Chapter 16
Semigroups and Families of Sobolev Spaces

Denote by H a Hilbert space. Denote by H' a Hilbert space whose points form a dense subset of H so that

$$\|x\|_{H'} \geq \|x\|_H, \; x \in H'.$$

Problem 240 *Show that*

$$H = L_2([0,1]), \; H' = H^{1,2}([0,1]) \tag{16.1}$$

provides an example of the above setting (see [1] for definitions of the spaces in (16.1)*) or see Definition 16 in Chapter 14.*

Problem 241 *Returning to the general case (not just the example in Problem 240), suppose $y \in H$ and*

$$f(x) = \langle x, y \rangle_H, \; x \in H.$$

Denote by g the restriction of f to H'. Show that g is in $(H')^*$, the dual space of H'.

Problem 242 *Referring to either Problem 99 or 205, show that there is $z \in H'$ so that*

$$g(x) = \langle x, z \rangle_{H'}, \; x \in H'.$$

Denote by M the transformation so that if $y \in H$ as in Problem 241, then

$$My = z$$

where z is as in Problem 242.

M is called the embedding map between the spaces H and H'.

J.W. Neuberger, *A Sequence of Problems on Semigroups*,
Problem Books in Mathematics, DOI 10.1007/978-1-4614-0430-9_16,

Problem 243 *Show that M above is linear.*

Problem 244 *Show that the range of M is dense in H.*

Problem 245 *Show that M^{-1} exists.*

Problem 246 *Show that*

$$\langle Mx, x\rangle_H > 0,\ x \in H, x \neq 0.$$

(That is, M is positive as a member of $L(H,H)$.)

Problem 247 *Show that*

$$|M|_{L(X,Y)} \leq 1$$

where

$$(X,Y) \text{ is any of the pairs } (H,H),(H,H'),(H',H),(H',H').$$

Problem 248 *Show that*

$$\langle x, My\rangle_H = \langle Mx, y\rangle_H,\ x, y \in H.$$

Problem 249 *Show that*

$$\langle x, My\rangle_{H'} = \langle Mx, y\rangle_{H'},\ x, y \in H'.$$

Problem 250 *Show that M, considered as a member of $L(H,H)$, has a unique positive symmetric square root (denoted by $M^{1/2}$) See [66], for example.*

Problem 251 *Show that M, considered as a member of $L(H',H')$, has a unique positive symmetric square root and this square root agrees, on H', with the square root in Problem 250.*

Problem 252 *Show that if $x \in H'$, then*

$$\|x\|_H = \|M^{1/2}x\|_{H'}.$$

Problem 253 *Show that if $x \in H$, then*

$$\|x\|_H = \|M^{1/2}x\|_{H'}.$$

Problem 254 *Using Problem 102, and perhaps some material from the Notes to Chapter 7, show that if $\lambda \geq 0$, then there is a unique symmetric, positive transformation*

$$M^\lambda$$

so that if $\alpha, \beta \geq 0$, then

$$M^\alpha M^\beta = M^{\alpha+\beta}$$

and if $\alpha = \frac{m}{n}$, then

$$M^\alpha = (M^m)^{1/n}.$$

Problem 255 *Suppose that*

$$T(\lambda) = M^{\lambda},\ \lambda \geq 0.$$

Show that T is a strongly continuous linear semigroup of contractions on H.

Problem 256 *Show that if $\lambda > 0$, then*

$$M^{-\lambda} \text{ exists}$$

(but perhaps is not continuous).

Problem 257 *If $\lambda > 0$, define*

$$H_{\lambda} = \text{ range of } M^{\lambda/2}$$

with

$$\|x\|_{H_{\lambda}} = \|M^{-\lambda/2}x\|_H,\ x \in H_{\lambda}.$$

Show that H_{λ} is a Hilbert space.

Problem 258 *For D as in Problem 235 show that*

$$(D^t D)^{-1} = M$$

where M is derived by the choice of

$$H = R^{n+1},\ H' = S_n.$$

S_n is as in Problem 234.

Problem 259 *Consider M derived from H, H' as in Problem 240. Show that the usual norm (see [1]) for $H^{2,2}([0,1])$ is equivalent to the norm of H_2, as defined above for the pair (H, H').*

Problem 260 *Make a substantial generalization of Problem 259.*

Problem 261 *Consider practical numerical consequences of Problem 260. For example, let if D as in Problem 258 be replaced by a discrete version of a second-order derivative operator E. Compare the coding effort for the resulting $(E^t E)^{-1}$ with $(D^t D)^{-2}$.*

Problem 262 *Suppose that $n \in Z^+$, $C \in L(R^n, R^n)$ and*

$$\phi(x) = \frac{1}{2}\|\begin{pmatrix} x \\ Cx \end{pmatrix}\|^2,\ x \in R^n.$$

Find

$$\phi'(x),\ x \in R^n.$$

Problem 263 *For ϕ as in Problem 262, find the gradient of ϕ, that is, the function*

$$\nabla\phi : R^n \to R^n$$

such that

$$\phi'(x)h = \langle h, (\nabla\phi)(x)\rangle_{R^n},\ x, h \in R^n.$$

Problem 264 *For ϕ as in Problem 263, find an explicit expression for $x \in R^n$ which minimizes $\phi(x)$.*

Problem 265 *Suppose that each of H, K is a Hilbert space and T is a closed, densely defined linear transformation on H into K. Show that the orthogonal projection onto*

$$\{\begin{pmatrix} x \\ Tx \end{pmatrix} : x \in D(T)\}$$

is given by

$$P = \begin{pmatrix} (I + T^tT)^{-1} & T^t(I + TT^t)^{-1} \\ T(I + T^tT)^{-1} & I - (I + TT^t)^{-1} \end{pmatrix}, \qquad (16.2)$$

a formula of von Neumann [70]. (Note Problems 103–107 in Chapter 7.)

Problem 266 *For T as in Problem 265 denote by H' the space whose points are exactly those of $D(T)$ with*

$$\|x\|_{H'} = \|\begin{pmatrix} x \\ Tx \end{pmatrix}\|_{H\times K},\ x \in H'.$$

Show that

$$T \in L(H', K)$$

provided T is considered as a linear transformation from $H' \to K$.

Problem 267 *For T as in Problem 266, show that for $x \in H', y \in K$,*

$$\langle Tx, y\rangle_K = \langle \begin{pmatrix} x \\ Tx \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\rangle_{H\times K}$$
$$= \langle \begin{pmatrix} x \\ Tx \end{pmatrix}, P\begin{pmatrix} 0 \\ y \end{pmatrix}\rangle_{H\times K}.$$

Problem 268 *Following Problem 267 conclude that*

$$T^*y = \pi P\begin{pmatrix} 0 \\ y \end{pmatrix},\ y \in K,$$

where P is as in Problem 265 and

$$\pi\begin{pmatrix} r \\ s \end{pmatrix} = r,\ r \in H, s \in K.$$

Also conclude that

$$\langle Tx, y\rangle_K = \langle x, T^*y\rangle_{H'}.$$

Problem 269 *For H, K, H' in Problem 266 define*

$$M = (I + T^t T)^{-1},$$

and show that M is the embedding operator for the pair (H, H').

Problem 270 *Review Chapter 14 discerning how developments of the present chapter can give more concrete expressions for the various Sobolev gradients in Chapter 14 and hence more concrete expressions for the semi-groups of steepest descent there.*

Problem 271 *Take T, H, K, H' as in Problem 266. Write an essay organiz-ing your thoughts on the following:*

With T a closed densely defined linear transformation on H to K, T has an adjoint T^t which is a closed densely defined linear transformation on K to H so that

$$\langle Tx, y\rangle_K = \langle x, T^t y\rangle_H, \; x \in D(T), y \in D(T^t).$$

On the other hand, this same transformation T, the same collection of ordered pairs, considered as a member of $L(H', K)$, T has an adjoint T^* so that

$$\langle Tx, y\rangle_K = \langle x, T^* y\rangle_{H'}, \; x \in H', y \in K.$$

In your essay, comment on how one transformation may have two adjoints, one everywhere discontinuous and the other continuous. Investigate how there may be a different adjoint for T corresponding to any space H'' with the same points as H' but with a norm $\|\cdot\|_{H''}$ different from that of H' but equivalent to it in the sense that there are $m, M > 0$ such that

$$m\|x\|_{H''} \le \|x\|_{H'} \le M\|x\|_{H''}, \; x \in H'.$$

Comment on how linguistic and notational ambiguities might present a hin-drance to someone trying to deal with Sobolev gradients and matters of the present chapter. Can you think of a more rational notation that presents less of a hindrance?