Chapter 15
Numerics for Semigroups of Steepest Descent

At first we work with some numerical problems. We use the same example as in Chapter 14 but in a discrete form.

Suppose $n > 2$ is an integer. Define

$$\delta \;=\; 1/n.$$

Suppose

$$\phi_n : R^{n+1} \to R$$

such that

$$\phi_n(u_0, u_1, \ldots, u_n) \;=\; \frac{1}{2}\sum_{k=0}^{n}(\frac{u_k - u_{k-1}}{\delta} - \frac{u_k + u_{k-1}}{2})^2,$$

with $(u_0, u_1, \ldots, u_n) \in R^{n+1}$.

Problem 228 *Find a formula for*

$$\nabla\phi_n,$$

the conventional gradient of ϕ_n, that is, a vector of length $n+1$ of partial derivatives of ϕ_n.

Problem 229 *Show that*

$$\phi_n'(u)h \;=\; \langle h, (\nabla\phi_n)(u)\rangle_{R^{n+1}},\; h, u \in R^{n+1}. \tag{15.1}$$

Problem 230 *If*

$$u \;=\; (u_0, u_1, \ldots, u_n) \in R^{n+1},$$

find the unique number $\alpha_{n,u}$ such that

$$\phi_n(u - \alpha_{n,u}(\nabla\phi_n)(u)) \text{ is minimum.}$$

(Find an explicit expression using the usual inner product in R^{n+1}.)

J.W. Neuberger, *A Sequence of Problems on Semigroups*,
Problem Books in Mathematics, DOI 10.1007/978-1-4614-0430-9_15,

Problem 231 *Show that the iteration*

$$u \;\to\; u - \alpha_{n,y}(\nabla\phi_n)(u) \tag{15.2}$$

converges to the limit u *and that*

$$\phi_n(u) \;=\; 0.$$

Problem 232 *Write a computer program which follows the iteration in Problem 231. (Choose* $n = 10, n = 20, n = 100$ *and print or graph your results.)*

Problem 233 *Run your code developed in Problem 232. Notice that your code requires many iterations if the integer* n *is even as much as* 20*. Reflect on the First Law of Numerical Analysis, just before Problem 89. Reflect also Problems 83, 84. What is going on?*

Definition 18 *Define a second norm on* R^{n+1}*, called* $\|\cdot\|_{S_n}$*:*

$$\|u\|_{S_n} \;=\; (\sum_{k=1}^{n}(\frac{u_k - u_{k-1}}{\delta})^2 + (\frac{u_k + u_{k-1}}{2})^2)^{1/2},$$

imitating the norm in $H^{1,2}([0,1])$ *in* (14.6).

Definition 19 *Define two linear transformations* D_0, D_1

$$D_0, D_1 : R^{n+1} \to R^n$$

such that if

$$u \;=\; (u_0, u_1, \ldots, u_n) \;\in R^{n+1},$$

then

$$D_0 u \;=\; \{\frac{u_1 + u_0}{2}, \ldots, \frac{u_n + u_{n-1}}{2}\}$$

and

$$D_1 u \;=\; \{\frac{u_1 - u_0}{\delta}, \ldots, \frac{u_n - u_{n-1}}{\delta}\}.$$

Definition 20

$$\nabla_{S_n}\phi_n$$

is the function $R^{n+1} \to R^{n+1}$ *so that*

$$\phi_n'(u)h \;=\; \langle h, (\nabla_{S_n}\phi_n)(u)\rangle_{S_n},\; h, u \in R^{n+1}.$$

(One can represent the linear function $R^{n+1} \to R$ *in any inner product defined on* R^{n+1}*.)*

Definition 21 *Suppose* D *is the transformation*

$$R^{n+1} \to (R^n)^2$$

such that

$$Du = \begin{pmatrix} D_0 u \\ D_1 u \end{pmatrix}, \; u \in R^{n+1}.$$

Problem 234 *Show that if $u, v \in R^{n+1}$, then*

$$\langle u, v \rangle_{S_n} \;=\; \langle Du, Dv \rangle_{(R^n)^2}.$$

Problem 235 *Suppose $u \in R^{n+1}$. For*

$$(\nabla_{S_n} \phi_n)(u)$$

given in Definition 20, show that

$$(\nabla_{S_n} \phi_n)(u) \;=\; (D^t D)^{-1} (\nabla \phi_n)(y),$$

where $(\nabla \phi_n)(u)$ is the conventional gradient of ϕ_n at u.

Problem 236 *Suppose that H is a Hilbert space, S is a closed subspace of H and $P \in L(H, H)$. Show that P is the orthogonal projection of H onto S if and only if the following four conditions hold:*

- $P^2 = P$.
- $\langle Px, y \rangle_H = \langle x, Py \rangle_H$ *for all* $x, y \in H$.
- *The range of P is a subset of S.*
- *If x is in the range of P, then $Px = x$.*

Problem 237 *Suppose P is the projection of $(R^n)^2$ onto the range of D. Show that*

$$P \;=\; D(D^t D)^{-1} D^t.$$

Problem 238 *Write a computer program which follows the iteration with the gradient in Problem 235. (Choose $n = 10, n = 20, n = 100$ and print your results.)*

Problem 239 *Make a comparison with the results from your code from Problem 238 with results from your code in Problem 232. Reflect further on the First Law of Numerical Analysis.*