Chapter 14
Semigroups of Steepest Descent for Differential Equations

The first problem in this chapter seeks to make the point that for a given linear transformation A on a finite-dimensional space to itself, an adjoint for A depends on a choice of inner products, one for the domain space and one for the range space.

Problem 204 *Suppose that* $A \in L(R^2, R^2)$ *defined by*

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 4y \end{pmatrix}, \; \begin{pmatrix} x \\ y \end{pmatrix} \in R^2.$$

Suppose also that in addition to the standard inner product $\langle \cdot, \cdot \rangle_{R^2}$*, one has a second inner product* $\langle \cdot, \cdot \rangle_S$ *defined by*

$$\langle \begin{pmatrix} r \\ s \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle_S = \langle \begin{pmatrix} r \\ s \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \rangle_{R^2} + (r - s)(u - v), \; \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} r \\ s \end{pmatrix} \in R^2.$$

Find a linear transformation $B \in L(R^2, R^2)$ *such that*

$$\langle A\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} r \\ s \end{pmatrix} \rangle_{R^2} = \langle \begin{pmatrix} u \\ v \end{pmatrix}, B\begin{pmatrix} r \\ s \end{pmatrix} \rangle_S, \; \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} r \\ s \end{pmatrix} \in R^2.$$

For the remainder of this chapter H denotes a Hilbert space. There are two objectives for the problems in this chapter. One is to describe an important class of semigroups. The other is to introduce a theory of steepest descent for partial differential equations.

Problem 205 *Show that if* f *is a continuous linear function from* H *to* R *(that is to say, a member of the dual space* H^* *of* H*), then there is a unique* $y \in H$ *so that*

$$f(x) \; = \; \langle x, y \rangle_H, \; x \in H. \tag{14.1}$$

J.W. Neuberger, *A Sequence of Problems on Semigroups*,
Problem Books in Mathematics, DOI 10.1007/978-1-4614-0430-9_14,

Definition 15 *Suppose that ϕ is a C^1 function from $H \to R$. The gradient of ϕ is the function $\nabla\phi : H \to R$ so that*

$$\phi'(x)h \;=\; \langle h, (\nabla\phi)(x)\rangle_H,\; x, h \in\; H.$$

For the rest of this chapter we suppose that the gradient $\nabla\phi$ is defined on all of H and $\nabla\phi$ is locally lipschitz, that is, if $x \in H$ there is $\delta, M > 0$ such that

$$\|(\nabla\phi)(w) - (\nabla\phi)(y)\|_H \;\leq\; M\|w - y\|_H$$

if $\|w - x\|, \|y - x\| \leq \delta$.

Problem 206 *Suppose that $w > 0$, $x \in H$ and $z : [0, w] \to H$ so that*

$$z(0) = x,\; z'(t) = -(\nabla\phi)(z(t)),\; t \in [0, w].$$

Show that

$$(\phi(z))'(t) \;=\; -\|(\nabla\phi)(z(t))\|^2,\; t \in [0, w].$$

Problem 207 *Show that, given $x \in H$, there is a unique function $z : [0, \infty) \to H$ such that*

$$z(0) = x,\; z'(t) \;=\; -(\nabla\phi)(z(t)),\; t \in [0, \infty). \qquad (14.2)$$

Denote by T_ϕ the semigroup generated by (14.2), i.e., if $x \in H$ and $s \geq 0$, then

$$T_\phi(s)x = z(s),$$

where z satisfies (14.2).

Problem 208 *Show that if $x \in H$ and*

$$u \;=\; \lim_{t\to\infty} T_\phi(t)x \text{ exists}, \qquad (14.3)$$

then

$$(\nabla\phi)(u) = 0.$$

The study of limits in (14.3) is very important in the theory of semigroups. For many problems in the theory of differential equations in variational form (represented by a function ϕ) the critical points of ϕ are the solutions to the problem. In the notes in the last chapter there are additional references to applications. If a system of equations does not arise from a conventional variational form, one may often construct a function ϕ such that its zeros are solutions. This is illustrated by means of the following sequence of problems devoted to one of the simplest possible examples cast into a variational form: Find u with domain $[0, 1]$ so that

$$u' - u = 0. \qquad (14.4)$$

We know that u satisfies (14.4) if and only if there is $c \in R$ such that

$$u(t) \;=\; ce^t,\; t \in [0,1]$$

but it is good to study a new method in a simple known case.

We can try to place (14.4) in a variational form by introducing ϕ such that

$$\phi(u) \;=\; \frac{1}{2}\int_0^1 (u' - u)^2,\; u \in H. \tag{14.5}$$

But how do we choose H?

Problem 209 *Show that if $H = L_2([0,1])$, then ϕ has as its domain a linear set only dense in H. In addition, show that ϕ is nowhere continuous.*

Problem 210 *What do you think about the choice $H = L_2([0,1])$ for a space on which to try to minimize Φ? Would a useful gradient be forthcoming if this choice is made?*

The next problems introduce a Sobolev space which will serve us well. It is the simplest example of a Sobolev space, but the ideas in these problems carry over to very general cases. This provides an alternate, but equivalent, definition for $H^{1,2}([0,1])$ given in Chapter 7.

Denote by G_1 the set

$$\{\begin{pmatrix} u \\ u' \end{pmatrix} : u \in C^1([0,1])\}.$$

Problem 211 *Show that G_1 is a linear subspace of $L_2([0,1])^2$ with norm*

$$\|\begin{pmatrix} f \\ g \end{pmatrix}\|_{L_2([0,1])^2} = (\|f\|^2 + \|g\|^2)^{1/2},\; f,g \in C^1([0,1]). \tag{14.6}$$

Problem 212 *Denote by G_2 the closure, in $L_2([0,1])^2$, of G_1. Show that there are not two members of G_2 with the same first term.*

Definition 16

$$H = H^{1,2}([0,1])$$

denotes the space of all first terms of members of G_2. If $f \in H$ with $\binom{f}{g} \in G_2$, write f' for g and say that g is the generalized derivative of f. For norm in this space H take

$$\|f\|_H \;=\; (\|f\|^2_{L_2([0,1])} + \|f'\|^2_{L_2([0,1])})^{1/2}.$$

Problem 213 *Can one justify defining g in the above definition as f'? Show that if $f \in C^1([0,1])$, then this definition is consistent with the usual one.*

Problem 214 *Show that if* $f \in H^{1,2}$*, then for some number* c*,*

$$f(x) = c + \int_0^x f', \ x \in [0,1].$$

Problem 215 *Show that* ϕ *in* (14.5) *is continuous on* H*.*

Problem 216 *Find* ϕ' *for* ϕ *as in* (14.5)*.*

We want to find an expression for $\nabla\phi$ where ϕ is defined in (14.5).

Problem 217 *Show that if*

$$\begin{pmatrix} w \\ v \end{pmatrix} \in (L_2([0,1]))^2$$

and

$$\langle \begin{pmatrix} u \\ u' \end{pmatrix}, \begin{pmatrix} w \\ v \end{pmatrix} \rangle_{L_2([0,1])^2} = 0, \ \begin{pmatrix} u \\ u' \end{pmatrix} \in G_2,$$

then

$$v \in H, \ w = v' \text{ and } v(0) = 0 = v(1).$$

Problem 218 *Construct the projection* P *of all of* $L_2([0,1])^2$ *onto*

$$\{ \begin{pmatrix} u \\ u' \end{pmatrix} : u \in H \}.$$

To do this, take $\binom{f}{g} \in L_2([0,1])^2$*. Seek* $u \in H$ *such that*

$$\| \begin{pmatrix} u \\ u' \end{pmatrix} - \begin{pmatrix} f \\ g \end{pmatrix} \|^2_{L_2([0,1])}$$

is minimum, that is, seek $u, v \in H$ *such that*

$$\begin{pmatrix} u \\ u' \end{pmatrix} + \begin{pmatrix} v' \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \text{ and } v(0) = 0 = v(1). \tag{14.7}$$

In the notes there are references which contain extensive information on projections which one encounters in the construction of Sobolev gradients.

Problem 219 *Solve the system* (14.7)*. Note that, thanks to Problem 217, there is only one pair* (u, v) *which satisfies* (14.7)*. Define*

$$S(t) = \sinh(t); \ C(t) = \cosh(t), \ t \in R,$$

show that

$$u(t) = [C(1-t)\int_0^t (C(r)f(r) + S(r)g(r))\,dr +$$
$$C(t)\int_t^1 (C(1-r)f(r) - S(1-r)g(r))\,dr]/S(1),\ t \in [0,1].$$

Problem 220 *With ϕ as in (14.5), P as in Problem 218 and*

$$\pi : L_2([0,1])^2 \to L_2([0,1])$$

defined by

$$\pi\begin{pmatrix} f \\ g \end{pmatrix} = f,\ \begin{pmatrix} f \\ g \end{pmatrix} \in L_2([0,1])^2,$$

show that

$$(\nabla\phi)(y) = \pi P\begin{pmatrix} y - y' \\ y' - y \end{pmatrix},\ y \in H. \qquad (14.8)$$

Problem 221 *Find a simple form for $\nabla\phi$ in Problem 220.*

Problem 222 *Search for an expression for the solution z of (14.2) using the gradient in Problem 221 and search for a form for u in (14.3).*

Problem 223 *Show that in Problem 221 a limit depends on the selection of $z(0)$ in (14.2).*

Problem 224 *Show that a limit u in Problem 221 is the nearest element to $z(0)$ in the norm of $H^{1,2}([0,1])$ where z is as in (14.2).*

Usually one cannot find an explicit form for the Sobolev gradient for ϕ. For many cases one can use the above ideas to try to prove that the limit u in (14.3) exists and is a zero of ϕ. In addition, one may try to follow a trajectory z numerically. Some of the problems which follow deal with existence of the limit u in (14.3).

Definition 17 *Suppose ϕ is a function from a Hilbert space H into $[0,\infty)$ of class C^1 with a locally lipschitzian gradient and $\Omega \subset H$. One says that ϕ satisfies a gradient inequality on Ω if there is $c > 0$ such that*

$$\|(\nabla\phi)(x)\|_H \geq c(\phi(x))^{1/2}, \quad \text{if } x \in \Omega. \qquad (14.9)$$

Problem 225 *Suppose H is a Hilbert space, ϕ a function from H to $[0,\infty)$ such that $\nabla\phi$ is locally lipschitz, $x \in H$, and z the unique solution of*

$$z(0) = x,\ z'(t) = -(\nabla\phi)(z(t)),\ t \geq 0.$$

Suppose also that $\Omega \subset H$ *is such that* ϕ *satisfies a gradient inequality* (14.9) *in* Ω *with constant c. Show that if*

$$range(z) \subset \Omega,$$

then

$$(\phi(z))'(t) \;\le\; -c^2\phi(z(t)),\; t \ge 0. \tag{14.10}$$

Problem 226 *Show that if* (14.10) *holds, then*

$$\phi(z(t)) \;\le \phi(z(a))e^{-c^2(t-a)},\; t \ge a.$$

Problem 227 *Show that for z as in Problem 225,*

$$u = \lim_{t\to\infty} z(t) \text{ exists}$$

and

$$\phi(u) \;=\; 0.$$