10 Compact operators

In this chapter we consider the properties of compact linear operators between Banach spaces. The space $\mathscr{K}(X;Y)$ of (linear) compact operators from X to Y was already defined in 5.5(2). Because here we are always concerned with linear operators, for convenience we simply speak of compact operators.

All the spaces in this chapter are assumed to be Banach spaces. We begin with a discussion of the elementary properties of compact operators and then give the most important examples of such operators. These include compact embeddings between function spaces and compact integral operators.

10.1 Compact operators. Let X and Y be Banach spaces over $\mathbb{K}$. Then a linear map $T : X \to Y$ is called a ***compact (linear) operator*** if one of the following equivalent properties is satisfied:

(1) $\overline{T(\mathrm{B}_1(0))} \subset Y$ is compact (see the definition 5.5(2)).

(2) $T(\mathrm{B}_1(0)) \subset Y$ is precompact.

(3) $M \subset X$ is bounded $\quad\Longrightarrow\quad T(M) \subset Y$ is precompact.

(4) For every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X, the sequence $(Tx_n)_{n\in\mathbb{N}}$ contains a subsequence that is convergent in Y.

It follows from (2) that $T(\mathrm{B}_1(0))$ is bounded (see 4.7(2)), and so $T \in \mathscr{L}(X;Y)$, by 5.1. Hence it holds for the set defined in 5.5(2) that

$$\begin{aligned}\mathscr{K}(X;Y) &:= \{T : X \to Y\,;\ T \text{ is a compact linear operator}\}\\ &= \{T \in \mathscr{L}(X;Y)\,;\ T \text{ satisfies (4)}\}\,.\end{aligned}$$

Moreover, let $\mathscr{K}(X) := \mathscr{K}(X;X)$.

Note: The fact that compact maps (with the property (1)) are continuous only holds for linear maps. General nonlinear maps which satisfy (1) need not be continuous.

Proof (1)$\Leftrightarrow$(2)*.* This follows from 4.7(5), as Y is complete. □

Proof (2)$\Leftrightarrow$(3)*.* The linearity of T implies that for every $R > 0$ statement (2) is equivalent to the precompactness of $T\big(\mathrm{B}_R(0)\big)$. Because every bounded set M is contained in a ball $\mathrm{B}_R(0)$, it then follows that the smaller set $T(M)$ is also precompact. □

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Proof (1)⇒(4). If $x_n \in X$ for $n \in \mathbb{N}$ with $\|x_n\|_X < R$, then $\frac{1}{R}Tx_n = T(\frac{1}{R}x_n)$ are elements of the compact, and hence (by 4.6) also sequentially compact, set $\overline{T(\mathrm{B}_1(0))}$. □

Proof (4)⇒(1). Let $y_n \in \overline{T\big(\mathrm{B}_1(0)\big)}$ for $n \in \mathbb{N}$. Then there exist $x_n \in \mathrm{B}_1(0)$ with $\|y_n - Tx_n\|_Y \le \frac{1}{n}$. It follows from (4) that there exists a $y \in Y$ such that $Tx_n \to y$ for a subsequence $n \to \infty$, and hence also $y_n \to y$. This shows that $\overline{T\big(\mathrm{B}_1(0)\big)}$ is sequentially compact, and so, by 4.6, is also compact. □

We now prove some basic results.

10.2 Lemma.

(1) If X is a reflexive space, then it holds for every linear map $T : X \to Y$ that

$$T \in \mathscr{K}(X;Y) \quad \Longleftrightarrow \quad T \text{ is } \textbf{\textit{completely continuous}}\text{, i.e.}$$

if $x_n \to x$ converges weakly in X as $n \to \infty$, then $Tx_n \to Tx$ converges strongly in Y.

(2) $\mathscr{K}(X;Y)$ is a closed subspace of $\mathscr{L}(X;Y)$.

(3) If $T \in \mathscr{L}(X;Y)$ with $\dim \mathscr{R}(T) < \infty$, then $T \in \mathscr{K}(X;Y)$.

(4) If Y is a Hilbert space and $T \in \mathscr{L}(X;Y)$, then

$$T \in \mathscr{K}(X;Y) \quad \Longleftrightarrow \quad \begin{array}{l} \text{there exist } T_n \in \mathscr{L}(X;Y) \text{ with } \dim \mathscr{R}(T_n) < \infty, \\ \text{such that } \|T - T_n\| \to 0 \text{ as } n \to \infty\,. \end{array}$$

(5) For projections $P \in \mathscr{P}(X)$ it holds that

$$P \in \mathscr{K}(X) \quad \Longleftrightarrow \quad \dim \mathscr{R}(P) < \infty\,.$$

Proof (1)⇒. (In the proof of this implication the reflexivity of X is not needed.) Let $x_n \to x$ weakly as $n \to \infty$. By 8.3(5), the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, and so 10.1(4) yields the existence of a $y \in Y$ such that $Tx_n \to y$ strongly in Y for a subsequence $n \to \infty$. For $y' \in Y'$ the map $z \mapsto \langle Tz\,,\, y'\rangle$ defines an element in X'. Therefore,

$$\langle Tx_n\,,\, y'\rangle \to \langle Tx\,,\, y'\rangle \qquad \text{as } n \to \infty\,.$$

This yields that $Tx_n \to Tx$ weakly in Y. As strong convergence implies weak convergence, one must have $y = Tx$. Hence $Tx_n \to Tx$ converges strongly for a subsequence $n \to \infty$. On noting that all of the above argumentation can be applied to every subsequence of $(x_n)_{n\in\mathbb{N}}$, it follows that the whole (!) sequence $(Tx_n)_{n\in\mathbb{N}}$ has only one cluster point Tx, i.e. it converges strongly to Tx. □

Proof (1)⇐. Being completely continuous implies that T is continuous, and so $T \in \mathscr{L}(X;Y)$. Moreover, it follows from theorem 8.10 that bounded sequences in reflexive spaces contain weakly convergent subsequences. □

Proof (2). In order to see that $\mathscr{K}(X;Y)$ is a subspace, let $T_1, T_2 \in \mathscr{K}(X;Y)$, let $\alpha \in \mathbb{K}$ and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. Then there exists a subsequence $(T_1 x_{n_k})_{k\in\mathbb{N}}$ that is convergent in Y. Similarly, we may then choose a convergent subsequence $\left(T_2 x_{n_{k_l}}\right)_{l\in\mathbb{N}}$. This implies that $\left((\alpha T_1 + T_2)(x_{n_{k_l}})\right)_{l\in\mathbb{N}}$ converges in Y, which shows that $\alpha T_1 + T_2 \in \mathscr{K}(X;Y)$.

To prove that $\mathscr{K}(X;Y)$ is closed, assume that $T_n \in \mathscr{K}(X;Y)$ converges in $\mathscr{L}(X;Y)$ as $n \to \infty$ to $T \in \mathscr{L}(X;Y)$. For $\varepsilon > 0$ first choose n_ε with $\|T - T_{n_\varepsilon}\| \le \varepsilon$ and then (recall 10.1(2)) balls $\mathrm{B}_\varepsilon(y_i)$, $i = 1, \ldots, m_\varepsilon$, such that

$$T_{n_\varepsilon}(\mathrm{B}_1(0)) \subset \bigcup_{i=1}^{m_\varepsilon} \mathrm{B}_\varepsilon(y_i)\,, \quad \text{which implies:} \quad T(\mathrm{B}_1(0)) \subset \bigcup_{i=1}^{m_\varepsilon} \mathrm{B}_{2\varepsilon}(y_i)\,.$$

Hence $T(\mathrm{B}_1(0))$ is precompact, and so T is compact. □

Proof (3). We have that $Z := \mathscr{R}(T) \subset Y$ is finite-dimensional, and so it follows from 4.9 that with the Y-norm it is a Banach space. On setting $R := \|T\|$ we have that

$$T\big(\mathrm{B}_1(0)\big) \subset K_R := \{y \in Z\,;\ \|y\|_Y \le R\} \subset Z\,.$$

By 4.10, we have that $K_R \subset Z$ is compact, and hence combining 4.7(5) and 4.7(1) yields that $\overline{T(\mathrm{B}_1(0))}$ is compact. □

Proof (4)⇐. We have from (3) that $T_n \in \mathscr{K}(X;Y)$. Then (2) yields that $T \in \mathscr{K}(X;Y)$. □

Proof (4)⇒. Let $\varepsilon > 0$. It follows from 10.1(2) that we can choose balls $\mathrm{B}_\varepsilon(y_i)$, $i = 1, \ldots, m_\varepsilon$, with

$$T\big(\mathrm{B}_1(0)\big) \subset \bigcup_{i=1}^{m_\varepsilon} \mathrm{B}_\varepsilon(y_i)\,.$$

Set $Y_\varepsilon := \operatorname{span}\{y_1, \ldots, y_{m_\varepsilon}\}$ and let P_ε denote the orthogonal projection onto Y_ε. Then we have from 9.18 that $\mathrm{Id} - P_\varepsilon$ is also an orthogonal projection (equivalence of 9.18(1) and 9.18(2)), with $\|\mathrm{Id} - P_\varepsilon\| \le 1$ (equivalence of 9.18(1) and 9.18(4)). Now $T_\varepsilon := P_\varepsilon T$ maps to Y_ε, and for $x \in \mathrm{B}_1(0)$ it holds that $Tx \in \mathrm{B}_\varepsilon(y_i)$ for some i and that

$$(T - T_\varepsilon)(x) = (\mathrm{Id} - P_\varepsilon)Tx = (\mathrm{Id} - P_\varepsilon)(Tx - y_i),$$

and hence $\|(T - T_\varepsilon)(x)\|_Y \le \varepsilon$. □

Proof (5)⇐. Follows from (3). □

Proof (5)⇒. It holds that $\mathrm{B}_1(0) \cap \mathscr{R}(P) \subset P\big(\mathrm{B}_1(0)\big)$ is precompact, and so it follows from 4.10 that $\mathscr{R}(P)$ is finite-dimensional. □

In applications compact operators often occur as a composition of a continuous map and an embedding which is compact (we prove this in 10.3). The compact part of such a composition is often a canonical embedding. That is, if X, Y are Banach spaces and if X as a vector space is contained in Y, then we ask whether the map $\mathrm{Id} : X \to Y$ is injective, continuous and compact, respectively. We will answer this question completely for the function spaces $C^{k,\alpha}(\overline{\Omega})$ and $W^{m,p}(\Omega)$ (see 10.6 – 10.13) and we call the corresponding theorems embedding theorems.

10.3 Lemma. For $T_1 \in \mathscr{L}(X;Y)$ and $T_2 \in \mathscr{L}(Y;Z)$ it holds that:

$$T_1 \text{ or } T_2 \text{ is compact} \quad \Longrightarrow \quad T_2T_1 \text{ is compact.}$$

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X. As T_1 is continuous, the sequence $(T_1x_n)_{n\in\mathbb{N}}$ is bounded in Y. If T_2 is compact, it follows that there exists a convergent subsequence $(T_2T_1x_{n_k})_{k\in\mathbb{N}}$. If T_1 is compact, there exists a convergent subsequence $(T_1x_{n_k})_{k\in\mathbb{N}}$, and the continuity of T_2 then yields that also $(T_2T_1x_{n_k})_{k\in\mathbb{N}}$ converges. □

Embedding theorems

The embedding theorem 10.6 for Hölder spaces depends on the Arzelà-Ascoli theorem and the first result in theorem 10.5. For the latter we need the following

10.4 Lemma. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary. If Ω, in addition, is path connected (see the remark preceding 8.16), then for any two points $x_0, x_1 \in \Omega$ there exists a smooth curve γ in Ω which connects x_0 and x_1 and whose length $L(\gamma)$ can be bounded by $|x_1 - x_0|$, i.e. there exists a $\gamma \in C^\infty([0,1];\Omega)$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$, such that, with a constant C_Ω depending only on Ω,

$$L(\gamma) := \int_0^1 |\gamma'(t)|\,\mathrm{d}t \le \sup_{0\le t\le 1} |\gamma'(t)| \le C_\Omega \cdot |x_1 - x_0|\,.$$

Proof. It is sufficient to find a $\gamma \in C^{0,1}([0,1];\Omega)$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$ and with Lipschitz constant $\mathrm{Lip}(\gamma) \le C \cdot |x_1 - x_0|$. To see this, observe that we can then let $\gamma(t) := x_0$ for $t < 0$ and $\gamma(t) := x_1$ for $t > 1$ and set $\gamma_\varepsilon := \varphi_\varepsilon * \gamma$, with a standard Dirac sequence $(\varphi_\varepsilon)_{\varepsilon>0}$. On noting that $\|\gamma_\varepsilon'\|_{\sup} \le \mathrm{Lip}(\gamma_\varepsilon) \le \mathrm{Lip}(\gamma)$, it follows that for $\varepsilon > 0$ sufficiently small γ_ε has all the desired properties on $[-\varepsilon, 1+\varepsilon]$, and hence we only need to map $[0,1]$ affine linearly to $[-\varepsilon, 1+\varepsilon]$.

We consider a cover $\left(U^j\right)_{j=1,\dots,k}$ of $\partial\Omega$ as in A8.2 and choose points $z^j \in U^j \cap \Omega$. Then we choose an open set D with $\overline{D} \subset \Omega$, such that $z^1, \dots, z^k \in \overline{D}$, and such that $\overline{\Omega}$ is covered by $D, U^1, \dots, U^k$. Moreover, we cover $\overline{D}$ with finitely many balls $U^j := \mathrm{B}_\varrho\left(z^j\right) \subset \Omega$ with $j = k+1, \dots, l$.

For general points x_0 and x_1 we can then define a γ as a composition of subpaths, such that for these subpaths only the following three cases can occur. Altogether, the number of subpaths is bounded by the given cover.

If $x_0, x_1 \in U^j$ for some $j > k$, then define $\gamma(t) := (1-t)x_0 + tx_1$.

If $x_0, x_1 \in U^j$ for some $j \leq k$, then define

$$\gamma(t) := \tau\left((1-t)\tau^{-1}(x_0) + t\tau^{-1}(x_1)\right),$$

where with the notations from A8.2 we set

$$\tau(y) := \sum_{i=1}^{n-1} y_i e_i^j + \left(y_n + g^j(y_{,n})\right)e_n^j\,.$$

This defines a Lipschitz continuous path γ in Ω from x_0 to x_1 with

$$\mathrm{Lip}(\gamma) \leq \mathrm{Lip}(\tau) \cdot \left|\tau^{-1}(x_1) - \tau^{-1}(x_0)\right| \leq \mathrm{Lip}(\tau) \cdot \mathrm{Lip}(\tau^{-1}) \cdot |x_1 - x_0|.$$

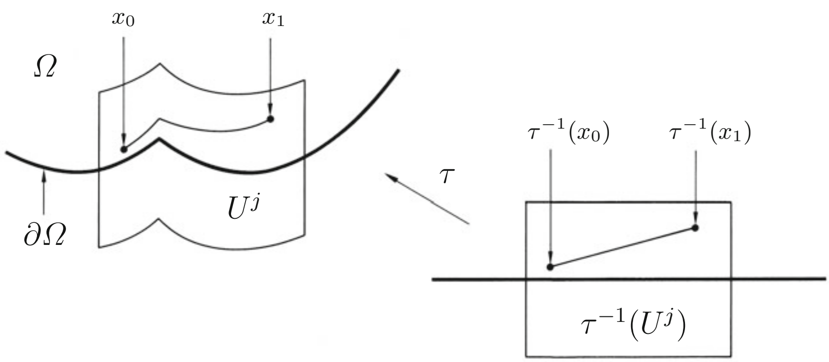

Fig. 10.1. *Construction of curves close to the boundary*

As a third case, let x_0 and x_1 be such that for no $j \in \{1, \dots, l\}$ they lie in the same set U^j of the above cover of $\overline{\Omega}$. Then there exists a constant $c > 0$, which depends only on the cover, such that

$$|x_0 - x_1| \geq c\,.$$

This follows from the fact that for every j and for points $x \in \overline{\Omega} \cap U^j$ that are sufficiently close to ∂U^j it must hold that $x \in U^k$ for some $k \neq j$.

We thus have to connect x_0 and x_1 by a curve with a bounded Lipschitz constant. We make use of the fact that Ω is connected, and so path connected. Hence for $j,k \in \{1,\dots,l\}$ there exists a $\gamma_{j,k} \in C^1([0,1];\Omega)$ with $\gamma_{j,k}(0) = z^j$ and $\gamma_{j,k}(1) = z^k$. Now let $x_0 \in U^{j_0}$ and $x_1 \in U^{j_1}$ with $j_0 \neq j_1$. First we connect x_0 with z^{j_0} inside U^{j_0} (as in the first two cases above) with a path such that the Lipschitz constant can be bounded by $C \cdot \left|z^{j_0} - x_0\right| \leq C \cdot \operatorname{diam} U^{j_0}$. Then we connect z^{j_0} with z^{j_1} by γ_{j_0,j_1}, and finally z^{j_1} with x_1 inside U^{j_1}. A reparametrization of the concatenated paths to the interval $[0,1]$ then yields the desired result. □

10.5 Theorem. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Then it holds for $k \geq 0$ that:

(1) The embedding

$$\mathrm{Id} : C^{k+1}(\overline{\Omega}) \to C^{k,1}(\overline{\Omega})$$

is well defined and continuous.

(2) The embedding

$$\mathrm{Id} : C^{k,1}(\overline{\Omega}) \to W^{k+1,\infty}(\Omega)$$

is well defined and an isomorphism, in the sense that for $u \in W^{k+1,\infty}(\Omega)$ there exists a unique $\widetilde{u} \in C^{k,1}(\overline{\Omega})$ such that $\widetilde{u} = u$ almost everywhere in Ω (i.e. $\widetilde{u} = u$ in $W^{k+1,\infty}(\Omega)$).

Proof. As Ω has a Lipschitz boundary, it consists of finitely many connected components, which all lie at positive distance from one another. Hence we may assume without loss of generality that Ω is connected. For two points $x_0, x_1 \in \Omega$ let γ be as in 10.4. Then for $v \in C^1(\Omega)$, with the notations as in 10.4, we have that

$$\begin{aligned} |v(x_1) - v(x_0)| &= \left|\int_0^1 (v\circ\gamma)'(t)\,\mathrm{d}t\right| \leq \int_0^1 |\nabla v(\gamma(t))| \cdot |\gamma'(t)|\,\mathrm{d}t \\ &\leq \sup_{0\leq t\leq 1} |\nabla v(\gamma(t))| \cdot L(\gamma) \leq C_\Omega \cdot |x_1 - x_0| \cdot \sup_{0\leq t\leq 1} |\nabla v(\gamma(t))|\,. \end{aligned} \tag{10-6}$$

This will be used in the following parts of the proof. □

Proof (1). For $u \in C^{k+1}(\overline{\Omega})$ consider derivatives $v := \partial^s u \in C^1(\overline{\Omega})$ with $|s| = k$. It follows from (10-6) that the Lipschitz constant of v can be bounded by the C^1-norm of v. The fact that this holds for all s of order k yields that $\|u\|_{C^{k,1}(\overline{\Omega})} \leq C \cdot \|u\|_{C^{k+1}(\overline{\Omega})}$ with a constant C. □

Proof (2) *well definedness.* First let $k = 0$. Let $u \in C^{0,1}(\overline{\Omega})$. If $\mathbf{e}_i$ denotes the i-th unit vector, and if $\zeta \in C_0^\infty(\Omega)$, then as $h \to 0$,

$$\begin{aligned} \left|\int_\Omega u(x)\partial_i\zeta(x)\,\mathrm{d}x\right| &\longleftarrow \left|\int_\Omega u(x)\frac{\zeta(x+h\mathbf{e}_i) - \zeta(x)}{h}\,\mathrm{d}x\right| \\ &= \left|\int_\Omega \frac{u(x-h\mathbf{e}_i) - u(x)}{h}\zeta(x)\,\mathrm{d}x\right| \leq \operatorname{Lip}(u)\int_\Omega |\zeta(x)|\,\mathrm{d}x\,. \end{aligned}$$

This implies (see E6.7) that $u \in W^{1,\infty}(\Omega)$ with $\|\partial_i u\|_{L^\infty} \le \mathrm{Lip}(u)$ for $i = 1, \dots, n$. For $k > 0$ apply this result to the derivatives $\partial^s u$ with $|s| = k$. $\square$

Proof (2) *surjectivity.* First let $k = 0$. Let $u \in W^{1,\infty}(\Omega)$. Consider $u_\varepsilon := \varphi_\varepsilon * (\mathcal{X}_\Omega u)$ for a standard Dirac sequence $(\varphi_\varepsilon)_{\varepsilon>0}$. Then it follows from (10-6) (with the notations as there) that

$$|u_\varepsilon(x_1) - u_\varepsilon(x_0)| \le C_\Omega \cdot |x_1 - x_0| \cdot \sup_{0 \le t \le 1} |\nabla u_\varepsilon(\gamma(t))|\,,$$

and, if ε is sufficiently small, for all $x = \gamma(t)$ with $0 \le t \le 1$ we have that

$$|\nabla u_\varepsilon(x)| = |\nabla(\varphi_\varepsilon * u)(x)| = |(\varphi_\varepsilon * \nabla u)(x)| \le \|\nabla u\|_{L^\infty(\Omega)}\,.$$

This implies

$$\frac{|u_\varepsilon(x_1) - u_\varepsilon(x_0)|}{|x_1 - x_0|} \le C_\Omega \cdot \|\nabla u\|_{L^\infty(\Omega)}\,. \tag{10-7}$$

Recalling from 4.15(2) that $u_\varepsilon \to u$ in $L^p(\Omega)$ for every $p < \infty$, there exists a subsequence $\varepsilon \to 0$ such that $u_\varepsilon \to u$ almost everywhere in Ω. Hence it follows from (10-7) that for almost all $x_0, x_1 \in \Omega$ (say, $x_0, x_1 \in \Omega \setminus N$),

$$\frac{|u(x_1) - u(x_0)|}{|x_1 - x_0|} \le C_\Omega \cdot \|\nabla u\|_{L^\infty(\Omega)}\,, \tag{10-8}$$

i.e. u is Lipschitz continuous outside of the null set N. Since $\overline{\Omega \setminus N} = \overline{\Omega}$, it follows from E4.18 that we can modify u on this null set so that $u \in C^{0,1}(\overline{\Omega})$. (After this modification u remains the same (!) element in $L^\infty(\Omega)$.) Since then $\|u\|_{C^0} = \|u\|_{L^\infty}$, we have shown that $\|u\|_{C^{0,1}} \le C \cdot \|u\|_{W^{1,\infty}}$.

If $u \in W^{k+1,\infty}(\Omega)$ with $k > 0$, then we can apply the above to the weak derivatives $v_s := \partial^s u$ for $|s| \le k$. In particular, upon modification on a null set we have that $v_s \in C^{0,1}(\overline{\Omega})$ with the above estimate in (10-8),

$$\mathrm{Lip}(v_s, \overline{\Omega}) \le C_\Omega \cdot \|\nabla v_s\|_{L^\infty(\Omega)} \le C_\Omega \cdot \|u\|_{W^{k+1,\infty}(\Omega)}\,,$$

since for the weak derivatives with $|s| \le k$ it holds that $\partial_i v_s = \partial_i \partial^s u = \partial^{s+\mathbf{e}_i} u \in L^\infty(\Omega)$. Hence we obtain the desired result. $\square$

10.6 Embedding theorem in Hölder spaces. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $k_1, k_2 \ge 0$ and $0 \le \alpha_1, \alpha_2 \le 1$, with

$$k_1 + \alpha_1 > k_2 + \alpha_2\,.$$

In the case $k_1 > 0$ we assume in addition that Ω has a Lipschitz boundary (see also E10.1). Then the embedding

$$\mathrm{Id} : C^{k_1,\alpha_1}(\overline{\Omega}) \to C^{k_2,\alpha_2}(\overline{\Omega})$$

is compact. Here $C^{k,0}(\overline{\Omega}) := C^k(\overline{\Omega})$ for $k \ge 0$.

Remark: For $k_1 = k_2 = 0$ the set $\overline{\Omega}$ can be replaced with an arbitrary compact set $S \subset \mathbb{R}^n$.

Proof. Let $(u_i)_{i\in\mathbb{N}}$ be a bounded sequence in $C^{k_1,\alpha_1}(\overline{\Omega})$. We need to show that a subsequence converges in $C^{k_2,\alpha_2}(\overline{\Omega})$.

First let $k_2 = k_1 = 0$, and so $0 \le \alpha_2 < \alpha_1 \le 1$. By the Arzelà-Ascoli theorem, there exist a $u \in C^0(\overline{\Omega})$ and a subsequence $i \to \infty$ such that u_i converges to u uniformly on $\overline{\Omega}$. Consider only this subsequence and $x, y \in \Omega$ with $x \ne y$. For $|y-x| \le \delta$ it then holds that

$$\begin{aligned}&\frac{|(u-u_i)(y)-(u-u_i)(x)|}{|y-x|^{\alpha_2}} = \lim_{j\to\infty} \frac{|(u_j-u_i)(y)-(u_j-u_i)(x)|}{|y-x|^{\alpha_2}}\\ &\le \delta^{\alpha_1-\alpha_2} \sup_j \|u_j-u_i\|_{C^{0,\alpha_1}} \le 2\delta^{\alpha_1-\alpha_2} \sup_j \|u_j\|_{C^{0,\alpha_1}}\,,\end{aligned}$$

while for $|y-x| \ge \delta$ we have that

$$\frac{|(u-u_i)(y)-(u-u_i)(x)|}{|y-x|^{\alpha_2}} \le 2\,\delta^{-\alpha_2}\|u-u_i\|_{C^0}\,.$$

Overall, there is a constant C such that

$$\sup_{\substack{x,\,y\,\in\,\overline{\Omega}\\ x\ne y}} \frac{|(u-u_i)(y)-(u-u_i)(x)|}{|y-x|^{\alpha_2}} \le \underbrace{C\,\delta^{\alpha_1-\alpha_2}}_{\to 0 \text{ as } \delta\to 0} + 2\,\delta^{-\alpha_2} \underbrace{\|u-u_i\|_{C^0}}_{\to 0 \text{ as } i\to\infty},$$

i.e. the Hölder constant for the exponent α_2 of $u-u_i$ converges to 0 as $i\to\infty$.

Now we consider the case $k_2 = k_1 \ge 1$, and so once again $0 \le \alpha_2 < \alpha_1 \le 1$. Then $(\partial^s u_i)_{i\in\mathbb{N}}$ for $|s| < k_1$ are bounded sequences in $C^1(\overline{\Omega})$, and hence, by 10.5(1), also in $C^{0,1}(\overline{\Omega})$, and for $|s| = k_1$ they are bounded sequences in $C^{0,\alpha_1}(\overline{\Omega})$. Applying the result shown above for the sequence $(\partial^s u_i)_{i\in\mathbb{N}}$ in $C^{0,\alpha_1}(\overline{\Omega})$ we can choose successively for s with $|s| \le k_1$ subsequences so that they converge in $C^{0,\alpha_2}(\overline{\Omega})$. Finally, one obtains a subsequence (which we again denote by $(u_i)_{i\in\mathbb{N}}$) which converges for all (!) s with $|s| \le k_1$

$$\partial^s u_i \to v_s \qquad \text{as } i\to\infty \text{ in } C^{0,\alpha_2}(\overline{\Omega})$$

with certain functions $v_s \in C^{0,\alpha_2}(\overline{\Omega})$. In particular, we obtain that $(u_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $C^{k_1}(\overline{\Omega})$. As this space is complete we necessarily have that $u := v_0 \in C^{k_1}(\overline{\Omega})$ with $\partial^s u = v_s$, i.e. u_i converges to u in $C^{k_1,\alpha_2}(\overline{\Omega})$.

Finally, let $k_1 > k_2$. By the results shown above, in the case $\alpha_2 < 1$ the embedding from $C^{k_2,1}(\overline{\Omega})$ to $C^{k_2,\alpha_2}(\overline{\Omega})$ is compact, and in the case $\alpha_1 > 0$ the embedding from $C^{k_1,\alpha_1}(\overline{\Omega})$ to $C^{k_1}(\overline{\Omega})$ is compact. In addition, we have from 10.5(1) that the embedding from $C^{k_1}(\overline{\Omega})$ to $C^{k_1-1,1}(\overline{\Omega})$ is continuous. Hence it remains to consider the map from $C^{k_1-1,1}(\overline{\Omega})$ to $C^{k_2,1}(\overline{\Omega})$, which in the case $k_1 = k_2+1$ is the identity. In this case we have that $1+\alpha_1 > \alpha_2$, and so $\alpha_2 < 1$ or $\alpha_1 > 0$, which means that the desired result follows from 10.3.

In the case $k_1 > k_2 + 1$ (e.g. when $\alpha_1 = 0$ and $\alpha_2 = 1$) it follows from the above result that the map from $C^{k_1-1,1}(\overline{\Omega})$ to $C^{k_1-1}(\overline{\Omega})$ is compact. Since

$k_1 - 1 \geq k_2 + 1$, the map from $C^{k_1-1}(\overline{\Omega})$ to $C^{k_2+1}(\overline{\Omega})$ is obviously continuous and the map from $C^{k_2+1}(\overline{\Omega})$ to $C^{k_2,1}(\overline{\Omega})$ is continuous thanks to 10.5(1). The desired result now follows on using 10.3. □

We now want to prove embedding theorems for Sobolev spaces. To this end, we consider on $\overline{\mathrm{B}_1(0)} \subset \mathbb{R}^n$ the function $x \mapsto |x|^\varrho$ with real ϱ and investigate to which Sobolev space $W^{m,p}\big(\mathrm{B}_1(0)\big)$, respectively, to which Hölder space $C^{k,\alpha}\big(\overline{\mathrm{B}_1(0)}\big)$ it belongs. The answer will motivate the formulation of the embedding theorems 10.9 and 10.13.

10.7 Sobolev number. Let $f_\varrho(x) := |x|^\varrho$ for $x \in \mathbb{R}^n \setminus \{0\}$, where $\varrho \in \mathbb{R}$. Then it holds that:

(1) f_ϱ is real analytic on $\mathbb{R}^n \setminus \{0\}$ and for $m \geq 0$ there exist positive numbers c_m, C_m, which depend also on n and ϱ, such that

$$c_m \left| \binom{\varrho}{m} \right| \cdot |x|^{\varrho - m} \leq \sum_{|s|=m} |\partial^s f_\varrho(x)| \leq C_m |x|^{\varrho - m}.$$

(2) For $k \geq 0$ and $0 < \alpha \leq 1$ it holds in the case $\varrho \notin \mathbb{N} \cup \{0\}$ that:

$$f_\varrho \in C^{k,\alpha}\big(\overline{\mathrm{B}_1(0)}\big) \quad \Longleftrightarrow \quad \varrho \geq k + \alpha.$$

(3) For $m \geq 0$ and $1 \leq p < \infty$ it holds in the case $\varrho \notin \mathbb{N} \cup \{0\}$ that:

$$f_\varrho \in W^{m,p}\big(\mathrm{B}_1(0)\big) \quad \Longleftrightarrow \quad \varrho > m - \frac{n}{p}.$$

Remark: If we consider the exponent ϱ as a measure of the regularity of the function f_ϱ, then it is natural to associate the following ***characteristic number*** (which we also call the ***Sobolev number*** or ***regularity number***) with the Hölder spaces and Sobolev spaces (where $C^{k,0}(\overline{\Omega}) := C^k(\overline{\Omega})$):

$$\begin{aligned} &k + \alpha &&\text{for } C^{k,\alpha}(\overline{\Omega}) &&\text{if } k \geq 0,\ 0 \leq \alpha \leq 1, \\ &m - \frac{n}{p} &&\text{for } W^{m,p}(\Omega) &&\text{if } m \geq 0,\ 1 \leq p \leq \infty. \end{aligned} \tag{10-9}$$

The fact that this Sobolev number does indeed characterize the regularity of the functions in these spaces is a consequence of the following embedding theorems.

Proof (1). The lower bound holds because on setting $e_x := \frac{x}{|x|}$ we have that

$$\pm \binom{\varrho}{m} |x|^{\varrho - m} = \frac{\pm 1}{m!} \partial_{e_x}^m f_\varrho(x) = \pm \sum_{|s|=m} \frac{\partial^s f_\varrho(x)}{s!} e_x^s \leq \sum_{|s|=m} |\partial^s f_\varrho(x)|$$

(with $c_m = 1$), and the upper bound follows from the fact that for all s

$$\partial^s f_\varrho(x) = p_s(x)|x|^{\varrho-2|s|} \tag{10-10}$$

with homogeneous polynomials p_s of degree $|s|$ or $p_s = 0$. This follows by induction on s, on noting that

$$\partial_i\partial^s f_\varrho(x) = \big(|x|^2\partial_i p_s(x) + (\varrho - 2|s|)x_i p_s(x)\big)\cdot|x|^{\varrho-2(|s|+1)}\,,$$

which yields the recurrence formula

$$p_{s+\mathbf{e}_i}(x) := |x|^2\partial_i p_s(x) + (\varrho - 2|s|)x_i p_s(x)\,. \tag{10-11}$$

$\square$

Proof (2). If $\varrho \ge k + \alpha$, then (1) yields that $|\partial^s f_\varrho(x)| \to 0$ as $|x| \to 0$ for $|s| \le k$, because $\varrho > k$. Hence $f_\varrho \in C^k\big(\overline{\mathrm{B}_1(0)}\big)$. If $|s| = k$, then it holds for $0 < |x_0| \le |x_1| \le 1$ in the case $|x_1 - x_0| \ge \frac{1}{2}|x_1|$ that

$$\begin{aligned}&|\partial^s f_\varrho(x_1) - \partial^s f_\varrho(x_0)| \le C_k\cdot\big(|x_0|^{\varrho-k} + |x_1|^{\varrho-k}\big)\\ &\le 2^{1+\varrho-k}C_k\cdot|x_1 - x_0|^{\varrho-k} \le 2^{1+2(\varrho-k)}C_k\cdot|x_1 - x_0|^{\alpha}\,.\end{aligned}$$

In the case $|x_1 - x_0| \le \frac{1}{2}|x_1|$ let $x_t := (1-t)x_0 + tx_1$ for $0 \le t \le 1$. Then $|x_t| \ge |x_1| - |x_1 - x_0| \ge |x_1 - x_0|$ and so

$$\begin{aligned}&|\partial^s f_\varrho(x_1) - \partial^s f_\varrho(x_0)| \le \int_0^1 |\nabla\partial^s f_\varrho(x_t)|\,\mathrm{d}t\cdot|x_1 - x_0|\\ &\le C_{k+1}\int_0^1 |x_t|^{\varrho-k-1}\,\mathrm{d}t\cdot|x_1 - x_0|\\ &\le C_{k+1}\int_0^1 |x_t|^{\alpha-1}\,\mathrm{d}t\cdot|x_1 - x_0| \le C_{k+1}|x_1 - x_0|^{\alpha}\,.\end{aligned}$$

Therefore, $f_\varrho \in C^{k,\alpha}\big(\overline{\mathrm{B}_1(0)}\big)$. Conversely, if this holds then (1) yields for $0 < |x| \le 1$ that

$$\infty > \|f_\varrho\|_{C^k} \ge c(n,k)\sum_{|s|=k}|\partial^s f_\varrho(x)| \ge c(n,k)\cdot c_k\cdot\left|\binom{\varrho}{k}\right|\cdot|x|^{\varrho-k}\,,$$

and so $\varrho > k$, because $\varrho \notin \mathbb{N}\cup\{0\}$. As before this means that (1) implies that $\partial^s f_\varrho(x) \to 0$ as $|x| \to 0$ for all $|s| \le k$. Hence it follows from (1) that for $0 < |x| \le 1$

$$\infty > \|f_\varrho\|_{C^{k,\alpha}} \ge c(n,k)\sum_{|s|=k}\frac{|\partial^s f_\varrho(x)|}{|x|^{\alpha}} \ge c(n,k)\cdot c_k\left|\binom{\varrho}{k}\right|\cdot|x|^{\varrho-k-\alpha}\,,$$

and so $\varrho \ge k + \alpha$. $\square$

Proof (3). Let $\varrho \notin \mathbb{N} \cup \{0\}$. It follows from (1) that

$$\big\| D^l f_\varrho \big\|^p_{L^p(\mathrm{B}_1(0)\setminus\{0\})} \text{ for } l \geq 0 \text{ and } 1 \leq p < \infty$$

is bounded from above and below by

$$\int_{\mathrm{B}_1(0)} |x|^{p(\varrho-l)}\,\mathrm{d}x = C(n)\int_0^1 r^{n-1+p(\varrho-l)}\,\mathrm{d}r\,.$$

Hence, $f_\varrho \in W^{m,p}(\mathrm{B}_1(0)\setminus\{0\})$ if and only if the integral on the right-hand side is finite for all $0 \leq l \leq m$. This holds if and only if $n+p(\varrho-m) > 0$. The fact that this then yields $f_\varrho \in W^{m,p}\big(\mathrm{B}_1(0)\big)$ follows upon observing that for $|s| < m$ and $\zeta \in C_0^\infty\big(\mathrm{B}_1(0)\big)$ with $0 < \varepsilon < 1$

$$\begin{aligned}
&\int_{\mathrm{B}_1(0)\setminus\mathrm{B}_\varepsilon(0)} \partial_i\zeta\partial^s f_\varrho\,\mathrm{dL}^n \\
&= -\int_{\partial\mathrm{B}_\varepsilon(0)} \nu_i\zeta\partial^s f_\varrho\,\mathrm{dH}^{n-1} - \int_{\mathrm{B}_1(0)\setminus\mathrm{B}_\varepsilon(0)} \zeta\partial^{s+\mathbf{e}_i} f_\varrho\,\mathrm{dL}^n\,,
\end{aligned}$$

where, by (1), the first integral on the right-hand side can be bounded by

$$C(n)\|\zeta\|_{\sup}\cdot\varepsilon^{n-1+\varrho-|s|} \to 0 \qquad \text{as } \varepsilon \to 0\,,$$

since

$$n-1+\varrho-|s| \geq n+\varrho-m > n\big(1-\frac{1}{p}\big) \geq 0\,.$$

$\square$

The Sobolev embedding theorem 10.9 rests on the following theorem and for the compactness result makes use of Rellich's embedding theorem (see A8.1 and A8.4).

10.8 Theorem (Sobolev). Let $1 \leq p, q < \infty$ with

$$1-\frac{n}{p} = -\frac{n}{q}\,. \tag{10-12}$$

Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ with $u \in L^s(\mathbb{R}^n)$ for an $s \in [1,\infty[$ and with $\nabla u \in L^p(\mathbb{R}^n;\mathbb{K}^n)$. Then $u \in L^q(\mathbb{R}^n)$, with

$$\|u\|_{L^q(\mathbb{R}^n)} \leq q\cdot\frac{n-1}{n}\|\nabla u\|_{L^p(\mathbb{R}^n)}\,. \tag{10-13}$$

In particular: The assumptions on u are satisfied for $u \in W^{1,p}(\mathbb{R}^n)$.
Remark: Since $q < \infty$ we must have $p < n$, and so $n \geq 2$. For the case $q = \infty$ see E10.7. For $n = 1$ it holds that $\|u\|_{L^\infty(\mathbb{R})} \leq \|\nabla u\|_{L^1(\mathbb{R})}$ for u as in the assumptions of the theorem (see also E3.6).

Proof. It is sufficient to establish the desired result for functions $u \in L^s(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$. To see this take u as in the assertion and set $u_\varepsilon := \varphi_\varepsilon * u \in C^\infty(\mathbb{R}^n)$ for a standard Dirac sequence $(\varphi_\varepsilon)_{\varepsilon>0}$. Then $u_\varepsilon \to u$ in $L^s(\mathbb{R}^n)$ and $\nabla u_\varepsilon = \varphi_\varepsilon * \nabla u \to \nabla u$ in $L^p(\mathbb{R}^n;\mathbb{K}^n)$. If the claim has been shown for smooth functions, then for $\varepsilon, \delta > 0$ we have

$$\|u_\varepsilon\|_{L^q} \le q \cdot \frac{n-1}{n} \|\nabla u_\varepsilon\|_{L^p}\,,$$
$$\|u_\varepsilon - u_\delta\|_{L^q} \le q \cdot \frac{n-1}{n} \|\nabla (u_\varepsilon - u_\delta)\|_{L^p}\,.$$

Hence the u_ε as $\varepsilon \searrow 0$ form a Cauchy sequence in $L^q(\mathbb{R}^n)$, which yields that $u_\varepsilon \to \widetilde{u}$ in $L^q(\mathbb{R}^n)$ as $\varepsilon \searrow 0$ for some $\widetilde{u} \in L^q(\mathbb{R}^n)$. It follows that

$$\|\widetilde{u}\|_{L^q} \le q \cdot \frac{n-1}{n} \|\nabla u\|_{L^p}\,.$$

Combining the above L^s-convergence and the L^q-convergence yields the existence of a subsequence $\varepsilon_k \searrow 0$ such that $u_{\varepsilon_k} \to u$ and $u_{\varepsilon_k} \to \widetilde{u}$ as $k \to \infty$ almost everywhere in $\mathbb{R}^n$. Consequently, $\widetilde{u} = u$ almost everywhere in $\mathbb{R}^n$ and we obtain the desired result.

Now let $u \in L^s(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$. In all of the following we will only make use of the fact that $u \in L^s(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$. First we consider the case

$$p = 1, \text{ and so } q = \frac{n}{n-1} \text{ (recall that } n \ge 2).$$

For $i \in \{1, \dots, n\}$ it follows from Fubini's theorem that $\xi \mapsto u(x', \xi)$ for almost all $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \in \mathbb{R}$ is an element of $L^s(\mathbb{R})$, where we use the notation

$$(x', \xi) := (x_1, \dots, x_{i-1}, \xi, x_{i+1}, \dots, x_n)\,.$$

Hence we have that $u(x', z_k) \to 0$ for a sequence $z_k \to \infty$ as $k \to \infty$. It follows for $x_i \in \mathbb{R}$ and sufficiently large k that

$$|u(x)| \le \int_{x_i}^{z_k} |\partial_i u(x', \xi)|\, \mathrm{d}\xi + |u(x', z_k)|\,,$$

and so

$$|u(x)| \le \int_{\mathbb{R}} |\partial_i u(x', \xi)|\, \mathrm{d}\xi\,.$$

For ease of exposition we will write this from now on in the compact notation

$$|u(x)| \le \int_{\mathbb{R}} |\partial_i u|\, \mathrm{d}\xi_i\,.$$

(Observe that the above already proves the remark for the case $n = 1$.) Upon multiplying these n inequalities we obtain that

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \Big(\int_{\mathbb{R}} |\partial_i u| \, \mathrm{d}\xi_i \Big)^{\frac{1}{n-1}} .$$

Integration over x_1 yields

$$\int_{\mathbb{R}} |u|^{\frac{n}{n-1}} \, \mathrm{d}\xi_1 \leq \Big(\int_{\mathbb{R}} |\partial_1 u| \, \mathrm{d}\xi_1 \Big)^{\frac{1}{n-1}} \cdot \int_{\mathbb{R}} \prod_{i=2}^{n} \Big(\int_{\mathbb{R}} |\partial_i u| \, \mathrm{d}\xi_i \Big)^{\frac{1}{n-1}} \mathrm{d}\xi_1$$

and applying the generalized Hölder inequality we obtain that this is

$$\leq \Big(\int_{\mathbb{R}} |\partial_1 u| \, \mathrm{d}\xi_1 \Big)^{\frac{1}{n-1}} \cdot \prod_{i=2}^{n} \Big(\int_{\mathbb{R}^2} |\partial_i u| \, \mathrm{d}(\xi_1, \xi_i) \Big)^{\frac{1}{n-1}} .$$

Now we integrate over x_2 and obtain in the case $n = 2$ the desired result. In the case $n \geq 3$ it follows once again with the help of the Hölder inequality that

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} |u|^{\frac{n}{n-1}} \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 \\
&\leq \Big(\int_{\mathbb{R}^2} |\partial_2 u| \, \mathrm{d}(\xi_1, \xi_2) \Big)^{\frac{1}{n-1}} \\
&\quad \cdot \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |\partial_1 u| \, \mathrm{d}\xi_1 \Big)^{\frac{1}{n-1}} \prod_{i=3}^{n} \Big(\int_{\mathbb{R}^2} |\partial_i u| \, \mathrm{d}(\xi_1, \xi_i) \Big)^{\frac{1}{n-1}} \mathrm{d}\xi_2 \\
&\leq \Big(\int_{\mathbb{R}^2} |\partial_2 u| \, \mathrm{d}(\xi_1, \xi_2) \Big)^{\frac{1}{n-1}} \\
&\quad \cdot \Big(\int_{\mathbb{R}^2} |\partial_1 u| \, \mathrm{d}(\xi_1, \xi_2) \Big)^{\frac{1}{n-1}} \cdot \prod_{i=3}^{n} \Big(\int_{\mathbb{R}^3} |\partial_i u| \, \mathrm{d}(\xi_1, \xi_2, \xi_i) \Big)^{\frac{1}{n-1}} .
\end{aligned}$$

Continuing this procedure inductively we obtain for $j = 1, \dots, n$ that

$$\begin{aligned}
&\int_{\mathbb{R}^j} |u|^{\frac{n}{n-1}} \, \mathrm{d}(\xi_1, \dots, \xi_j) \\
&\leq \prod_{i=1}^{j} \Big(\int_{\mathbb{R}^j} |\partial_i u| \, \mathrm{d}(\xi_1, \dots, \xi_j) \Big)^{\frac{1}{n-1}} \\
&\quad \cdot \prod_{i=j+1}^{n} \Big(\int_{\mathbb{R}^{j+1}} |\partial_i u| \, \mathrm{d}(\xi_1, \dots, \xi_j, \xi_i) \Big)^{\frac{1}{n-1}} ,
\end{aligned}$$

and hence for $j = n$ that

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} \, \mathrm{dL}^n \leq \prod_{i=1}^{n} \Big(\int_{\mathbb{R}^n} |\partial_i u| \, \mathrm{dL}^n \Big)^{\frac{1}{n-1}} \leq \Big(\int_{\mathbb{R}^n} |\nabla u| \, \mathrm{dL}^n \Big)^{\frac{n}{n-1}} ,$$

i.e. the desired result

$$\|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le \|\nabla u\|_{L^1(\mathbb{R}^n)} \,. \tag{10-14}$$

For $p > 1$ we want to apply this result to $v = |u|^{\frac{q(n-1)}{n}}$, where on letting p' denote the dual exponent to p it holds that

$$\frac{n-1}{n} - \frac{1}{p'} = -\frac{1}{n} + \frac{1}{p} = \frac{1}{q}, \quad \text{and so} \quad \frac{q(n-1)}{n} = 1 + \frac{q}{p'} > 1\,.$$

In order to avoid unnecessary difficulties, we consider for $\varepsilon > 0$ the functions

$$v_\varepsilon(x) := \psi_\varepsilon(|u(x)|)^{\frac{q(n-1)}{n}}\,,$$

where $\psi_\varepsilon : [0,\infty[\; \to [0,\infty[$ is continuously differentiable, with

$$\psi_\varepsilon(z) \le z\,, \quad \psi_\varepsilon'(z) \le 1\,, \quad \psi_\varepsilon(z) \nearrow z \text{ as } \varepsilon \searrow 0\,.$$

As $u \in C^1(\mathbb{R}^n)$, we also have that $v_\varepsilon \in C^1(\mathbb{R}^n)$, with

$$|\nabla v_\varepsilon| \le \tfrac{q(n-1)}{n}\, w_\varepsilon \cdot |\nabla u|\,, \quad \text{where } w_\varepsilon := \psi_\varepsilon\big(|u|\big)^{\frac{q}{p'}}\,.$$

For $\varrho > 1$ we choose in particular

$$\psi_\varepsilon(z) := \Big(\varepsilon^\varrho + \Big(\tfrac{z}{1+\varepsilon z}\Big)^\varrho\Big)^{\frac{1}{\varrho}} - \varepsilon\,,$$

which means that there exists a constant C_ε depending on ε such that

$$\psi_\varepsilon(z) \le C_\varepsilon \cdot \min(1, z^\varrho)\,.$$

It follows that

$$w_\varepsilon \in L^{p'}(\mathbb{R}^n) \text{ and } v_\varepsilon \in L^1(\mathbb{R}^n), \quad \text{if } \varrho\, q\frac{n-1}{n} \ge s\,.$$

The Hölder inequality then yields that $\nabla v_\varepsilon \in L^1(\mathbb{R}^n;\mathbb{R}^n)$. It follows from inequality (10-14) that $v_\varepsilon \in L^{\frac{n}{n-1}}(\mathbb{R}^n)$, i.e. $\psi_\varepsilon\big(|u|\big) \in L^q(\mathbb{R}^n)$, with

$$\begin{aligned}
\Big(\int_{\mathbb{R}^n} \psi_\varepsilon(|u|)^q \,\mathrm{dL}^n\Big)^{\frac{n-1}{n}} &= \Big(\int_{\mathbb{R}^n} v_\varepsilon^{\frac{n}{n-1}} \,\mathrm{dL}^n\Big)^{\frac{n-1}{n}} \le \int_{\mathbb{R}^n} |\nabla v_\varepsilon| \,\mathrm{dL}^n \\
&\le \frac{q(n-1)}{n} \int_{\mathbb{R}^n} w_\varepsilon \cdot |\nabla u| \,\mathrm{dL}^n \\
&\le \frac{q(n-1)}{n} \Big(\int_{\mathbb{R}^n} \psi_\varepsilon(|u|)^q \,\mathrm{dL}^n\Big)^{\frac{1}{p'}} \|\nabla u\|_{L^p}\,,
\end{aligned}$$

and hence

$$\Big(\int_{\mathbb{R}^n} \psi_\varepsilon(|u|)^q \,\mathrm{dL}^n\Big)^{\frac{1}{q}} \le \frac{q(n-1)}{n} \|\nabla u\|_{L^p}\,.$$

Letting $\varepsilon \searrow 0$ we obtain the desired result from the monotone convergence theorem. $\square$

10.9 Embedding theorem in Sobolev spaces. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Further, let $m_1 \geq 0$, $m_2 \geq 0$ be integers, and let $1 \leq p_1 < \infty$ and $1 \leq p_2 < \infty$. Then the following holds:

(1) If

$$m_1 - \frac{n}{p_1} \geq m_2 - \frac{n}{p_2}\,, \quad \text{and} \quad m_1 \geq m_2\,, \tag{10-15}$$

then the embedding

$$\mathrm{Id} : W^{m_1,p_1}(\Omega) \to W^{m_2,p_2}(\Omega)$$

exists and is continuous. Here $W^{0,p}(\Omega) = L^p(\Omega)$. The following estimate holds: There exists a constant C, which depends on n, Ω, m_1, p_1, m_2, p_2, such that for $u \in W^{m_1,p_1}(\Omega)$

$$\|u\|_{W^{m_2,p_2}(\Omega)} \leq C\|u\|_{W^{m_1,p_1}(\Omega)}\,. \tag{10-16}$$

(2) If

$$m_1 - \frac{n}{p_1} > m_2 - \frac{n}{p_2}\,, \quad \text{and} \quad m_1 > m_2\,,$$

then the embedding

$$\mathrm{Id} : W^{m_1,p_1}(\Omega) \to W^{m_2,p_2}(\Omega)$$

exists and is continuous and compact.

(3) For arbitrary open, bounded sets $\Omega \subset \mathbb{R}^n$ assertions (1) and (2) hold with the spaces $W^{m_i,p_i}(\Omega)$ replaced by $W_0^{m_i,p_i}(\Omega)$. Here $W_0^{0,p}(\Omega) = L^p(\Omega)$.

Proof (1). We also prove the corresponding result in (3), i.e. we let $\Omega \subset \mathbb{R}^n$ be open and bounded. For $m_1 = m_2$ the claim follows from the Hölder inequality. For $m_1 = m_2 + 1$ we have that

$$1 - \frac{n}{p_1} \geq -\frac{n}{p_2}\,.$$

Let $u \in W_0^{m_1,p_1}(\Omega)$. For $|s| \leq m_2$ it holds that $v := \partial^s u \in W_0^{1,p_1}(\Omega)$. As Ω is bounded, it follows from the Hölder inequality that then v is also an element of $W_0^{1,p}(\Omega)$ for $1 \leq p \leq p_1$. Extending v by 0 on $\mathbb{R}^n \setminus \Omega$ yields that $v \in W^{1,p}(\mathbb{R}^n)$ (see 3.29). If $n = 1$, choose $p = 1$ and obtain from the remark in 10.8 that with $\varrho := \mathrm{L}^n(\Omega)$

$$\|v\|_{L^{p_2}(\Omega)} \;\leq\; \varrho^{\frac{1}{p_2}}\|v\|_{L^\infty(\mathbb{R})} \;\leq\; \varrho^{\frac{1}{p_2}}\|\nabla v\|_{L^1(\mathbb{R})} = \varrho^{\frac{1}{p_2}}\|\nabla v\|_{L^1(\Omega)}$$

and in the case $p_1 > 1$, with p_1' denoting the dual exponent to p_1, that

$$\|\nabla v\|_{L^1(\Omega)} \;\leq\; \varrho^{\frac{1}{p_1'}}\|\nabla v\|_{L^{p_1}(\Omega)}\,.$$

If $n \geq 2$, choose $1 \leq p \leq p_1 < \infty$ and $1 \leq p_2 \leq q < \infty$ with

$$1 - \frac{n}{p_1} \geq 1 - \frac{n}{p} = -\frac{n}{q} \geq -\frac{n}{p_2}\,,$$

e.g. $q = \max\bigl(\frac{n}{n-1}, p_2\bigr)$, and obtain from 10.8 that $v \in L^{p_2}(\Omega)$, with

$$\|v\|_{L^{p_2}(\Omega)} \;\leq\; \varrho^{\frac{1}{p_2} - \frac{1}{q}} \|v\|_{L^q(\mathbb{R}^n)} \;\leq\; \varrho^{\frac{1}{p_2} - \frac{1}{q}} \cdot q\frac{n-1}{n} \|\nabla v\|_{L^p(\mathbb{R}^n)}$$

and

$$\|\nabla v\|_{L^p(\mathbb{R}^n)} = \|\nabla v\|_{L^p(\Omega)} \leq \varrho^{\frac{1}{p} - \frac{1}{p_1}} \|\nabla v\|_{L^{p_1}(\Omega)}\,.$$

If Ω has a Lipschitz boundary, and if $u \in W^{m_1,p_1}(\Omega)$, then we have that $v := \partial^s u \in W^{1,p_1}(\Omega)$ for $|s| \leq m_2$. Then let $\widetilde{v} := E(v)$, where $E : W^{1,p_1}(\Omega) \to W_0^{1,p_1}(\widetilde{\Omega})$ with $\widetilde{\Omega} = \mathrm{B}_1(\Omega)$ is the extension operator from A8.12. Similarly to the above we then obtain the bound

$$\|\widetilde{v}\|_{L^{p_2}(\widetilde{\Omega})} \leq \widetilde{C} \cdot \|\nabla \widetilde{v}\|_{L^{p_1}(\widetilde{\Omega})}\,,$$

and hence, since $\widetilde{v} = v$ on Ω,

$$\|v\|_{L^{p_2}(\Omega)} \leq \|\widetilde{v}\|_{L^{p_2}(\widetilde{\Omega})} \leq \widetilde{C} \cdot \|\widetilde{v}\|_{W^{1,p_1}(\widetilde{\Omega})} \leq \widetilde{C} \cdot \|E\| \cdot \|v\|_{W^{1,p_1}(\Omega)}\,.$$

Now we consider the case $m_1 = m_2 + k$ with $k \geq 2$. Then let $\widetilde{m}_i := m_2 + i$ for $i = 0, \ldots, k$. Choose $1 \leq \widetilde{p}_i < \infty$ with $\widetilde{p}_0 = p_2$ and $\widetilde{p}_k = p_1$, such that

$$\widetilde{m}_i - \frac{n}{\widetilde{p}_i} \geq \widetilde{m}_{i-1} - \frac{n}{\widetilde{p}_{i-1}} \qquad \text{for } i = 1, \ldots, k, \tag{10-17}$$

e.g. $\widetilde{p}_i$ for $1 \leq i < k$ with $\frac{1}{\widetilde{p}_i} = \min\bigl(1, \frac{1}{n} + \frac{1}{\widetilde{p}_{i-1}}\bigr)$. Now apply the above proof successively for $i = k, \ldots, 1$. ☐

Proof (2). Once again we also prove the corresponding result in (3). For $m_1 = m_2 + 1$ choose $p_2 < p < \infty$ with

$$1 - \frac{n}{p_1} \;\geq -\frac{n}{p} \;> -\frac{n}{p_2}\,.$$

Let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $W^{m_1,p_1}(\Omega)$ (for (3) in $W_0^{m_1,p_1}(\Omega)$). For $|s| \leq m_2$ it then holds that $v_k := \partial^s u_k$ are bounded in $W^{1,p_1}(\Omega)$ (or $W_0^{1,p_1}(\Omega)$). By (1), the sequence $(v_k)_{k\in\mathbb{N}}$ is bounded in $L^p(\Omega)$. Since $L^p(\Omega)$ is reflexive, theorem 8.10 yields the existence of a subsequence $(v_{k_i})_{i\in\mathbb{N}}$, which can be chosen as the same subsequence for all $|s| \leq m_2$, that converges weakly in $L^p(\Omega)$ to $v \in L^p(\Omega)$. As Ω is bounded, $v_{k_i} \to v$ converges weakly in $L^1(\Omega)$ as $i \to \infty$ and $(v_{k_i})_{i\in\mathbb{N}}$ is bounded in $W^{1,1}(\Omega)$ (or $W_0^{1,1}(\Omega)$). Hence it follows from Rellich's embedding theorem (A8.1 and A8.4) that $v_{k_i} \to v$ strongly in $L^1(\Omega)$. Noting that $1 \leq p_2 < p$ then yields the strong convergence also in $L^{p_2}(\Omega)$ (see E10.11).

For $m_1 = m_2 + k$ with $k \geq 2$ we again choose $\widetilde{m}_i$, $\widetilde{p}_i$ as in the proof of (1), where now (10-17) needs to be a strict inequality for an $i_0 \in \{1, \ldots, k\}$. Then for i_0 we can apply the above proof, and for $i \neq i_0$ the result (1). ☐

Now we consider the embedding of Sobolev spaces into Hölder spaces. The proof of theorem 10.13 rests on two results: a bound on the supremum norm and a bound on the Hölder constant.

10.10 Theorem. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $1 < p < \infty$, with

$$1 - \frac{n}{p} > 0 \quad (\text{and so } p > n \geq 1)\,.$$

For every function $u \in W_0^{1,p}(\Omega)$ it then holds that $u \in L^\infty(\Omega)$ with

$$\|u\|_{L^\infty(\Omega)} \leq C(n, p, \operatorname{diam}\Omega)\|\nabla u\|_{L^p(\Omega)}\,.$$

Proof. Analogously to the proof of 10.8, it is sufficient to establish the desired result for functions $u \in C_0^\infty(\Omega)$. Further, let $R := \operatorname{diam}\Omega$, so that $\Omega \subset \mathrm{B}_R(x_0)$ for all $x_0 \in \Omega$. Then it holds for all $\xi \in \partial\mathrm{B}_1(0)$ that

$$|u(x_0)| = \left| \int_0^R \frac{\mathrm{d}}{\mathrm{d}r}\big(u(x_0 + r\xi)\big)\,\mathrm{d}r \right| \leq \int_0^R |\nabla u(x_0 + r\xi)|\,\mathrm{d}r\,.$$

Integrating this inequality over ξ with respect to the surface measure H^{n-1} and denoting the surface area of the unit sphere by $\sigma_n := \mathrm{H}^{n-1}(\partial\mathrm{B}_1(0))$ we get

$$\sigma_n |u(x_0)| \leq \int_0^R \int_{\partial\mathrm{B}_1(0)} |\nabla u(x_0 + r\xi)|\,\mathrm{dH}^{n-1}(\xi)\,\mathrm{d}r\,.$$

A transformation to Euclidean coordinates shows that the right-hand side is

$$= \int_{\mathrm{B}_R(x_0)} \frac{|\nabla u(x)|}{|x - x_0|^{n-1}}\,\mathrm{d}x\,,$$

and the Hölder inequality yields that this can be bounded by

$$\leq \Big(\int_{\mathrm{B}_R(x_0)} \frac{\mathrm{d}x}{|x - x_0|^{p'(n-1)}} \Big)^{\frac{1}{p'}} \cdot \|\nabla u\|_{L^p(\Omega)}\,.$$

The first factor is independent of x_0 and finite if $p'(n-1) < n$, i.e. if $p' < n'$ (where n' is the dual exponent to n), which is equivalent to $p > n$. But this was part of the assumption. $\square$

10.11 Theorem (Morrey). Let $\Omega \subset \mathbb{R}^n$ be open, let $0 < \alpha \leq 1$ and let $u \in W_0^{1,1}(\Omega)$ satisfy

$$\int_{\mathrm{B}_r(x_0)\cap\Omega} |\nabla u|\,\mathrm{dL}^n \leq M \cdot r^{n-1+\alpha} \tag{10-18}$$

for all $x_0 \in \Omega$ and $r > 0$. Then for almost all $x_1, x_2 \in \Omega$,

$$\frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha} \leq C(n, \alpha) \cdot M\,. \tag{10-19}$$

Note: A p-version of the result is given in 10.12(1).

Proof. We may assume that $u \in W^{1,1}(\mathbb{R}^n)$, because u can be extended by 0 on $\mathbb{R}^n \setminus \Omega$ to yield a function in $W^{1,1}(\mathbb{R}^n)$ (see 3.29). Then for every ball $\mathrm{B}_r(x_0)$ with $x_0 \in \mathbb{R}^n$ we have that

$$\int_{\mathrm{B}_r(x_0)} |\nabla u| \,\mathrm{dL}^n \le M(2r)^{n-1+\alpha}\,, \tag{10-20}$$

on noting that in the case $\mathrm{B}_r(x_0) \cap \Omega = \emptyset$ this is trivially true, and that otherwise there exists an $x_1 \in \mathrm{B}_r(x_0) \cap \Omega$ and then $\mathrm{B}_r(x_0) \subset \mathrm{B}_{2r}(x_1)$, and for this latter ball we can apply (10-18).

We begin by proving the bound on the Hölder constant for the case where u is a C^1-function. Given two points $x_1, x_2 \in \mathbb{R}^n$, let

$$x_0 := \frac{1}{2}(x_1 + x_2) \quad \text{and} \quad \varrho := \frac{1}{2}|x_2 - x_1|\,.$$

Denoting the volume of the n-dimensional unit ball by κ_n, we have that

$$\begin{aligned} &\kappa_n \varrho^n |u(x_1) - u(x_2)| = \int_{\mathrm{B}_\varrho(x_0)} |u(x_1) - u(x_2)| \,\mathrm{d}x \\ &\le \int_{\mathrm{B}_\varrho(x_0)} |u(x_1) - u(x)| \,\mathrm{d}x + \int_{\mathrm{B}_\varrho(x_0)} |u(x_2) - u(x)| \,\mathrm{d}x\,. \end{aligned} \tag{10-21}$$

Because of symmetry we only need to bound the first integral. Now it holds for $x \in \mathrm{B}_\varrho(x_0)$ that

$$\begin{aligned} |u(x) - u(x_1)| &= \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big(u(x_1 + t(x - x_1))\big) \,\mathrm{d}t \right| \\ &\le |x - x_1| \int_0^1 |\nabla u(x_1 + t(x - x_1))| \,\mathrm{d}t\,. \end{aligned}$$

Since $|x - x_1| \le 2\varrho$, integration over x yields

$$\int_{\mathrm{B}_\varrho(x_0)} |u(x) - u(x_1)| \,\mathrm{d}x \le 2\varrho \int_0^1 \int_{\mathrm{B}_\varrho(x_0)} |\nabla u(x_1 + t(x - x_1))| \,\mathrm{d}x \,\mathrm{d}t\,.$$

With the transformation of variables $y(x) := x_1 + t(x - x_1)$ this is

$$\begin{aligned} &= 2\varrho \int_0^1 t^{-n} \int_{\mathrm{B}_{t\varrho}(x_1 + t(x_0 - x_1))} |\nabla u(y)| \,\mathrm{d}y \,\mathrm{d}t \\ &\le 2\varrho \int_0^1 t^{-n} M(2t\varrho)^{n-1+\alpha} \,\mathrm{d}t = \frac{M}{\alpha}(2\varrho)^{n+\alpha}\,, \end{aligned}$$

where we used (10-20). Hence it follows from (10-21) that

$$|u(x_1) - u(x_2)| \le \frac{2^{n+1}M}{\alpha\kappa_n}(2\varrho)^\alpha = \frac{2^{n+1}M}{\alpha\kappa_n}|x_1 - x_2|^\alpha\,. \tag{10-22}$$

For an arbitrary $u \in W^{1,1}(\mathbb{R}^n)$ one can consider the convolution with a standard Dirac sequence $(\varphi_\varepsilon)_{\varepsilon>0}$. Then the functions $u_\varepsilon := \varphi_\varepsilon * u$ are in $C^\infty(\mathbb{R}^n)$ and satisfy (10-20). Indeed,

$$\nabla u_\varepsilon(x) = \int_{\mathbb{R}^n} \varphi_\varepsilon(y) \nabla u(x-y)\,\mathrm{d}y$$

and so, using (10-20) for u, we have

$$\begin{aligned}\int_{\mathrm{B}_r(x_0)} |\nabla u_\varepsilon(x)|\,\mathrm{d}x &\leq \int_{\mathbb{R}^n} \left(\int_{\mathrm{B}_r(x_0)} |\nabla u(x-y)|\,\mathrm{d}x \right) \varphi_\varepsilon(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathrm{B}_r(x_0-y)} |\nabla u(x)|\,\mathrm{d}x \right) \varphi_\varepsilon(y)\,\mathrm{d}y \\ &\leq M(2r)^{n-1+\alpha} \int_{\mathbb{R}^n} \varphi_\varepsilon(y)\,\mathrm{d}y = M(2r)^{n-1+\alpha}\,.\end{aligned}$$

Hence we obtain (10-22) for u_ε, and noting that $u_\varepsilon \to u$ almost everywhere for a subsequence as $\varepsilon \to 0$ then yields the desired result. □

10.12 Remarks. The inequality (10-19) states that u is Hölder continuous outside of a null set N. But then the function u restricted to $\Omega \setminus N$ can be uniquely extended to a $C^{0,\alpha}$-function on $\overline{\Omega}$. Hence the given function $u \in W_0^{1,1}(\Omega)$ has a unique Hölder continuous representative. Moreover, it holds that:

(1) Theorem 10.11 can also be applied in the general case where $u \in W_0^{1,p}(\Omega)$ with $1 \leq p < \infty$. If u then satisfies for $0 < \alpha \leq 1$ the inequality

$$\|\nabla u\|_{L^p(\mathrm{B}_r(x_0)\cap\Omega)} \leq M \cdot r^{\frac{n}{p}-1+\alpha} \tag{10-23}$$

for all $x_0 \in \Omega$ and $r > 0$, then the conclusion of 10.11 holds true.

(2) If $u \in W_0^{1,p}(\Omega)$ with $1 - \frac{n}{p} > 0$, then (1) holds with $\alpha := 1 - \frac{n}{p}$.

(3) Theorem 10.11 also holds for $\Omega = \mathbb{R}^n$ and $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof (1). The Hölder inequality yields that

$$\int_{\mathrm{B}_r(x_0)\cap\Omega} |\nabla u|\,\mathrm{dL}^n \leq C(n) r^{\frac{n}{p'}} \left(\int_{\mathrm{B}_r(x_0)\cap\Omega} |\nabla u|^p\,\mathrm{dL}^n \right)^{\frac{1}{p}} \leq C(n) M r^{n-1+\alpha}.$$

□

10.13 Embedding theorem of Sobolev spaces into Hölder spaces. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Moreover, let $m \geq 1$ be an integer and let $1 \leq p < \infty$. In addition, let $k \geq 0$ be an integer and let $0 \leq \alpha \leq 1$. Then the following holds:

(1) If

$$m - \frac{n}{p} = k + \alpha \quad \text{and} \quad 0 < \alpha < 1 \text{ (and so } \alpha \neq 0, 1), \tag{10-24}$$

then the embedding

$$\mathrm{Id} : W^{m,p}(\Omega) \to C^{k,\alpha}(\overline{\Omega})$$

exists and is continuous. In particular, for $u \in W^{m,p}(\Omega)$ there exists a unique continuous function that agrees almost everywhere with u (and which we again denote by u) such that

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} \leq C(\Omega, n, m, p, k, \alpha)\|u\|_{W^{m,p}(\Omega)} . \tag{10-25}$$

(2) If

$$m - \frac{n}{p} > k + \alpha \,,$$

then the embedding

$$\mathrm{Id} : W^{m,p}(\Omega) \to C^{k,\alpha}(\overline{\Omega})$$

exists and is continuous and compact. Here $C^{k,0}(\overline{\Omega}) := C^k(\overline{\Omega})$ for $k \geq 0$.

(3) For arbitrary open, bounded sets $\Omega \subset \mathbb{R}^n$ assertions (1) and (2) hold with the space $W^{m,p}(\Omega)$ replaced by $W_0^{m,p}(\Omega)$.

Proof (1). We also prove the corresponding result in (3). We may assume that $k = 0$. Otherwise apply the following argument to all functions $\partial^s u \in W^{m-k,p}(\Omega)$ (or $W_0^{m-k,p}(\Omega)$) for $|s| \leq k$, on noting that $m - k \geq 1$.

Next we reduce the proof to the case $m = 1$. If $m > 1$, we may choose $1 \leq q < \infty$ such that

$$\alpha - 1 = -\frac{n}{q} \,, \qquad \text{and so} \qquad m - \frac{n}{p} = \alpha = 1 - \frac{n}{q} \,.$$

It then follows from 10.9(1) that the embedding from $W^{m,p}(\Omega)$ into $W^{1,q}(\Omega)$ is continuous (use 10.9(3) for the embedding from $W_0^{m,p}(\Omega)$ into $W_0^{1,q}(\Omega)$). Thus we have to consider only functions in $W^{1,q}(\Omega)$ (or $W_0^{1,q}(\Omega)$).

Hence we consider only the case where in the statement of the theorem $k = 0$ and $m = 1$, i.e.

$$1 - \frac{n}{p} \;=\; \alpha \,.$$

For the case in (3), the desired result follows upon combining theorem 10.10 and theorem 10.11 (see 10.12(2)). Otherwise we consider the continuous extension operator $E : W^{1,p}(\Omega) \to W_0^{1,p}\big(\mathrm{B}_1(\Omega)\big)$ from A8.12 and then apply the theorems 10.10 and 10.11 to the functions Eu. $\square$

Proof (2). We also prove the corresponding result in (3). Choose $\widetilde{m} \leq m$ and $1 < \widetilde{p} < \infty$, as well as $\widetilde{k} \geq 0$ and $0 < \widetilde{\alpha} < 1$, such that

$$m - \frac{n}{p} \geq \widetilde{m} - \frac{n}{\widetilde{p}} = \widetilde{k} + \widetilde{\alpha} > k + \alpha\,,$$

where we can set $\widetilde{m} = m$ and $\widetilde{p} = p$ if $\frac{n}{p}$ is not an integer. Then, by 10.9(1) and (1), the embeddings from $W^{m,p}(\Omega)$ into $W^{\widetilde{m},\widetilde{p}}(\Omega)$ and from $W^{\widetilde{m},\widetilde{p}}(\Omega)$ into $C^{\widetilde{k},\widetilde{\alpha}}(\Omega)$ are continuous, respectively (for (3) we argue correspondingly with 10.9(3)). Finally, by 10.6, the embedding from $C^{\widetilde{k},\widetilde{\alpha}}(\overline{\Omega})$ into $C^{k,\alpha}(\overline{\Omega})$ is compact. $\square$

Laplace operator

We now present a typical application of the embedding theorems for the Laplace operator. This is essential for the treatment of the corresponding eigenvalue problem (see 12.16).

10.14 Inverse Laplace operator. We consider the homogeneous Dirichlet problem from 6.5(1) with the assumptions stated there and with

$$h_i = 0\,, \quad b \geq 0\,.$$

For $u \in W_0^{1,2}(\Omega)$ and $f \in L^2(\Omega)$ let $A(u)$ and $J(f)$ be the functionals in $W_0^{1,2}(\Omega)'$ defined by

$$\begin{aligned}
\langle v\,,\, A(u)\rangle_{W_0^{1,2}} &:= \int_\Omega \Big(\sum_{i,j=1}^n \partial_i v \cdot a_{ij} \partial_j u + vbu \Big)\,\mathrm{dL}^n\,, \\
\langle v\,,\, J(f)\rangle_{W_0^{1,2}} &:= \int_\Omega vf\,\mathrm{dL}^n
\end{aligned}$$

for $v \in W_0^{1,2}(\Omega)$. Then it holds that:

(1) $J : L^2(\Omega) \to W_0^{1,2}(\Omega)'$ is continuous and injective.

(2) $A : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)'$ is an isomorphism. We call A the ***weak differential operator*** corresponding to the boundary value problem 6.5(1). For $a_{ij} = \delta_{i,j}$ and $b = 0$ this is the ***weak Laplace operator*** with respect to homogeneous Dirichlet boundary conditions.

(3) $A^{-1}J : L^2(\Omega) \to L^2(\Omega)$ is compact.

(4) $A^{-1}J : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is compact, and for domains Ω with Lipschitz boundary the operator $A^{-1}J : W^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is also compact.

(5) $JA^{-1} : W_0^{1,2}(\Omega)' \to W_0^{1,2}(\Omega)'$ is compact.

Proof (1),(2). We have that $\langle v\,,\,Au\rangle = a(v,u)$, where a is defined as in (6-11). The fact that J and A are well defined and continuous follows as in the proof of 6.6. It follows from 4.22 that J is injective. A is injective thanks of the coercivity of a, as shown in the proof of 6.8. Recalling 6.3(1) with $X := W_0^{1,2}(\Omega)$ yields that for $u' \in X'$ there exists a unique $u \in X$ such that

$$\langle v\,,\,Au\rangle = a(v,u) = \langle v\,,\,u'\rangle \qquad \text{for all } v \in X,$$

where $\|u\|_X$ can be bounded by $\|u'\|_{X'}$. □

Proof (3). We recall from (1) and (2) that $J : L^2(\Omega) \to W_0^{1,2}(\Omega)'$ and $A^{-1} : W_0^{1,2}(\Omega)' \to W_0^{1,2}(\Omega)$, respectively, are continuous. The embedding $\mathrm{Id} : W_0^{1,2}(\Omega) \to L^2(\Omega)$ is compact, by 10.1(4) and A8.1. The desired result then follows from 10.3.

Remark: If Ω has a Lipschitz boundary, then it follows from 10.9 that $\mathrm{Id} : W^{1,2}(\Omega) \to L^2(\Omega)$ is also compact. □

Proof (4),(5). We can argue with the above maps in the order Id, J, A^{-1} and A^{-1}, Id, J, respectively. □

Integral operators

As a second class of compact maps we now investigate some integral operators. Such operators occur, for example, when boundary value problems are reformulated as integral equations with the help of a Green's function (see 10.18). First we prove the compactness of Hilbert-Schmidt operators and of integral operators with a weakly singular kernel.

10.15 Hilbert-Schmidt integral operator. We have defined in 5.12 an integral operator $T : L^p(\Omega_2) \to L^q(\Omega_1)$, which we claim is compact.

Proof. We recall from 5.12 that T is continuous with $\|T\| \le \|K\|$. In order to prove the compactness of T we extend K by 0 outside $\Omega_1 \times \Omega_2$, i.e. $K(x,y) := 0$ if $x \notin \Omega_1$ or $y \notin \Omega_2$. Then it follows for $h \in \mathbb{R}^{n_1}$ and $f \in L^p(\Omega_2)$ with $\|f\|_{L^p(\Omega_2)} \le 1$, in the same way as in the proof of 5.12, that

$$\begin{aligned}&\int_{\mathbb{R}^{n_1}} |Tf(x+h) - Tf(x)|^q \,\mathrm{d}x \\ &\le \int_{\mathbb{R}^{n_1}} \Big(\int_{\mathbb{R}^{n_2}} |K(x+h,y) - K(x,y)|^{p'} \,\mathrm{d}y\Big)^{\frac{q}{p'}} \,\mathrm{d}x\end{aligned} \tag{10-26}$$

and

$$\int_{\mathbb{R}^{n_1}\setminus \mathrm{B}_R(0)} |Tf(x)|^q \,\mathrm{d}x \le \int_{\mathbb{R}^{n_1}\setminus \mathrm{B}_R(0)} \Big(\int_{\mathbb{R}^{n_2}} |K(x,y)|^{p'} \,\mathrm{d}y\Big)^{\frac{q}{p'}} \,\mathrm{d}x\,. \tag{10-27}$$

The right-hand side in (10-27) converges to 0 as $R \to \infty$, since $\|K\| < \infty$. If, in addition, the right-hand side in (10-26) converges to 0 as $h \to 0$, then the compactness of T follows from the Riesz compactness criterion in theorem 4.16. To show this let $K^h(x,y) := K(x+h,y)$. We need to consider $\|K^h - K\|$, where here the norm of the kernel is defined by integrating over all of $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. We begin by approximating K by bounded kernels with compact support

$$K_R(x,y) := \begin{cases} K(x,y) & \text{if } |x| \le R,\ |y| \le R,\ |K(x,y)| \le R, \\ 0 & \text{otherwise.} \end{cases}$$

Then, on setting $E_R := \{ (x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \ ;\ K(x,y) \neq K_R(x,y) \}$, we have that

$$|K^h - K| \le |(K_R)^h - K_R| + |(\mathcal{X}_{E_R} K)^h| + |\mathcal{X}_{E_R} K|,$$

which yields that

$$\|K^h - K\| \le C\big(\|(K_R)^h - K_R\| + \|\mathcal{X}_{E_R} K\|\big).$$

Noting that $E_{R'} \subset E_R$ for $R' > R$ and that $\bigcap_{R>0} E_R$ is a null set we see that the second term on the right-hand side converges to 0 as $R \to \infty$ (analogously to (10-27) consider the monotone convergence of $(1 - \mathcal{X}_{E_R})|K|$). Since K_R is bounded with compact support, the first term in the case $\frac{q}{p'} \ge 1$ obeys the inequality

$$\|(K_R)^h - K_R\|^q \le C(R, \frac{q}{p'}) \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} |(K_R)^h - K_R|^{p'}(x,y)\,\mathrm{d}y\,\mathrm{d}x\,,$$

while in the case $r := \frac{p'}{q} > 1$ the Hölder inequality with exponent r gives

$$\begin{aligned} \|(K_R)^h - K_R\|^{p'} &= \Big(\int_{\mathbb{R}^{n_1}} \Big(\int_{\mathbb{R}^{n_2}} |(K_R)^h - K_R|^{p'}(x,y)\,\mathrm{d}y\Big)^{\frac{1}{r}} \mathrm{d}x\Big)^r \\ &\le C(R,r) \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} |(K_R)^h - K_R|^{p'}(x,y)\,\mathrm{d}y\,\mathrm{d}x\,. \end{aligned}$$

Now we use the fact that $(K_R)^h \to K_R$ in $L^{p'}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ as $h \to 0$, recall 4.15(1).

In the Hilbert space case $p = 2$, $q = 2$ the compactness can also be shown as follows: Choose an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $L^2(\Omega_2)$ (see 9.8). Then, by the completeness relation 9.7(5),

$$\begin{aligned} \|K\|^2 &= \int_{\Omega_1} \big\|\overline{K(x,\cdot)}\big\|^2_{L^2(\Omega_2)}\,\mathrm{d}x = \int_{\Omega_1} \sum_{n\in\mathbb{N}} \Big|\big(\overline{K(x,\cdot)}\,,\, e_n\big)_{L^2(\Omega_2)}\Big|^2\,\mathrm{d}x \\ &= \int_{\Omega_1} \sum_{n\in\mathbb{N}} |Te_n(x)|^2\,\mathrm{d}x = \sum_{n\in\mathbb{N}} \|Te_n\|^2_{L^2(\Omega_1)}\,. \end{aligned}$$

We define the continuous projections P_n by

$$P_n f := \sum_{k=1}^{n} (f\,,\,e_k)_{L^2(\Omega_2)}\,e_k\,.$$

Then using 9.7(3) and the continuity of T, we see that

$$\|Tf - TP_n f\|_{L^2(\Omega_1)} = \left\| T\left(\sum_{k>n} (f\,,\,e_k)_{L^2(\Omega_2)}\,e_k\right)\right\|_{L^2(\Omega_1)}$$

$$= \left\|\sum_{k>n} (f\,,\,e_k)_{L^2(\Omega_2)}\,Te_k\right\|_{L^2(\Omega_1)} \leq \sum_{k>n} \left|(f\,,\,e_k)_{L^2(\Omega_2)}\right| \|Te_k\|_{L^2(\Omega_1)}\,.$$

On applying the Cauchy-Schwarz inequality in $\ell^2(\mathbb{R})$ we find that this is

$$\leq \underbrace{\left(\sum_{k>n} \left|(f\,,\,e_k)_{L^2(\Omega_2)}\right|^2\right)^{\frac{1}{2}}}_{\leq \|f\|_{L^2(\Omega_2)}} \cdot \underbrace{\left(\sum_{k>n} \|Te_k\|^2_{L^2(\Omega_1)}\right)^{\frac{1}{2}}}_{\to 0 \text{ as } n\to\infty}\,.$$

Hence, $TP_n \to T$ in $\mathscr{L}\big(L^2(\Omega_2)\,;L^2(\Omega_1)\big)$ as $n \to \infty$. Since $\mathscr{R}(P_n)$, and hence also $\mathscr{R}(TP_n) = T\big(\mathscr{R}(P_n)\big)$, are finite-dimensional, it follows from 10.2(4) that $T \in \mathscr{K}\big(L^2(\Omega_2)\,;L^2(\Omega_1)\big)$. $\square$

We now discuss operators with ***weakly singular integral kernels***, i.e. kernel functions $(x,y) \mapsto K(x,y)$ that for x fixed are locally integrable in y.

10.16 Schur integral operators. Let $\Omega \subset \mathbb{R}^n$ be open and bounded (!). Let $K : (\overline{\Omega}\times\overline{\Omega}) \setminus D \to \mathbb{K}$ be continuous, where $D := \{(x,x)\,;\; x \in \overline{\Omega}\}$ is the diagonal of $\overline{\Omega}\times\overline{\Omega}$. Assume that

$$|K(x,y)| \leq \frac{C}{|x-y|^\alpha} \qquad \text{with } \alpha < n.$$

Then it holds that:

(1) The definition

$$(Tf)(x) := \int_\Omega K(x,y)f(y)\,\mathrm{d}y$$

yields a map $T \in \mathscr{K}\big(C^0(\overline{\Omega})\big)$.

(2) The composition of operators of Schur type is again a Schur operator. In particular, the iterated operators T^m are integral operators of the above type, with exponent

$$\alpha_m = \begin{cases} n - m(n-\alpha) & \text{if } 1 \le m < \frac{n}{n-\alpha}, \\ \varepsilon & \text{for every } \varepsilon > 0, \text{ if } m = \frac{n}{n-\alpha}, \\ 0 & \text{if } m > \frac{n}{n-\alpha}. \end{cases}$$

(3) If $1 \le p < \infty$ with $\alpha < \frac{n}{p'}$, then T is a Hilbert-Schmidt operator on $L^p(\Omega)$ and $T \in \mathscr{K}\big(L^p(\Omega); C^0(\overline{\Omega})\big)$.

Proof (1) *and* (3). We can always ensure that $\alpha < \frac{n}{p'}$, on choosing p sufficiently large. Moreover, the boundedness of Ω yields that the embedding from $C^0(\overline{\Omega})$ into $L^p(\Omega)$ is continuous for all p. Hence it follows from 10.3 that we only need to show the compactness of $T : L^p(\Omega) \to C^0(\overline{\Omega})$. We have that $Tf(x)$ exists for all x and

$$|Tf(x)| \le C \cdot \Big(\int_\Omega \frac{\mathrm{d}y}{|x-y|^{\alpha p'}} \Big)^{\frac{1}{p'}} \|f\|_{L^p(\Omega)} \,.$$

Since $\alpha p' < n$ and Ω is bounded, the integral on the right-hand side is bounded uniformly in x. Hence the functions Tf with $\|f\|_{L^p(\Omega)} \le 1$ are uniformly bounded. It follows from the Arzelà-Ascoli theorem that it is sufficient to show that they are also equicontinuous, since then 10.1(2) is satisfied. It holds that

$$|Tf(x_1) - Tf(x_2)| \le \|f\|_{L^p(\Omega)} \cdot \Big(\int_\Omega |K(x_1,y) - K(x_2,y)|^{p'} \,\mathrm{d}y \Big)^{\frac{1}{p'}}$$

and the integral on the right-hand side can be bounded for every $\delta > 0$ by

$$\begin{aligned} &\le \int_{\Omega \setminus \mathrm{B}_\delta(x_1)} |K(x_1,y) - K(x_2,y)|^{p'} \,\mathrm{d}y \\ &\quad + C \cdot \int_{\mathrm{B}_\delta(x_1)} \Big(\frac{1}{|y-x_1|^{\alpha p'}} + \frac{1}{|y-x_2|^{\alpha p'}} \Big) \,\mathrm{d}y \,. \end{aligned}$$

For $|x_1 - x_2| \le \frac{\delta}{2}$ the first term is

$$\begin{aligned} &\le C \sup \big\{ \, |K(x_1,y) - K(x_2,y)|^{p'} \;;\; (x_1,y),(x_2,y) \notin \mathrm{B}_{\frac{\delta}{4}}(D) \, \big\} \\ &\longrightarrow 0 \quad \text{ as } |x_1 - x_2| \to 0 \text{ and for every } \delta \,, \end{aligned}$$

thanks to the continuity of K away from the diagonal D, and the second term is

$$\le C \int_{\mathrm{B}_{2\delta}(0)} \frac{\mathrm{d}y}{|y|^{\alpha p'}} \le C\delta^{n-\alpha p'} \longrightarrow 0 \quad \text{ as } \delta \to 0.$$

Here we assume the usual ***convention on constants***, which states that constants that occur in a chain of inequalities may all be denoted by C, even though the constant will in general change after each step. In addition, this convention states that large positive constants are denoted by C, while small positive constants are denoted by c.

The bound above proves the equicontinuity of the functions Tf with $\|f\|_{L^p(\Omega)} \le 1$, and hence we have shown that $T \in \mathscr{K}\big(L^p(\Omega); C^0(\overline{\Omega})\big)$. $\square$

Proof (2). Now let T_1, T_2 be two such integral operators with kernels K_1, K_2 and exponents $\alpha_1 < n$ and $\alpha_2 < n$. By Fubini's theorem, for $f \in C^0(\overline{\Omega})$ we have that

$$\begin{aligned} T_1T_2f(x) &= \int_\Omega K_1(x,z)\Big(\int_\Omega K_2(z,y)f(y)\,\mathrm{d}y\Big)\,\mathrm{d}z \\ &= \int_\Omega \underbrace{\Big(\int_\Omega K_1(x,z)K_2(z,y)\,\mathrm{d}z\Big)}_{=:\,K(x,y)} f(y)\,\mathrm{d}y\,, \end{aligned}$$

if we can show that for each fixed x the function

$$y \longmapsto \widetilde{K}(x,y) := \int_\Omega |K_1(x,z)K_2(z,y)|\,\mathrm{d}z$$

is in $L^1(\Omega)$. To this end, we show that for $x \ne y$ (with the usual convention on constants)

$$\begin{aligned} |K(x,y)| \le \widetilde{K}(x,y) &\le C\int_\Omega \frac{\mathrm{d}z}{|z-x|^{\alpha_1}|z-y|^{\alpha_2}} \\ &\le \begin{cases} \dfrac{C}{|x-y|^{\alpha_1+\alpha_2-n}} & \text{if } \alpha_1+\alpha_2 > n, \\ C_R\log\dfrac{R}{|x-y|} \le \dfrac{C_{R,\varepsilon}}{|x-y|^\varepsilon} & \text{if } \alpha_1+\alpha_2 = n \\ & \text{for large } R \text{ and every } \varepsilon > 0, \\ C & \text{if } \alpha_1+\alpha_2 < n, \end{cases} \end{aligned}$$

where in the last case K is bounded. In order to prove these bounds, we replace z by $\frac{x+y}{2} - |x-y|z$ and set

$$e := \frac{x-y}{2|x-y|}\,, \quad \Omega_{x,y} := \{z \in \mathbb{R}^n\,;\ \frac{x+y}{2} - |x-y|z \in \Omega\}\,.$$

Then

$$\int_\Omega \frac{\mathrm{d}z}{|z-x|^{\alpha_1}|z-y|^{\alpha_2}} = |x-y|^{n-\alpha_1-\alpha_2}\int_{\Omega_{x,y}} \frac{\mathrm{d}z}{|z+e|^{\alpha_1}|z-e|^{\alpha_2}} \tag{10-28}$$

and

$$\frac{1}{|z+e|^{\alpha_1}|z-e|^{\alpha_2}} \le \begin{cases} 2^{\alpha_1}|z-e|^{-\alpha_2} & \text{for } |z-e| \le \frac{1}{2}, \\ 2^{\alpha_2}|z+e|^{-\alpha_1} & \text{for } |z+e| \le \frac{1}{2}, \\ (|z|-\frac{1}{2})^{-\alpha_1-\alpha_2} & \text{for } |z| \ge 1, \\ 2^{\alpha_1+\alpha_2} & \text{otherwise.} \end{cases} \tag{10-29}$$

We distinguish between the three stated cases.

For $\alpha_1 + \alpha_2 > n$ it follows that

$$\int_{\Omega_{x,y}} \frac{dz}{|z+e|^{\alpha_1}|z-e|^{\alpha_2}} \le \int_{\mathbb{R}^n} \frac{dz}{|z+e|^{\alpha_1}|z-e|^{\alpha_2}} \,. \tag{10-30}$$

Since $\alpha_1 < n$, $\alpha_2 < n$ and $\alpha_1 + \alpha_2 > n$, the integral on the right-hand side exists and its value is independent of e and depends only on n, α_1, α_2. To see this, let $e_1, e_2 \in \partial \mathrm{B}_{\frac{1}{2}}(0)$ and choose a linear orthogonal transformation which maps e_1 to e_2. It follows from the transformation (change-of-variables) theorem that the integrals for e_1 and e_2 are equal. This proves that the last integral in (10-30) depends only on n, α_1, α_2.

For $\alpha_1 + \alpha_2 = n$ we choose a radius R with $\Omega \subset \mathrm{B}_{\frac{R}{2}}(0)$. Then it follows that $|z-x|^{-\alpha_1} \le C_{R,\varepsilon}|z-x|^{-\alpha_1-\varepsilon}$ for $z, x \in \Omega$ for every fixed $\varepsilon > 0$. Hence for ε sufficiently small we can apply the first case to $\alpha_1 + \varepsilon$ and α_2. This is the second estimate. It follows that $\Omega \subset \mathrm{B}_R\big(\frac{x+y}{2}\big)$ for $x, y \in \Omega$, hence (10-28) implies

$$\begin{aligned} \int_{\Omega} \frac{dz}{|z-x|^{\alpha_1}|z-y|^{\alpha_2}} &\le \int_{\mathrm{B}_{\frac{R}{|x-y|}}(0)} \frac{dz}{|z+e|^{\alpha_1}|z-e|^{\alpha_2}} \\ &\le C_R \cdot \Big(1 + \int_{\mathrm{B}_{\frac{R}{|x-y|}}(0) \setminus \mathrm{B}_1(0)} \frac{dz}{|z|^{\alpha_1+\alpha_2}}\Big) \\ &\le C_R \cdot \big(1 + \log \frac{R}{|x-y|}\big) \,, \end{aligned}$$

hence the desired first estimate.

For the case $\alpha_1 + \alpha_2 < n$ we decompose the integral over Ω into integrals over $\mathrm{B}_\delta(x)$, $\mathrm{B}_\delta(y)$ and $\Omega \setminus \big(\mathrm{B}_\delta(x) \cup \mathrm{B}_\delta(y)\big)$, where $\delta := \frac{3}{4}|x-y|$. On noting that in the latter set it holds that $|z-x| \ge c\,\big|z - \frac{x+y}{2}\big|$ and $|z-y| \ge c\,\big|z - \frac{x+y}{2}\big|$ with a small constant c, we obtain that

$$\begin{aligned} \int_{\Omega} \frac{dz}{|z-x|^{\alpha_1}|z-y|^{\alpha_2}} &\le C\,\delta^{-\alpha_2} \int_{\mathrm{B}_\delta(x)} \frac{dz}{|z-x|^{\alpha_1}} \\ &\quad + C\,\delta^{-\alpha_1} \int_{\mathrm{B}_\delta(y)} \frac{dz}{|z-y|^{\alpha_2}} + C \int_{\Omega} \frac{dz}{\big|z - \frac{x+y}{2}\big|^{\alpha_1+\alpha_2}} \\ &\le C \cdot (\delta^{n-\alpha_1-\alpha_2} + 1) \,. \end{aligned}$$

This ends the three cases.

It remains to show that K is continuous outside of the diagonal D. For $(x_2, y_2) \to (x_1, y_1)$ with $x_1 \neq y_1$ we have that

$$\begin{aligned}
&|K(x_2, y_2) - K(x_1, y_1)| \\
&\le C \int_\Omega \frac{|K_1(x_2, z) - K_1(x_1, z)|}{|z - y_2|^{\alpha_2}}\, \mathrm{d}z + C \int_\Omega \frac{|K_2(z, y_2) - K_2(z, y_1)|}{|z - x_1|^{\alpha_1}}\, \mathrm{d}z\,.
\end{aligned}$$

We decompose the first integral (the second integral can be bounded correspondingly) into the parts over $\Omega \setminus \mathrm{B}_\delta(x_1)$ and $\mathrm{B}_\delta(x_1)$. The former part is

$$\le C \underbrace{\sup_{|z - x_1| \ge \delta} |K_1(x_2, z) - K_1(x_1, z)|}_{\to 0 \text{ as } x_2 \to x_1 \text{ for every } \delta} \cdot \underbrace{\int_\Omega \frac{\mathrm{d}z}{|z - y_2|^{\alpha_2}}}_{\text{bounded in } y_2}\,.$$

Since $|z - y_2| \ge \frac{1}{2}|x_1 - y_1| > 0$ for $z \in \mathrm{B}_\delta(x_1)$ if y_2 is close to $y_1 \neq x_1$ and if δ is sufficiently small, the second part is

$$\le \frac{C}{|x_1 - y_1|^{\alpha_2}} \underbrace{\int_{\mathrm{B}_\delta(x_1)} \Big(\frac{1}{|z - x_2|^{\alpha_1}} + \frac{1}{|z - x_1|^{\alpha_1}} \Big)\, \mathrm{d}z}_{\le C\delta^{n - \alpha_1} \to 0 \text{ as } \delta \to 0}\,.$$

In the case $\alpha_1 + \alpha_2 < n$ it holds that $K(x_2, y_2) \to K(x_1, y_1)$ even if $x_1 = y_1$, because the part of the integral over $\Omega \setminus \mathrm{B}_\delta(x_1)$ converges to 0 as before, while the integral over $\mathrm{B}_\delta(x_1)$ is

$$\begin{aligned}
&\le C \int_{\mathrm{B}_\delta(x_1)} \Big(\frac{1}{|z - x_2|^{\alpha_1}} + \frac{1}{|z - x_1|^{\alpha_1}} \Big) \frac{\mathrm{d}z}{|z - y_2|^{\alpha_2}} \\
&\longrightarrow C \int_{\mathrm{B}_\delta(x_1)} \frac{2\, \mathrm{d}z}{|z - x_1|^{\alpha_1 + \alpha_2}} \qquad \text{as } x_2 \to x_1, y_2 \to y_1 = x_1 \\
&\le C\, \delta^{n - \alpha_1 - \alpha_2} \longrightarrow 0 \qquad \text{as } \delta \to 0.
\end{aligned}$$

This proves the result on the composition of T_1 with T_2. □

The fundamental solution

For integral kernels K as in 10.16 with $\alpha = n$ the induced T is no longer compact, and even the existence of the operator T is no longer guaranteed. That is because the function $y \mapsto |x - y|^{-n}$ is no longer integrable in a neighbourhood of x. However, such kernels play an essential role in the potential theoretic approach to partial differential equations, as we will see in 10.18.

10.17 Fundamental solution of the Laplace operator. For $x \in \mathbb{R}^n \setminus \{0\}$ let

$$F(x) := \begin{cases} \dfrac{1}{\sigma_n(n-2)}|x|^{2-n} & \text{for } n \geq 3, \\ \dfrac{1}{2\pi}\log\dfrac{1}{|x|} & \text{for } n = 2, \\ -\dfrac{1}{2}|x| & \text{for } n = 1, \end{cases}$$

where σ_n denotes the surface area of $\partial \mathrm{B}_1(0) \subset \mathrm{I\!R}^n$ ($\sigma_3 = 4\pi$, $\sigma_2 = 2\pi$, $\sigma_1 = 2$, $\sigma_n = n\kappa_n$, with κ_n the volume of $\mathrm{B}_1(0) \subset \mathrm{I\!R}^n$).

(1) It holds that $F \in C^\infty\big(\mathrm{I\!R}^n \setminus \{0\}\big)$ and

$$\partial_i F(x) = -\frac{1}{\sigma_n}\frac{x_i}{|x|^n}\ , \ \ \partial_{ij}F(x) = -\frac{1}{\sigma_n|x|^n}\Big(\delta_{i,j} - n\frac{x_i}{|x|}\frac{x_j}{|x|}\Big)\ , \ \ \Delta F = 0\ .$$

(2) It holds that $F \in W^{1,1}_{\mathrm{loc}}(\mathrm{I\!R}^n)$ and with the notations as in 5.15 we have that

$$-\Delta[F] = -\sum_{i=1}^n \partial_i[\partial_i F] = [\delta_0] \quad \text{in } \mathscr{D}'(\mathrm{I\!R}^n).$$

Note: F is the fundamental solution for $-\Delta$.

(3) If $f : \mathrm{I\!R}^n \to \mathrm{I\!R}$ is measurable and bounded with compact support, then

$$u(x) := \int_{\mathrm{I\!R}^n} F(x-y)f(y)\,\mathrm{d}y = (F * f)(x)$$

defines a $u \in C^1(\mathrm{I\!R}^n)$ which satisfies

$$-\Delta[u] = -\sum_{i=1}^n \partial_i[\partial_i u] = [f] \quad \text{in } \mathscr{D}'(\mathrm{I\!R}^n),$$

i.e. u is a weak solution of the differential equation $-\Delta u = f$ in $\mathrm{I\!R}^n$.

Proof (1). By direct calculation. □

Proof (2). We have that $F \in W^{1,1}\big(\mathrm{B}_R(0) \setminus \{0\}\big)$ for $R > 0$. Similarly to the end of the proof of 10.7(3) (or on recalling the corollary in A8.9) it then follows that $F \in W^{1,1}\big(\mathrm{B}_R(0)\big)$, where outside of the null set $\{0\}$ the weak derivatives coincide with the classical ones. Hence, $\partial_i[F] = [\partial_i F]$, which yields for $\zeta \in C_0^\infty\big(\mathrm{B}_R(0)\big)$ that as $\varepsilon \searrow 0$

$$\begin{aligned} \int_{\mathrm{I\!R}^n} (-\Delta\zeta)F\,\mathrm{dL}^n &= \int_{\mathrm{I\!R}^n} \nabla\zeta \bullet \nabla F\,\mathrm{dL}^n \longleftarrow \int_{\mathrm{I\!R}^n \setminus \mathrm{B}_\varepsilon(0)} \nabla\zeta \bullet \nabla F\,\mathrm{dL}^n \\ &= -\int_{\partial \mathrm{B}_\varepsilon(0)} \zeta\nu_{\mathrm{B}_\varepsilon(0)} \bullet \nabla F\,\mathrm{dH}^{n-1} = \frac{1}{\sigma_n}\int_{\partial \mathrm{B}_1(0)} \zeta(\varepsilon y)\,\mathrm{dH}^{n-1}(y) \longrightarrow \zeta(0)\,, \end{aligned}$$

since $\Delta F = 0$ in $\mathrm{I\!R}^n \setminus \{0\}$. □

Proof (3). Applying 10.16(3) for the kernel $(x,y) \mapsto F(x-y)$ shows that $u \in C^0(\mathbb{R}^n)$. For $\zeta \in C_0^\infty(\mathbb{R}^n)$ it follows, since $F \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, that

$$\int_{\mathbb{R}^n} \big(\partial_i\zeta u + \zeta v_i\big)\,\mathrm{dL}^n = 0 \quad \text{with} \quad v_i(x) := \int_{\mathbb{R}^n} \partial_i F(x-y) f(y)\,\mathrm{d}y\,. \tag{10-31}$$

By 10.16(3) it follows that $v_i \in C^0(\mathbb{R}^n)$, whence $u \in C^1(\mathbb{R}^n)$, with $\partial_i u = v_i$. Moreover, it follows from (2) that

$$\begin{aligned}\int_{\mathbb{R}^n} (-\Delta\zeta(x))u(x)\,\mathrm{d}x &= \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} (-\Delta\zeta(x+y))F(x)\,\mathrm{d}x\Big) f(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \Big(-\Delta[F]\big(\zeta(\bullet+y)\big)\Big) f(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \zeta(y) f(y)\,\mathrm{d}y\,.\end{aligned}$$

□

10.18 Singular integral operators. For motivational purposes we continue the considerations in 10.17. We approximate $\partial_i u = v_i$ in (10-31) for $\varepsilon > 0$ by

$$v_i^\varepsilon(x) := \int_{\mathbb{R}^n \setminus \mathrm{B}_\varepsilon(x)} \partial_i F(x-y) f(y)\,\mathrm{d}y\,.$$

If $f \in C_0^0(\mathbb{R}^n)$, then $v_i^\varepsilon \in C^1(\mathbb{R}^n)$, with

$$\begin{aligned}\partial_j v_i^\varepsilon(x) &= \int_{\mathbb{R}^n \setminus \mathrm{B}_\varepsilon(x)} \partial_{ji} F(x-y) f(y)\,\mathrm{d}y - w_{ji}^\varepsilon(x)\,, \qquad \text{where} \\ w_{ji}^\varepsilon(x) &:= \int_{\partial \mathrm{B}_\varepsilon(x)} \nu_{\mathrm{B}_\varepsilon(x)}(y) \bullet \mathbf{e}_j \partial_i F(x-y) f(y)\,\mathrm{dH}^{n-1}(y) \\ &= \frac{1}{\sigma_n} \int_{\partial \mathrm{B}_1(0)} y_j y_i\, f(x+\varepsilon y)\,\mathrm{dH}^{n-1}(y)\,.\end{aligned}$$

We note that as $\varepsilon \searrow 0$

$$w_{ji}^\varepsilon(x) \longrightarrow \frac{1}{\sigma_n} \cdot \int_{\partial \mathrm{B}_1(0)} y_j y_i\,\mathrm{dH}^{n-1}(y) \cdot f(x) = \frac{1}{n}\,\delta_{i,j} f(x)\,. \tag{10-32}$$

Hence, if we want to show that u in 10.17(3) belongs to the space $C^2(\mathbb{R}^n)$, then we have to investigate whether the limit

$$\big(T_{ji} f\big)(x) := \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n \setminus \mathrm{B}_\varepsilon(x)} \partial_{ji} F(x-y)\, f(y)\,\mathrm{d}y$$

exists, and whether T_{ji} is well defined as a continuous operator on appropriate function spaces. On recalling the identity for the second derivatives $\partial_{ji}F(x-y)$ of the fundamental solution from 10.17(1), we note that the above kernel $(x,y) \mapsto K(x,y) := \partial_{ji}F(x-y)$ is a ***singular integral kernel***, i.e. a kernel as in 10.16 but with $\alpha = n$. However, we recall from 10.17(1) that this kernel has the particular form

$$K(x,y) = \frac{\omega(\frac{x-y}{|x-y|})}{|x-y|^n} \quad \text{with} \quad \omega(\xi) := -\frac{1}{\sigma_n}(\delta_{j,i} - n\xi_j\xi_i) \text{ for } |\xi| = 1,$$

where the mean value of $\omega : \partial\mathrm{B}_1(0) \to \mathbb{R}$ vanishes (see (10-32)), i.e.

$$\int_{\partial\mathrm{B}_1(0)} \omega(\xi)\,\mathrm{dH}^{n-1}(\xi) = 0\,. \tag{10-33}$$

Now we consider arbitrary kernels K of the above type with the property (10-33) and prove that for certain functions f the limit

$$(Tf)(x) := \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n \setminus \mathrm{B}_\varepsilon(x)} K(x,y)f(y)\,\mathrm{d}y$$

exists. This limit is also referred to as the ***Cauchy principal value*** of $\int_{\mathbb{R}^n} K(x,y)f(y)\,\mathrm{d}y$ at the point x (observe that $y \mapsto K(x,y)f(y)$ in general is not integrable!). Classes of functions on which T can still be shown to be a continuous operator include C^α-spaces (see 10.19) and L^p-spaces (see 10.20). In both cases T is not (!) a compact operator. For ease of presentation we also define

$$\omega(x) := \omega\Big(\frac{x}{|x|}\Big) \qquad \text{for } x \in \mathbb{R}^n \setminus \{0\}. \tag{10-34}$$

10.19 Hölder-Korn-Lichtenstein inequality. Let $\omega : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a Lipschitz continuous function on $\partial\mathrm{B}_1(0)$ which satisfies (10-33) and (10-34). Then for $0 < \alpha < 1$ and $f \in C^{0,\alpha}(\overline{\mathrm{B}_R(0)})$ with $f = 0$ on $\partial\mathrm{B}_R(0)$ the limit

$$(Tf)(x) := \lim_{\varepsilon \searrow 0} \int_{\mathrm{B}_R(0) \setminus \mathrm{B}_\varepsilon(x)} \frac{\omega(x-y)}{|x-y|^n} f(y)\,\mathrm{d}y$$

exists pointwise for $x \in \mathbb{R}^n$, and for all $\widetilde{R} > 0$ it holds that

$$\|Tf\|_{C^{0,\alpha}(\overline{\mathrm{B}_{\widetilde{R}}(0)})} \le C(n,R,\alpha) \cdot \|\omega\|_{C^{0,1}(\partial\mathrm{B}_1(0))} \cdot \|f\|_{C^{0,\alpha}(\overline{\mathrm{B}_R(0)})}\,.$$

Proof. We extend f by 0 on $\mathbb{R}^n \setminus \mathrm{B}_R(0)$. As the mean value of ω is equal to 0, for $|x| \le 2R$ we have

$$\int_{\mathrm{B}_R(0)\setminus\mathrm{B}_\varepsilon(x)} \frac{\omega(x-y)}{|x-y|^n} f(y)\,\mathrm{d}y = \int_{\mathrm{B}_{3R}(x)\setminus\mathrm{B}_\varepsilon(x)} \frac{\omega(x-y)}{|x-y|^n}\big(f(y) - f(x)\big)\,\mathrm{d}y\,,$$

because a transformation to polar coordinates yields that

$$\int_{\mathrm{B}_{3R}(x)\setminus\mathrm{B}_\varepsilon(x)} \frac{\omega(x-y)}{|x-y|^n}\,\mathrm{d}y = \int_\varepsilon^{3R} r^{n-1} \int_{\partial\mathrm{B}_1(0)} \frac{\omega(\xi)}{r^n}\,\mathrm{dH}^{n-1}(\xi)\,\mathrm{d}r = 0\,.$$

Noting that with a constant C depending on ω it holds that

$$\left|\frac{\omega(x-y)}{|x-y|^n}\big(f(y)-f(x)\big)\right| \le C\cdot|x-y|^{\alpha-n}\|f\|_{C^{0,\alpha}}\ ,$$

we see that the integrand is integrable over $\mathrm{B}_{3R}(x)$, and hence

$$Tf(x)=\int_{\mathrm{B}_{3R}(x)}\frac{\omega(x-y)}{|x-y|^n}\big(f(y)-f(x)\big)\,\mathrm{d}y$$

and

$$|Tf(x)|\le C\int_{\mathrm{B}_{3R}(0)}|y|^{\alpha-n}\,\mathrm{d}y\cdot\|f\|_{C^{0,\alpha}}=C(\omega,n,R,\alpha)\|f\|_{C^{0,\alpha}}\ .$$

For $|x|\ge 2R$,

$$|Tf(x)|\le C\,\|f\|_{C^0}\int_{\mathrm{B}_R(0)}\frac{\mathrm{d}y}{|x-y|^n}\le\frac{C(\omega,n,R)}{(|x|-R)^n}\|f\|_{C^0}\ .$$

Similarly, for $x_1,x_2\in\mathbb{R}^n$ and $\varrho\ge R+\max(|x_1|,|x_2|)$,

$$\begin{aligned}
&Tf(x_1)-Tf(x_2)\\
&=\int_{\mathrm{B}_\varrho(x_1)}\frac{\omega(x_1-y)}{|x_1-y|^n}\big(f(y)-f(x_1)\big)\,\mathrm{d}y\\
&\quad-\int_{\mathrm{B}_\varrho(x_2)}\frac{\omega(x_2-y)}{|x_2-y|^n}\big(f(y)-f(x_2)\big)\,\mathrm{d}y\\
&=\int_{\mathrm{B}_\varrho(x_1)}\Big(\frac{\omega(x_1-y)}{|x_1-y|^n}\big(f(y)-f(x_1)\big)-\frac{\omega(x_2-y)}{|x_2-y|^n}\big(f(y)-f(x_2)\big)\Big)\,\mathrm{d}y\\
&\quad+\int_{\mathbb{R}^n}\frac{\omega(x_2-y)}{|x_2-y|^n}\big(f(y)-f(x_2)\big)\big(\mathcal{X}_{\mathrm{B}_\varrho(x_1)}(y)-\mathcal{X}_{\mathrm{B}_\varrho(x_2)}(y)\big)\,\mathrm{d}y\,.
\end{aligned}$$

The second integral can be bounded by

$$\begin{aligned}
&\le C\,\|f\|_{C^0}\int_{\mathbb{R}^n}\big|\mathcal{X}_{\mathrm{B}_\varrho(0)}(y)-\mathcal{X}_{\mathrm{B}_\varrho(x_2-x_1)}(y)\big|\frac{\mathrm{d}y}{|y|^n}\\
&=C\,\|f\|_{C^0}\int_{\mathbb{R}^n}\Big|\mathcal{X}_{\mathrm{B}_1(0)}(\widetilde{y})-\mathcal{X}_{\mathrm{B}_1\left(\frac{1}{\varrho}(x_2-x_1)\right)}(\widetilde{y})\Big|\frac{\mathrm{d}\widetilde{y}}{|\widetilde{y}|^n}
\end{aligned}$$

(with the variable transformation $y=\varrho\widetilde{y}$), which converges to 0 for every x_1 and x_2 as $\varrho\to\infty$. Setting $\delta:=|x_2-x_1|$, the first integral from above can be bounded on $\mathrm{B}_{2\delta}(x_1)$ by (we employ the usual convention on constants)

$$\begin{aligned}
&\le C\,\|f\|_{C^{0,\alpha}}\cdot\int_{\mathrm{B}_{2\delta}(x_1)}\big(|y-x_1|^{\alpha-n}+|y-x_2|^{\alpha-n}\big)\,\mathrm{d}y\\
&\le C\,\|f\|_{C^{0,\alpha}}\cdot\int_{\mathrm{B}_{3\delta}(0)}|y|^{\alpha-n}\,\mathrm{d}y\le C\,\|f\|_{C^{0,\alpha}}\cdot\delta^\alpha\,.
\end{aligned}$$

On the remaining domain $\mathrm{B}_\varrho(x_1) \setminus \mathrm{B}_{2\delta}(x_1)$ we write the integrand as

$$\frac{\omega(x_1 - y)}{|x_1 - y|^n}\big(f(x_2) - f(x_1)\big) + \Big(\frac{\omega(x_1 - y)}{|x_1 - y|^n} - \frac{\omega(x_2 - y)}{|x_2 - y|^n}\Big)\big(f(y) - f(x_2)\big)\,.$$

Recalling that the mean value of ω is equal to 0 yields that the integral of the first term vanishes. The Lipschitz continuity of ω implies that

$$\begin{aligned}
&|\omega(x_1 - y) - \omega(x_2 - y)| \le C\left|\frac{x_1 - y}{|x_1 - y|} - \frac{x_2 - y}{|x_2 - y|}\right| \\
&= C\frac{\big||x_2 - y|(x_1 - y) - |x_1 - y|(x_2 - y)\big|}{|x_1 - y|\,|x_2 - y|} \le C\frac{|x_1 - x_2|}{|x_2 - y|}\,,
\end{aligned}$$

and we have

$$\begin{aligned}
&\left|\frac{1}{|x_1 - y|^n} - \frac{1}{|x_2 - y|^n}\right| \\
&\le \frac{|x_1 - x_2|}{|x_1 - y|^n\,|x_2 - y|^n}\sum_{i=0}^{n-1}|x_1 - y|^i\,|x_2 - y|^{n-1-i} \\
&\le n|x_1 - x_2|\Big(\frac{1}{|x_2 - y|\,|x_1 - y|^n} + \frac{1}{|x_1 - y|\,|x_2 - y|^n}\Big)\,.
\end{aligned}$$

Together this gives

$$\begin{aligned}
&\left|\frac{\omega(x_1 - y)}{|x_1 - y|^n} - \frac{\omega(x_2 - y)}{|x_2 - y|^n}\right| \\
&\le C\cdot|x_1 - x_2|\Big(\frac{1}{|x_2 - y|\,|x_1 - y|^n} + \frac{1}{|x_1 - y|\,|x_2 - y|^n}\Big)\,.
\end{aligned}$$

On noting that $\frac{1}{2}|x_1 - y| \le |x_2 - y| \le 2|x_1 - y|$ for $|y - x_1| \ge 2\delta$, it follows that the remaining integral over $\mathrm{B}_\varrho(x_1) \setminus \mathrm{B}_{2\delta}(x_1)$ is bounded uniformly in ϱ by

$$\begin{aligned}
&\le C\,\|f\|_{C^{0,\alpha}}\cdot\delta\int_{\mathbb{R}^n\setminus\mathrm{B}_{2\delta}(x_1)}|x_1 - y|^{\alpha-n-1}\,\mathrm{d}y \\
&\le C\,\|f\|_{C^{0,\alpha}}\cdot\delta\int_{2\delta}^{\infty}r^{\alpha-2}\,\mathrm{d}r \le C\,\|f\|_{C^{0,\alpha}}\cdot\delta^\alpha\,.
\end{aligned}$$

□

10.20 Calderón-Zygmund inequality. Let $\omega : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ on $\partial\mathrm{B}_1(0)$ be measurable with respect to the measure H^{n-1} and bounded and such that it satisfies (10-33) and (10-34). Then for $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$ and $0 < \varepsilon \le 1$ the integral

$$(T_\varepsilon f)(x) := \int_{\mathbb{R}^n\setminus\mathrm{B}_\varepsilon(x)}\frac{\omega(x - y)}{|x - y|^n}f(y)\,\mathrm{d}y$$

exists for almost all $x \in \mathbb{R}^n$. This defines operators $T_\varepsilon \in \mathscr{L}\big(L^p(\mathbb{R}^n)\big)$ and for $f \in L^p(\mathbb{R}^n)$ there exists

$$\begin{aligned} Tf &:= \lim_{\varepsilon \searrow 0} T_\varepsilon f \quad \text{ in } L^p(\mathbb{R}^n) \text{ with} \\ \|Tf\|_{L^p(\mathbb{R}^n)} &\le C(n,p) \cdot \|\omega\|_{L^\infty(\partial \mathrm{B}_1(0))} \cdot \|f\|_{L^p(\mathbb{R}^n)} \,. \end{aligned}$$

Proof. See Appendix A10. □

Remark: For $n = 1$ we have that $\omega(-1) = -\omega(+1)$, hence up to a multiplicative constant $\omega(1) = 1$ and $\omega(-1) = -1$. Then

$$(Tf)(x) = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R} \setminus]\, x-\varepsilon, x+\varepsilon\, [} \frac{f(y)}{x-y} \, \mathrm{d}y$$

is called the ***Hilbert transform*** of f.

E10 Exercises

E10.1 Counterexample to embedding theorems. Show that theorem 10.6 in the case $k_1 > 0$ does not (!) hold for arbitrary open bounded sets $\Omega \subset \mathbb{R}^n$.

Solution. A characteristic counterexample is the following: Let $e \in \mathbb{R}^n$ with $|e| = 1$ and set

$$\Omega := \bigcup_{k \in \mathbb{N}} \mathrm{B}_{r_k}(x_k) \qquad \text{with } x_k = \frac{1}{k} e, \ r_k = \frac{1}{4k^2} \,,$$

so that the closed balls $\overline{\mathrm{B}_{r_k}(x_k)}$ are pairwise disjoint. Now if $(a_k)_{k \in \mathbb{N}}$ is a sequence that converges in $\mathbb{R}$ to a, then

$$u(x) := \begin{cases} a_k & \text{for } |x - x_k| \le r_k, \ k \in \mathbb{N}, \\ a & \text{for } x = 0, \end{cases}$$

defines a $u \in C^0(\overline{\Omega})$. Since $\nabla u = 0$ in Ω it follows that also $u \in C^1(\overline{\Omega})$ (see definition 3.6). Note that for $0 < \alpha \le 1$

$$\sup_{x \in \overline{\Omega},\ x \ne 0} \frac{|u(x) - u(0)|}{|x|^\alpha} \ge \sup_k \Big(\Big(\frac{k}{2}\Big)^\alpha |a_k - a| \Big) \,,$$

and $a_k = a + (1 + \log k)^{-1}$ yields that u lies in none of the spaces $C^{0,\alpha}(\overline{\Omega})$. Hence the embedding in 10.6 for $(k_1, \alpha_1) = (1, 0)$ and $(k_2, \alpha_2) = (0, \alpha)$ does not even exist for the above Ω. □

E10.2 Ehrling's lemma. Let X, Y, Z be Banach spaces. Assume $K \in \mathscr{K}(X;Y)$ and let $T \in \mathscr{L}(Y;Z)$ be injective. Then for every $\varepsilon > 0$ there exists a $C_\varepsilon < \infty$, such that for all $x \in X$

$$\|Kx\|_Y \leq \varepsilon \|x\|_X + C_\varepsilon \|TKx\|_Z .$$

Solution. Otherwise for an $\varepsilon > 0$ there exist points $\widetilde{x}_n \in X$ with

$$\|K\widetilde{x}_n\|_Y > \varepsilon \|\widetilde{x}_n\|_X + n \|TK\widetilde{x}_n\|_Z .$$

Then $x_n := \frac{\widetilde{x}_n}{\|\widetilde{x}_n\|_X}$ are bounded in X and

$$\|Kx_n\|_Y > \varepsilon + n \|TKx_n\|_Z . \tag{E10-1}$$

Since K is compact, there exists a subsequence (which we again denote by $(x_n)_{n\in\mathbb{N}}$) such that $Kx_n \to y \in Y$ as $n \to \infty$, and so

$$\|Ty\|_Z \longleftarrow \|TKx_n\|_Z \leq \frac{1}{n}\|Kx_n\|_Y \longrightarrow 0 .$$

As T is injective, it follows that $y = 0$ and hence $\|Kx_n\|_Y \to 0$, which contradicts (E10-1). □

E10.3 Application of Ehrling's lemma. Let $\Omega \subset \mathbb{R}^n$ be open and bounded, let $1 < p < \infty$ and let $m \geq 2$. Show that:

(1) For every $\varepsilon > 0$ there exists a constant C_ε such that for all $u \in W_0^{m,p}(\Omega)$

$$\|u\|_{W_0^{m-1,p}(\Omega)} \leq \varepsilon \|u\|_{W_0^{m,p}(\Omega)} + C_\varepsilon \|u\|_{L^p(\Omega)} .$$

(2) An equivalent norm on $W_0^{m,p}(\Omega)$ is given by

$$\|u\| := \|D^m u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} .$$

Solution (1). This follows from Ehrling's lemma, on noting that the embedding from $W_0^{m,p}(\Omega)$ into $W_0^{m-1,p}(\Omega)$ is compact (either on recalling 8.11(3), 8.10, Rellich's embedding theorem A8.1 and 10.1(4), or on recalling Sobolev's embedding theorem 10.9). □

Solution (2). We have from (1) that

$$\begin{aligned}\|u\|_{W^{m-1,p}} &\leq \varepsilon \|u\|_{W^{m,p}} + C_\varepsilon \|u\|_{L^p} \\ &\leq \varepsilon \|D^m u\|_{L^p} + \varepsilon \|u\|_{W^{m-1,p}} + C_\varepsilon \|u\|_{L^p} ,\end{aligned}$$

which for $\varepsilon \leq \frac{1}{2}$ yields the bound

$$\|u\|_{W^{m-1,p}} \leq 2\varepsilon \|D^m u\|_{L^p} + 2C_\varepsilon \|u\|_{L^p} .$$

Consequently,

$$\|u\| \leq \|u\|_{W^{m,p}} \leq \max(1 + 2\varepsilon, 2C_\varepsilon) \cdot \|u\| .$$

□

E10.4 On Ehrling's lemma. Let $\Omega = \mathrm{B}_R(0) \subset \mathbb{R}^n$. Show that: For $\varepsilon > 0$ there exists a constant C_ε such that for all $u \in C^2(\overline{\Omega})$

$$\|\nabla u\|_{C^0(\overline{\Omega})} \leq \varepsilon \left\|D^2 u\right\|_{C^0(\overline{\Omega})} + C_\varepsilon \|u\|_{C^0(\overline{\Omega})} ,$$

and obtain an explicit bound for the constant C_ε.

Solution. First let $R = 1$ and $\varepsilon \leq 1$. For $x_0 \in \Omega$ with $\nabla u(x_0) \neq 0$ we choose $y_0, y_1 \in \overline{\Omega} \cap \overline{\mathrm{B}_\varepsilon(x_0)}$ such that $y_1 - y_0$ points in the direction of $\nabla u(x_0)$ and $|y_1 - y_0| \geq \frac{\varepsilon}{2}$.
Remark: This is possible because $\Omega = \mathrm{B}_1(0)$. If $\mathrm{B}_\varepsilon(x_0) \subset \Omega$, then we can choose $y_0 = x_0$ and $y_1 = x_0 + \varepsilon \frac{\nabla u(x_0)}{|\nabla u(x_0)|}$.

Then, setting $y_t := (1-t)y_0 + ty_1$, it holds that

$$\begin{aligned} &u(y_1) - u(y_0) = \int_0^1 \nabla u(y_t) \cdot (y_1 - y_0) \,\mathrm{d}t \\ &= \nabla u(x_0) \cdot (y_1 - y_0) \\ &\quad + \int_0^1 \int_0^1 \sum_{i,j=1}^n \partial_{ij} u\big((1-s)x_0 + sy_t\big)(y_t - x_0)_i (y_1 - y_0)_j \,\mathrm{d}s \,\mathrm{d}t \end{aligned}$$

and

$$\nabla u(x_0) \cdot (y_1 - y_0) = |\nabla u(x_0)|\, |y_1 - y_0| .$$

It follows that

$$\begin{aligned} |\nabla u(x_0)| &\leq \left\|D^2 u\right\|_{C^0(\overline{\Omega})} \cdot \sup_{0 \leq t \leq 1} |y_t - x_0| + \frac{|u(y_1) - u(y_0)|}{|y_1 - y_0|} \\ &\leq \varepsilon \left\|D^2 u\right\|_{C^0(\overline{\Omega})} + \frac{4}{\varepsilon} \|u\|_{C^0(\overline{\Omega})} , \end{aligned}$$

and hence the desired bound with $C_\varepsilon = \frac{4}{\varepsilon}$. (For $\varepsilon \geq 1$ the claim follows with $C_\varepsilon = 4$.) If R is arbitrary, then define

$$v(x) := u(\tfrac{x}{R}) .$$

The established bound for v

$$\|\nabla v\|_{C^0(\overline{\mathrm{B}_1(0)})} \leq \varepsilon \left\|D^2 v\right\|_{C^0(\overline{\mathrm{B}_1(0)})} + \frac{4}{\min(\varepsilon, 1)} \|v\|_{C^0(\overline{\mathrm{B}_1(0)})}$$

transforms to

$$\|\nabla u\|_{C^0(\overline{\mathrm{B}_R(0)})} \leq \frac{\varepsilon}{R} \left\|D^2 u\right\|_{C^0(\overline{\mathrm{B}_R(0)})} + \frac{4R}{\min(\varepsilon, 1)} \|u\|_{C^0(\overline{\mathrm{B}_R(0)})} .$$

Now replace ε by $R\varepsilon$ and set $C_\varepsilon = 4\left(\min(\varepsilon, \frac{1}{R})\right)^{-1}$. $\square$

E10.5 An a priori estimate. Let $u \in C^2([0,1])$ be a solution of the linear differential equation

$$au'' + bu' + du = 0 \quad \text{in }]0,1[\,,$$

where $a, b, d \in C^0([0,1])$ and $a \geq c_0$ with a positive constant c_0. Then there exists a constant C, which depends only on the coefficients, such that

$$\|u\|_{C^2} \leq C \cdot \|u\|_{C^0}\,.$$

Solution. The differential equation implies that

$$c_0\|u''\|_{C^0} \leq C\big(\|u'\|_{C^0} + \|u\|_{C^0}\big) \quad \text{with } C := \|b\|_{C^0} + \|d\|_{C^0}\,,$$

and so

$$c_0\big(\|u''\|_{C^0} + \|u'\|_{C^0}\big) \leq (C + c_0)\big(\|u'\|_{C^0} + \|u\|_{C^0}\big)\,.$$

It follows from E10.4 that this can be bounded by

$$\leq (C + c_0)\varepsilon\|u''\|_{C^0} + (C + c_0) \cdot (C_\varepsilon + 1)\|u\|_{C^0}\,.$$

On choosing ε with $(C + c_0)\varepsilon = \frac{c_0}{2}$, we obtain, with a new constant C, that

$$\|u''\|_{C^0} + \|u'\|_{C^0} \leq C\|u\|_{C^0}\,.$$

□

E10.6 Equivalent norm. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary and let $m \geq 2$. Then an equivalent norm on $C^m(\overline{\Omega})$ is given by

$$\|u\| := \|D^m u\|_{C^0(\overline{\Omega})} + \|u\|_{C^0(\overline{\Omega})}\,.$$

E10.7 Counterexample to embedding theorems. Let Ω be as in theorem 10.9 and let

$$1 - \tfrac{n}{p} = 0\,.$$

Then $W^{1,p}(\Omega)$ is not (!) embedded in $L^\infty(\Omega)$, except in the case $n = 1$.
Note: In theorem 10.9 the case $m_1 = m_2 + 1$, $p_2 = \infty$ is not allowed, while theorem 10.8 does not permit $q = \infty$.

Solution. The case $n = 1$ (we then have $p = 1$) was solved in E3.6. For $n \geq 2$ a counterexample is

$$u(x) := \log|\log|x|| \quad \text{for } 0 < |x| < \tfrac{1}{2}.$$

Let $\Omega := \mathrm{B}_{\frac{1}{2}}(0)$. Then $u \in L^s(\Omega)$ for $1 \leq s < \infty$, but u is not bounded. Moreover, $u \in W^{1,n}(\Omega \setminus \{0\})$, because $u \in C^\infty(\Omega \setminus \{0\})$, with

$$\int_{\Omega} |\nabla u|^n \, \mathrm{dL}^n = \int_{\Omega} \frac{\mathrm{d}x}{\big(|x||\log|x||\big)^n} = C(n) \int_0^{\frac{1}{2}} \frac{\mathrm{d}r}{r|\log r|^n}$$
$$= \widetilde{C}(n) \left[\frac{1}{|\log r|^{n-1}} \right]_{r=0}^{r=\frac{1}{2}} < \infty \,.$$

It follows that $u \in W^{1,n}(\Omega)$, similarly to the end of the proof of 10.7(3) (or alternatively by using the corollary in A8.9). □

E10.8 Sobolev spaces on $\mathbb{R}^n$. For $m \geq 1$ and $1 \leq p < \infty$,

$$W^{m,p}(\mathbb{R}^n) = W_0^{m,p}(\mathbb{R}^n) \,.$$

Proof. Recalling that $C^\infty(\mathbb{R}^n) \cap W^{m,p}(\mathbb{R}^n)$ is dense in $W^{m,p}(\mathbb{R}^n)$ (see 4.24), it is sufficient to approximate functions $u \in C^\infty(\mathbb{R}^n) \cap W^{m,p}(\mathbb{R}^n)$ in the $W^{m,p}$-norm by functions in $C_0^\infty(\mathbb{R}^n)$. To this end, choose a function $\eta \in C^\infty(\mathbb{R}^n)$ with

$$\eta(x) = \begin{cases} 1 & \text{for } |x| \leq 1, \\ 0 & \text{for } |x| \geq 2 \end{cases}$$

(see 4.19), and define $\eta_R(x) := \eta\big(\frac{x}{R}\big)$. Then for all multi-indices s with $|s| \leq m$,

$$\partial^s(u - \eta_R u) = (1 - \eta_R)\partial^s u - \sum_{\substack{0 \leq r \leq s \\ r \neq s}} \binom{s}{r} (\partial^{s-r}\eta_R)\partial^r u \,.$$

Noting that $1 - \eta_R = 0$ on $\mathrm{B}_R(0)$ and that $|\partial^{s-r}\eta_R| \leq C\,R^{-|s-r|}$ in $\mathbb{R}^n$ yields that

$$\begin{aligned} &\|\partial^s(u - \eta_R u)\|_{L^p(\mathbb{R}^n)} \\ &\leq \|\partial^s u\|_{L^p(\mathbb{R}^n \setminus \mathrm{B}_R(0))} + C \sum_{\substack{0 \leq r \leq s \\ r \neq s}} R^{-|s-r|} \|\partial^r u\|_{L^p(\mathbb{R}^n)} \,, \end{aligned}$$

which converges to 0 as $R \to \infty$. □

E10.9 Embedding theorem. Let $m_1, m_2 \geq 0$ and $1 \leq p_1, p_2 < \infty$ with

$$m_1 - \frac{n}{p_1} = m_2 - \frac{n}{p_2} \,, \qquad \text{where } m_1 \geq m_2 \,.$$

Then the embedding $\mathrm{Id} : W^{m_1,p_1}(\mathbb{R}^n) \to W^{m_2,p_2}(\mathbb{R}^n)$ exists and is continuous.

Observe: In theorem 10.9 this result was shown for bounded open sets $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary. (Theorem 10.9 also holds for an inequality between the Sobolev numbers.) Here we prove the theorem for $\Omega = \mathbb{R}^n$, where it is essential that the two Sobolev numbers are equal, which is also the case in theorem 10.8.

Solution. For $m_1 = m_2$ the result is trivial. For $m_1 = m_2 + 1$ let $u \in W^{m_1,p_1}(\mathbb{R}^n)$. Then $\partial^s u$ is in $W^{1,p_1}(\mathbb{R}^n)$ for all multi-indices s with $|s| \le m_1 - 1 = m_2$. Sobolev's theorem 10.8 then yields that $\partial^s u \in L^{p_2}(\mathbb{R}^n)$ with

$$\|\partial^s u\|_{L^{p_2}(\mathbb{R}^n)} \le C(p_2,n)\|\nabla\partial^s u\|_{L^{p_1}(\mathbb{R}^n)} \le C(p_2,n)\|u\|_{W^{m_1,p_1}(\Omega)}\,.$$

For $m_1 = m_2 + k$ with $k \ge 2$ define $\widetilde{m}_i$ and $\widetilde{p}_i$ for $i = 0,\dots,k$ by

$$\widetilde{m}_i := m_2 + i\,, \quad \widetilde{m}_i - \frac{n}{\widetilde{p}_i} = m_2 - \frac{n}{p_2}\,, \quad \text{i.e. } \frac{1}{\widetilde{p}_i} = \frac{i}{n} + \frac{1}{p_2}\,.$$

Then $\widetilde{p}_0 = p_2$ and $\widetilde{p}_i$ is monotonically decreasing in i with $\widetilde{p}_k = p_1$, and hence $1 \le \widetilde{p}_i < \infty$ for $i = 0,\dots,k$. The desired result now follows from successive applications of theorem 10.8. ☐

E10.10 Poincaré inequalities. Let $1 \le p,q < \infty$ with $1 - \frac{n}{p} = -\frac{n}{q}$ and let $u \in W^{1,p}(\mathbb{R}^n)$. Then

$$\|u\|_{L^r(\mathbb{R}^n)} \le C(n,p)\mathrm{L}^n(\{u \ne 0\})^{\frac{1}{r}-\frac{1}{q}} \cdot \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

for $1 \le r < q$, and

$$\|u\|_{L^p(\mathbb{R}^n)} \le C(n,p)\mathrm{L}^n(\{u \ne 0\})^{\frac{1}{n}} \cdot \|\nabla u\|_{L^p(\mathbb{R}^n)}\,.$$

Solution. If $\{u \ne 0\} := \{x \in \mathbb{R}^n\,;\ u(x) \ne 0\}$ has finite Lebesgue measure then it follows from the Hölder inequality for $1 \le r < q$ that

$$\int_{\mathbb{R}^n} |u|^r \,\mathrm{dL}^n = \int_{\mathbb{R}^n} \mathcal{X}_{\{u\ne 0\}} \cdot |u|^r \,\mathrm{dL}^n \le \mathrm{L}^n(\{u \ne 0\})^{1-\frac{r}{q}} \Big(\int_{\mathbb{R}^n} |u|^q \,\mathrm{dL}^n\Big)^{\frac{r}{q}}\,,$$

and so 10.8 yields the first inequality. Setting $r = p$, and noting that $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$, we obtain the second inequality. ☐

E10.11 Convergence in L^p-spaces. Let $1 \le p_0 < p_1 < \infty$, and suppose $u_k \in L^{p_0}(\mu) \cap L^{p_1}(\mu)$ for $k \in \mathbb{N}$ and $u \in L^{p_0}(\mu)$. Then it holds for $p_0 \le p < p_1$ that

$$\begin{array}{lcl} \{u_k\,;\ k \in \mathbb{N}\} \text{ bounded in } L^{p_1}(\mu), & & u_k, u \in L^p(\mu), \\ u_k \to u \text{ strongly in } L^{p_0}(\mu) & \Longrightarrow & u_k \to u \text{ strongly in } L^p(\mu) \\ \text{as } k \to \infty & & \text{as } k \to \infty. \end{array}$$

Solution. We have for all $\varepsilon > 0$ the elementary inequality

$$a^p \le \varepsilon\, a^{p_1} + C_\varepsilon a^{p_0} \quad \text{for all } a \ge 0,$$

where C_ε is a constant depending on ε, p, p_1, p_0. It follows that

$$\int_\Omega |u_k - u_l|^p \,\mathrm{d}\mu \le \varepsilon \underbrace{\int_\Omega |u_k - u_l|^{p_1} \,\mathrm{d}\mu}_{\text{bounded in } k,l} + C_\varepsilon \underbrace{\int_\Omega |u_k - u_l|^{p_0} \,\mathrm{d}\mu}_{\to 0 \text{ as } k,l\to\infty},$$

which implies that $\{u_k\,;\; k \in \mathbb{N}\}$ is a Cauchy sequence in $L^p(\mu)$ as well. Hence there exists a $\widetilde{u} \in L^p(\mu)$ with $u_k \to \widetilde{u}$ in $L^p(\mu)$. It follows for a subsequence $k \to \infty$ that $u_k \to u$ and $u_k \to \widetilde{u}$ μ-almost everywhere, and so $u = \widetilde{u}$ in $L^p(\mu)$. □

E10.12 Compact sets in c_0. Let c_0 be the space of null sequences, equipped with the supremum norm $\|\bullet\|_{\sup}$.

(1) Show that $M \subset c_0$ is precompact if and only if M is bounded and for every $\varepsilon > 0$ there exists an index n_ε such that $|x_n| \le \varepsilon$ for all $n \ge n_\varepsilon$ and all $x \in M$.

(2) Let $F : c_0 \to c_0$ be defined by $F(x) = \{x_i^3\,;\; i \in \mathbb{N}\}$. Prove that $F(\mathrm{B}_1(0))$ is not precompact, but $DF(x)(\mathrm{B}_1(0))$ is for every $x \in c_0$.

E10.13 Nuclear operators. Let X, Y be Banach spaces and let $T : X \to Y$ be ***nuclear***, i.e. there exist $\lambda_k \in \mathbb{K}$, $x'_k \in X'$, $y_k \in Y$ for $k \in \mathbb{N}$ with

$$\sum_{k=1}^\infty |\lambda_k| < \infty\,, \quad \|x'_k\|_{X'} = 1\,, \quad \|y_k\|_Y = 1\,,$$

such that

$$Tx = \sum_{k=1}^\infty \lambda_k \,\langle x\,,\, x'_k\rangle_X \, y_k \qquad \text{for all } x \in X.$$

Then T is compact.

Solution. The operators

$$T_n x := \sum_{k=1}^n \lambda_k \,\langle x\,,\, x'_k\rangle_X \, y_k$$

are compact on recalling 10.2(3). Moreover,

$$\|(T - T_n)x\| \le \Big(\sum_{k=n+1}^\infty |\lambda_k|\Big)\|x\|\,,$$

and so $T_n \to T$ in $\mathscr{L}(X;Y)$. Hence 10.2(2) yields that T is compact. □

E10.14 Compact operator without eigenvalues. Setting

$$Tx := \sum_{k=1}^\infty \frac{x_k}{k+1} e_{k+1} \qquad \text{for } x = (x_k)_{k\in\mathbb{N}}$$

defines an operator $T : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$. Show that T is compact, but that T has no eigenvalues (see 11.2(2)).

Solution. Noting that

$$T\big(\mathrm{B}_1(0)\big) \subset \left\{ x \in \ell^2(\mathbb{C}) \ ; \ |x_i| \le \tfrac{1}{i} \text{ for all } i \right\}$$

and recalling E4.13, we have that T is compact. If we assume that $\lambda \in \mathbb{C}$ is an eigenvalue, then $Tx = \lambda x$ for an $x \neq 0$. If $\lambda = 0$, then $Tx = 0$, and so $x = 0$, a contradiction. If $\lambda \neq 0$, it follows that $x_1 = 0$ and $x_{k+1} = \frac{1}{\lambda(k+1)} x_k$ for $k \ge 1$, and so again $x = 0$, a contradiction. □

E10.15 Bound on the dimension of eigenspaces. Let $\Omega \subset \mathbb{R}^n$, let $K \in L^2(\Omega \times \Omega; \mathbb{C})$ and let $T \in \mathscr{L}\big(L^2(\Omega; \mathbb{C})\big)$ be the Hilbert-Schmidt integral operator defined by

$$(Tf)(x) := \int_\Omega K(x,y) f(y) \, \mathrm{d}y \, .$$

Show that

$$\dim \mathscr{N}(\mathrm{Id} - T) \le \|K\|^2_{L^2(\Omega\times\Omega)} \, .$$

Solution. By 10.15, $T \in \mathscr{K}\big(L^2(\Omega;\mathbb{C})\big)$. This implies, on noting that $(\mathrm{Id} - T)(x) = 0$ is equivalent to $x = Tx \in \mathscr{R}(T)$, that $\mathscr{N}(\mathrm{Id} - T) \cap \mathrm{B}_1(0) \subset T\big(\mathrm{B}_1(0)\big)$ is precompact, and hence, by 4.10, that $\mathscr{N}(\mathrm{Id} - T)$ is finite-dimensional. Choose an orthonormal system $f_1, \ldots, f_n$ in $\mathscr{N}(\mathrm{Id} - T)$, where $n := \dim \mathscr{N}(\mathrm{Id} - T)$. Then

$$n = \sum_{i=1}^n \|f_i\|^2_{L^2(\Omega)} = \sum_{i=1}^n \|Tf_i\|^2_{L^2(\Omega)} = \int_\Omega \sum_{i=1}^n \left| \int_\Omega K(x,y) f_i(y) \, \mathrm{d}y \right|^2 \mathrm{d}x \, .$$

Setting $K_x(y) := \overline{K(x,y)}$ and using Bessel's inequality 9.6 we obtain that

$$n = \int_\Omega \sum_{i=1}^n (K_x \, , \, f_i)^2_{L^2(\Omega)} \, \mathrm{d}x \le \int_\Omega \|K_x\|^2_{L^2(\Omega)} \, \mathrm{d}x = \|K\|^2_{L^2(\Omega\times\Omega)} \, .$$

□

E10.16 Norm of Hilbert-Schmidt operators. Under the same assumptions as in E10.15 show that

$$\|T\| = \|K\|_{L^2(\Omega\times\Omega)} \quad \Longleftrightarrow \quad \begin{array}{l} \text{There exist } K_1, K_2 \in L^2(\Omega) \text{ with} \\ K(x,y) = K_1(x) K_2(y) \text{ for almost all } x, y \in \Omega. \end{array}$$

Remark: In this case T is a nuclear operator as in E10.13, with only a single term in the sum.

Solution $\Rightarrow$. Let $K \neq 0$. The assumption yields that for $\varepsilon > 0$ there exist functions $f_\varepsilon \in L^2(\Omega)$ with $\|f_\varepsilon\|_{L^2(\Omega)} = 1$ such that

$$
\begin{aligned}
(1-\varepsilon)\|K\|^2_{L^2(\Omega\times\Omega)} &\le \|Tf_\varepsilon\|^2_{L^2(\Omega)} \\
&= \int_\Omega\Big(\int_\Omega K(z,x)f_\varepsilon(x)\,\mathrm{d}x\Big)\Big(\int_\Omega \overline{K(z,y)}\,\overline{f_\varepsilon(y)}\,\mathrm{d}y\Big)\,\mathrm{d}z \\
&= \int_\Omega\int_\Omega f_\varepsilon(x)\overline{f_\varepsilon(y)}\Big(\int_\Omega K(z,x)\overline{K(z,y)}\,dz\Big)\,\mathrm{d}x\,\mathrm{d}y \\
&\le \underbrace{\Big(\int_\Omega\int_\Omega |f_\varepsilon(x)|^2|f_\varepsilon(y)|^2\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{2}}}_{=1} \\
&\quad\cdot\Big(\int_\Omega\int_\Omega\Big|\int_\Omega K(z,x)\overline{K(z,y)}\,\mathrm{d}z\Big|^2\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Letting $\varepsilon \to 0$ we obtain the inequality

$$
\int_\Omega\int_\Omega |K(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \le \Big(\int_\Omega\int_\Omega\Big|\int_\Omega K(z,x)\overline{K(z,y)}\,\mathrm{d}z\Big|^2\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{2}}. \tag{E10-2}
$$

Moreover, the Cauchy-Schwarz inequality yields that for almost all $x, y \in \Omega$ we have that

$$
\Big|\int_\Omega K(z,x)\overline{K(z,y)}\,\mathrm{d}z\Big|^2 \le \int_\Omega |K(z,x)|^2\,\mathrm{d}z \cdot \int_\Omega |K(z,y)|^2\,\mathrm{d}z\,. \tag{E10-3}
$$

Integrating over x and y, we obtain the opposite inequality (E10-2). This implies that in fact equality holds in (E10-2), and therefore for almost all $(x, y) \in \Omega \times \Omega$ also equality holds in (E10-3). On recalling the remark in 2.3(3), the functions $K_x(z) := K(z, x)$ and $K_y(z) := K(z, y)$ are linearly dependent in $L^2(\Omega)$ for almost all $(x, y) \in \Omega \times \Omega$. In other words (see A6.9), there exists a null set $N_0 \subset \Omega$ such that for all $x \in \Omega \setminus N_0$ it holds that: for almost all $y \in \Omega$ the functions K_x and K_y are linearly dependent. Since we assumed that $K \neq 0$ in $L^2(\Omega \times \Omega)$, we can choose $x_0 \in \Omega \setminus N_0$ such that $K_{x_0} \neq 0$ in $L^2(\Omega)$. Then there exists a null set $N \subset \Omega$ such that for $y \in \Omega \setminus N$ the function K_y is a multiple of K_{x_0}, i.e. there exists a function $\alpha : \Omega \setminus N \to \mathbb{C}$ such that for $y \in \Omega \setminus N$

$$
K(z, y) = \alpha(y)K(z, x_0) \quad \text{for almost all } z \in \Omega.
$$

Setting $K_1(z) := K(z, x_0)$ and $K_2(y) := \alpha(y)$, it follows that

$$
K(z, y) = K_1(z)K_2(y) \quad \text{for almost all } (z, y) \in \Omega \times \Omega.
$$

Fubini's theorem then yields that $K_1, K_2 \in L^2(\Omega)$. $\square$

A10 Calderón-Zygmund inequality

We present a proof of the L^p-estimate in 10.20. To this end, we begin with the following

A10.1 Definition. Let $D \subset \mathbb{C}$ be open and let $f : D \to Y$ be (real) continuously differentiable, where Y is a Banach space over $\mathbb{C}$. Then we define

$$\partial_{\overline{z}} f := \frac{1}{2}(\partial_x f + \mathrm{i}\,\partial_y f) \quad \text{and} \quad \partial_z f := \frac{1}{2}(\partial_x f - \mathrm{i}\,\partial_y f)\,,$$

where we denote complex numbers by $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$.

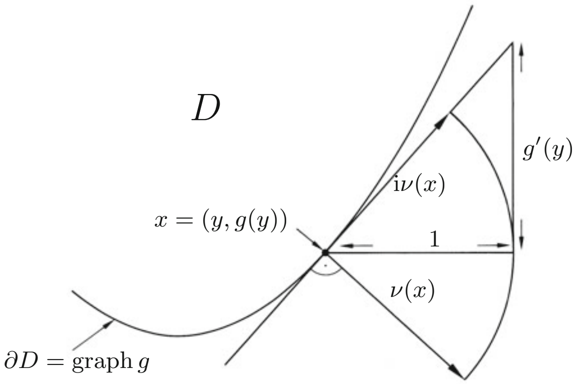

Fig. 10.2. *Outer normal and oriented tangent in* $\mathbb{C}$

Now let $D \subset \mathbb{C}$ be open and bounded with Lipschitz boundary (see A8.2). For functions $f \in C^0(\overline{D}; Y)$ we define the ***oriented boundary integral***

$$\int_{\partial D} f(z)\,\mathrm{d}z := \mathrm{i} \int_{\partial D} \nu(x) f(x)\,\mathrm{dH}^1(x)\,,$$

where $\nu : \partial D \to \mathbb{C}$ is the outer normal to D (see A8.5(3) and Fig. 10.2) and $\nu(x)f(x)$ denotes the complex product of $\nu(x)$ and $f(x)$. Then ***Cauchy's integral theorem*** states that for $f \in C^1(\overline{D}; Y)$

$$\int_{\partial D} f(z)\,\mathrm{d}z = 2\mathrm{i} \int_D \partial_{\overline{z}} f(z)\,\mathrm{dL}^2(z)\,.$$

In the special case where $\partial_{\overline{z}} f = 0$ in D, the function f is called ***holomorphic*** in D.

Proof. Let $y' \in Y'$ and set $g(z) := \langle f(z)\,,\, y' \rangle_Y$. Then (see 5.11)

$$\left\langle \int_{\partial D} f(z)\,\mathrm{d}z\,,\,y'\right\rangle_Y = \int_{\partial D} g(z)\,\mathrm{d}z = \mathrm{i}\int_{\partial D} (\operatorname{Re} g)\nu\,\mathrm{dH}^1 - \int_{\partial D} (\operatorname{Im} g)\nu\,\mathrm{dH}^1\,.$$

It follows from Gauß's theorem (see A8.8) that this is

$$\begin{aligned}
&= \int_D \big(\mathrm{i}\nabla(\operatorname{Re} g) - \nabla(\operatorname{Im} g)\big)\,\mathrm{dL}^2 \\
&= \int_D \Big(\mathrm{i}(\partial_x + \mathrm{i}\partial_y)\frac{g+\overline{g}}{2} - (\partial_x + \mathrm{i}\partial_y)\frac{g-\overline{g}}{2\mathrm{i}}\Big)\,\mathrm{dL}^2 \\
&= 2\mathrm{i}\int_D \partial_{\overline{z}} g\,\mathrm{dL}^2 = \left\langle 2\mathrm{i}\int_D \partial_{\overline{z}} f\,\mathrm{dL}^2\,,\,y'\right\rangle_Y\,.
\end{aligned}$$

□

First we consider the case $n = 1$ in Theorem 10.20.

A10.2 Theorem. If $f \in C_0^\infty(\mathrm{I\!R})$ and $1 < p < \infty$, then

$$T_1 f(x) := \int_{\mathrm{I\!R}\setminus \mathrm{B}_1(x)} \frac{f(s)}{x-s}\,\mathrm{d}s$$

defines a function $T_1 f$ in $L^p(\mathrm{I\!R})$ and there exists a constant $C(p)$ such that for all f

$$\|T_1 f\|_{L^p(\mathrm{I\!R})} \le C(p)\|f\|_{L^p(\mathrm{I\!R})}\,.$$

Therefore 10.20 holds in the case $n = 1$.

Proof. As $f \in C_0^0(\mathrm{I\!R})$ we have that $|T_1 f(x)| \le \frac{C}{|x|}$ for large x, and so $T_1 f \in L^p(\mathrm{I\!R})$. In addition, the representation

$$T_1 f(x) = \int_{\mathrm{I\!R}\setminus \mathrm{B}_1(0)} \frac{f(x-s)}{s}\,\mathrm{d}s$$

shows that $T_1 f \in C^0(\mathrm{I\!R})$. For the proof of the bound we may assume without loss of generality that $f \ge 0$, otherwise consider $\max(f,0)$ and $\max(-f,0)$. We extend $T_1 f$ to the upper half-plane

$$D := \{z \in \mathbb{C}\,;\ \operatorname{Im} z > 0\}\,.$$

To this end we define

$$\varphi(z) := \frac{1}{z}\big(\log(1+z) - \log(1-z)\big) \qquad \text{for } z \in D,$$

where

$$\begin{aligned}
&\log(z) := \log(|z|) + \mathrm{i}\arg(z) \qquad \text{for } z \in \mathbb{C}\setminus\,]-\infty,0]\,, \\
&\arg\big(r\mathrm{e}^{\mathrm{i}\theta}\big) := \theta \qquad \text{for } r > 0,\ |\theta| < \pi.
\end{aligned}$$

Consider the function

$$F(z) := \int_{\mathbb{R}} \varphi(z-s)f(s)\,\mathrm{d}s \qquad \text{for } z \in D.$$

Let $x \neq 0, \pm 1$ and $y \searrow 0$. Then

$$\operatorname{Re}\varphi(x+\mathrm{i}y) \longrightarrow \frac{1}{x}\big(\log|1+x| - \log|1-x|\big) =: \psi(x) \geq 0\,,$$

and, examining how $1 \pm (x+\mathrm{i}y)$ approaches the positive and negative real axis, respectively,

$$\operatorname{Im}\varphi(x+\mathrm{i}y) \longrightarrow \begin{cases} \dfrac{\pi}{x} & \text{if } |x| > 1, \\ 0 & \text{if } |x| < 1. \end{cases}$$

On noting in addition that $|\varphi(x+\mathrm{i}y)| \leq C \cdot \log|x \pm 1|$ for $|x \pm 1| \leq \frac{1}{2}$, and that otherwise φ is a bounded function, it follows from Lebesgue's convergence theorem that

$$F(x+\mathrm{i}y) \to (\psi * f)(x) + \mathrm{i}\pi T_1 f(x) \qquad \text{as } y \searrow 0,$$

locally uniformly in x, i.e. $\operatorname{Im}(F)$ is a continuous extension of $\pi T_1 f$ to D. Since $\psi \in L^1(\mathbb{R})$ (observe that $0 \leq \psi(x) \leq \frac{C}{x^2}$ for large $|x|$), it holds that $\psi * f \in L^p(\mathbb{R})$ with the convolution estimate

$$\|\psi * f\|_{L^p(\mathbb{R})} \leq \|\psi\|_{L^1(\mathbb{R})} \cdot \|f\|_{L^p(\mathbb{R})}\,.$$

In addition we have that $\operatorname{Re}F(z) \geq 0$ for all $z \in D$, because for $z = x + \mathrm{i}y$

$$\operatorname{Re}\varphi(z) = \frac{1}{|z|^2}\big(x(\log|1+z| - \log|1-z|) + y(\arg(1+z) - \arg(1-z))\big)$$

is nonnegative, and f is assumed to be nonnegative. Hence $z \mapsto F(z)^p$ is a well-defined function that is continuous in $\overline{D}$, where

$$z^p := \mathrm{e}^{p \log z} \qquad \text{for } z \in \mathbb{C} \setminus \,]-\infty, 0]\,.$$

As φ is holomorphic in D, and hence so is F, and then also F^p, it follows from Cauchy's integral theorem for $R > 0$ that

$$0 = \int_{\partial(D \cap \mathrm{B}_R(0))} F(z)^p\,\mathrm{d}z = \int_{-R}^{R} F(x)^p\,\mathrm{d}x + \int_{D \cap \partial \mathrm{B}_R(0)} F(z)^p\,\mathrm{d}z\,.$$

Since f has compact support, we have that $|F(z)| \leq C\frac{\log|z|}{|z|}$ for large $|z|$, and so as $R \to \infty$

$$\left|\int_{D \cap \partial \mathrm{B}_R(0)} F(z)^p\,\mathrm{d}z\right| \leq CR\Big(\frac{\log R}{R}\Big)^p \longrightarrow 0\,.$$

This shows that

$$\int_{\mathbb{R}} F(x)^p \,\mathrm{d}x = 0\,.$$

Writing $F(x) = F_1(x) + \mathrm{i}F_2(x)$, it follows from the identity

$$F(x)^p - \big(\mathrm{i}F_2(x)\big)^p = p\int_0^1 \big(tF_1(x) + \mathrm{i}F_2(x)\big)^{p-1}\,\mathrm{d}t \cdot F_1(x)$$

that

$$\left|\int_{\mathbb{R}} \big(\mathrm{i}F_2(x)\big)^p \,\mathrm{d}x\right| \le C(p)\int_{\mathbb{R}} \big(|F_1(x)|^{p-1} + |F_2(x)|^{p-1}\big)|F_1(x)|\,\mathrm{d}x\,.$$

From the generalized Young's inequality it follows for $0 < \delta \le 1$ that this is

$$\le \delta\int_{\mathbb{R}} |F_2(x)|^p\,\mathrm{d}x + \frac{C(p)}{\delta^{p-1}}\int_{\mathbb{R}} |F_1(x)|^p\,\mathrm{d}x\,.$$

Since $\mathrm{Re}(\mathrm{i}F_2(x))^p = \cos(p\frac{\pi}{2})|F_2(x)|^p$, we have

$$\begin{aligned}
&\left|\cos(p\frac{\pi}{2})\right|\int_{\mathbb{R}} |F_2(x)|^p\,\mathrm{d}x = \left|\mathrm{Re}\int_{\mathbb{R}} (\mathrm{i}F_2(x))^p\,\mathrm{d}x\right| \\
&\le \left|\int_{\mathbb{R}} (\mathrm{i}F_2(x))^p\,\mathrm{d}x\right| \le \delta\int_{\mathbb{R}} |F_2(x)|^p\,\mathrm{d}x + \frac{C(p)}{\delta^{p-1}}\int_{\mathbb{R}} |F_1(x)|^p\,\mathrm{d}x\,.
\end{aligned}$$

In the case $\cos(p\frac{\pi}{2}) \ne 0$, on choosing $\delta = \frac{1}{2}\left|\cos(p\frac{\pi}{2})\right|$, it then follows (employing the usual convention on constants) that

$$\int_{\mathbb{R}} |F_2|^p\,\mathrm{dL}^1 \le C(p)\int_{\mathbb{R}} |F_1|^p\,\mathrm{dL}^1 = C(p)\|\psi * f\|^p_{L^p(\mathbb{R})} \le C(p)\|f\|^p_{L^p(\mathbb{R})}\,.$$

This is the desired result when $\cos(p\frac{\pi}{2}) \ne 0$, which for example is satisfied for $1 < p \le 2$. For $2 \le p < \infty$ the claim follows with a duality argument. In particular, it then holds that $1 < p' \le 2$, and so for all $g \in C_0^0(\mathbb{R})$ we have that

$$\begin{aligned}
&\left|\int_{\mathbb{R}} gT_1f\,\mathrm{dL}^1\right| = \left|\int_{\mathbb{R}} fT_1g\,\mathrm{dL}^1\right| \\
&\le \|f\|_{L^p(\mathbb{R})}\|T_1g\|_{L^{p'}(\mathbb{R})} \le C(p')\|f\|_{L^p(\mathbb{R})}\|g\|_{L^{p'}(\mathbb{R})}\,,
\end{aligned}$$

which together with 6.13 implies that

$$\|T_1f\|_{L^p(\mathbb{R})} \le C(p')\|f\|_{L^p(\mathbb{R})}\,.$$

□

In conjunction with the following lemma, we obtain 10.20 in the case $n = 1$.

A10.3 Lemma. The result in 10.20 holds true if there exists a constant $C(n,p)$ such that

$$\|T_1 f\|_{L^p(\mathbb{R}^n)} \le C(n,p)\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all } f \in C_0^\infty(\mathbb{R}^n)\,.$$

Remark: For $f \in C_0^\infty(\mathbb{R}^n)$ it holds that $T_1 f \in L^\infty(\mathbb{R}^n)$. Moreover $|T_1 f(x)| \le C\|f\|_{\sup} \cdot |x|^{-n}$ for large $|x|$, and so $T_1 f \in L^p(\mathbb{R}^n)$.

Proof. Let $f \in L^p(\mathbb{R}^n)$ and $f_k \in C_0^\infty(\mathbb{R}^n)$ with $\|f - f_k\|_{L^p} \to 0$ as $k \to \infty$. It follows from the Hölder inequality that for $x \in \mathbb{R}^n$

$$\begin{aligned}
&|T_1 f(x) - T_1 f_k(x)| \le C \cdot \int_{\mathbb{R}^n \setminus \mathrm{B}_1(x)} \frac{|f(y) - f_k(y)|}{|x-y|^n}\,\mathrm{d}y \\
&\le C \cdot \|f - f_k\|_{L^p} \Big(\int_{\mathbb{R}^n \setminus \mathrm{B}_1(0)} \frac{\mathrm{d}y}{|y|^{np'}} \Big)^{\frac{1}{p'}} \longrightarrow 0 \quad \text{as } k \to \infty,
\end{aligned}$$

and, in addition, if C_0 denotes the constant $C(n,p)$ from the assumptions, that

$$\|T_1 f_k - T_1 f_l\|_{L^p} = \|T_1(f_k - f_l)\|_{L^p} \le C_0 \|f_k - f_l\|_{L^p} \longrightarrow 0 \quad \text{as } k,l \to \infty.$$

Hence $(T_1 f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R}^n)$ with limit $T_1 f$, and so the assumed L^p-estimate also holds for f, i.e.

$$\|T_1 f\|_{L^p} \le C_0 \|f\|_{L^p}\,.$$

Now let $\varepsilon > 0$ and set $f_\varepsilon(y) := f(\varepsilon y)$. Then

$$\begin{aligned}
T_\varepsilon f(x) &= \int_{\mathbb{R}^n \setminus \mathrm{B}_\varepsilon(x)} \frac{\omega(x-y)}{|x-y|^n} f(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n \setminus \mathrm{B}_1(\frac{x}{\varepsilon})} \frac{\omega(\frac{x}{\varepsilon} - y)}{|\frac{x}{\varepsilon} - y|^n} f_\varepsilon(y)\,\mathrm{d}y = T_1 f_\varepsilon(\frac{x}{\varepsilon})\,.
\end{aligned}$$

This yields that $T_\varepsilon f \in L^p(\mathbb{R}^n)$, with

$$\begin{aligned}
\|T_\varepsilon f\|_{L^p} &= \Big(\int_{\mathbb{R}^n} \Big| T_1 f_\varepsilon\big(\frac{x}{\varepsilon}\big)\Big|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} = \Big(\varepsilon^n \int_{\mathbb{R}^n} |T_1 f_\varepsilon(x)|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \\
&\le C_0 \Big(\varepsilon^n \int_{\mathbb{R}^n} |f_\varepsilon(x)|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} = C_0 \|f\|_{L^p}\,.
\end{aligned}$$

It follows for $0 < \varepsilon_1 < \varepsilon_2$ that

$$\begin{aligned}
&\|T_{\varepsilon_1} f - T_{\varepsilon_2} f\|_{L^p} \\
&\le \|T_{\varepsilon_1}(f - f_k)\|_{L^p} + \|T_{\varepsilon_2}(f - f_k)\|_{L^p} + \|T_{\varepsilon_1} f_k - T_{\varepsilon_2} f_k\|_{L^p} \\
&\le \underbrace{2C_0 \|f - f_k\|_{L^p}}_{\to 0 \text{ as } k \to \infty} + \|T_{\varepsilon_1} f_k - T_{\varepsilon_2} f_k\|_{L^p}\,.
\end{aligned}$$

Since the mean of ω vanishes, for all x

$$\begin{aligned}
&|T_{\varepsilon_1} f_k(x) - T_{\varepsilon_2} f_k(x)| \\
&= \left| \int_{\mathrm{B}_{\varepsilon_2}(x)\setminus \mathrm{B}_{\varepsilon_1}(x)} \frac{\omega(x-y)}{|x-y|^n} f_k(y)\,\mathrm{d}y \right| \\
&= \left| \int_{\mathrm{B}_{\varepsilon_2}(x)\setminus \mathrm{B}_{\varepsilon_1}(x)} \frac{\omega(x-y)}{|x-y|^n} \big(f_k(y) - f_k(x)\big)\,\mathrm{d}y \right| \\
&\le C \cdot \int_{\mathrm{B}_{\varepsilon_2}(x)\setminus \mathrm{B}_{\varepsilon_1}(x)} \frac{\mathrm{d}y}{|x-y|^{n-1}} \cdot \|\nabla f_k\|_{\sup} \\
&\le C(n)\varepsilon_2 \|\nabla f_k\|_{\sup} \,.
\end{aligned}$$

Since in addition $T_{\varepsilon_1} f_k(x) = T_{\varepsilon_2} f_k(x)$ for $x \in \mathbb{R}^n \setminus \mathrm{B}_{\varepsilon_2}(\operatorname{supp} f_k)$, we obtain for every k that

$$\begin{aligned}
&\|T_{\varepsilon_1} f_k - T_{\varepsilon_2} f_k\|_{L^p} \\
&\le C(n)\varepsilon_2 \|\nabla f_k\|_{\sup} \mathrm{L}^n\big(\mathrm{B}_{\varepsilon_2}(\operatorname{supp} f_k)\big)^{\frac{1}{p}} \longrightarrow 0 \quad \text{ as } \varepsilon_2 \to 0.
\end{aligned}$$

This proves that the functions $T_\varepsilon f$ for $\varepsilon \to 0$ form a Cauchy sequence in $L^p(\mathbb{R}^n)$. Hence it also holds that

$$\left\| \lim_{\varepsilon \searrow 0} T_\varepsilon f \right\|_{L^p} \le C_0 \|f\|_{L^p} \,.$$

□

A10.4 Theorem. Theorem 10.20 also holds in the case $n > 1$.

Proof. We need to prove a bound for T_1 similarly to A10.3. Since we can decompose ω as

$$\omega(\xi) = \frac{\omega(\xi) + \omega(-\xi)}{2} + \frac{\omega(\xi) - \omega(-\xi)}{2} \,,$$

it is sufficient to consider separately the two cases: ω is an even function, i.e. $\omega(-\xi) = \omega(\xi)$, or an odd function, i.e. $\omega(-\xi) = -\omega(\xi)$.

We begin with the case when ω is odd. (Observe that odd kernels always satisfy the vanishing mean value property (10-33).) For $f \in C_0^\infty(\Omega)$ it then holds, upon using polar coordinates, that

$$\begin{aligned}
T_1 f(x) &= \int_{\mathbb{R}^n \setminus \mathrm{B}_1(0)} \frac{\omega(y)}{|y|^n} f(x-y)\,\mathrm{d}y \\
&= \int_{\partial \mathrm{B}_1(0)} \omega(\xi) \int_1^\infty \frac{f(x - r\xi)}{r}\,\mathrm{d}r\,\mathrm{d}\mathrm{H}^{n-1}(\xi) \,.
\end{aligned}$$

As ω is odd, this is

$$
\begin{aligned}
&= \frac{1}{2}\int_{\partial \mathrm{B}_1(0)} \omega(\xi) \int_1^\infty \frac{f(x-r\xi)-f(x+r\xi)}{r}\,\mathrm{d}r\,\mathrm{dH}^{n-1}(\xi) \\
&= \frac{1}{2}\int_{\partial \mathrm{B}_1(0)} \omega(\xi) \left(\int_{\{|t|\geq 1\}} \frac{f(x-t\xi)}{t}\,\mathrm{d}t\right) \mathrm{dH}^{n-1}(\xi)\,,
\end{aligned}
$$

and so the Hölder inequality yields that

$$
\begin{aligned}
&|T_1 f(x)|^p \leq 2^{-p}\left(\int_{\partial \mathrm{B}_1(0)} |\omega(\xi)|^{\frac{1}{p'}+\frac{1}{p}} \left|\int_{\{|t|\geq 1\}} \frac{f(x-t\xi)}{t}\,\mathrm{d}t\right| \mathrm{dH}^{n-1}(\xi)\right)^p \\
&\leq 2^{-p}\|\omega\|_{L^1(\partial \mathrm{B}_1(0))}^{\frac{p}{p'}} \int_{\partial \mathrm{B}_1(0)} |\omega(\xi)| \left|\int_{\{|t|\geq 1\}} \frac{f(x-t\xi)}{t}\,\mathrm{d}t\right|^p \mathrm{dH}^{n-1}(\xi)\,.
\end{aligned}
$$

For every $\xi \in \partial \mathrm{B}_1(0)$ we decompose the space $\mathbb{R}^n$ as

$$
\mathbb{R}^n = Z_\xi \perp \operatorname{span}\{\xi\}\,.
$$

For $z \in Z_\xi$ it then follows from A10.2 that

$$
\varPhi_\xi(z) := \int_{\mathbb{R}} \left|\int_{\{|t|\geq 1\}} \frac{f(z+(s-t)\xi)}{t}\,\mathrm{d}t\right|^p \mathrm{d}s \leq C(p)\int_{\mathbb{R}} |f(z+s\xi)|^p\,\mathrm{d}s\,,
$$

and so, setting $M_\omega := \|\omega\|_{L^1(\partial \mathrm{B}_1(0))}$, that

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |T_1 f(x)|^p\,\mathrm{d}x \\
&\leq 2^{-p} M_\omega^{p-1} \int_{\partial \mathrm{B}_1(0)} |\omega(\xi)| \left(\int_{Z_\xi} \varPhi_\xi(z)\,\mathrm{dL}^{n-1}(z)\right) \mathrm{dH}^{n-1}(\xi) \\
&\leq C(p) M_\omega^{p-1} \int_{\partial \mathrm{B}_1(0)} |\omega(\xi)| \int_{Z_\xi}\int_{\mathbb{R}} |f(z+s\xi)|^p\,\mathrm{d}s\,\mathrm{dL}^{n-1}(z)\,\mathrm{dH}^{n-1}(\xi)\,.
\end{aligned}
$$

This shows that

$$
\|T_1 f\|_{L^p} \leq C(p)\|\omega\|_{L^1(\partial \mathrm{B}_1(0))} \cdot \|f\|_{L^p}\,,
$$

which proves 10.20 for odd ω. Observe that the proof did not use the boundedness of ω: it suffices to assume that ω is integrable over $\partial \mathrm{B}_1(0)$.

We now assume that ω is even and reduce this case to the odd case. To this end we define the convolution operator

$$
S_\varepsilon g(x) := \int_{\mathbb{R}^n \setminus \mathrm{B}_\varepsilon(x)} g(y)\frac{x-y}{|x-y|^{n+1}}\,\mathrm{d}y \quad \text{and} \quad Sg(x) := \lim_{\varepsilon \searrow 0} S_\varepsilon g(x)\,.
$$

As the vector-valued integral kernel of S_ε is odd, what was shown above implies that for $g \in L^q(\mathbb{R}^n)$ with $1 < q < \infty$,

$$\begin{aligned} &S_\varepsilon g \longrightarrow Sg \quad \text{ in } L^q(\mathbb{R}^n;\mathbb{R}^n) \\ &\text{with} \quad \|S_\varepsilon g\|_{L^q} \le C(n,q)\|g\|_{L^q}\,. \end{aligned} \tag{A10-1}$$

We begin by establishing that there exists a $c_0 > 0$ such that

$$\sum_{i=1}^n S_{i\varepsilon} S_{i\varepsilon} g \longrightarrow -c_0 g \quad \text{ in } L^q(\mathbb{R}^n) \text{ for } g \in C_0^\infty(\mathbb{R}^n), \tag{A10-2}$$

where $S_{i\varepsilon}$ denotes the i-th coordinate of the operator S_ε. We will use this property to bound $T_1 f$ in a first step in terms of $ST_1 f$. In the last part of the proof we then show that ST_1 is also a singular integral operator with an odd kernel.

In order to prove (A10-2) we write

$$\sum_{i=1}^n S_{i\varepsilon} S_{i\varepsilon} g(x) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n \setminus \mathrm{B}_\varepsilon(x) \setminus \mathrm{B}_\varepsilon(y)} \frac{(x-z)\cdot(z-y)}{|x-z|^{n+1}|z-y|^{n+1}}\,\mathrm{d}z \right) g(y)\,\mathrm{d}y\,.$$

With the change of variables $z = -z' + \frac{x+y}{2}$ this becomes

$$= -\int_{\mathbb{R}^n} \varphi_\varepsilon\big(\frac{x-y}{2}\big) g(y)\,\mathrm{d}y\,,$$

where

$$\varphi_\varepsilon(x) := \int_{\{|z \pm x| \ge \varepsilon\}} \frac{(z+x)\cdot(z-x)}{|z+x|^{n+1}|z-x|^{n+1}}\,\mathrm{d}z\,.$$

With the change of variables $z = \varepsilon z'$ we obtain that $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi_1\big(\frac{x}{\varepsilon}\big)$. Hence assertion (A10-2) follows from 4.15(2), if we show that φ_1 is a non-negative integrable function. If $D \subset \mathbb{R}^n$ is open and invariant under the reflection in $\partial\mathrm{B}_{|x|}(0)$, i.e. $\frac{|x|^2}{|z|^2} z \in D$ for $z \in D$, then the change of variables $z = \frac{|x|^2}{|z'|^2} z'$ yields, on noting that

$$\mathrm{d}z = \Big(\frac{|x|}{|z'|}\Big)^{2n} \mathrm{d}z' \quad \text{and} \quad |z \pm x| = \frac{|x|}{|z'|}|z' \pm x|\,,$$

that

$$\int_D \frac{|z|^2 - |x|^2}{|z+x|^{n+1}|z-x|^{n+1}}\,\mathrm{d}z = \int_D \frac{|x|^2 - |z'|^2}{|z'+x|^{n+1}|z'-x|^{n+1}}\,\mathrm{d}z'\,,$$

i.e. this integral vanishes. Applying this result to the domain $D = \{z;\ |z \pm x| > 1,\ |z' \pm x| > 1\}$, we obtain that

$$\varphi_1(x) = \int_E \frac{|z|^2 - |x|^2}{|z+x|^{n+1}|z-x|^{n+1}}\,\mathrm{d}z$$

with $E := \{|z \pm x| \geq 1\} \cap \left(\{|z+x| \leq \frac{|z|}{|x|}\} \cup \{|z-x| \leq \frac{|z|}{|x|}\}\right)$, and so $|z| \geq |x|$ for $z \in E$, which implies that $\varphi_1 \geq 0$. Clearly φ_1 is continuous on $\mathbb{R}^n \setminus \{0\}$, and for $|x| \leq \frac{1}{2}$

$$\varphi_1(x) \leq C \int_{\{|z| \geq \frac{1}{2}\}} \frac{\mathrm{d}z}{|z|^{2n}} < \infty\,,$$

while for $|x| \geq 2$

$$\varphi_1(x) \leq \int_E \frac{\mathrm{d}z}{|z+x|^n |z-x|^n}\,.$$

We partition E into $\{z \in E\,;\ z \bullet x \geq 0\}$ and $\{z \in E\,;\ z \bullet x \leq 0\}$. For z in the first set it holds that $|z+x| \geq |x|$ and with $z' := z - x$ we have that $1 \leq |z'| \leq \frac{|x|}{|x|-1}$. An analogous result holds for the second set. Overall we obtain that

$$\varphi_1(x) \leq \frac{2}{|x|^n} \int_{\{1 \leq |z'| \leq \frac{|x|}{|x|-1}\}} \frac{\mathrm{d}z'}{|z'|^n} \leq \frac{C}{|x|^{n+1}}\,.$$

The last inequality follows from the fact that we integrate over an annular region of width $\frac{1}{|x|-1}$. This shows that φ_1 is integrable and the result (A10-2) is shown.

Now let $f \in C_0^\infty(\mathbb{R}^n)$ as before. It follows from (A10-2) and the L^p-bound for S_ε that for $\zeta \in C_0^\infty(\mathbb{R}^n)$ and as $\varepsilon \searrow 0$

$$\begin{aligned} c_0 \left| \int_{\mathbb{R}^n} \zeta T_1 f \,\mathrm{dL}^n \right| &\longleftarrow \left| \int_{\mathbb{R}^n} \Big(\sum_{i=1}^n S_{i\varepsilon} S_{i\varepsilon} \zeta \Big) T_1 f \,\mathrm{dL}^n \right| \\ &= \left| \int_{\mathbb{R}^n} \sum_{i=1}^n S_{i\varepsilon}\zeta \cdot S_{i\varepsilon} T_1 f \,\mathrm{dL}^n \right| \leq \|S_\varepsilon \zeta\|_{L^{p'}} \|S_\varepsilon T_1 f\|_{L^p}\,, \end{aligned}$$

with

$$\|S_\varepsilon \zeta\|_{L^{p'}} \leq C(n,p') \|\zeta\|_{L^{p'}}\,.$$

As $T_1 f \in L^p(\mathbb{R}^n)$ (see the remark in A10.3), it holds in addition that

$$\|S_\varepsilon T_1 f\|_{L^p} \longrightarrow \|S T_1 f\|_{L^p} \quad \text{as } \varepsilon \searrow 0.$$

Hence, on recalling 6.13, we obtain the bound

$$\|T_1 f\|_{L^p} \leq C(n,p) \|S T_1 f\|_{L^p}\,.$$

Now we show that ST_1, too, is essentially a singular integral operator with an odd kernel. It holds that

$$\begin{aligned} S_\varepsilon T_1 f(x) &= \int_{\mathbb{R}^n} \left(\int_{\{|z-x| \geq \varepsilon,\ |z-y| \geq 1\}} \frac{x-z}{|x-z|^{n+1}} \frac{\omega(z-y)}{|z-y|^n} \,\mathrm{d}z \right) f(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \Phi_\varepsilon(x-y) f(y)\,\mathrm{d}y\,, \end{aligned}$$

where

$$\Phi_\varepsilon(x) := \int_{\{|z-x|\geq\varepsilon\,,\,|z|\geq 1\}} \frac{x-z}{|x-z|^{n+1}} \frac{\omega(z)}{|z|^n}\,\mathrm{d}z\,.$$

Since

$$\Phi_\varepsilon(x) = S_\varepsilon h(x) \quad \text{with} \quad h(z) := \mathcal{X}_{\mathbb{R}^n\setminus \mathrm{B}_1(0)}(z)\frac{\omega(z)}{|z|^n}\,,$$

and since $h \in L^q(\mathbb{R}^n)$ for every $1 < q < \infty$ (not for $q = 1$!), it follows from the previously shown convergence in (A10-1) that

$$\begin{aligned} &\Phi_\varepsilon = S_\varepsilon h \to Sh =: \Phi \quad \text{in } L^q(\mathbb{R}^n;\mathbb{R}^n)\,,\\ &\text{with} \quad \|\Phi\|_{L^q} \leq C(n,q)\|h\|_{L^q}\,. \end{aligned}$$

Here we have that

$$\begin{aligned} \|h\|_{L^q} &= \left(\int_1^\infty r^{n-1-nq} \int_{\partial\mathrm{B}_1(0)} |\omega(\xi)|^q \,\mathrm{dH}^{n-1}(\xi)\,\mathrm{d}r\right)^{\frac{1}{q}}\\ &= c_1(n,q)\|\omega\|_{L^q(\partial\mathrm{B}_1(0))}, \end{aligned}$$

with

$$c_\varrho(n,q) := \left(\int_\varrho^\infty r^{-1-n(q-1)}\,\mathrm{d}r\right)^{\frac{1}{q}}\,.$$

In addition,

$$ST_1 f(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,\mathrm{d}y\,.$$

Similarly to Φ_ε, for every $\delta > 0$

$$\psi_\delta(x) := \int_{\{\delta\leq|z|\leq 1\}} \frac{x-z}{|x-z|^{n+1}} \frac{\omega(z)}{|z|^n}\,\mathrm{d}z$$

defines a function $\psi_\delta \in L^p(\mathbb{R}^n;\mathbb{R}^n)$. Moreover, the limit

$$\psi(x) := \lim_{\delta\searrow 0} \psi_\delta(x) = \int_{\{|z|\leq 1\}} \frac{x-z}{|x-z|^{n+1}} \frac{\omega(z)}{|z|^n}\,\mathrm{d}z$$

exists pointwise for $x \neq 0$. In order to prove this, choose for $\varrho > 0$ an $\eta \in C_0^\infty(\mathrm{B}_\varrho(0))$ with $\eta = 1$ in $\mathrm{B}_{\frac{\varrho}{2}}(0)$ and decompose $\psi_\delta(x)$ for $|x| > \varrho$ as induced by the decomposition

$$\frac{x-z}{|x-z|^{n+1}} = \eta(z)\frac{x-z}{|x-z|^{n+1}} + \big(1-\eta(z)\big)\frac{x-z}{|x-z|^{n+1}}\,.$$

The first term is a Lipschitz continuous (in fact smooth) function of z. Hence the corresponding integral converges as $\delta \searrow 0$ (see the first part of the proof of the Hölder-Korn-Lichtenstein inequality 10.19). The integral over the second term is independent of δ for $\delta < \frac{\varrho}{2}$.

On employing the change of variables $z = |x|z'$ we now see that there exists a measurable function $\omega_0 : \partial \mathrm{B}_1(0) \to \mathbb{R}^n$ such that

$$\Phi(x) + \psi(x) = \frac{\omega_0(x)}{|x|^n}\,, \tag{A10-3}$$

where $\omega_0(x) := \omega_0\big(\frac{x}{|x|}\big)$ and

$$\omega_0(\xi) := \int_{\mathbb{R}^n} \frac{\xi - z}{|\xi - z|^{n+1}} \frac{\omega(z)}{|z|^n}\,\mathrm{d}z \qquad \text{for almost all } \xi \in \partial \mathrm{B}_1(0)\,.$$

As ω is an even function, ω_0 must be odd. Moreover, for $|x| \geq 2$ and $|z| \leq 1$ (cf. the proof of 10.19)

$$\left| \frac{x-z}{|x-z|^{n+1}} - \frac{x}{|x|^{n+1}} \right| \leq C|z|\Big(\frac{1}{|z-x||x|^n} + \frac{1}{|x|\,|z-x|^n} \Big) \leq \frac{C|z|}{|x|^{n+1}}\,,$$

which in view of the mean value property of ω implies that

$$\begin{aligned} |\psi(x)| &= \left| \int_{\{|z|\leq 1\}} \Big(\frac{x-z}{|x-z|^{n+1}} - \frac{x}{|x|^{n+1}} \Big) \frac{\omega(z)}{|z|^n}\,\mathrm{d}z \right| \\ &\leq \frac{C}{|x|^{n+1}} \int_{\{|z|\leq 1\}} \frac{|\omega(z)|}{|z|^{n-1}}\,\mathrm{d}z \leq \frac{C}{|x|^{n+1}} \|\omega\|_{L^1(\partial \mathrm{B}_1(0))}\,. \end{aligned}$$

Hence $\psi \in L^q(\mathbb{R}^n \setminus \mathrm{B}_2(0); \mathbb{R}^n)$ for $1 \leq q < \infty$ (here the case $q = 1$ is included (!)), with

$$\|\psi\|_{L^q(\mathbb{R}^n \setminus \mathrm{B}_2(0))} \leq C(n,q)\|\omega\|_{L^1(\partial \mathrm{B}_1(0))}\,.$$

Therefore, on recalling (A10-3), we obtain for $1 < q < \infty$ that

$$\begin{aligned} c_2(n,q)\|\omega_0\|_{L^q(\partial \mathrm{B}_1(0))} &= \left\| \frac{\omega_0}{|\bullet|^n} \right\|_{L^q(\mathbb{R}^n \setminus \mathrm{B}_2(0))} \\ &\leq \|\Phi\|_{L^q(\mathbb{R}^n)} + \|\psi\|_{L^q(\mathbb{R}^n \setminus \mathrm{B}_2(0))} \\ &\leq C(n,q)\big(c_1(n,q)\|\omega\|_{L^q(\partial \mathrm{B}_1(0))} + \|\omega\|_{L^1(\partial \mathrm{B}_1(0))} \big)\,, \end{aligned}$$

and so

$$\|\omega_0\|_{L^1(\partial \mathrm{B}_1(0))} \leq C(n)\|\omega\|_{L^\infty(\partial \mathrm{B}_1(0))} < \infty\,.$$

Hence the previously shown L^p-bound for kernels induced by odd ω can be applied to the kernel induced by ω_0. We note from (A10-3) that

$$\Phi(x) = \mathcal{X}_{\mathbb{R}^n \setminus \mathrm{B}_2(0)}(x) \frac{\omega_0(x)}{|x|^n} - \widetilde{\Phi}(x)\,,$$

where

$$\widetilde{\Phi}(x) := \mathcal{X}_{\mathbb{R}^n \setminus \mathrm{B}_2(0)}(x)\psi(x) - \mathcal{X}_{\mathrm{B}_2(0)}(x)\Phi(x)\,,$$

and we note that for every $1 < q < \infty$

$$\left\| \widetilde{\Phi} \right\|_{L^1(\mathbb{R}^n)} \leq \|\psi\|_{L^1(\mathbb{R}^n \setminus \mathrm{B}_2(0))} + C(n,q)\|\Phi\|_{L^q(\mathbb{R}^n)} < \infty\,.$$

We obtain using the L^p-bound for the kernel induced by ω_0 and the convolution estimate that

$$\begin{aligned}
&\|ST_1 f\|_{L^p(\mathbb{R}^n)} \\
&\leq \left(\int_{\mathbb{R}^n} \left| \int_{\{|y|\geq 2\}} \frac{\omega_0(x-y)}{|x-y|^n} f(y)\,\mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} + \left\| \widetilde{\Phi} * f \right\|_{L^p(\mathbb{R}^n)} \\
&\leq \left(C(p)\|\omega_0\|_{L^1(\partial \mathrm{B}_1(0))} + \left\| \widetilde{\Phi} \right\|_{L^1(\mathbb{R}^n)} \right) \|f\|_{L^p(\mathbb{R}^n)}\,.
\end{aligned}$$

This proves 10.20 also for even kernels. $\square$