1 Introduction

Functional analysis deals with the structure of function spaces and the properties of continuous mappings between these spaces. Linear functional analysis, in particular, is confined to the analysis of linear mappings of this kind. Its development was based on the fundamental observation that the topological concepts of the Euclidean space $\mathbb{R}^n$ can be generalized to function spaces as well. To this end, functions are interpreted as points in a given space (see the cover page, where a part of the orthonormal system in 9.9 is shown). Given a set S, we consider the set of all maps $f : S \to \mathbb{R}$. Denoting this set by $\mathscr{F}(S;\mathbb{R})$ means that any point $f \in \mathscr{F}(S;\mathbb{R})$ defines a mapping $x \mapsto f(x)$ that assigns to each element $x \in S$ a unique $f(x) \in \mathbb{R}$. Then the set $\mathscr{F}(S;\mathbb{R})$ becomes a vector space if we define for all $f_1, f_2, f \in \mathscr{F}(S;\mathbb{R})$ and $\alpha \in \mathbb{R}$

$$(f_1 + f_2)(x) := f_1(x) + f_2(x)\ ,\ (\alpha f)(x) := \alpha f(x) \quad \text{for } x \in S\,.$$

With the help of characteristic examples we now investigate similarities and differences between the Euclidean space $\mathbb{R}^n$ and some function spaces. The function spaces will be covered in more detail later on in the book.

First we consider the space $C^0(S)$ (see 3.2) of continuous functions $f : S \to \mathbb{R}$, where S is a bounded, closed set in $\mathbb{R}^n$. The supremum norm on $C^0(S)$ is defined by

$$\|f\|_{C^0} := \sup\{|f(x)|\,;\ x \in S\} \quad \text{for } f \in C^0(S)\,.$$

It satisfies the same norm axioms (see 2.4) as the Euclidean norm on $\mathbb{R}^n$,

$$\|x\|_{\mathbb{R}^n} := \Big(\sum_{i=1}^{n} x_i^2\Big)^{\frac{1}{2}} \quad \text{for } x = (x_i)_{i=1,\dots,n} = (x_1, \dots, x_n) \in \mathbb{R}^n\,.$$

One difference between the two spaces is that $C^0(S)$, in contrast to $\mathbb{R}^n$, is an infinite-dimensional space, when S contains infinitely many points. This can be seen as follows. Let $x_i \in S$ for $i \in \mathbb{N}$ be pairwise distinct. Then for each $n \in \mathbb{N}$ we can find functions $\varphi_{n,i} \in C^0(S)$ for $i = 1, \dots, n$, such that $\varphi_{n,i}(x_j) = \delta_{i,j}$ for $i, j = 1, \dots, n$. Here

$$\delta_{i,j} := \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{otherwise} \end{cases}$$

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denotes the ***Kronecker symbol***. Now if $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, n$ are such that

$$f := \sum_{i=1}^{n} \alpha_i \varphi_{n,i} = 0 \quad \text{in } C^0(S)\,,$$

then it follows that $0 = f(x_j) = \alpha_j$ for $j = 1, \ldots, n$. Hence $\varphi_{n,1}, \ldots, \varphi_{n,n}$ are linearly independent and, since $n \in \mathbb{N}$ was chosen arbitrarily, the dimension of $C^0(S)$ cannot be finite. This changes the properties of the space significantly. For instance, while in $\mathbb{R}^n$ all bounded closed sets are compact (see the Heine-Borel theorem 4.7(7)), this is not the case in $C^0(S)$ (see the Arzelà-Ascoli theorem 4.12).

Also, the scalar product in $\mathbb{R}^n$,

$$(x\,,\,y)_{\mathbb{R}^n} := \sum_{i=1}^{n} x_i y_i \quad \text{for } x = (x_i)_{i=1,\ldots,n}\ ,\ y = (y_i)_{i=1,\ldots,n} \in \mathbb{R}^n\,,$$

has an analogue for function spaces; indeed, define (cf. 3.16(3))

$$(f\,,\,g)_{L^2} := \int_S f(x)g(x)\,\mathrm{d}x \quad \text{for } f, g \in C^0(S)\,.$$

The corresponding norm $\|f\|_{L^2} := \sqrt{(f\,,\,f)_{L^2}}$ is bounded from above by the supremum norm, that is, there exists a constant $C < \infty$ such that

$$\|f\|_{L^2} \le C\|f\|_{C^0} \quad \text{for all } f \in C^0(S)$$

(this follows from 3.18, if C denotes the square root of the Lebesgue measure of S). In general, a similar bound from below cannot be derived. To see this, consider the interval $S = [\,-1, 1\,] \subset \mathbb{R}$ and for $0 < \varepsilon < 1$ the functions f_ε defined by $f_\varepsilon(x) := \max\bigl(0, \frac{1}{\varepsilon}\bigl(1 - \frac{|x|}{\varepsilon}\bigr)\bigr)^{\frac{1}{2}}$, for which $\|f_\varepsilon\|_{C^0} = \varepsilon^{-\frac{1}{2}}$, but $\|f_\varepsilon\|_{L^2} = 1$. That means that the C^0-norm and the L^2-norm on $C^0(S)$ are not equivalent to each other (see 2.15); the C^0-norm is stronger than the L^2-norm. That is, the space $C^0(S)$, equipped with the L^2-norm, is not complete. For example, the functions g_k for $k \in \mathbb{N}$, $g_k(x) := (1-x)^k$ for $x \ge 0$, $g_k(x) := 1$ for $x \le 0$, form a Cauchy sequence with respect to the L^2-norm, but there exists no function $g \in C^0(S)$ such that $\|g_k - g\|_{L^2} \to 0$ as $k \to \infty$.

In a situation like this we can apply a general principle in mathematics: completion (see 2.24). Similarly to defining the real numbers $\mathbb{R}$ as the completion of the rational numbers $\mathbb{Q}$, we can complete the space $C^0(S)$ with respect to the L^2-norm. Thus we obtain the complete space $L^2(S)$ of all square integrable, Lebesgue measurable functions on S (see 3.15 and 4.15(3)). In this space fundamental assertions hold, such as Lebesgue's convergence theorem (see 3.25).

We encounter a similar situation in a further generalization from the finite-dimensional case to the infinite-dimensional one. For the finite-dimensional case, let $E : S \to \mathbb{R}$ be a continuous function defined on a bounded closed set $S \subset \mathbb{R}^n$. We now look for a minimum of this function over S. The compactness of S and the continuity of E yield that such a minimum exists: E has an ***absolute minimum*** on S, that is, there exists an $x_0 \in S$ such that

$$E(x_0) = \inf_{x \in S} E(x) .$$

The same holds true if we only assume that S is closed and if in addition we require that $E(x) \to \infty$ for $x \in S$ as $\|x\|_{\mathbb{R}^n} \to \infty$.

As an infinite-dimensional analogue we consider the following Dirichlet boundary value problem on an open, bounded set $\Omega \subset \mathbb{R}^n$. The given datum is a continuous function u_0 defined on the boundary $\partial\Omega$ of Ω, i.e. $u_0 \in C^0(\partial\Omega)$, and we want to find a continuous function $u : \overline{\Omega} \to \mathbb{R}$ that is twice continuously differentiable in Ω, such that

$$\begin{aligned} \Delta u(x) := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u(x) = 0 \quad &\text{for } x \in \Omega, \\ u(x) = u_0(x) \quad &\text{for } x \in \partial\Omega. \end{aligned}$$

In applications, u is, for example, a stationary temperature distribution or the potential of a charge-free electric field. One approach to find a solution is to consider the corresponding ***energy functional*** (here identical to the ***Dirichlet integral***)

$$E(u) := \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x ,$$

where $\nabla u(x) := \Big(\frac{\partial}{\partial x_1} u(x), \dots, \frac{\partial}{\partial x_n} u(x) \Big)$. Here we use the term ***functional***, because E acts on functions, that is, E is a function defined on functions. In order to guarantee that $E(u) < \infty$, we initially define the domain of E to be

$$M := \big\{ v \in C^1(\overline{\Omega}) \; ; \; v = u_0 \text{ on } \partial\Omega \big\} , \qquad \text{so} \qquad E : M \to \mathbb{R} ,$$

where we assume that M is nonempty. If we now assume that $u \in M$ is an absolute minimum of E on M, then $E(u) \le E(u + \varepsilon\zeta)$ for all $\varepsilon \in \mathbb{R}$ and all $\zeta \in C^1(\overline{\Omega})$ such that $\zeta = 0$ in a neighbourhood of $\partial\Omega$. On noting that $\varepsilon \mapsto E(u+\varepsilon\zeta)$ is differentiable in ε (this function is quadratic in ε), it follows that

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} E(u + \varepsilon\zeta)\big|_{\varepsilon=0} = \int_\Omega \nabla\zeta(x) \bullet \nabla u(x) \, \mathrm{d}x .$$

The fact that this identity holds for all functions ζ with the above mentioned properties contains all the information needed in order to derive a differential equation for u. That is why the functions ζ are also called ***test functions***, and $u \in M$ is called a ***weak solution*** of the boundary value problem if the integral identity holds for all test functions. Introducing this solution concept allows the treatment of partial differential equations by means of functional analysis (see 6.5–6.8). We obtain the corresponding classical differential equation on assuming that $u \in C^2(\Omega)$, as integration by parts then yields that

$$0 = \int_\Omega \nabla\zeta(x) \bullet \nabla u(x)\,\mathrm{d}x = -\int_\Omega \zeta(x)\Delta u(x)\,\mathrm{d}x$$

for all test functions ζ. This implies $\Delta u = 0$ in Ω (cf. 4.22), and hence u is a solution of the original Dirichlet problem.

However, the existence of an absolute minimum $u \in M$ for a functional $E : M \to \mathbb{R}$ with $M \subset C^1(\overline{\Omega})$ is not established as easily as in the finite-dimensional case. For instance, if $\Omega =]0,1[$,

$$M_1 := \left\{ u \in C^1\big([0,1]\big)\ ;\ u(0) = 0,\ u'(1) = 1 \right\}$$

with $\|u\|_{C^1} := \|u\|_{C^0} + \|u'\|_{C^0}$ and

$$E_1(u) := \|u'\|_{C^0}^2 + \int_0^1 |u'(x)|^2\,\mathrm{d}x\,,$$

then M_1 is closed in $C^1\big([0,1]\big)$ and E_1 is continuous with respect to the C^1-norm. Moreover, $E_1(u) \geq \|u'\|_{C^0}^2 \to \infty$ for $u \in M_1$ as $\|u\|_{C^1} \to \infty$, since for $u \in M_1$ and $x \in [0,1]$ we have

$$|u(x)| = \left| \int_0^x u'(y)\,\mathrm{d}y \right| \leq \|u'\|_{C^0}\,,$$

and hence $\|u\|_{C^1}^2 \leq 4\,\|u'\|_{C^0}^2$. Consequently, all the assumptions are satisfied which lead in the above finite-dimensional case to the existence of an absolute minimum.

But E_1 does not have an absolute minimum on M_1. To see this, note that $E_1(u) \geq \|u'\|_{C^0}^2 \geq |u'(1)|^2 = 1$ for all $u \in M_1$. This lower bound also represents the infimum of E_1 over M_1, since the functions $u_\varrho(x) := \frac{1}{\varrho}x^\varrho$ for $\varrho > 1$ satisfy

$$\left\|u_\varrho'\right\|_{C^0}^2 = 1 \quad \text{and} \quad \int_0^1 \left|u_\varrho'(x)\right|^2 \mathrm{d}x = \frac{1}{2\varrho - 1} \to 0 \text{ as } \varrho \to \infty.$$

Now, if $u \in M_1$ was an absolute minimum, i.e. $E_1(u) = 1$, then

$$\|u'\|_{C^0}^2 = 1 \quad \text{and} \quad \int_0^1 |u'(x)|^2\,\mathrm{d}x = 0\,.$$

But the second equality implies $u' = 0$, which contradicts the first equality.

In conclusion, we note that the main difficulty in proving the existence of an absolute minimum lies in the fact that $C^1(\overline{\Omega})$ is equipped with a supremum norm, while the functional $E(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2$ corresponds to an integral norm, which cannot be used to bound the C^1-norm (similarly to our first example). If, on the other hand, we equip $C^1(\overline{\Omega})$ with the integral norm

$$\|u\|_{W^{1,2}} := \|u\|_{L^2} + \|\nabla u\|_{L^2}\,,$$

then (similarly to the first example) the space is no longer complete. But the completeness of the space under consideration is a crucial property in all existence proofs. Hence at times it becomes necessary to seek solutions to boundary value problems, or minima of functionals, in a larger class of functions. For instance, on completing the space $C^1(\overline{\Omega})$ with respect to the above $W^{1,2}$-norm (see 3.27), and thus obtaining the ***Sobolev space*** $W^{1,2}(\Omega)$, we can consider the functional E to be defined on $W^{1,2}(\Omega)$ rather than on $C^1(\overline{\Omega})$. In this new space, the above variational problem admits a solution (see 8.17).

As a third example we consider the infinite-dimensional analogue of matrices. The set of all sequences with only finitely many nonzero terms is defined by

$$\begin{aligned} c_* := \Big\{ x = (x_k)_{k\in\mathbb{N}}\ ;\ & x_k \in \mathbb{R} \text{ for } k \in \mathbb{N}, \text{ and there exists an } n \in \mathbb{N}, \\ & \text{such that } x_k = 0 \text{ for all } k > n \Big\}. \end{aligned}$$

A linear map $T : c_* \to c_*$ is characterized by the values T_{ij}, the i-th coordinate of $T(\mathbf{e}_j)$. Here $\mathbf{e}_j$ corresponds to the j-th unit vector of the Euclidean space, that is, $\mathbf{e}_j := (\delta_{j,k})_{k\in\mathbb{N}} \in c_*$. In other words

$$Tx = \sum_{i\in\mathbb{N}} \Big(\sum_{j\in\mathbb{N}} T_{ij} x_j \Big) \mathbf{e}_i\,,$$

where in each sum only finitely many terms are nonzero, with their number depending on x. Hence T can be represented by a matrix $(T_{ij})_{i,j\in\mathbb{N}}$ with infinitely many rows and columns.

For finite matrices, i.e. in the finite-dimensional case, a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is injective if and only if it is surjective. However, if we consider the ***shift operator*** $T : c_* \to c_*$, defined by

$$T(x_1, x_2, x_3, \dots) := (0, x_1, x_2, x_3, \dots)\,,$$

then T is injective, but not surjective. Nevertheless, later on we will see that the above property of finite matrices carries over to certain maps, namely to compact perturbations of the identity (see the Fredholm alternative 11.11). Chapters 11 and 12 are devoted to the spectral theory of such operators. There we will generalize results from linear algebra that provide normal forms for finite-dimensional matrices. For instance, the Jordan normal form of matrices corresponds to the spectral theorem for compact operators (see 11.9

and 11.13), while the fact that every symmetric matrix is diagonalizable corresponds to the spectral theorem for compact normal operators (see 12.11 and 12.12). In function spaces such operators occur in the analysis of differential and integral equations.

As a final example we consider a ***Sturm-Liouville problem***. A solution to the Sturm-Liouville problem is given by a function $u \in C^2\big([0,1]\big)$ satisfying the differential equation

$$Tu := -(pu')' + qu = f$$

and, for instance, satisfying the boundary conditions

$$u(0) = 0\,, \quad u'(1) = 0\,.$$

We assume that the right-hand side of the differential equation satisfies $f \in C^0\big([0,1]\big)$, while for the coefficients we assume e.g. $q \in C^0\big([0,1]\big)$ and $p \in C^1\big([0,1]\big)$, with p being a strictly positive function, i.e. there exists a number $c > 0$, such that $p(x) \geq c$ for all $x \in [0,1]$.

The Sturm-Liouville problem can be formulated as an integral equation. Then one looks for a function $u \in C^0\big([0,1]\big)$ such that $u = K_f u$, where

$$(K_f u)(x) := \int_0^x \frac{1}{p(y)} \int_y^1 (f - qu)(z)\,\mathrm{d}z\,\mathrm{d}y\,.$$

If $u \in C^0\big([0,1]\big)$ is a solution to this integral equation, i.e. $u = K_f u$, then the integral representation and the assumptions on p, q, f yield that $u \in C^2\big([0,1]\big)$, and that both the differential equation and the boundary conditions are satisfied.

It follows from the Banach fixed point theorem that the integral equation admits a unique solution. This is true whenever K_f is a ***contraction mapping***, i.e. if there exists a number $\theta < 1$, such that

$$\|K_0 u\| \leq \theta \|u\| \quad \text{for all } u \in C^0\big([0,1]\big)\,,$$

where $\|\bullet\|$ denotes the supremum norm (it is also possible to use other, equivalent norms, which can lead to improved contraction factors). For instance, for $p = 1$ we have

$$\begin{aligned} |K_0 u(x)| &= \left| \int_0^x \int_y^1 (qu)(z)\,\mathrm{d}z\,\mathrm{d}y \right| = \left| \int_0^1 (qu)(z) \min(z,x)\,\mathrm{d}z \right| \\ &\leq \|u\| \int_0^1 |zq(z)|\,\mathrm{d}z\,, \end{aligned}$$

and hence the boundary value problem has a unique solution if

$$p = 1 \quad \text{and} \quad \int_0^1 z|q(z)|\,\mathrm{d}z < 1\,.$$

However, this unduly restricts the class of admissible functions q. In order to be able to treat more general q, we reformulate the problem and attempt to solve an infinite-dimensional system of linear equations. To this end, let $\{e_i\,;\ i \in \mathbb{N}\}$ be a linearly independent set in the function space

$$V := \left\{ v \in C^2\big([0,1]\big)\ ;\ v(0)=0,\ v'(1)=0 \right\}$$

and define

$$a_{ij} := \int_0^1 e_i(x)(Te_j)(x)\,\mathrm{d}x \quad \text{and} \quad f_i := \int_0^1 e_i(x)f(x)\,\mathrm{d}x\,.$$

Using the formal ansatz $u = \sum_{j\in\mathbb{N}} u_j e_j$ it then follows from $Tu = f$ that formally

$$\sum_{j\in\mathbb{N}} a_{ij}u_j = f_i \quad \text{for all } i \in \mathbb{N}\,.$$

If the e_i form a Schauder basis (see 9.3) with respect to the L^2-norm, then this infinite-dimensional system of equations is even formally equivalent to the differential equation. For, with an arbitrary function $\zeta = \sum_{i\in\mathbb{N}} \alpha_i e_i \in V$ and since $Tu = \sum_{j\in\mathbb{N}} u_j Te_j$, it follows from the system of equations that

$$\begin{aligned}
0 &= \sum_{i\in\mathbb{N}} \alpha_i \Big(\sum_{j\in\mathbb{N}} a_{ij}u_j - f_i \Big) \\
&= \sum_{i\in\mathbb{N}} \alpha_i \Big(\int_0^1 \sum_{j\in\mathbb{N}} u_j e_i(x)(Te_j)(x)\,\mathrm{d}x - \int_0^1 e_i(x)f(x)\,\mathrm{d}x \Big) \\
&= \int_0^1 \Big(\sum_{i\in\mathbb{N}} \alpha_i e_i(x) \Big) \Big(\sum_{j\in\mathbb{N}} u_j (Te_j)(x) - f(x) \Big)\,\mathrm{d}x \\
&= \int_0^1 \zeta(x)\big((Tu)(x) - f(x)\big)\,\mathrm{d}x\,,
\end{aligned}$$

and hence (similarly to the Dirichlet problem above) that the differential equation is fulfilled. Remember, that this conclusion was formal.

We now assume that we can choose for each i the e_i as normalized eigenvector of T corresponding to the eigenvalue λ_i, i.e.

$$Te_i = \lambda_i e_i\,, \quad \int_0^1 e_i(x)^2\,\mathrm{d}x = 1\,.$$

It follows for $i, j \in \mathbb{N}$ that

$$\begin{aligned}
&(\lambda_i - \lambda_j) \int_0^1 e_i(x)e_j(x)\,\mathrm{d}x \\
&= \int_0^1 (Te_i)(x)e_j(x)\,\mathrm{d}x - \int_0^1 e_i(x)(Te_j)(x)\,\mathrm{d}x = 0\,.
\end{aligned}$$

For the last identity we have used the fact that T is a self-adjoint operator. To see this, note that, for $u, v \in V$,

$$\begin{aligned}
&\int_0^1 v(x)(Tu)(x)\,\mathrm{d}x \\
&= -\int_0^1 v(x)(pu')'(x)\,\mathrm{d}x + \int_0^1 q(x)v(x)u(x)\,\mathrm{d}x \\
&= -\underbrace{\Big[\, v(x)p(x)u'(x)\,\Big]_{x=0}^{x=1}}_{=0} + \int_0^1 \big(p(x)v'(x)u'(x) + q(x)v(x)u(x)\big)\,\mathrm{d}x
\end{aligned}$$

is symmetric in v and u. Hence for $\lambda_i \neq \lambda_j$ it follows that

$$a_{ij} = \int_0^1 e_i(x)(Te_j)(x)\,\mathrm{d}x = \lambda_j \int_0^1 e_i(x)e_j(x)\,\mathrm{d}x = 0\,.$$

Moreover, setting $N := \{i \in \mathbb{N}\,;\ \lambda_i = 0\}$ and assuming that all eigenvalues λ_i with $i \notin N$ are pairwise distinct, yields that

$$a_{ij} = \lambda_i \delta_{i,j} \quad \text{for all } i, j \in \mathbb{N}.$$

Hence the (formal) infinite-dimensional system of linear equations is reduced to diagonal form and reads

$$\lambda_i u_i = f_i \quad \text{for all } i \in \mathbb{N}.$$

We obtain the solvability condition

$$f_i = 0 \quad \text{for } i \in N,$$

and, formally, the solution

$$u = \sum_{i \notin N} \frac{1}{\lambda_i}\Big(\int_0^1 e_i(x)f(x)\,\mathrm{d}x\Big)e_i + \sum_{i \in N} \alpha_i e_i\,,$$

where the α_i, $i \in N$, can be chosen arbitrarily. Moreover we see that, analogously to linear algebra, the number of linearly independent functions corresponding to the eigenvalue 0, i.e. the number of degrees of freedom for the solution u, agrees with the number of side constraints for the datum f (cf. 11.6 and 12.8).

Thus we have reduced the Sturm-Liouville problem to an eigenvalue problem for the operator T. Here we note that we employed arguments which are analogous to matrix calculus, but which are merely formal for infinite matrices. Of course, these need to be justified and this will be the subject of Chapters 11 and 12.