Chapter 13
Semigroups of Steepest Descent, Abstract Linear Case

Problem 193 *Suppose that* $m, n \in Z^+$ *and* $A \in L(R^n, R^m)$*. Suppose also that*

$$\begin{pmatrix} r \\ s \end{pmatrix} \in R^n \times R^m$$

and

$$\phi(x) = \frac{1}{2}\|\begin{pmatrix} x \\ Ax \end{pmatrix} - \begin{pmatrix} r \\ s \end{pmatrix}\|^2_{(R^n, R^m)}.$$

Find the minimum of ϕ*.*

Problem 194 *Suppose that* H, K *are two Hilbert spaces,* $G \in L(H, K)$ *and* $h \in K$*. Define*

$$\phi(x) \ = \ \frac{1}{2}\|Gx - h\|^2_K, \ x \in H.$$

Denote by G^* *the element of* $L(K, H)$ *such that*

$$\langle Gx, y\rangle_K \ = \ \langle x, G^* y\rangle_H, \ x \in H, \ y \in K.$$

Show that

$$(\nabla\phi)(x) \ = \ G^*Gx - G^*h, \ x \in H$$

where $(\nabla\phi)(x)$ *is the element of* H *so that*

$$\phi'(x)h = \langle h, (\nabla\phi)(x)\rangle_H, \ x, h \in H.$$

Problem 195 *For* $\nabla\phi$ *as in Problem 194,* $x \in H$ *and* z *the unique solution of*

$$z(0) \ = x, \ z'(t) \ = \ -(\nabla\phi)(z(t)), \ t \geq 0,$$

show that

$$z(t) \ = \ e^{-tG^*G}x \ + \ \int_0^t e^{-(t-s)G^*G}G^*h \, ds. \tag{13.1}$$

(Variation of parameters for ordinary differential equations.)

J.W. Neuberger, *A Sequence of Problems on Semigroups*,
Problem Books in Mathematics, DOI 10.1007/978-1-4614-0430-9_13,

Denote by Q the orthogonal projection of H onto the null space of G and denote by P the orthogonal projection of K onto the range of G.

Problem 196 *In the setting of Problem 195 show that*

$$\lim_{t\to\infty} e^{-tG^*G}x = Qx.$$

Problem 197 *In the setting of Problem 195 show that if* $t \in R$, *then*

$$Ge^{tG^*G} = e^{tGG^*}G \text{ and } G^*e^{GG^*} = e^{G^*G}G^*.$$

Problem 198 *For* z *as in Problem 195 show that*

$$\lim_{t\to\infty} G(z(t)) \;=\; g$$

where g *is the orthogonal projection of* h *onto the range of* G.

Problem 199 *For* z *as in Problem 195 show that if*
$h \in$ *range* G, *then*

$$u \;=\; \lim_{t\to\infty} z(t) \text{ exists and } Gu \;= h.$$

Problem 200 *Suppose that* z *is as in Problem 199, and* $y \in H$ *is such that* $Gy = h$. *Show that*

$$\|u - z(t)\| \le \|y - z(t)\|,\ t \ge 0.$$

Problem 201 *Consider an alternative to using continuous steepest descent for finding a zero of* G:

Suppose that ϕ, h, G *are as in Problem 194 and* h *is in the range of* G. *Suppose also that* $u_0 \in X$, *and*

$$u_k = u_{k-1} - \delta_k(\nabla\phi)(u_{k-1}),\ k = 1, 2, \ldots$$

where at step $k \in Z^+$,

$$\nabla\phi_k \text{ is from } (13.1)$$

and δ_k *is chosen to minimize* q:

$$q(t) = \min \|\phi(u_{k-1} - t(\nabla\phi)(u_{k-1}))\|_X,\ t \ge 0.$$

Show that the sequence $\delta_1, \delta_2, \ldots$ *is unique. Show that* $u_0, u_1, \ldots$ *converges to* u *such that*

$$\phi(u) \text{ is minimum.}$$

This illustrates discrete steepest descent for linear problems.

The following is both an application to the developments of this chapter and a preview of Chapters 14 and 15.

Problem 202 *Suppose that*

$$\Omega = [-1, 1]^2$$

and X, Y *are the Sobolev spaces*

$$H^{2,2}(\Omega), L_2(\Omega),$$

respectively. Denote by $G : X \to Y$ *the transformation so that*

$$(Gu)(x, y) = yu_{1,1}(x, y) + u_{2,2}(x, y),\ (x, y) \in \Omega.$$

Show that the problem of finding

$$Gu = 0 \tag{13.2}$$

is elliptic, hyperbolic in

$$[-1, 1] \times (0, 1] \text{ and } [-1, 1] \times [-1, 0),$$

respectively. The problem of solving (13.2) *is called a Tricomi problem.*

Do you know what boundary or supplementary conditions to impose are necessary and sufficient in order that there is one and only one solution to this problem on Ω*? It is my understanding that this problem has not been solved.*

Problem 203 *After considering Chapter 15 write a code for an appropriate discretization of Problem 202. Observe how an answer to an iteration as in Problem 202 depends on the choice of* u_0.