Chapter 12
Splitting Method, Numerics

Developments in the preceding chapters suggest a practical technique for time-dependent partial differential equations. Suppose $w > 0$ and $X = L_2([0,1])$ and each of A, B, C is a linear transformation from a dense subset of X into X. For some $w > 0$ one might seek

$$u : [0, w] \times X \to X$$

so that

$$u_1(t,x) = A(u(t,\cdot))(x) + B(u(t,\cdot))(x) + C(u(t,\cdot))(x), \tag{12.1}$$

$t \in [0,w],\ x \in [0,1]$, where u_1 denotes the partial derivative of u in its first argument.

Problem 183 *Show how, with $B = C = 0$, A may be chosen so that (12.1) is the heat equation of Chapter 6.*

Problem 184 *Show how to choose A, B, C in (12.1) so that the equation becomes*

$$u_1(t,x) = u_{2,2}(t,x) + u_2(t,x) + u(t,x),\ t \in [0,w], x \in [0,1]. \tag{12.2}$$

Problem 185 *Devise a numerical scheme for solving*

$$u_1(t,x) = u_{2,2}(t,x),\ t \in [0,w],\ x \in [0,1].$$

Problem 186 *Devise a numerical scheme for solving*

$$u_1(t,x) = u_2(t,x),\ t \in [0,w],\ x \in [0,1].$$

Problem 187 *Devise a numerical scheme for solving*

$$u_1(t,x) = u(t,x),\ t \in [0,w],\ x \in [0,1].$$

J.W. Neuberger, *A Sequence of Problems on Semigroups*,
Problem Books in Mathematics, DOI 10.1007/978-1-4614-0430-9_12,

Problem 188 *Suppose one seeks* u *satisfying* (12.2) *with the boundary condition*

$$u(t,0) = 0,\ t \in [0,w].$$

Define semigroups T_1, T_2, T_3 *from Problems 185, 186, 187, respectively. Determine if the extension of Problem 173 to the combination of three semigroups might allow a formula for solving* (12.2) *by means of a finite-dimensional version of*

$$(T_1(\frac{t}{n})T_2(\frac{t}{n})T_3(\frac{t}{n}))^n f,\ t \in [0,w] \tag{12.3}$$

where

$$u(0,x) = f(x),\ x \in [0,1],$$

f *being a given member of* $L_2([0,1])$ *specifying an initial condition for* (12.2).

Problem 189 *Code your solutions to Problems 185, 186, 187. Consider using an implicit scheme for the part coming from Problem 185.*

Problem 190 *Use the main parts of your codes in Problem 189 to implement* (12.3)*. Test for various choices of* n *and various discretizations of* $[0,1]$.

Equation (12.2) is a linear example of a reaction–diffusion–convection equation. The term corresponding to A is the diffusion term, the one corresponding to B is the convection term and the one corresponding to C is the reaction term.

This splitting method is often used in cases where there is not yet a proof of convergence, but is often used to great success in a practical way. A typical example is indicated by the following:

Problem 191 *Suppose that* $w > 0$*. Devise a code to implement a splitting method to find a numerical approximation to* u, v *such that*

$$\begin{aligned}
u_1(t,x,y) &= b_1 u_{2,2}(t,x,y) + b_2 u_{3,3}(t,x,y) \\
&\quad + c_1 u_2(t,x,y) + d_1 u_3(t,x,y) + m_1 u(t,x,y)v(t,x,y), \\
v_1(t,x,y) &= b_2 v_{2,2}(t,x,y) + v_{3.3}(t,x,y) \\
&\quad + c_2 v_2(t,x,y) + d_2 v_3(t,x,y) + m_2 u(t,x,y)v(t,x,y), \\
&\quad t \in [0,w],\ x,y \in [0,1] \times [0,1],
\end{aligned}$$

where $b_1, b_2, c_1, c_2 > 0, \quad d_1, d_2, m_1, m_2 \in R$ *and*

$$u(t,0,0) = u(t,1,0) = u(t,0,1) = u(t,1,1),\ t \geq 0,$$

$$u(0,x,y) = h(x,y),\ (x,y) \in [0,1]^2$$

for some given function h *on* $[0,1]$ *specifying initial conditions.*

Problem 192 *Define A, B, C so that with*

$$z = \begin{pmatrix} u \\ v \end{pmatrix}$$

the equation in Problem 191 becomes

$$z' = Az + Bz + Cz.$$