Chapter 10
Some Connections Between Resolvents and Linear Semigroups

This group of problems may be thought of as a continuation of those in Chapter 5. A review of Chapter 5 might be in order.

Suppose that X is a Banach space.

Problem 146 *Suppose that each of*

$$Q,\ \{Q_k\}_{k=1}^{\infty} \in L(X,X)\ \textit{with}\ |Q_k| \le 1, k \in Z^+$$

so that for each $x \in X$,

$$\lim_{k\to\infty} Q_k x = Qx.$$

Show that if $m \in Z^+$, *then*

$$\lim_{k\to\infty} Q_k^m x = Q^m x,\ x \in X.$$

Problem 147 *Suppose that* F *is a function whose domain is a dense subset of* X *so that* $G = F^{-1}$ *exists, has domain all of* X *and*

$$\|G(x) - G(y)\| \le \|x - y\|\ \textit{for all}\ x, y \in X.$$

Show that if W *is a dense subset of the domain of* F, *then*

$$F(W)\ \textit{is dense in}\ X.$$

Problem 148 *Suppose that each of*

$$T, \{T_k\}_{k=1}^{\infty} \tag{10.1}$$

is a linear strongly continuous semigroup of transformations such that

$$|T(t)|, |T_k(t)| \le 1,\ t \ge 0, k \in Z^+,$$

J.W. Neuberger, *A Sequence of Problems on Semigroups*,
Problem Books in Mathematics, DOI 10.1007/978-1-4614-0430-9_10,

and that

$$A, \{A_k\}_{k=1}^{\infty}, \tag{10.2}$$

respectively, are the generators of the members of (10.1)*. Show that, following the notation of Problem 65,*

$$\|(I - \lambda A)^{-k} x - (I - \lambda A_n)^{-k} x\| \leq \int_0^{\infty} d\phi_{k,\lambda} \, \|T(\cdot)x - T_n(\cdot)x\|,$$

for $n, k \in Z^+, \ x \in X$*.*

Problem 149 *In addition to the hypothesis of Problem 148, suppose that if* $x \in X$*, then*

$$T_n(\cdot)x$$

converges to

$$T(\cdot)x \text{ as } n \to \infty,$$

uniformly on each bounded subinterval of $[0, \infty)$*. Show that*

$$\lim_{n\to\infty} (I - \lambda A_n)^{-k} x = (I - \lambda A)^{-k} x, \ k \in Z^+, x \in X.$$

Problem 150 *In addition to the hypothesis of Problem 148, suppose that* W *is a dense subset of* X *which is a subset of the domain of each of the transformations in* (10.2)*. Suppose finally that for each* $x \in W$*,*

$$\lim_{k\to\infty} A_k x = Ax.$$

Show that for each $\lambda > 0$ *and* y *in the range of* $(I - \lambda A)$*,*

$$\lim_{k\to\infty} (I - \lambda A_k)^{-1} y = (I - \lambda A)^{-1} y.$$

Problem 151 *Under the hypothesis of Problem 150, show that if* $m \in Z^+$*,* $x \in X$ *and* $\lambda > 0$*, then*

$$\lim_{k\to\infty} (I - \lambda A_k)^{-m} x = (I - \lambda A)^{-m} x.$$

Problem 152 *Under the hypothesis of Problem 150, show that if* $x \in W$*, then*

$$\{A_n x\}_{n=1}^{\infty}$$

is a bounded sequence.

Problem 153 *Under the hypothesis of Problem 148, show that if* $x \in W$*, then*

$$(T_n(\cdot)x)'(t) = AT_n(t)x = T_n(t)Ax, \ t \geq 0, \ n \in Z^+.$$

Problem 154 *Under the hypothesis of Problem 150, show that if $x \in W$, then*

$$\{T_n(\cdot)x\}_{n=1}^{\infty}$$

is a bounded and equicontinuous collection on each bounded subinterval of $[0, \infty)$.

Problem 155 *Show that if $\lambda > 0$, there is $M \in Z^+$ so that if $m > M$, then there are maximal $a_m < \lambda$, minimal $b_m > \lambda$ so that*

$$0 < \frac{1}{2} - \int_{a_m}^{\lambda} d\phi_{m,\lambda} \leq \epsilon \text{ and } 0 < \frac{1}{2} - \int_{\lambda}^{b_m} d\phi_{m,\lambda} \leq \epsilon.$$

(See Problem 65 for notation.)

Problem 156 *Show that for Problem 155,*

$$b_m - a_m \to 0 \text{ as } m \to \infty.$$

Problem 157 *Suppose $x \in W$. Assuming the denial of the conclusion to Problem 160 show that there is*

- $\epsilon > 0$, *a bounded subinterval $[a, b]$ of $[0, \infty)$*
- *a member λ of (a, b)*
- *a sequence $\{\lambda_j\}_{j=1}^{\infty}$ converging to λ*
- *an increasing sequence $\{n_j\}$ of positive integers*

so that

$$\|T_j(\lambda_{n_j})x - T(\lambda_{n_j})x\| > \epsilon, \ j \in Z^+.$$

Problem 158 *Using the assumptions, conclusions of Problem 157 and the result of Problem 154, show that there are*

- *a sequence $\{a_j\}_{j=1}^{\infty}$ converging monotonically increasing to λ*
- *a sequence $\{b_j\}_{j=1}^{\infty}$ converging monotonically decreasing to λ*

so that

$$\|\int_{a_j}^{b_j} (T_{n_j}(\cdot)x - T(\cdot)x)\| \geq \epsilon, \ j \in Z^+.$$

Problem 159 *Continuing on with Problem 158 and using the results of Problem 154, reach a contradiction.*

Problem 160 *Under the hypothesis of Problem 150, show that for each $x \in W$,*

$$T_n(\cdot)x$$

converges uniformly on every bounded subinterval of $[0, \infty)$ to

$$T(\cdot)x \text{ as } n \to \infty.$$

Problem 161 *Under the hypothesis of Problem 150, show that for each* $x \in X$,

$$\{T_n(\cdot)x\}_{n=1}^{\infty}$$

converges uniformly on every bounded subinterval of $[0, \infty)$ *to*

$$T(\cdot)x \text{ as } n \to \infty.$$

Problem 162 *Study [17] Chapter III, Section 5 to see an alternate path to Problem 161.*