Chapter 11
Combining Semigroups, Strongly Continuous Linear Case

Suppose that X is a Banach space.

Problem 163 *Suppose that each of*

$$\{T_k\}_{k=1}^{\infty} \tag{11.1}$$

is a linear strongly continuous semigroup of transformations such that

$$|T_k(t)| \leq 1,\ t \geq 0, k \in Z^+.$$

Suppose also that

$$\{A_k\}_{k=1}^{\infty},$$

respectively, are the generators of the members of (11.1). *Suppose also that A is a densely defined linear transformation on X such that*

$$\lim_{k\to\infty} A_k x = Ax,\ x \text{ in some dense subset of the range of } A.$$

Suppose finally that

$$\text{the range of } I - \lambda A \text{ is dense in } X$$

for all $\lambda > 0$. Show that if $\lambda > 0$, then

$$(I - \lambda A_k)^{-1} x \text{ converges, as } k \to \infty, \tag{11.2}$$

for all x in the range of $I - \lambda A$.

Problem 164 *Under the hypothesis of Problem 163 show that*

$$(I - \lambda A_k)^{-1} x \text{ converges for all } x \in X \text{ as } k \to \infty. \tag{11.3}$$

Problem 165 *Under the hypothesis of Problem 163 show that if $m \in Z^+$, then*

$$(I - \lambda A_k)^{-m} x \text{ converges for all } x \in X \text{ as } k \to \infty. \tag{11.4}$$

J.W. Neuberger, *A Sequence of Problems on Semigroups*,
Problem Books in Mathematics, DOI 10.1007/978-1-4614-0430-9_11,

Problem 166 *Under the hypothesis of Problem 163 show that if $j, k, m \in Z^+$, then*

$$(I - \frac{\lambda}{m}A_j)^{-m}x - (I - \frac{\lambda}{m}A_k)^{-m}x = \frac{1}{\lambda}\int_0^\infty d\phi_{m,\lambda}\,(T_j(\cdot)x - T_k(\cdot)x)$$

for all $x \in X$, $\lambda > 0$.

Problem 167 *Show that under the hypothesis of Problem 166, if $x \in X$, then*

$$T_1(\cdot)x, T_2(\cdot)x, \ldots \tag{11.5}$$

converges uniformly on each bounded subinterval of $[0, \infty)$. (An argument by way of contradiction might be considered here.)

Under the hypothesis of Problem 167, denote by T the function with domain $[0, \infty)$ and range the collection of functions from X to X so that if $x \in X$ and $\lambda \geq 0$, then

$$T(\lambda)x = \lim_{k\to\infty} T_k(\lambda)x.$$

Problem 168 *Show that*

$$T(\lambda) \in L(X, X),\ \lambda \geq 0.$$

Problem 169 *Show that if $x \in X$, then*

$$T(\cdot)x$$

is continuous.

Problem 170 *Show that*

$$|T(\lambda)| \leq 1,\ \lambda \geq 0.$$

Problem 171 *Show that*

$$T(0) = I,\ T(t)T(s) = T(t+s),\ t, s \geq 0.$$

Problem 172 *Conclude that T is a linear, nonexpansive, strongly continuous semigroup on X.*

Problem 173 *Suppose that each of K and S is a linear, nonexpansive, strongly continuous semigroup on X with generators B, C, respectively. Suppose that*

- *$W = Domain(B) \bigcap Domain(C)$ is dense in X.*
- *$I - \lambda(B + C)$ has range dense in X if $\lambda \geq 0$.*
- $$A_n = n((K(\frac{1}{n})S(\frac{1}{n}) - I),\ n \in Z^+.$$

Find A so that

$$Ax = \lim_{n \to \infty} A_n x, \ x \in W.$$

Problem 174 *Under the conditions of Problem 173 show that the sequence*

$$A_1, A_2, \ldots$$

gives rise to a semigroup T as in Problem 167.

Problem 175 *Investigate relationships between the generators B, C of Problem 173 and the generator A of T in Problem 174.*

Problem 176 *Note that the above problems in this chapter involve nonexpansive semigroups. Develop a corresponding theory for general strongly continuous linear semigroups T for which there is $M > 0$ so that*

$$|T(t)| \le M, \ t \ge 0.$$

Problem 177 *Note that the above problems in this chapter involve nonexpansive semigroups. Develop a corresponding theory for general strongly continuous linear semigroups T for which there are $M, \omega > 0$ so that*

$$|T(t)| \le M e^{wt}, \ t \ge 0.$$

Problem 178 *Read Chapter III, Section 5 of [17] for stronger results than given above in this chapter. Also read the discussion in [19] concerning Trotter–Kato formulae and related developments concerning the Feynman–Kac formula. Examine the development in this chapter and make comparisons with [17],[19]. See Notes for this chapter for additional comments in this regard.*

Problem 179 *Suppose that*

$$\{T_n\}_{k=n}^{\infty}$$

is a sequence of nonexpansive linear strongly continuous semigroups on the Banach space X for which there is a dense subset W of X common to its corresponding sequence of generators

$$\{A_n\}_{n=1}^{\infty}.$$

Suppose also that if $x \in W$, then

$$\lim_{n \to \infty} A_n x$$

exists. Try to determine if there is a strongly continuous linear nonexpansive semigroup T such that if $x \in X$, then

$$\{T_n(\cdot)x\}_{n=1}^{\infty}$$

converges uniformly on each closed and bounded subinterval of $[0, \infty)$.

Problem 180 *Suppose that each of T and S is a strongly continuous linear nonexpansive semigroup on the Banach space X. Try to determine if there is a strongly continous linear nonexpansive semigroup U on X so that*

$$U(t)x = \lim_{n\to\infty}(T(\frac{t}{n})S(\frac{t}{n}))^n x,\ x \in X.$$

Problem 181 *In addition to the conditions of Problem 180 suppose that A, B are the generators of T, S respectively, $\lambda > 0$ and*

$$A_n = \frac{n}{\lambda}((I - \frac{\lambda}{n}A)^{-1} - I), B_n = \frac{n}{\lambda}((I - \frac{\lambda}{n}B)^{-1} - I),$$

for $n \in Z^+$. Denote by

$$\{T_n\}_{n=1}^{\infty},\ \{S_n\}_{n=1}^{\infty}$$

two sequences of continuous linear nonexpansive semigroups generated

$$\{A_n\}_{n=1}^{\infty},\ \{B_n\}_{n=1}^{\infty},$$

respectively. Finally define

$$\{U_n\}_{n=1}^{\infty}$$

by

$$U_n(t)x = \lim_{k\to\infty}((T_n(\frac{t}{k})(S_n(\frac{t}{k}))^k x,\ x \in X, n \in Z^+.$$

Is it true that

$$\{U_n\}_{n=1}^{\infty}$$

is such that there is a strongly continuous linear semigroup U on X such that if $x \in X$, then

$$\{U_n(\cdot)x\}_{n=1}^{\infty}$$

converges uniformly to

$$U(\cdot)x$$

on each closed and bounded subset of $[0, \infty)$?

Problem 182 *Find and read [23] for generalization of Trotter-Kato results to nonlinear semigroups.*