R$ is such that*

$$\mu(f) = f(z),\ f \in C(X).$$

Show that μ *is linear,* β*-continuous and that*

$$\mu(f)\mu(g) = \mu(fg),\ f, g \in C(X),$$

i.e., that μ *is a* β*-continuous multiplicative linear functional on* $C(X)$*.*

Problem 294 *For* S *as in Problem 290,* $t \geq 0$*,* $x \in X$ *and* μ *with domain* $C(X)$ *such that*

$$\mu(f) \;=\; (S(t)f)(x),\ f \in C(X),$$

show that μ *is a transformation as in Problem 293.*

Problem 295 *Show that if* μ *is a nonzero* β*-continuous multiplicative linear functional on* $C(X)$*, then there is a unique point* $z \in X$ *such that*

$$\mu(f) = f(z) \text{ for all } f \in C(X).$$

See [14], in particular the result there due to J. Lawson.

Problem 296 *Define* T *with domain* $[0, \infty)$ *so that if* $t \geq 0$*, then* $T(t)$ *is the transformation with domain* X *and range in* X *such that if* $x \in X$*, then* $T(t)x$ *is the element* $z \in X$ *such that*

$$(S(t)f)(x) = f(z),\ f \in C(X).$$

Show that T *is a semigroup on* X*.*

Problem 297 *Show that* T *in Problem 296 is jointly continuous.*

Problem 298 *Show that the Lie generator* A *of* T *satisfies the equation in Definition 25.*

Problem 299 *Write an essay summarizing results of this chapter.*