Chapter 1
Introduction

1.1 What Is Cooperative Control?

Multiple robots or sensors have a number of advantages over a single agent, including robustness to failures of individual agents, reconfigurability, and the ability to perform challenging tasks such as environmental monitoring, target localization, that cannot be achieved by a single agent. A cooperative control system consists of a group of autonomous agents with sensing or communication capabilities, for example, robots with camera sensors and vehicles with communication devices. The goal of the group is to achieve prescribed agent and group behaviors using only local information available to each agent from sensing or communication devices. Such local information may include relative configuration and motion obtained from sensing or communication between agents, agent's sensor measurements, and so on. The relative sensing and communication dictates the architecture of information flow between agents. Thus, a cooperative system has four basic elements: group objective, agents, information topology, and control algorithms governing the motion of the agents.

Examples of group objectives in cooperative control include flocking, schooling, cohesion, guarding, escorting [112, 98, 131, 76, 51, 107], agreement [110, 75, 102, 63, 10, 103, 5], vehicle formation maintenance [124, 47, 36, 5], gradient climbing in an environmental field [8, 98, 27, 18], cooperative load transport [14, 145, 106, 129, 128], distributed estimation and optimal sensing [97, 99, 86, 153, 33], source seeking [151] and coverage control [34, 78]. Some of the cooperative control objectives involve only relative variables (e.g., relative positions) between agents while others depend on absolute variables (e.g., inertial positions) of agents. We illustrate below some examples of cooperative control.

Formation Control. A major focus in cooperative control is formation stability, where the group objective is to stabilize the relative distances/positions between the agents to prescribed desired values. Formation maintenance finds natural applications in coverage control, drag force reduction, and space interferometry. We take space interferometry as a motivating example. As shown in Fig. 1.1, space in-

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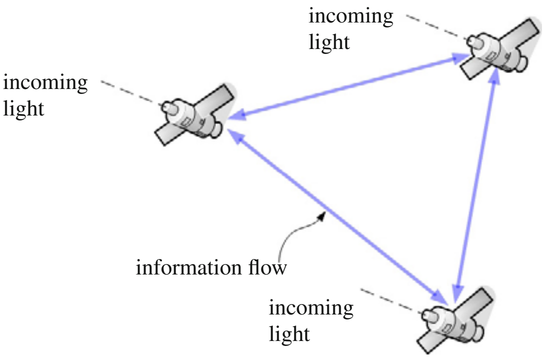

Fig. 1.1 Space interferometry using multiple spacecraft. Spacecraft must maintain precise relative formation and the same attitude towards the incoming light to generate an interferometry pattern. The information flow between spacecraft is setup by optical sensors, which measure relative positions between spacecraft.

terferometry uses multiple spacecraft pointing towards an object of interest. Each spacecraft collects light from the object. If the relative positions between spacecraft are maintained precisely, an interferometry pattern is generated and measurement of the magnitude and phase of this pattern can be obtained by coherently mixing the light from individual spacecraft [73, 118, 123]. Multiple such measurements allow reconstruction of the image of the object, which has a much finer resolution than any single spaceborne telescope achieves. In space, position information of individual spacecraft in the earth frame is imprecise or unavailable, whereas relative position between spacecraft can be measured precisely by optical sensors [74]. Thus, maintaining spacecraft formation in space may make use of only relative position information while maneuvering the formation to point towards objects of interest.

Agreement. In the agreement problem, the group objective is the convergence of distributed variables of interest (agents' positions or headings, phase of oscillations, etc.) to a common value. Given different contexts, the agreement problem is also called *consensus*, or *synchronization*. As shown in Fig. 1.1, space interferometry requires spacecraft to align their attitudes with each other, which is an agreement problem. Agreement problem also has potential applications in schooling and flocking in distributed robotics and biological systems [100, 112, 127, 24], distributed estimation and sensor fusion [99, 104], fire surveillance [25] and distributed computing [148, 140, 16], among others.

Optimal Sensing. The group objective for optimal sensing is to optimally place the agents' positions so that certain meaningful utility functions are maximized. Examples of utility functions include probability of detecting a target [34, 78] and information obtained from a sensor network [86, 33]. In this case, the utility functions usually depend on the absolute positions of the agents.

Most cooperative control problems concern coordinated motion of agents in different scenarios. Therefore, agent dynamics become important in achieving different group objectives. Small mobile sensors or robots can be controlled by directly manipulating their velocities. Such agents are commonly modeled by a first order kinematic model. Depending on the group objective, the agent model can be a simple integrator or a set of integrators subject to nonholonomic constraints, such as a unicycle model. If the group objective is to control the position of the sensor to improve sensing capability, it may suffice to model the sensors as massless points with single integrator kinematics. When the velocities of agents are not directly manipulatable or the masses of the agents are significant, double integrator agent dynamics are more appropriate. In numerous applications, the attitudes of the agents play an important role, which means that agent models in Euler-Lagrangian form or in Hamiltonian form may be required.

To achieve the group objective, each agent may need information from other agents. If agent i has access to agent j's information, the information of agent j flows to agent i and agent j is a neighbor of agent i. The abstract structure of the information flows in the group is then represented as a graph, where each agent is a node and the information flows between them are represented as links [1]. In many applications, the information flow between agents is achieved by direct sensing or communication. For example, to control the relative distances between the agents, the agents obtain their relative distance information by sensing or by communicating their inertial positions. In some applications, the information flow is realized through a physical medium. For example, consider a group of agents transporting a payload. By interacting with the payload, the agents can obtain the relative information between them without explicit communication. We will illustrate such an example in Chapter 8.

One of the main challenges in cooperative control design is to achieve prescribed group objectives by *distributed* feedback laws. The distributed laws make use of information available only to individual agents. Such information includes the information flow from neighbors, and sensor measurements from agent itself. Take the agreement problem as an example. When modeled as a first order integrator, the agent can aggregate the differences between its own state and its neighbors' and take that aggregated value as a feedback control. In this case, the control algorithm is distributed since it only employs information from neighboring agents. If the control algorithm and the agent model are both linear, stability can be analyzed by examining the eigenvalues of the closed-loop system matrix with the help of algebraic graph theory. This approach leads to simple stability criteria for the agreement problem, e.g., [63, 111, 102, 103, 47, 76, 87].

However, for some applications of cooperative control, only nonlinear algorithms can achieve the objective. Consider the following formation control problem: The group objective is to stabilize relative distances (the Euclidean norms of relative positions) between agents to desired values. In this case, the desired equilibria are spheres, which are compact sets containing more than one point. When each agent is modeled as a linear system, such as a double integrator, there is no linear feedback law globally stabilizing the desired equilibria. This is because a linear agent model with linear feedback results in a linear system, whose equilibria can simply

[1] A brief introduction to graph theory will be presented later in this chapter.

be a point or a subspace. Thus, only nonlinear feedback laws may solve this formation control problem. Indeed, most of the formation control algorithms have been proposed in the form of nonlinear artificial attraction and repulsion forces between neighboring agents. The design and analysis of such rules make use of graph theory and potential function methods.

1.2 What Is in This Book?

For different cooperative control problems, there are different control design methods. In this book, we introduce a *unifying* passivity-based framework for cooperative control problems. Under this passivity-based framework, we develop robust, adaptive, and scalable design techniques that address a broad class of cooperative control problems, including the formation control and the agreement problem discussed above.

This framework makes explicit the passivity properties used implicitly in the Lyapunov analysis of several earlier results, including [131, 98, 102], and simplifies the design and analysis of a complex network of agents by exploiting the network structure and inherent passivity properties of agent dynamics. With this simplification, the passivity approach further overcomes the simplifying assumptions of existing designs and offers numerous advantages, including:

1. Admissibility of complex and heterogenous agent dynamics: Unlike some of the existing cooperative control literature where the agent is modeled as a point robot, the passivity approach allows high order and nonlinear dynamics, including Lagrangian and Hamiltonian systems. As illustrated in Chapter 5, attitude coordination among multiple rigid bodies can be studied under this passivity framework. In this case, the agent dynamics are in the Hamiltonian form. Chapter 6 discusses the agreement of multiple Lagrangian systems. The passivity approach is further applicable to *heterogenous* systems in which the agent dynamics and parameters, such as masses, dampings, vary across the group.

2. Design flexibility, robustness and adaptivity: The passivity approach abstracts the common core of several multi-agent coordination problems, such as formation stabilization, group agreement, and attitude coordination. Because passivity involves only input-output variables, it has inherent robustness to unknown model parameters. Since passivity is closely related to Lyapunov stability, this passivity approach lends itself to systematic adaptive designs that enhance robustness of cooperative systems. Such design flexibility and adaptivity will be demonstrated in this book by the adaptive designs in Chapters 3, 4 and 6.

3. Modularity and scalability: The passivity framework yields decentralized controllers which allow the agents to make decisions based on relative information with respect to their neighbors, such as relative distance. A key advantage of the passivity-based design is its *modularity*, which means that the control laws do not rely on the knowledge of number of other agents, the communication structure of the network, or any other global network parameters.

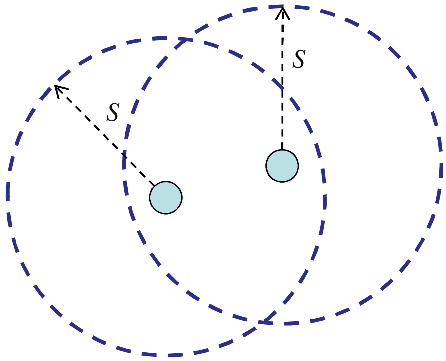

Fig. 1.2 If the sensing/communication ranges for both robots are chosen to be the same and one agent is within the sensing/communication range of the other agent, the information flow between them is symmetric. $S > 0$ denotes the sensing or communication radius.

Our major assumptions for this passivity-based approach are:

Bidirectional Information Topology: Control algorithms with bidirectional information topology tend to have inherent stability properties as we explicate with the help of passivity arguments in this book. Although directional information topology can render stability for first order agents [103, 109], it may lead to instability for high order agents, as we illustrate in Example 2.3 in Chapter 2.

Bidirectional information topology also appears naturally in a number of cooperative control applications. For example, as shown in Fig. 1.2, the information topology of agents with the same sensing range can be modeled as bidirectional. In the load transport problem studied in Chapter 8, the agents exert force on the payload and receive reaction forces from the payload. The exerted force and the reaction force contain implicitly the relative motion information between the agents and the payload. Thus, the information flows are bidirectional.

Static Information Topology: Our design assumes that the information topology remains unchanged. This is not a restrictive assumption since in most practical situations, the information topology remains static for a certain period of time. If that period of time is long enough, by standard dwell time arguments [91, 56], the closed-loop system remains stable. Note that for first order linear consensus protocols, such as those studied in [103, 63], robustness to arbitrary switching topology has been justified. However, for higher order systems, it is well known that switching may lead to instability [80, 81]. We will show that for first order protocols, the passivity-based framework can handle a broad class of switching topology whereas for higher order cooperative systems, topology switching improperly may result in instability.

1.3 Notation and Definition

• We denote by $\mathbb{R}$ and by $\mathbb{C}$ the set of real numbers and complex numbers, respectively. The notation $\mathbb{R}_{\geq 0}$ denotes the set of all real nonnegative numbers. The real part and the imaginary part of a complex number $x \in \mathbb{C}$ are given by $Re[x]$ and $Im[x]$, respectively.
• All the vectors in this book are column vectors. The set of p by 1 real vectors is denoted by $\mathbb{R}^p$ while the set of p by q real matrices is denoted by $\mathbb{R}^{p\times q}$. The transpose of a matrix $A \in \mathbb{R}^{p\times q}$ is given by $A^T \in \mathbb{R}^{q\times p}$.
•$\mathcal{N}(A)$ and $\mathcal{R}(A)$ are the null space (kernel) and the range space of a matrix A, respectively. I_p and $\mathbf{0}_p$ denote the $p\times p$ identity and zero matrices, respectively. The $p\times q$ zero matrix is denoted by $\mathbf{0}_{p\times q}$. Likewise, 1_N and 0_N denote the $N\times 1$ vector with each entry of 1 and 0, respectively. Without confusion, we will also use 0 to denote a vector of zeros with a compatible dimension.
• The Kronecker product of matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is defined as

$$A\otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}, \tag{1.1}$$

and satisfies the properties

$$(A\otimes B)^T = A^T \otimes B^T \tag{1.2}$$

$$(A\otimes I_p)(C\otimes I_p) = (AC)\otimes I_p \tag{1.3}$$

where A and C are assumed to be compatible for multiplication.
• The maximum and minimum eigenvalues of a symmetric matrix A are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively.
• For a vector $x \in \mathbb{R}^p$, $|x|$ denotes its 2-norm, that is $|x| = \sqrt{x^T x}$.
• The norm of a matrix A is defined as its induced norm $||A|| = \sqrt{\lambda_{\max}(A^T A)}$.
• We use the notation $\mathrm{diag}\{K_1, K_2, \cdots, K_n\}$ to denote the block diagonal matrix

$$\begin{pmatrix} K_1 & \mathbf{0}_{p\times q} & \cdots & \mathbf{0}_{p\times q} \\ \mathbf{0}_{p\times q} & K_2 & \cdots & \mathbf{0}_{p\times q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{p\times q} & \mathbf{0}_{p\times q} & \cdots & K_n \end{pmatrix} \tag{1.4}$$

where $K_i \in \mathbb{R}^{p\times q}$, $i = 1, \cdots, n$.
• The notation $K = K^T > 0$ means that K is a symmetric positive definite matrix while $k > 0$ implies k is a positive scalar.
• Given a vector $v \in \mathbb{R}^3$, the cross product $v\times$ is a linear operator, and can be represented in a coordinate frame as left-multiplication by the skew-symmetric matrix:

$$\widehat{v} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \tag{1.5}$$

where $(v_1,\ v_2,\ v_3)$ are the components of v in the given coordinate frame. The inverse operation of cross product is given by $^\vee$, that is

$$(\widehat{v})^\vee = v. \tag{1.6}$$

• For the coordinate frame representation of a vector, the leading superscript indicates the reference frame while the subscript i denotes the agent i. The superscript d means the desired value. As an illustration, ${}^j v_i^d$ means the desired velocity of the ith agent in the jth frame.
• A function is said to be C^k if its partial derivatives exist and are continuous up to order k.
• Given a C^2 function $P : \mathbb{R}^p \to \mathbb{R}$ we denote by ∇P its gradient vector, and by $\nabla^2 P$ its Hessian matrix.
• A function $\alpha : [0, a) \to \mathbb{R}_{\geq 0}$ is of class K if it is continuous, strictly increasing and satisfies $\alpha(0) = 0$. It is said to belong to class K_∞ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class KL if, for each fixed s, the function $\beta(r,s)$ belongs to class K with respect to r and, for each fixed r, the function $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$. An example of class KL functions is shown in Fig. 1.3.

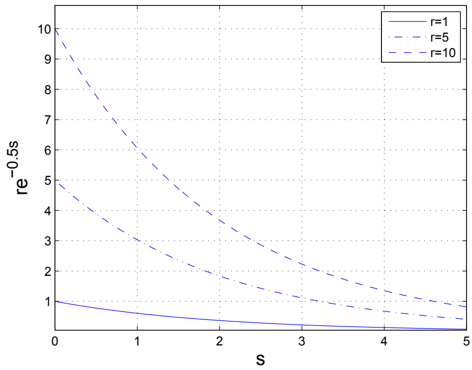

Fig. 1.3 The function $\beta(r,s) = re^{-0.5s}$ is of class KL because for fixed r, $re^{-0.5s}$ is decreasing and converges to zero as s converges to ∞ and for fixed s, $re^{-0.5s}$ is monotonically increasing with respect to r.

• The system $\dot{x} = f(x,u)$ is said to be Input-to-State Stable (ISS) [125, 126] if there exist functions $\beta \in KL$, $\rho \in K$ such that for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq 0$ and satisfies

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \rho(\sup_{t_0 \leq \tau \leq t} |u(\tau)|). \tag{1.7}$$

• For a closed set $\mathscr{A}$, $|\chi|_{\mathscr{A}}$ denotes the distance from the point χ to $\mathscr{A}$, defined as

$$|\chi|_{\mathscr{A}} = \inf_{\eta \in \mathscr{A}} |\chi - \eta|. \tag{1.8}$$

• Given the dynamics of the state $\chi(t)$, a closed invariant set $\mathscr{A}$ is uniformly asymptotically stable with region of attraction $\mathscr{G}$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\chi(t_0)|_{\mathscr{A}} \leq \delta \quad \Rightarrow \quad |\chi(t)|_{\mathscr{A}} \leq \varepsilon \quad \forall t \geq t_0 \tag{1.9}$$

and, if for each $\varepsilon > 0$ and $r > 0$, there exists $T > 0$ such that for every initial condition $\chi(t_0) \in \mathscr{G}$ the resulting trajectory satisfies

$$|\chi(t_0)|_{\mathscr{A}} \leq r \Rightarrow \quad |\chi(t)|_{\mathscr{A}} \leq \varepsilon \quad \forall t \geq T. \tag{1.10}$$

Several results on set stability and, in particular, converse Lyapunov theorems are presented in [82] and [137].

• We use the notation $\chi \to \partial\mathscr{G}^{\infty}$ to indicate a sequence of points χ in $\mathscr{G}$ converging to a point on the boundary of $\mathscr{G}$, or if $\mathscr{G}$ is unbounded, having the property $|\chi| \to \infty$.

1.4 Basic Graph Theory

In this book, we will make use of basic result from algebraic graph theory to facilitate our analysis. The results presented in this section are standard in the literature and will be well known to readers familiar with graph theory.

A graph is an abstract representation of a group of nodes where some of them are connected by links. More formally, a graph G is an ordered pair $G = (\mathscr{V}, E)$ consisting of a set $\mathscr{V}$ of nodes and a set $E \subset \mathscr{V} \times \mathscr{V}$ of links. Thus, a link is an ordered pair of two *distinct* nodes.

A directed link (i, j) is an *incoming link* to node j and an *outgoing link* from node i. We then draw an arrow from node i to node j. We call node i (respectively, j) the *negative* (respectively, *positive*) end of link (i, j). If both links (i, j) and (j, i) belong to E, we combine these two links as one *undirected* link and use a bidirectional arrow to denote this link.

Depending on the directions of the links, a graph may be categorized as *directed* or *undirected*. If a graph G consists of only undirected links, it is *undirected*. Otherwise, the graph is *directed*.

We say node i is a *neighbor* of node j if the link (i, j) exists in the graph G. This means that for each directional link, the negative end is the neighbor of the positive end. Note that for undirected graphs, if node i is a neighbor of node j, then node j is also a neighbor of node i. We denote by $\mathscr{N}_j$ the *set of neighbors* of node j.

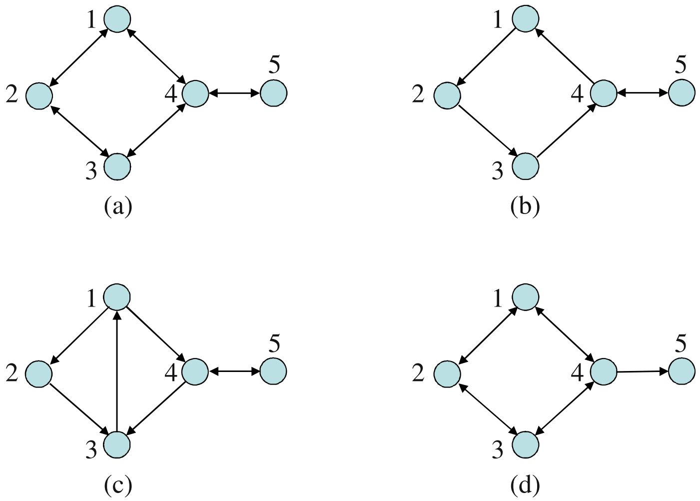

Fig. 1.4 Different types of graphs of five nodes. (a): an undirected connected graph. (b): a balanced and strongly connected graph. (c): a strongly connected graph. (d): a weakly connected graph. A directed link is denoted by a line with a directional arrow while an undirected link is denoted by a bidirectional arrow. The node number is beside each node.

For $i \in \mathscr{V}$, if the number of incoming links to i is the same as the number of outgoing links from i, the graph is *balanced*. Clearly, an undirected graph is a special balanced graph.

A *directed path* is a sequence of p nodes $1, \cdots, p$, such that $(i, i+1) \in E$, $\forall i = 1, \cdots, p-1$. A *cycle* is a directed path such that the starting and the ending nodes of the path are the same. A graph is called *strongly connected* if there exists a directed path from any one node to another. Note that for an undirected graph, strong connectedness is simply termed connectedness. A graph is called *weakly connected* if replacing all the directed links in E with undirected ones gives a connected undirected graph. In Fig. 1.4 are several examples of five nodes illustrating connectedness of different graphs.

Definition 1.1 (Graph Laplacian matrix L).

Consider a directed graph G with N nodes. The Laplacian matrix of a graph G, denoted by $L \in \mathbb{R}^{N \times N}$, is given by

$$\ell_{ij} := \begin{cases} |\mathscr{N}_i| & \text{if } i = j \\ -1 & \text{if } j \in \mathscr{N}_i \\ 0 & \text{otherwise,} \end{cases} \tag{1.11}$$

where $|\mathscr{N}_i|$ is the cardinality of the set $\mathscr{N}_i$. □

The definition in (1.11) results in the following property of L:

Property 1.1. The graph Laplacian matrix L has an eigenvalue of zero associated with an eigenvector 1_N, i.e., $L1_N = 0_N$. □

Example 1.1. Following (1.11), we compute the graph Laplacian matrices for the graphs in Fig. 1.4 as

$$L_a = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad L_b = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{1.12}$$

$$L_c = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \text{and} \quad L_d = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}. \tag{1.13}$$

It is easy to see all these four Laplacian matrices satisfy Property 1.1. □

In particular, the Laplacian matrix for undirected graphs satisfies Properties 1.2 and 1.3 below.

Property 1.2. The Laplacian matrix L of an undirected graph is symmetric and positive semidefinite. □

Property 1.3. [17, Item 4e and Corollary 6.5]

An undirected graph is connected if and only if the second smallest eigenvalue of its Laplacian matrix is strictly positive. □

We verify the positive semidefiniteness of L in Property 1.2 by showing

$$y^T L y \geq 0 \quad \forall y \in \mathbb{R}^N. \tag{1.14}$$

To see this, we let y_i be the ith element of y and note from (1.11) that

$$y^T L y = \sum_{i=1}^N y_i \sum_{j \in \mathcal{N}_i} (y_i - y_j) = \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} (y_i^2 - y_i y_j) \tag{1.15}$$

$$= \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} (y_i^2 - 2y_i y_j + y_j^2) + \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} (y_i y_j - y_j^2) \tag{1.16}$$

$$= \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} (y_i - y_j)^2 + \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} (y_i y_j - y_j^2). \tag{1.17}$$

Because the graph is undirected, we have

$$\sum_{i=1}^N \sum_{j \in \mathcal{N}_i} y_j^2 = \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} y_i^2 \tag{1.18}$$

which implies that the last term in (1.17) is indeed $-y^T Ly$. Therefore, it follows from (1.17) that

$$y^T Ly = \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}(y_i - y_j)^2 \geq 0. \tag{1.19}$$

For a general directed graph, the graph Laplacian matrix L is not symmetric and $y^T Ly$ can be sign-indefinite. However, if the directed graph is balanced and strongly connected, $y^T Ly \geq 0$ holds for any y due to the following property:

Property 1.4. [103] The graph Laplacian matrix L of a *balanced and strongly connected* graph G satisfies

$$L + L^T = \frac{1}{2}L_{\mathrm{sym}} \tag{1.20}$$

where L_{sym} represents the graph Laplacian matrix of the *undirected* graph obtained by replacing the directed edges in G with undirected ones. □

For an undirected graph G, we may assign an orientation to G by considering one of the two nodes of a link to be the positive end. We denote by $\mathcal{L}_i^+$ ($\mathcal{L}_i^-$) the set of links for which node i is the positive (negative) end.

Definition 1.2 (Graph Incidence matrix D).

Denoting by ℓ the total number of links, we define the $N \times \ell$ *incidence matrix* D of an undirected graph G as

$$d_{ik} := \begin{cases} +1 \text{ if } k \in \mathcal{L}_i^+ \\ -1 \text{ if } k \in \mathcal{L}_i^- \\ \;\;0 \;\;\text{otherwise.} \end{cases} \tag{1.21}$$

□

Property 1.5. We obtain from (1.21) an incidence matrix D corresponding to a particular orientation assignment to the undirected graph G. Independently of how we assign the orientation to G, the resulting incidence matrix D has the following properties:

1. The rank of D is at most $N-1$ and the rank of D is $N-1$ if and only if the graph G is connected;
2. The columns of D are linearly independent when no cycles exist in the graph;
3. If the graph G is connected, the only null space of D^T is spanned by 1_N;
4. The graph Laplacian matrix L of G satisfies

$$L = DD^T. \tag{1.22}$$

□

Example 1.2. We verify the last item in Property 1.5 by considering the graph G in Fig. 1.5. We obtain from (1.11) that

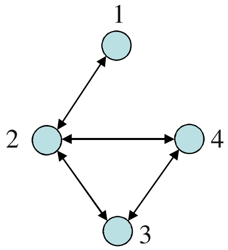

Fig. 1.5 An undirected graph of four agents whose Laplacian matrix is in (1.23). The agent number is illustrated at each node.

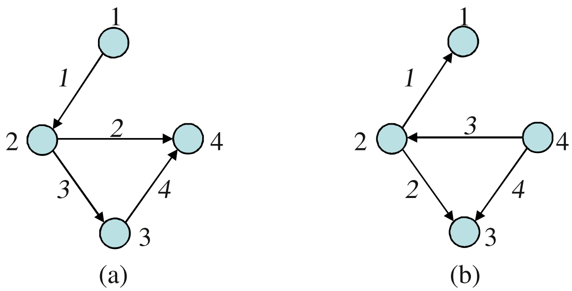

Fig. 1.6 Two different orientation assignments to the graph in Fig. 1.5 yields two different graph incidence matrices D in (1.24). However, both incidence matrices give the same Laplacian matrix (1.23) using (1.22). The arrow points to the positive end of each link. The link number is denoted in *italic* at each link.

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}. \tag{1.23}$$

To show that the choice of D does not affect L, we assign different orientations to G as in Fig. 1.6 and obtain the two incidence matrices D as

$$D_a = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}. \tag{1.24}$$

A simple computation shows that $L = D_a D_a^T = D_b D_b^T$. Thus, the choice of orientation assignment to the graph does not affect the graph Laplacian matrix. □

1.5 Passivity and Passivity-preserving Structures

In this section, we briefly review the definition of passivity and its relation to stability. We also present four passivity-preserving structures that will be utilized in the rest of the book. Some of the results in this section are based on [69, 116].

Definition 1.3 (Passivity of Static Nonlinearity).
A static nonlinearity $y = h(u)$, where $h : \mathbb{R}^p \to \mathbb{R}^p$, is *passive* if, for all $u \in \mathbb{R}^p$,

$$u^T y = u^T h(u) \geq 0; \tag{1.25}$$

and *strictly passive* if (1.25) holds with strict inequality $\forall u \neq 0$. □

Definition 1.4 (Passivity and Strict Passivity of Dynamical Systems).
The dynamical system

$$\mathscr{H}: \quad \begin{cases} \dot{\xi} = f(\xi, u) \\ y = h(\xi, u), \quad \xi \in \mathbb{R}^n,\ u, y \in \mathbb{R}^p, \end{cases} \tag{1.26}$$

is said to be *passive* if there exists a C^1 *storage function* $S(\xi) \geq 0$ such that

$$\dot{S} = \nabla S(\xi)^T f(\xi, u) \leq -W(\xi) + u^T y \tag{1.27}$$

for some positive semidefinite function $W(\xi)$. We say that (1.26) is *strictly passive* if $W(\xi)$ is positive definite. □

Definition 1.5 (Strict Input and Output Passivity).
For the dynamic system (1.26), if S in (1.27) satisfies

$$\dot{S} \leq -u^T \psi(u) + u^T y \tag{1.28}$$

for some function $\psi(u)$ such that $u^T \psi(u) > 0$, then (1.26) is *input strictly passive*. Likewise, if

$$\dot{S} \leq -y^T \psi(y) + u^T y \tag{1.29}$$

holds for some function $\psi(y)$ where $y^T \psi(y) > 0$, (1.26) is *output strictly passive*.
□

Example 1.3 (Passivity of Euler-Lagrange Systems).
A standard model of mechanical systems with n degrees of freedom is given by the *Euler-Lagrange equation*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}(x, \dot{x})\right) - \frac{\partial L}{\partial x}(x, \dot{x}) = \tau \tag{1.30}$$

where $x = [x_1, \cdots, x_n]^T$ are the generalized coordinates of the system and $\tau = [\tau_1, \cdots, \tau_n]^T$ is the generalized torque acting on the system. The Lagrangian function $L(x, \dot{x})$ satisfies

$$L(x,\dot{x}) = K(x,\dot{x}) - P(x) \tag{1.31}$$

where $P(x)$ is the potential energy of the system and is bounded from below, i.e.,

$$P(x) \geq \bar{P} := \min_x P(x), \tag{1.32}$$

and $K(x,\dot{x})$ is the kinetic energy of the system which is assumed to be of the form

$$K(x,\dot{x}) = \frac{1}{2}\dot{x}^T M(x)\dot{x}, \tag{1.33}$$

in which $M(x) = M(x)^T$ is the positive definite generalized inertia matrix.

A further computation from (1.30) and (1.33) leads to

$$M(x)\ddot{x} + C(x,\dot{x})\dot{x} + g(x) = \tau \tag{1.34}$$

where $g(x) = \frac{\partial P(x)}{\partial x}$. A well known property of (1.34) is that $\dot{M}(x) - 2C(x,\dot{x})$ is skew-symmetric [6, 71], i.e.,

$$\dot{M}(x) - 2C(x,\dot{x}) = -(\dot{M}(x) - 2C(x,\dot{x}))^T. \tag{1.35}$$

The Euler-Lagrange system (1.34) is passive from the generalized torque input τ to the generalized velocity $\dot{x}$. Such a result is established by using (1.35) and taking the total energy of the system $V = K(x,\dot{x}) + P(x) - \bar{P}$ as the storage function. The derivative of V is given by

$$\begin{aligned}
\dot{V} &= \dot{x}^T M(x)\ddot{x} + \frac{1}{2}\dot{x}^T \dot{M}(x)\dot{x} + g(x) && (1.36)\\
&= \dot{x}^T \tau + \frac{1}{2}\dot{x}^T(\dot{M}(x) - 2C(x,\dot{x}))\dot{x} && (1.37)\\
&= \dot{x}^T \tau. && (1.38)
\end{aligned}$$

If τ is chosen as

$$\tau = -R\dot{x} + \tau_e, \quad R = R^T > 0, \tag{1.39}$$

we immediately verify the strict output passivity from τ_e to $\dot{x}$. □

Passivity of a linear time invariant dynamic system is closely related to positive realness of the transfer function of that system.

Definition 1.6. [Positive Realness]

A scalar transfer function $g(s)$ is called *positive real* if

- poles of $g(s)$ have nonpositive real parts;
- for all $\omega \in \mathbb{R}$ for which $j\omega$ is not a pole of $g(s)$, $Re[g(j\omega)] \geq 0$;
- any pure imaginary pole $j\omega$ of $g(s)$ is a simple pole and the associated residues are nonnegative. □

The second condition in Definition 1.6 means that the Nyquist plot of $g(j\omega)$ lies in the closed right-half complex plane, which implies that the phase shift of $g(s)$ cannot exceed $\pm 90°$.

Definition 1.7. [Strict Positive Realness [61, 142]]
A transfer function $g(s)$ is called *strictly positive real* if $g(s-\varepsilon)$ is positive real for some $\varepsilon > 0$. □

The strict positive realness of $g(s)$ can also be characterized in the following lemma:

Lemma 1.1. *A scalar transfer function $g(s)$ is strictly positive real if and only if*

- *poles of $g(s)$ have negative real parts;*
- *for all $\omega \in \mathbb{R}$, $Re[g(j\omega)] > 0$;*
- *either $g(\infty) > 0$ or $g(\infty) = 0$ and $\lim_{\omega\to\infty} \omega^2 Re[g(j\omega)] > 0$.* □

Example 1.4. The first-order integrator $g(s) = \frac{1}{s}$ is positive real since it has a simple pole at $\omega = 0$, associated with residue 1, and

$$Re\left[\frac{1}{j\omega}\right] = 0, \quad \forall \omega \neq 0. \tag{1.40}$$

The second-order integrator $g(s) = \frac{1}{s^2}$ is not positive real since the phase shift of $g(s)$ is $-180°$.

The transfer function $g(s) = \frac{1}{as+c}$ for $a, c > 0$ is strictly positive real since $g(s-\varepsilon)$ is positive real for $\varepsilon = c/a > 0$. □

When a transfer function $g(s)$ is realized by a minimal state space representation

$$\mathscr{H}: \quad \begin{cases} \dot{\xi} = A\xi + Bu \\ y = C\xi + Du, \end{cases} \tag{1.41}$$

the positive realness of $g(s)$ means that (1.41) is passive.

Lemma 1.2. *Let $\mathscr{H}$ in (1.41) be a minimal state space representation of $g(s)$. Then,*

- *$\mathscr{H}$ is passive if $g(s)$ is positive real;*
- *$\mathscr{H}$ is strictly passive if $g(s)$ is strictly positive real.* □

The passivity property of a dynamical system remains unchanged when the input and output variables are transformed in a "symmetric" fashion as in Fig. 1.7.

Structure 1 (Symmetric Input-Output Transformation) *Let the system H in Fig. 1.7 be passive and let E be a matrix with a compatible dimension. Then the system in Fig. 1.7 is passive from $\bar{u}$ to $\bar{y}$.* □

Proof. Note that $u^T y = (E^T \bar{u})^T y = \bar{u}^T \bar{y}$. Thus, the passivity from u to y translates to the passivity from $\bar{u}$ to $\bar{y}$. □

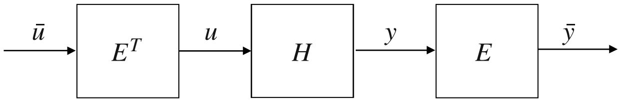

Fig. 1.7 Pre- and post- multiplication of a matrix and its transpose preserves the passivity of H.

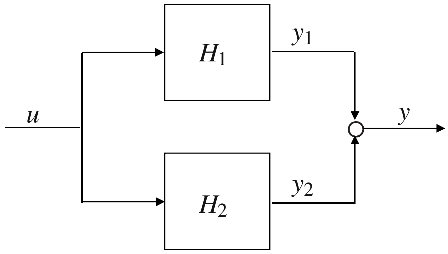

Fig. 1.8 Parallel interconnection of two passive systems.

The definition of passivity closely relates to the stability of (1.26) when $u = 0$. In fact, when the storage function S is positive definite, (1.27) implies that for $u = 0$,

$$\dot{S} \le -W(\xi) \le 0. \tag{1.42}$$

Assume that $f(0,0) = 0$. Using standard Lyapunov arguments, we conclude that the unforced system $\dot{\xi} = f(\xi,0)$ has a stable equilibrium $\xi = 0$. If, in addition, $W(\xi)$ is positive definite, $\xi = 0$ is asymptotically stable. If S is also proper, i.e., $S(\xi) \to \infty$ as $|\xi| \to \infty$, the asymptotic stability of $\xi = 0$ is global.

The stability properties are preserved if two or more passive systems are interconnected properly. Among all possible passivity-preserving structures, the following three structures are employed in our cooperative control design.

Structure 2 (Parallel Interconnection) *Consider the parallel interconnection of two passive systems H_1 and H_2 in Fig. 1.8. Then the interconnected system is passive from u to y.* □

Structure 3 (Negative Feedback Interconnection) *Consider the negative feedback interconnection of two passive systems H_1 and H_2 in Fig. 1.9. Then the interconnected system is passive from u to y.* □

Replacing H_1 in Structure 3 with Structure 1, we obtain Structure 4 below:

Structure 4 (Symmetric Interconnection) *Consider the interconnection structure of two passive systems H_1 and H_2 in Fig. 1.10. Then the interconnected system is passive from u to y.* □

We will demonstrate in the next chapter that Structure 4 arises naturally in cooperative control with bidirectional information flow. In particular, the matrices E

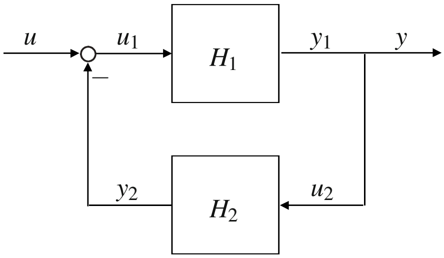

Fig. 1.9 Negative feedback interconnection of two passive systems.

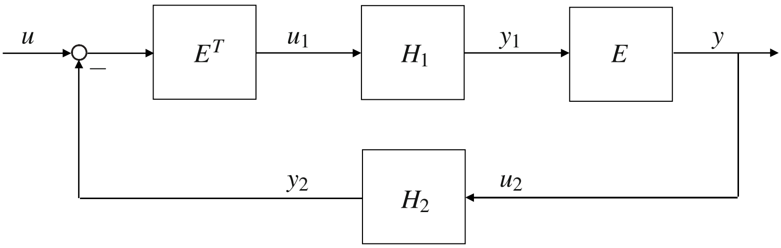

Fig. 1.10 Symmetric Interconnection of two passive systems H_1 and H_2 is still passive.

and E^T are dictated by the undirected information topology between the agents. The H_1 and H_2 blocks in Structure 4, being block diagonal, represent the dynamics of individual agents and their relative configuration, respectively. We will then apply passivation designs to H_1 and H_2 such that the closed-loop stability is guaranteed by Structure 4.