Univariate and Multivariate Process Capability Analysis for Different Types of Specification Limits

Ashis Kumar Chakraborty and Moutushi Chatterjee

1 Introduction

In the context of statistical quality control, process capability analysis is one of the widely accepted approaches for assessing the ability of a process to produce what it is supposed to produce. Normally, an index known as the process capability index, abbreviated as PCI henceforth, is used to judge the health of the process vis-a-vis the given specification. In this context, the concept of PCI is generally applied in manufacturing industries. PCI mostly gives single valued assessment of the ability of a process to produce items within the pre-assigned specification limits. It is, generally, a higher the better type of index with the 'high' value indicating that the process is capable of producing item that in all likelihood will meet or exceed customers' requirement.

According to Kotz and Johnson (2002), before computing the PCI of a process, one has to ensure that the following two assumptions are satisfied:

1. The quality characteristic under consideration follows normal distribution;
2. The process is under statistical control.

In this context, the assumption of normality is made only due to the fact that such assumption gives some computational advantages. On the other hand, the second assumption is comparatively stronger because, absence of stability in the process makes it unpredictable and hence in that situation, PCI values may not be able to reflect the actual capability level of the process correctly.

A.K. Chakraborty (✉) · M. Chatterjee
SQC & OR Unit, Indian Statistical Institute, Kolkata, India
e-mail: akchakraborty123@rediffmail.com

M. Chatterjee
e-mail: tushi.stats@gmail.com

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Kane (1986), in his seminal paper, first documented some of the process capability indices, which were already being used in industries for quite some times and discussed about the importance of using those PCIs for assessing capability of the process. Due to the unquestionable significance of the concept of PCIs, specially in the context of manufacturing industries, Kane's (1986) paper motivated a huge number of statisticians as well as industrial engineers, working in the field of statistical quality control, to carry out further research work in this field. Kotz and Johnson (2002) have reviewed about 170 of such high quality research papers published within just 15 years of Kane's (1986) paper.

The quality characteristics, which are generally encountered in practice, belong to either of the following three categories viz.,

1. The nominal the best, i.e., processes with both upper specification limit (USL) and lower specification limit (LSL), e.g. height, length;
2. The smaller the better, i.e., processes with only USL, e.g. surface roughness, flatness;
3. The larger the better, i.e., processes with only LSL, e.g. tensile strength, compressive strength.

Moreover, for the quality characteristics of nominal the better type, the corresponding bi-lateral specification limits may be symmetric or asymmetric (with respect to the target) in nature. The consequences of asymmetric bi-lateral specification limits are discussed in detail in Sect. 3.

The four classical PCIs, for symmetric bi-lateral specification limits, which are commonly used, are,

$$\left.\begin{aligned} C_p &= \frac{U-L}{6\sigma} \\ C_{\mathrm{pk}} &= \frac{d-|\mu-M|}{3\sigma} \\ C_{\mathrm{pm}} &= \frac{d}{3\sqrt{\sigma^2+(\mu-T)^2}} \\ C_{\mathrm{pmk}} &= \frac{d-|\mu-M|}{3\sqrt{\sigma^2+(\mu-T)^2}} \end{aligned}\right\} \tag{1}$$

Here, 'U' and 'L' denote the USL and LSL respectively; $d = (U-L)/2$, $M = (U+L)/2$ and 'T' denotes the targeted value of the quality characteristic under consideration. Also, suppose, 'X' is a random variate corresponding to the measurable quality characteristic under consideration. Then, μ and σ are such that, $X \sim N(\mu, \sigma^2)$.

Vannman (1995) unified these PCIs and proposed the following super-structure of PCIs for symmetric bi-lateral specification limits:

$$C_p(u, v) = \frac{d - u|\mu - M|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}, \quad u, v \geq 0. \tag{2}$$

Note that the PCIs defined in Eq. (1) involve parameters like μ and σ which are often unobservable and consequently, the actual values of these PCIs are also difficult to obtain. To address this problem, the common practice is to compute the values of the plug-in or natural estimators of these PCIs. Such estimators are obtained by replacing the parameters like μ and σ by their corresponding estimators viz., $\overline{X}$ and 's' respectively, based on the random sample(s) drawn from the process. However, such plug-in estimators are subject to sampling fluctuation and hence can not be considered as the substitute of the original PCIs unless their distributional and inferential properties are studied extensively. The properties of the PCIs in Eq. (1) have been studied extensively in literature (refer Kotz and Johnson 2002, Pearn et al. 1992 and the references there in).

Although most of the quality characteristics of nominal the better type have symmetric bi-lateral specification limits, there are some practical situations, where due to some design aspect or to control production cost without compromising with the quality level of the product, asymmetry with respect to the target is solicited in the bi-lateral specification limits. For example, in the context of manufacturing iron rods of specific length, it is easier to cut a longer rod into a smaller one; than to make a shorter rod longer. Accordingly, the specifications should be set such that the distance between USL and T is more than the distance between LSL and T. Similarly, quality characteristics like hole diameter, should have asymmetric specification limits, as it is easier to make a hole with smaller diameter to a larger one through drilling, whereas, turning a larger hole into a smaller one, without compromising with its circularity, requires lot more effort.

A number of remarkable attempts have been made to define PCIs for processes with asymmetric bi-lateral specification limits (see Kane 1986; Boyles 1994; Franklin and Wasserman 1992; Kushlar and Hurley 1992; Vannman 1997 and the references there-in). Chen and Pearn (2001) defined a super-structure of PCIs called $C''_p(u,v)$ for asymmetric specification limits, which is similar to $C_p(u,v)$ of symmetric specification limits. Latter, Chatterjee and Chakraborty (2014) have established exact relationship between the proportion of non-conforming items produced by the process and some member indices of $C''_p(u,v)$, viz., C''_p and C''_{pk}. Chatterjee and Chakraborty (2014) have also studied some other interesting properties of $C''_p(u,v)$ including the inter-relationships between the member indices of $C''_p(u,v)$, threshold value of $C''_p(u,v)$ and optimality of $C''_p(u,v)$ on target. These are discussed in more detail in Sect. 3 along with a numerical example.

Apart from the bilateral specification limits, there are also some processes involving larger the better or smaller the better types of quality characteristics which require unilateral or one sided specification limits. In such situations, as the name 'unilateral' suggests, either of USL or LSL exist. For example, quality characteristics like surface roughness and flatness are of smaller the better type in a sense that their values should be as minimum as possible. Hence, only an USL is set for such quality characteristics. On the other hand, tensile strength and compressive strength are the examples of larger the better type of quality characteristics, where, the corresponding quality characteristic values should be as high as possible but should have at least a minimum value, decided by the LSL, for proper functioning of the concerned item.

Among the PCIs defined specifically for unilateral specification limits (see Kane 1986, Vannman 1998, Grau 2009 and the references there-in), the member indices of the super-structures of PCIs called $C_p^U(u, v)$ and $C_p^L(u, v)$, which are defined similar to $C_p(u, v)$, are closer to the practical situations. Chatterjee and Chakraborty (2012) have made an extensive review of the PCIs for unilateral specification region.

Despite the fact that Grau's (2009) super-structure performs better than the other available PCIs for unilateral specification limits, there was some problem in its practical implementation. In fact Grau's (2009) super-structure involves a term 'k' whose purpose is to penalize the deviation of the quality characteristic value from the target towards the opposite side of the existing specification limit. However, no mathematical formulation of 'k' was provided and this left room for favourable manipulation. Chatterjee and Chakraborty (2012) have proposed a formulation of 'k' based on the concept of loss of profit due to the deviation of the quality characteristic value from the target towards the opposite side of the available specification limit. A brief discussion on the PCIs for unilateral specification limits and the formulation of 'k' along with a numerical example are given in Sect. 4.

Although the bi-lateral and unilateral specification limits cover most of the quality characteristics encountered in practice, as has already been discussed earlier in this section, there is another type of quality characteristics which do correspond to neither of these types specification limits. The center of a drilled hole (in case of manufacturing processes) or the case of hitting a target (in ballistics) are some examples of such quality characteristics and the corresponding specification region is circular in nature.

Krishnamoorthi (1990) and Bothe (2006) have defined PCIs for circular specification regions. However, both of them have assumed equal variances (homoscedasticity) and independence of the two axes of the specification region—which may not be practically viable due to several technical reasons. To address these problems, Chatterjee and Chakraborty (2015) have defined a super-structure of PCIs for circular specification region, called $C_{p,c}(u, v)$, which does not require these assumptions. Besides, the authors have studied some important properties of $C_{p,c}(u, v)$, like, inter-relationship among the member indices, the threshold value, relationship with proportion of non-conforming items produced by the process and so on. Moreover, Chatterjee and Chakraborty (2015) have derived the expressions for the expectations and variances of the plug-in estimators of $C_{p,c}(u, v)$ based on the concept of circular normal distribution (see Scheur 1962). Section 5 contains a more detail discussion on the PCIs for circular specification limits.

The PCIs discussed so far, deal with one characteristic of a process at a time. However, with the increasing complexity in the technology, this may not be a valid assumption. In fact, often processes with multiple correlated characteristics are encountered in practice. For example (refer Taam et al. 1993), in an automated paint application process, there are more than one important quality characteristics viz., paint thickness, paint thinner levels, paint lot differences, temperature and so on which are interrelated among themselves. Use of univariate PCIs may not be able to assess the actual capability of the process efficiently, in such situations. One needs to use appropriate multivariate process capability indices (MPCI) in such cases.

Although the literature of statistical quality control is enriched with some mathematically sound MPCIs (see Taam et al. 1993, Chen 1994, Shinde and Khadse 2009, Shahriari et al. 2009 and the references there-in), most of these are difficult to interpret. Moreover, shop-floor people are more conversant with the univariate PCIs $C_p, C_{\text{pk}}, C_{\text{pm}}$ and C_{pmk} and hence MPCIs which function similar to these PCIs should be easily acceptable to them.

Chakraborty and Das (2007) defined an MPCI called $C_G(u, v)$ which functions similar to $C_p(u, v)$ but takes into account 'p' correlated quality characteristics simultaneously under consideration. Moreover, for $p = 1$, $C_G(u, v)$ boils down to $C_p(u, v)$ which is highly desirable. Later Chatterjee and Chakraborty (2013) have studied some of the properties of $C_G(u, v)$ like interrelationship between member indices and relationship with proportion of non-conforming items produced by the process and observed that these properties are similar to those of $C_p(u, v)$ from multivariate perspective.

Chatterjee and Chakraborty (2011) have also proposed a multivariate analogue of $C_p''(u, v)$, called $C_M(u, v)$, for assessing capability of processes having multiple correlated quality characteristics and asymmetric specification region with respect to the target vector. They have also studied the inter-relationship between the member indices of $C_M(u, v)$. The details of these MPCIs are given in Sect. 6.

Most of the PCIs, available in literature, are based on the common assumption that, the underlying statistical distribution of the concerned quality characteristic is normal. However, this assumption may not always be valid in practice. For example, McCormack et al. (2000) have observed that, in the context of high purity manufacturing, often, the particle count distribution and the distributions of process yield data are found to be non-normal. Some very interesting research work have been carried out in literature, to deal with the impact of such non-normality in the capability assessment of a process. A more detail discussion, in this regard, is made in Sect. 7.

Although, normality is an important, though not indispensable, assumption for process capability assessment, it is often difficult to check the same. However, the situation has somewhat improved in recent times and a number of statistical softwares are now available for testing univariate and multivariate normality. A brief discussion, in this regard, is made in Sect. 8.

Finally, the chapter concludes in Sect. 9 with a brief summarization of the PCIs for different types for specification limits, as have been discussed here.

2 List of Notations

Before going into an elaborate discussion about the univariate and multivariate process capability indices for different types of specification limits, let us first consider the following notations which are used in process capability studies time and again.

1. U: Upper specification limit (USL);
2. L: Lower specification limit (LSL);
3. n: Sample size;
4. $M = \frac{U+L}{2}$;
5. $d = \frac{U-L}{2}$;
6. T: Target;
7. 'X' is a random variable corresponding to the measurable quality characteristic under consideration, such that, $X \sim N(\mu, \sigma^2)$.
8. $D_U = U - T$;
9. $D_L = T - L$
10. $d^* = \min(D_U, D_L)$;
11. $S(x, y) = \frac{1}{3} \times \Phi^{-1}\left\{\frac{\Phi(x) + \Phi(y)}{2}\right\}$;
12. $F^* = \max\left(\frac{d^*(\mu-T)}{D_U}, \frac{d^*(T-\mu)}{D_L}\right)$;
13. $F = \max\left(\frac{d(\mu-T)}{D_U}, \frac{d(T-\mu)}{D_L}\right)$;
14. $k = \max\{\frac{D_U}{D_L}, \frac{D_L}{D_U}\}$;
15. $R_U = \frac{\mu-T}{D_U}$;
16. $R_L = \frac{T-\mu}{D_L}$;
17. $k_1 = \frac{U-T}{\sigma}$;
18. $k_2 = \frac{T-L}{\sigma}$;
19. $A_U^* = \max\{(\mu - T), \frac{T-\mu}{k_U^*}\}$;
20. $A_L^* = \max\{\frac{\mu-T}{k_L^*}, (T - \mu)\}$;
21. D: Diameter of circular specification region;
22. $r_{C,i} = \sqrt{(x_i - \overline{x})^2 + (y_i - \overline{y})^2}$;
23. $\widehat{\mu_C} = \frac{\sum_{i=1}^{n} r_{C,i}}{n}$;
24. $\overline{\mathrm{MR}} = \frac{\sum_{i=2}^{n} \mathrm{MR}_i}{n-1}$, where, MR_i's are obtained from moving range chart;
25. $\widehat{\sigma}_{\mathrm{ST}} = \frac{\overline{\mathrm{MR}}}{d_2}$. Since for MR chart, information from two samples are considered at a times, we have $d_2 = 1.128$;
26. $r_i = \sqrt{x_i^2 + y_i^2}$;
27. $\widehat{\sigma_{\mathrm{LT}}} = \frac{1}{c_4}\sqrt{\frac{\sum_{i=1}^{n}(r_i - \overline{r})^2}{n-1}}$;

28. $\widehat{\mu}_r = \frac{\sum_{i=1}^{n} r_i}{n}$;
29. $\widehat{\sigma}_{\mathrm{ST,C}} = \frac{\overline{\mathrm{MR}_C}}{d_2}$, where, MR_C values are obtained from the moving range chart of the data set after the target hole center is shifted to the middle of the cluster of actual hole centers;
30. $\widehat{\sigma}_{\mathrm{LT,C}} = \frac{1}{c_4}\sqrt{\frac{\sum_{i=1}^{n}(r_{C,i}-\overline{r_C})^2}{n-1}}$;
31. c_4, d_2 and d_3 are the common constants of the literature of control chart which are expressed as functions of the sample size 'n';
32. $d_i^{**} = \sqrt{(X_{1_i} - \mu_1)^2 + (X_{2_i} - \mu_2)^2}$;
33. $\mu^* = \frac{1}{n}\sum_{i=1}^{n} d_i^{**}$;
34. 'p' denotes the number of characteristics under consideration;
35. $\boldsymbol{X} = (X_1, X_2, \ldots, X_p)\prime$: Random vector characterizing the 'p' correlated quality characteristics under consideration (Note that now onwards vectors will be denoted by bold-faced letters);
36. $\boldsymbol{D} = (|\mu_1 - M_1|, |\mu_2 - M_2|, \ldots, |\mu_p - M_p|)'$;
37. $\boldsymbol{d} = ((\mathrm{USL}_1 - \mathrm{LSL}_1)/2, ((\mathrm{USL}_2 - \mathrm{LSL}_2)/2, \ldots, ((\mathrm{USL}_p - \mathrm{LSL}_p)/2)'$;
38. $\boldsymbol{T} = (T_1, T_2, \ldots, T_p)'$;
39. $\boldsymbol{M} = (M_1, M_2, \ldots, M_p)'$;
40. T_i is the target value, M_i is the nominal value for the ith characteristic of the item, for $i = 1(1)p$;
41. u and v are the scalar constants that can assume any non-negative integer value;
42. $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)'$: Mean vector of a '$p$' variate process;
43. $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{1p} & \sigma_{2p} & \cdots & \sigma_{pp} \end{pmatrix}$ is the dispersion matrix of a 'p' variate process;

3 Univariate Process Capability Indices for Asymmetric Specification Limits

Often for the quality characteristics of nominal the best type, the respective upper specification limit (USL) and lower specification limit (LSL) are symmetric with respect to the corresponding target (T). However, this may not always be the case—asymmetry in specification limits with respect to 'T' is also quite common in manufacturing industry. Such asymmetry may generate from a number of very practical situations some of which have been discussed by Boyles (1994). Sometimes, for particular quality characteristic of a product, the customer and or the design engineer is ready to allow more deviation from target towards a particular specification limit than towards the other; generating asymmetry in the specification limits. For example, in the context of drilling holes with hole diameter being the quality characteristic of interest, it is easier to increase the diameter of a hole through repeating the drilling operation than to shorten the existing hole diameter. Therefore, here USL should be closure to target than LSL. Again, it may so happen that, although initially a process starts with symmetric specification limits, after some times, the customer and/ or the manufacturer opts for asymmetric specification limits, to avoid unnecessary increase in production cost or due to some technical or financial issues. Finally, while transforming non-normal data into the normal one, often the symmetric specification limits get converted into asymmetric, owing to the same transformation.

Thus, the quality characteristics having asymmetric specification limits are not rare in industries, though most of the PCIs, available in literature are only applicable to quality characteristics with symmetric specification limits (Sect. 1). To address this problem, Kane (1986) modified C_p and C_{pk} by shifting one of USL and LSL so that the new specification limits are symmetric with respect to the target and defined $C_p^* = \min(\frac{T-\text{LSL}}{3\sigma}, \frac{\text{USL}-T}{3\sigma})$ and $C_{\text{pk}}^* = \min(\text{CPL}^*, \text{CPU}^*)$, where, $\text{USL} - T \neq T - \text{LSL}$, $\text{CPL}^* = \frac{T-\text{LSL}}{3\sigma}(1 - \frac{|T-\mu|}{T-\text{LSL}})$ and $\text{CPU}^* = \frac{\text{USL}-T}{3\sigma}(1 - \frac{|T-\mu|}{\text{USL}-T})$. Later, Franklin and Wasserman (1992) and Kushlar and Hurley (1992) proposed shifting both the specification limits $(T - D_L, T + D_U)$ to obtain symmetric ones $(T \pm \frac{D_L+D_U}{2})$, where, $D_U = \text{USL} - T$, $D_L = T - \text{LSL}$. However, the revised specification limits obtained by such shifting are subsets of the original specification limits and hence assessment of process capability based on these revised limits are often misleading.

Boyles (1994) proposed a new index as $S_{\text{pk}} = S(\frac{\text{USL}-\mu}{\sigma}, \frac{\mu-\text{LSL}}{\sigma})$, where, S($x$,$y$) is a smooth function which is defined as $S(x, y) = \frac{1}{3}\Phi^{-1}\{\frac{\Phi(x)+\Phi(y)}{2}\}$. Chen and Pearn (2001) generalized this index as $S_p(v) = S(\frac{\text{USL}-\mu}{\sqrt{\sigma^2+v(\mu-T)^2}}, \frac{\mu-\text{LSL}}{\sqrt{\sigma^2+v(\mu-T)^2}})$, where, $v \geq 0$. Although the properties of S_{pk} were studied by Ho (2003), but due to its very complicated nature, it has found very limited application in practice.

Similar to $C_p(u, v)$ of symmetric specification limits, for asymmetric specification limits, Vannman (1997) defined the following two super-structures of PCIs:

$$C_{\text{pv}}(u, v) = \frac{d - |T - M| - u|\mu - T|}{3\sqrt{\sigma^2 + v(\mu - T)^2}},$$

and

$$C_{\text{pa}}(u, v) = \frac{d - |\mu - M| - u|\mu - T|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}.$$

However, $C_{\mathrm{pv}}(u,v)$ fails to capture the asymmetry of the loss function with respect to 'T'; while, $C_{\mathrm{pa}}(u,v)$ is not optimum on target.

To address these drawbacks of the PCIs defined so far for asymmetric specification limits, Pearn (1998) proposed a new index analogous to C_{pk} for asymmetric tolerances which is given by

$$C^{*}_{\mathrm{pk}} = \frac{d^{*} - F^{*}}{3\sigma} \tag{3}$$

where, $d^{*} = \min(D_L, D_U)$ and $F^{*} = \max\{\frac{d^{*}(\mu - T)}{D_U}, \frac{d^{*}(T-\mu)}{D_L}\}$. Pearn and Lin (2000) studied some properties of C^{*}_{pk} and proposed a consistent and asymptotically unbiased estimator which converges to a mixture of two normal distributions. Later Chen and Pearn (2001) generalized C^{*}_{pk} to a super-structure which is defined as

$$C''_{p}(u,v) = \frac{d^{*} - uF^{*}}{3\sqrt{\sigma^2 + vF^2}} \tag{4}$$

where, $F = \max\{\frac{d(\mu - T)}{D_U}, \frac{d(T-\mu)}{D_L}\}$. $C''_{p}(u,v)$ is optimum on target and also, high value of $C''_{p}(u,v)$ indicates high process yield—these are two of the most important properties of any PCI irrespective of the nature of the respective specification limits.

Now, $C''_{p}(u,v)$ involve parameters of the quality characteristics, viz., μ and σ^2, which are often unobservable. Hence, the plug-in estimator called $\hat{C}''_{p}(u,v)$ is used for all practical purposes, where $\hat{C}''_{p}(u,v)$ is obtained by replacing μ and σ^2 in Eq. (4) by the sample mean $(\overline{X})$ and the sample variance s^2 respectively. However, indiscriminate use of such plug-in estimators is not solicited as that may lead to wrong assessment of the process capability. One needs to study the statistical properties of these plug-in estimators. Pearn et al. (2001, 2004) have made thorough studies of some of the distributional and inferential properties of $\hat{C}''_{\mathrm{pk}}$ and $\hat{C}''_{\mathrm{pmk}}$.

Proportion of non-conformance (PNC) is another measure for assessing the performance of a process apart from PCI. PNC measures the probability of producing items which are non-conforming with respect to the preassigned specification limits. Thus, ability of establishing relationship between these two parallel concepts of process performance analysis, is considered to be an added advantage of using a particular PCI. For symmetric specification limits, PNC is expressed in terms of C_p and C_{pk} as follows:

$$p = 2\Phi(-3C_p) \tag{5}$$

$$p' = \Phi[-3(2C_p - C_{pk})] + \Phi[-3C_{pk}] \tag{6}$$

Note that since C_p measures only the potential capability of a process, 'p' fails to measure the actual PNC unless $\mu = T$; whereas 'p'' measures the observable PNC. In this context, potential capability is the capability a process that can at most be attained given the current dispersion level and specification scenario. Chatterjee and Chakraborty (2014) have explored analogous relationship between C_p'', C_{pk}'' and PNC, where, $C_p'' = \frac{d^*}{3\sigma}$.

3.1 Relationship Between C_p'' and Proportion of Non-conformance

When the process is on target and the distribution of the quality characteristic is normal, the proportion of non-conformance can be defined as

$$P_{NC} = P[X > U|X \sim N(T, \sigma^2)] + P[X < L|X \sim N(T, \sigma^2)] = P_1 + P_2, \text{ say} \quad (7)$$

For establishing relationship between C_p'' and PNC, the following two situations are considered based on the relative position of 'T' with respect to μ, USL and LSL.

Case I: $d^* = D_U = U - T$

Here, $C_p'' = \frac{D_U}{3\sigma}$. From Eq. (7), $P_1 = 1 - \Phi(C_p'')$ and $P_2 = 1 - \Phi[3kC_p'']$, where, $k = \max\{\frac{D_U}{D_L}, \frac{D_L}{D_U}\}$. Hence from Eq. (7), when $\mu = T$, the expression for proportion of non-conformance is,

$$P_{NC} = 2 - \Phi(C_p'') - \Phi[3kC_p''] \quad (8)$$

Case II: $d^* = D_L = T - L$

Here, $C_p'' = \frac{D_L}{3\sigma}$. From Eq. (7), $P_1 = 1 - \Phi(3kC_p'')$, $P_2 = 1 - \Phi[3C_p'']$ and consequently, the expression of P_{NC} is given by Eq. (8).

Thus, when $\mu = T$, the expression for PNC remains same irrespective of the position of 'T' with respect to μ, USL and LSL. Also, for $k = 1$ and $C_p'' = 1$, we have, $P_{NC} = 0.0027$ which is same as the value of 'p' obtained from Eq. (5), when $C_p = 1$. This is due to the fact that, for $k = 1$, the specification limits become symmetric and hence $C_p = C_p''$.

However, P_{NC} measures the proportion of non-conformance only when $\mu = T$ and hence it is required to explore the relationship between PNC, C_p'' and C_{pk}'' (similar to the case of symmetric specification limits) from a more general perspective.

3.2 Relationship Between C''_{pk} and Proportion of Non-conformance

When $\mu \neq T$, PNC can be formulated as

$$\begin{aligned} P_{\text{NC}}^{E} &= 1 - P[L < X < U | X \sim N(\mu, \sigma^2)] \\ &= 2 - \Phi[\frac{D_U}{\sigma}(1 - R_U)] - \Phi[\frac{D_L}{\sigma}(1 - R_L)] \\ &= 2 - I_1 - I_2, \text{ say} \end{aligned} \tag{9}$$

where, P_{NC}^{E} denotes the expected/ observed PNC, $R_U = \frac{\mu - T}{D_U}$, $R_L = \frac{T - \mu}{D_L}$, $I_1 = \Phi[\frac{D_U}{\sigma}(1 - R_U)]$ and $I_2 = \Phi[\frac{D_L}{\sigma}(1 - R_L)]$. Based on the position of 'T' with respect to μ, USL and LSL, there can be four mutually exclusive and collectively exhaustive situations (see Wu et al. 2009) for each of which Chatterjee and Chakraborty (2014) have established exact relationship between C''_p, C''_{pk} and P_{NC}^{E} as follows:

Case I: $d^* = D_U$ and $R_U < R_L$, i.e. $\mu < T$:

$$P_{\text{NC}}^{E} = 2 - \Phi[3kC''_{\text{pk}}] - \Phi[3\{C''_{\text{pk}} + (k+1)R_L C''_p\}] \tag{10}$$

Case II: $d^* = D_U$ and $R_U > R_L$, i.e. $\mu > T$:

$$P_{\text{NC}}^{E} = 2 - \Phi[3C''_{\text{pk}}] - \Phi[3\{kC''_{\text{pk}} + (k+1)R_U C''_p\}] \tag{11}$$

Case III: $d^* = D_L$ and $R_U > R_L$, i.e. $\mu > T$:

$$P_{\text{NC}}^{E} = 2 - \Phi[3kC''_{\text{pk}}] - \Phi[3\{C''_{\text{pk}} + (k+1)R_U C''_p\}] \tag{12}$$

Case IV: $d^* = D_L$ and $R_U < R_L$, i.e. $\mu < T$:

$$P_{\text{NC}}^{E} = 2 - \Phi[3C''_{\text{pk}}] - \Phi[3\{kC''_{\text{pk}} + (k+1)R_L C''_p\}] \tag{13}$$

In this context, $R_U = R_L$ implies $\mu = T$ and hence the specification limits become symmetric about 'T'. Here, one interesting point to note is that, unlike 'p' in Eq. (5), here, P_{NC} does not ensure providing minimum observable proportion of non-conformance; rather, it only measures the observed proportion of non-conformance of the process when $\mu = T$. In particular, the value of $(P_{\text{NC}} - P_{\text{NC}}^{E})$ increases with the increase in the value of 'k'. Thus, contradicting the usual convention, it may so happen that, a process, with asymmetric specification limits, produces more non-conforming items when it is on target compared to the situation when $\mu = M$ and this is more clearly described in Fig. 1.

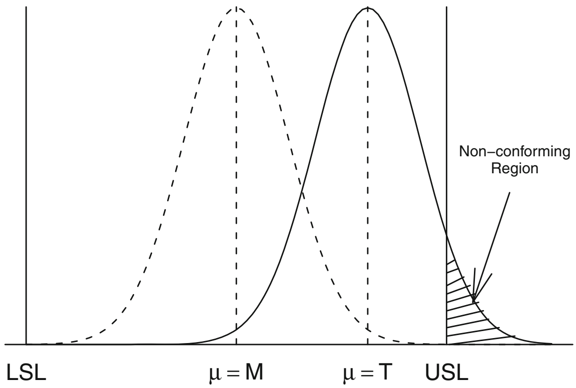


Fig. 1 Asymmetric specification limits with $P_{NC} > P_{NC}^{E}$

Chatterjee and Chakraborty (2014) have extensively studied the interrelationship between the member indices of $C''_p(u, v)$ and have observed that $C''_p \geq C''_{pk} \geq C''_{pmk}$ and $C''_p \geq C''_{pm} \geq C''_{pmk}$, where equality is attained for $\mu = M = T$. Moreover, there is no clear-cut relationship between C''_{pk} and C''_{pm}. These are analogous to the inter-relationship between the member indices of $C_p(u, v)$, as have been observed by Kotz and Johnson (2002).

A mathematical expression for the threshold value of C''_p has also been developed by Chatterjee and Chakraborty (2014). In this context, threshold value is one of the most important features of a PCI from the interpretational view point. A process with a PCI value beyond the threshold value is considered to be capable of producing items within the pre-assigned specification limits; while that with a smaller value of PCI with respect to the said threshold value is likely to be incapable. Usually, threshold values are computed for the PCIs like C_p, measuring potential capability of a process. The common industrial practice is to consider '1' as the threshold value of a PCI, irrespective of the nature of the corresponding specification limits. Chatterjee and Chakraborty (2014) have formulated the threshold value of C''_p as,

$$C_p^{''(T)}(0,0) = C_p^{''(T)} = \begin{cases} \frac{2k_1}{k_1 + k_2} & \text{if } D_U < D_L \\ \frac{2k_2}{k_1 + k_2} & \text{if } D_L < D_U \end{cases} \tag{14}$$

where, k_1 and k_2 are positive real numbers with $k_1 \neq k_2$, such that $k_1 = \frac{U-T}{\sigma}$ and $k_2 = \frac{T-L}{\sigma}$. From Eq. (13), it is evident that, $C_p^{''(T)}$, the threshold value of C''_p, is a function of the degree of asymmetry of the specification limits and hence, considering '1' as the threshold value of C''_p, without properly investigating the nature of the specification limits leave room for over/under estimation of the actual capability of a process.

3.3 Example

In order to illustrate the theoretical aspects of the PCIs for asymmetric specification limits discussed so far, we now consider a numerical example based on the data on a high-end audio speaker component called Pulux edge manufactured in Taiwan (Lin and Pearn 2002). For a particular model of Pulux edge, $U = 5.950$, $L = 5.650$ and $T = 5.835$. Lin and Pearn (2002) have collected 90 observations with the corresponding summary statistics found to be as follows:

Sample size (n) = 90, sample mean ($\overline{X}$) = 5.83 and sample standard deviation (s) = 0.023. Moreover, here $D_U \neq D_L$ indicating asymmetry in the specification limits with respect to T. Based on this data, we compute the values of some of the PCIs and the corresponding PNC values for both the symmetric and asymmetric specification limits to make a comparative study of their performances when the actual specification limits are asymmetric. Thus, $\widehat{C}_p = 2.17, \widehat{p} = 7.515 \times 10^{-11}$, $\widehat{C}_{\text{pk}} = 0.870$, $\widehat{p}' = 0.004527$, $\widehat{C}''_p = 1.6667$, $\widehat{P}_{\text{NC}} = 0.0477$, $\widehat{C}''_{\text{pk}} = 1.6217$ and $\widehat{P}^E_{\text{NC}} = 9.0784 \times 10^{-8}$. Following the standard notations, here ‘hat’ $((\widehat{\ }))$ is added to the usual PCIs and others to denote their estimated values.

Thus, excluding $\widehat{C}_{\text{pk}}$, all the PCIs consider the process to be capable. The threshold value of $\widehat{C}''_p$ is found to be 0.7667 and since both $\widehat{C}''_p$ and $\widehat{C}''_{\text{pk}}$ have values higher than $\widehat{C}''^{(T)}_p$, the process is likely to be capable. This assessment of the process is also supported by P^E_{NC} as the $\widehat{P}^E_{\text{NC}}$ value is found to be considerably small. Also, since here, $\mu \neq T$, P_{NC} is not applicable here.

Therefore $\widehat{C}_p$ is not applicable here as $\mu \neq T$ while $\widehat{C}_{\text{pk}}$ makes an incorrect assessment of the process and hence they are not suitable here. On the other hand, $\widehat{C}''_p$, $\widehat{C}''_{\text{pk}}$ and $\widehat{P}^E_{\text{NC}}$ assess the capability of the process correctly. These argue in favour of the selection of appropriate PCIs based on the nature of the specification limits and other related aspects of a process.

4 Process Capability Indices for Unilateral (One Sided) Specification Limits

The PCIs discussed so far, are primarily meant for quality characteristics of nominal the best type and having bi-lateral specification limits. Quality characteristics of smaller the better type (e.g., surface roughness, degree of radiation and so on) and larger the better type (e.g., tensile strength, compressive strength and so on) requiring unilateral (one-sided) specification limits are also common in various manufacturing industries. However, there are only a few PCIs available in literature to assess capability of such processes.

Kane (1986) discussed about two such PCIs viz., $C_{\mathrm{PU}} = \frac{U-\mu}{3\sigma}$ and $C_{\mathrm{PL}} = \frac{\mu-L}{3\sigma}$. As is the relationship between C_p and 'p', given in Eq. (5), for unilateral specification limits also, analogous relationships, like $p^U = \Phi(3C_{\mathrm{PU}})$ and $p^L = \Phi(3C_{\mathrm{PL}})$, hold good between PNC, C_{PU} and C_{PL}, where, p^U and p^L denote the proportions of non-conformance generated due to exceeding USL and LSL respectively, when $\mu = T$. Also, 1 is usually considered as the threshold value of C_{PU} and C_{PL}. The distributional as well as inferential properties of these two PCIs, for both the single and multiple sample information, have also been studied extensively (see Lin and Pearn 2002, Pearn and Chen 2002, Shu et al. 2006). In fact, most of the research works on PCIs for unilateral specification limits are based on C_{PU} and C_{PL} only, due to their computational simplicity. Chatterjee and Chakraborty (2012) have made a thorough review of these PCIs for unilateral specification limits.

However, C_{PU} and C_{PL} suffer from the following critical drawbacks:

1. Neither of C_{PU} and C_{PL} incorporate the concept of 'T', the target value for the corresponding variable under consideration, in their respective definitions. As a result, they fail to measure the proximity of the process centering towards the target.
2. Unlike C_p, C_{PU} and C_{PL} can not be considered as the potential PCIs either, due to the presence of the mean μ in their definitions.

Therefore, despite being easy to compute, C_{PU} and C_{PL} are difficult to interpret.

Like $C_p(u, v)$, defined in Eq. (2), Vannman (1998) has defined the following two sets of superstructures of PCIs for unilateral specification limits:

$$\left.\begin{aligned} C_{\mathrm{pau}}(u, v) &= \frac{USL-\mu-u|\mu-T|}{3\sqrt{\sigma^2+v(\mu-T)^2}} \\ C_{\mathrm{pal}}(u, v) &= \frac{\mu-LSL-u|\mu-T|}{3\sqrt{\sigma^2+v(\mu-T)^2}} \end{aligned}\right\} \tag{15}$$

and

$$\left.\begin{aligned} C_{\mathrm{pvu}}(u, v) &= \frac{USL-T-u|\mu-T|}{3\sqrt{\sigma^2+v(\mu-T)^2}} \\ C_{\mathrm{pvl}}(u, v) &= \frac{T-LSL-u|\mu-T|}{3\sqrt{\sigma^2+v(\mu-T)^2}} \end{aligned}\right\} \tag{16}$$

Later, Grau (2009) observed some drawbacks in these two superstructures. The values of $C_{\mathrm{pvu}}(u, v)$ and $C_{\mathrm{pvl}}(u, v)$ are symmetric with respect to T which is not desirable for an ideal PCI for unilateral specification limits. In this context, the basic difference in the nature of asymmetric and unilateral specification limits is that for asymmetric specification limits, deviation from T towards USL and LSL are not of equal importance. However, in the context of the unilateral specification limits, deviation from T towards the existing specification limit (USL or LSL, depending upon the situation) is considered to be serious; while, deviation from T on the opposite side of the said specification limit can not be considered as undesirable, at least from the point of view of the quality of the product. Rather, such products are actually having better quality. Thus for both of these two types of specification limits, the corresponding loss function can by no means be considered as symmetric.

Again, for $0 \le u < 1$, $C_{\text{pau}}(u, v)$ and $C_{\text{pal}}(u, v)$ are not optimum on target. Also, for $u \ge 1$, $C_{\text{pvu}}(u, v)$ and $C_{\text{pvl}}(u, v)$ values become negative even before μ reaches U or L, which is highly undesirable. Therefore, neither of the superstructures of PCIs for the processes with unilateral specification limits, defined in Eqs. (15) and (16), are suitable for practical applications.

Grau (2009) has proposed the following superstructure of PCIs for unilateral specification limits, which is free from these drawbacks, and have also studied some of its distributional properties.

$$\left.\begin{aligned} C_p^U(u, v) &= \frac{U - T - uA_U^*}{3\sqrt{\sigma^2 + vA_U^{*2}}} \\ C_p^L(u, v) &= \frac{T - L - uA_L^*}{3\sqrt{\sigma^2 + vA_L^{*2}}} \end{aligned}\right\} \tag{17}$$

where, $A_U^* = \max\{(\mu - T), \frac{T-\mu}{k_U^*}\}$, $A_L^* = \max\{\frac{\mu-T}{k_L^*}, (T - \mu)\}$. Also, $k_I^*(>1)$ quantifies the amount of loss incurred due to deviation from T towards the opposite side of the existing specification limit, where k_I^* stands for k_U^* or k_L^* depending upon the situation. Note that $C_p^U(u, v)$ and $C_p^L(u, v)$ are defined in such a way that the corresponding PCIs will be free from k_U^* and k_L^* respectively, when the quality characteristic value deviates from T towards the existing specification limit. Since Grau (2009) did not suggest any mathematical formulation of k_I^*, its choice becomes subjective, increasing the scopes for favourable manipulation in the values of $C_p^U(u, v)$ and $C_p^L(u, v)$. In order to eliminate such subjectivity in the definitions of $C_p^U(u, v)$ and $C_p^L(u, v)$, Chatterjee and Chakraborty (2012) have proposed a mathematical formulation of k_I^*.

It is interesting to note that for unilateral specification limits, target is set to maximize profit or to minimize loss. Thus, although deviation of μ from target towards the other side of the existing specification limit will definitely produce items of better quality; the manufacturer is likely to incur a loss of profit per item under constant selling price, since such production will require larger amount of ingredient or higher degree of expertise or more sophisticated machinery. Chatterjee and Chakraborty (2012) have applied this concept of loss of profit to formulate k_I^*.

For the purpose of illustration, suppose the quality characteristic under consideration is of smaller the better type and hence the corresponding process has only USL. Also suppose for this process, there are ‘m’ stages through which loss of profit can be incurred and let C_i^U is the corresponding loss of profit for the ith stage, where, $i = 1(1)m$. Here, one possible choice for the stages of loss of profit may be per unit or some convenient fraction of the unit of measurement. Moreover, let ‘n’ denotes the total number of produced items among which n_1 items have the values of the quality characteristic less than ‘T’, and the remaining n_2 items have the quality characteristic value greater than or equal to the target value such that $n = n_1 + n_2$ with $n_1 > 0$. Also, 'C' is the constant selling price.

Then k_U^* can be formulated as

$$k_U^* = \frac{\text{Selling Price Per Item}}{\text{Average Loss of Profit Per Item}}$$

$$= \frac{C}{\frac{1}{n_1} C_{\text{LP},U}^{\text{Total}}} \tag{18}$$

where, $C_{\text{LP},U}^{\text{Total}}$, the total loss of profit due to deviation from 'T' towards left, can be defined as

$$C_{\text{LP},U}^{\text{Total}} = \sum_{j=1}^{n_1} \sum_{i=1}^{m} C_i^U I_{ij} \tag{19}$$

with

$$I_{ij} = \begin{cases} 1 & \text{if } j\text{th item belongs to the } i\text{th stage of loss of profit}, \forall i = 1(1)m, j = 1(1)n_1, \\ 0 & \text{otherwise}. \end{cases},$$

Similarly, for quality characteristics of the larger-the-better type, k_L^* can be formulated as

$$k_L^* = \frac{C}{\frac{1}{n_1} C_{\text{LP},L}^{\text{Total}}} \tag{20}$$

where, $C_{\text{LP},L}^{\text{Total}} = \sum_{j=1}^{n_1} \sum_{i=1}^{m} C_i^L I_{ij}$; C_i^L is the loss of profit at the ith stage, when, there exists 'm' such stages through which loss of profit (due to deviation of process mean form 'T' towards right i.e. towards the direction opposite to LSL with respect to 'T') can be incurred and I_{ij} has the same interpretation as before.

4.1 Example

To illustrate the impact of k_I^* on $C_p^U(u, v)$ and $C_L^p(u, v)$, we consider the data set on polarized dependent loss (PDL) of wavelength multiplexer (see Pearn et al. 2009). Here only the data corresponding to supplier I is considered, for which $n = 105$, $\hat{\mu} = 0.061$ decibel (dB), $\hat{\sigma} = 0.0049$ dB and USL = 0.08 dB. Moreover, although the original data set did not take into account the values of T, constant selling price per item and stage of loss of profit per item; following Chatterjee and

Chakraborty (2012), we have, $T = 0.064$ dB, C = \$1.00 and loss of profit for per 0.001 dB deviation from T towards left is 0.02 dB. Then, for the present data set, $k_U^* = 8.5337$.

Thus, $\hat{C}_p^U(0,0) = 1.078$, $\hat{C}_p^U(1,0) = 1.053$, $\hat{C}_p^U(0,1) = 1.075$, $\hat{C}_p^U(1,1) = 1.050$, $\hat{C}_{\mathrm{pvu}}(0,0) = 0.86$ and $\hat{C}_{\mathrm{pvu}}(0,1) = 0.90$. Here, it is easy to observe that out of the total number of 105 observations, 74 have the values of the quality characteristic less than $T = 0.064$. As has already been discussed, these 74 items can not be considered as having inferior quality—the only problem here is in terms of loss of profit. Now, since for $u = 0, 1$ and $v = 0, 1$, all the $\hat{C}_p^U(u, v)$ values are found to be greater than 1 which indicates that the process is performing satisfactorily and the loss of profit is also under control. However, $\hat{C}_{\mathrm{pvu}}(u, v)$ does not take into account this aspect of unilateral specification limits. Since, $C_{\mathrm{pvu}}(0,0)$ and $C_{\mathrm{pvu}}(0,1)$ merely measure the proximity of μ towards T, irrespective of the direction of such deviation, these PCIs fail to assess actual process performance and consider the process to be incapable which is not actually the case.

5 Process Capability Indices for Circular Specification Region

Apart from the bi-lateral (both symmetric and asymmetric) and unilateral specification limits, there is another type of specification limit which is known as circular specification limit. Such specification limits can be observed in processes like drilling holes (in manufacturing industries) or hitting a target (in ballistics). The uniqueness of circular specification limits is that the so called USL and/or LSL do not exist and consequently, the conventional PCIs are not applicable here.

Krishnamoorthi (1990) first proposed PCIs for processes with circular specification limits which are defined below:

$$\left.\begin{aligned} \mathrm{PC}_p &= \frac{\frac{\pi}{4}D^2}{9\pi\sigma^2} = \frac{1}{36} \times \frac{D^2}{\sigma^2} \\ \mathrm{PC}_{\mathrm{pk}} &= \frac{D^2}{4\left[\sqrt{(\overline{X}-a)^2 + (\overline{Y}-b)^2} + 3\sigma\right]^2} \end{aligned}\right\} \tag{21}$$

where, D is the diameter of the circular specification region, (a, b) is the targeted center of the process and σ is the common standard deviation along the two axes X_1 and X_2, such that, when $\sigma_1 \neq \sigma_2$, $\sigma = \max(\sigma_1, \sigma_2)$. It is assumed that $(X_1, X_2) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho = 0)$. Note that, PC_p is defined analogous to C_p and it measures the potential capability of a process; while $\mathrm{PC}_{\mathrm{pk}}$ measures the actual process capability when the specification region under consideration is circular in nature.

Bothe (2006) has proposed another set of PCIs for circular specification region based on the concept of radial distance. He has considered average radial distance between the centers of various drilled holes or the average radial distance of centers of the drilled holes from the target center as the quality characteristic of interest and has defined the following PCIs analogous to C_p and C_{pk} based on these radial distances:

$$\left.\begin{array}{l} \widehat{C}_P = \frac{\mathrm{USL}-\widehat{\mu_C}}{3\widehat{\sigma}_{\mathrm{ST},C}} \\ \widehat{P}_P = \frac{\mathrm{USL}-\widehat{\mu_C}}{3\widehat{\sigma}_{\mathrm{LT},C}} \\ \widehat{C}_{\mathrm{PK}} = \frac{\mathrm{USL}-\widehat{\mu_r}}{3\widehat{\sigma}_{\mathrm{ST}}} \\ \widehat{P}_{\mathrm{PK}} = \frac{\mathrm{USL}-\widehat{\mu_r}}{3\widehat{\sigma}_{\mathrm{LT}}} \end{array}\right\} \tag{22}$$

Here, $\widehat{\mu_C} = \frac{\sum_{i=1}^{n} r_{C,i}}{n}$; $r_{C,i} = \sqrt{(x_i - \bar{x})^2 + (y_i - \bar{y})^2}$ and 'n' is the sample size. Also, $\overline{\mathrm{MR}} = \frac{\sum_{i=2}^{n} \mathrm{MR}_i}{n-1}$, MR_i's are obtained from moving range chart; $\widehat{\sigma_{\mathrm{ST}}} = \frac{\overline{\mathrm{MR}}}{1.128}$; $\widehat{\sigma_{\mathrm{LT}}} = \frac{1}{c_4}\sqrt{\frac{\sum_{i=1}^{n}(r_i - \bar{r})^2}{n-1}}$; $r_i = \sqrt{x_i^2 + y_i^2}$; $\widehat{\mu}_r = \frac{\sum_{i=1}^{n} r_i}{n}$; $\widehat{\sigma}_{\mathrm{ST},C} = \frac{\overline{\mathrm{MR}_C}}{d_2}$, where, MR_C values are obtained from the moving range chart of the data set after the target hole location is shifted to the middle of the cluster of actual hole centers and d_2 is a function of the sample size 'n' and $\widehat{\sigma}_{\mathrm{LT},C} = \frac{1}{c_4}\sqrt{\frac{\sum_{i=1}^{n}(r_{C,i} - \overline{r_C})^2}{n-1}}$ and c_4 is a constant based on the sample size 'n'.

Note that, for both these two sets of PCIs defined in Eqs. (21) and (22), it is assumed that the variation in the values of the quality characteristics along the two axes are the same (homoscedastic) and also, these two axes are mutually independent. However, in reality, due to several practical reasons such assumptions of homoscedasticity and independence of X_1 and X_2 are seldom valid. As a result, even if the specification region is circular, the process region is elliptical in nature. Neither of the PCIs defined so far take care of this problem. Moreover, under the assumption of bivariate normality of (X_1, X_2), the distribution of radial distance is no more normal—rather it follows circular normal distribution (Scheur 1962). Thus, PCIs like C_p and C_{pk} are not suitable for assessing capability of such processes. From this view point also, the PCIs defined in Eq. (22) are not suitable for circular specification regions.

Chatterjee and Chakraborty (2015) have addressed these problems by defining a superstructure of PCIs for circular specification region. Suppose $\boldsymbol{X} = (X_1, X_2)' \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$. Also, without loss of generality, suppose the target center of the process is set at $(0, 0)$ point of the co-ordinate axes. Then, like $C_p(u, v)$ defined in Eq. (2), Chatterjee and Chakraborty (2015) have defined a superstructure of PCIs for circular specification region as

$$C_{p,c}(u,v) = \frac{\frac{D}{2} - \frac{u}{\sqrt{\pi}}\mu^*}{\sqrt{\chi^2_{\alpha,2}\sigma_1\sigma_2\sqrt{1-\rho^2}}} \times \frac{1}{1+v\mu'\Sigma^{-1}\mu} \tag{23}$$

where, $\mu = (\mu_1, \mu_2)'$ and $\mu^* = \frac{1}{n}\sum_{i=1}^{n} d_i^{**}$, with $d_i^{**} = \sqrt{(X_{1_i} - \mu_1)^2 + (X_{2_i} - \mu_2)^2} = \sqrt{(\boldsymbol{X}_i - \mu)'(\boldsymbol{X}_i - \mu)}$; $\boldsymbol{X}_i = (X_{1i}, X_{2i})'$ for $i = 1, 2, \ldots, n$ and 'n' is the number of sample observations randomly drawn from a process. Note that bold faced letters are used to denote vector valued variables.

Here $C_{p,c} = C_{p,c}(0,0), C_{\mathrm{pk},c} = C_{p,c}(1,0), C_{\mathrm{pm},c} = C_{p,c}(0,1)$ and $C_{\mathrm{pmk},c} = C_{p,c}(1,1)$ are, by definition, analogous to $C_p, C_{\mathrm{pk}}, C_{\mathrm{pm}}$ and C_{pmk} respectively.

Note that $C_{p,c}(u,v)$ is defined from a more general perspective compared to the PCIs defined in Eqs. (21) and (22) and hence it does not require the so called assumptions of homoscedasticity and independence along the two axes. $C_{p,c}(u,v)$ is optimum on target as well which is a desirable property of a good PCI. Moreover, for a fixed value of ρ, the values of all the member indices of $C_{p,c}(u,v)$ decrease with the increase in at least one of σ_1^2 and σ_2^2. Similar to the inter-relationships between the member indices of $C_p(u,v)$ and $C_p''(u,v)$ with $u = 0, 1$ and $v = 0, 1$, here also, it is easy to check that

$$C_{p,c}(u,v) \leq C_{p,c}(u,0) \leq C_{p,c}(0,0)$$

$$C_{p,c}(u,v) \leq C_{p,c}(0,v) \leq C_{p,c}(0,0), \quad \forall u \geq 0, v \geq 0$$

and there is no clear-cut relationship between $C_{\mathrm{pk},c}$ and $C_{\mathrm{pmk},c}$.

Since, similar to C_p, $C_{p,c}$ measures the potential capability of a process, Chatterjee and Chakraborty (2015) have derived the expression for the threshold value of $C_{p,c}$ as

$$C_{p,c}^T = \sqrt{\frac{D}{2 \times \chi^2_{\alpha,2} \times \sigma_{\min} \times \sqrt{1-\rho^2}}} \tag{24}$$

Thus, the threshold value of $C_{p,c}$ is a function of σ_1, σ_2 and ρ and hence is not unique.

However, for $\rho = 0$ and $\sigma_1 = \sigma_2 = \frac{D}{2}$, although the process region becomes circular and coincides with the specification region, $C_{p,c}^T \neq 1$ and for this reason, $C_{p,c}(u,v)$ is not suitable when the correlation between the two axes is very low.

Again, similar to C_{pk}, $C_{\text{pk},c}$ is a yield based PCI and Chatterjee and Chakraborty (2015) have established exact relationships between $C_{p,c}$, $C_{\text{pk},c}$ and PNC as follows.

When the process is on target, PNC can be formulated as

$$\begin{aligned} P_{\text{NC}} &= P\left[(\boldsymbol{X}-\boldsymbol{0})'\Sigma^{-1}(\boldsymbol{X}-\boldsymbol{0}) > \begin{pmatrix}\frac{D}{2} 0\end{pmatrix} I_2 \begin{pmatrix}\frac{D}{2}\\ 0\end{pmatrix}\right] \\ &= P\left[\chi^2_{\alpha,2n-4} > 2(n-1)\sqrt{|S|}C^2_{p,c}\right], \end{aligned} \tag{25}$$

where, $A = (n-1)S$, S being the sample variance-covariance matrix.

However, in practice, often the assumption that $\hat{\mu} = (0,0)'$ may not hold and in such cases, P_{NC} measures only the minimum attainable PNC. Considering the more general case, i.e. when $\hat{\mu} \neq (0,0)'$, Chatterjee and Chakraborty (2012) have derived the expression for observed PNC as

$$\begin{aligned} P^E_{\text{NC}} &= P\left[\boldsymbol{X}'\Sigma^{-1}\boldsymbol{X} > \frac{D^2}{4} | \boldsymbol{X} \sim N_2(\mu,\Sigma)\right] \\ &= P\left[2(n-1)\sqrt{|S|}C^2_{\text{pk},c} < \sqrt{F_{\alpha,2,2}(\lambda) \times \chi^2_{\alpha,2n-4}} - \frac{\mu^*}{\sqrt{\pi}} \times \sqrt{F_{\alpha,2n-4,2}}\right] \end{aligned} \tag{26}$$

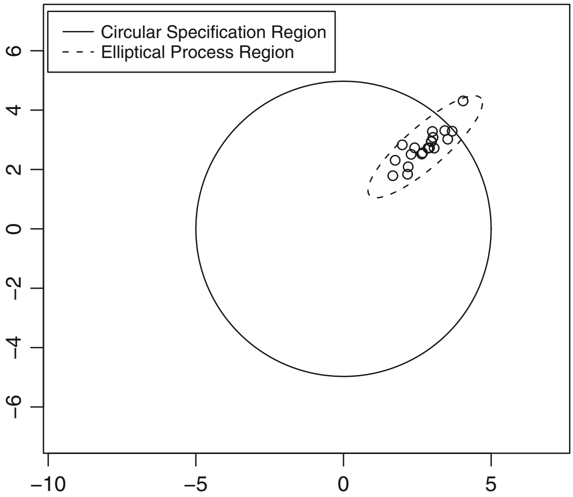
where, $\chi^2_{\alpha,2n-4}$ and $F_{\alpha,2,2}$ denote respectively the upper α% point of a χ^2 distribution with $(2n-4)$ degrees of freedom and a F distribution with (2, 2) degrees of freedom.

Moreover, based on the properties of circular normal distribution (refer Scheur 1962), Chatterjee and Chakraborty (2015) have derived the expressions for the expectations and variances of the member indices of $C_{p,c}(u,v)$ for $u = 0, 1$ and $v = 0, 1$.

5.1 Example

To investigate the performance of $C_{p,c}(u,v)$ for assessing capability of processes having circular specification limits, we now consider a manufactured product and we are concerned about the holes drilled subject to some specifications. 20 holes were drilled and for each hole, the values of the corresponding X_1 and X_2 co-ordinates of the centers of the holes were noted. Here, $D = 10$, $\overline{X}_1 = 2.766$, $\overline{X}_2 = 2.776$, $\hat{\sigma}_1^2 = 0.408$, $\hat{\sigma}_2^2 = 0.321$ and $\hat{\rho} = 0.856$. The complete data set is available in Chatterjee and Chakraborty (2015) and Fig. 2 provides a

Fig. 2 Circular specification region and the elliptical process region



diagrammatic representation of the process region and the corresponding specification region.

Since $\hat{\sigma}_1^2 \neq \hat{\sigma}_2^2$ and the value of $\hat{\rho}$ is also considerably high, the PCIs defined in Eqs. (21) and (22) are not applicable here. Hence values of the member indices of the superstructure $C_{p,c}(u,v)$ are computed as follows:

$\widehat{C}_{p,c} = 3.8097$, $\widehat{C}_{\mathrm{pk,c}} = 3.5184$, $\widehat{C}_{\mathrm{pm},c} = 0.7605$, $\widehat{C}_{\mathrm{pmk},c} = 0.7024$, $\hat{P}_{\mathrm{NC}} = 2\times 10^{-8}$, $\hat{P}^{E}_{\mathrm{NC}} = 0.0598$ and $\hat{C}^{T}_{p,c} = 1.3613$.

From the above computations it can be observed that $C_{p,c}$ considers the process to be potentially capable and this is also supported by the low value of $\hat{P}_{\mathrm{NC}}$. However, all the other PCIs except $C_{\mathrm{pk},c}$ consider the process to be incapable. Now, from Fig. 2 it is evident that, the data points lie far away from the target center $(0,0)'$ and this is correctly reflected by the low values of $C_{\mathrm{pm},c}$ and $C_{\mathrm{pmk},c}$. Also, a considerable part of the process region lies outside the circular specification region and this increases the value of $\hat{P}^{E}_{\mathrm{NC}}$. In fact, $\hat{C}_{\mathrm{pk},c}$, being a yield based PCI, has rightly reflected the incapability of the process though the high value of $\hat{P}^{E}_{\mathrm{NC}}$. Thus, although the process is potentially capable, since it is highly off-centered, it is not actually performing satisfactorily. Moreover, the apparent contradiction between the values of $\hat{C}_{\mathrm{pk},c}$, $\hat{C}_{\mathrm{pm},c}$ and $\hat{C}_{\mathrm{pmk},c}$ argue for the judgemental use of PCIs as well as the importance of diagrammatic representations of the process and specification region to have a prima-facie impression about the health of the process.

6 Multivariate Process Capability Indices

The common assumption of all the PCIs discussed so far, irrespective of the nature of the specification limits, is that, there is only one measurable quality characteristic of a manufactured product based on which the capability of the corresponding process is to be assessed. However, the practical scenario is not that much simplified. In fact, often it is seen that there are a number of measurable quality characteristics corresponding to a particular item and these quality characteristics are inter-related among themselves. For example, in an automated paint application process, one of the major quality characteristics is paint thickness. However, capability analysis of the said process, based on only paint thickness, may not reveal the true capability of the process. In fact, in an automated paint application process, there are a number of other quality characteristics like ability of surface preparation and part location, paint thinner levels, paint lot differences, temperature and so on which are inter-related to paint thickness at different degrees. Common industrial practice is to apply univariate PCIs for each of these quality characteristics separately and summarize the process capability as the arithmetic or geometric mean of these individual PCI values. However, this approach may not be able to assess the capability of the process accurately as it ignores the correlation structure among the quality characteristics. This necessitates the application of multivariate process capability indices (MPCI).

Despite having ample scope of industrial applications, there are only a few MPCIs available in literature. Following Shinde and Khadse (2009), the MPCIs, defined so far, are either of the following types:

1. MPCIs defined as the ratio of tolerance region and process region; e.g., Taam et al. (1993), Goethals and Cho (2011) and so on;
2. MPCIs expressed as the probability of non-conforming products; e.g., Chen (1994), Khadse and Shinde (2006), Pearn et al. (2006), Shiau et al. (2013) and so on;
3. MPCIs based on principal component analysis; e.g., Wang and Chen (1998), Wang and Du (2000), Shinde and Khadse (2009), Perakis and Xekalaki (2012), Tano and Vannman (2012) and so on;
4. MPCIs based on the concept of non-parametric statistics; refer Polansky (2001);
5. Other approaches including vector representation of MPCIs; e.g., Kirmani and Polansky (2009), Shahriari et al. (2009); MPCIs based on lowner ordering; refer Kirmani and Polansky (2009) and so on.

6.1 $C_G(u, v)$—*A Multivariate Process Capability Index for Symmetric Specification Region*

Most of the MPCIs defined so far are difficult to compute and hence are meant for theoreticians. Moreover, since shop-floor people are very much conversant with classical univariate indices, some multivariate analogue of $C_p(u, v)$ would be more palatable to them. Chakraborty and Das (2007) have defined a MPCI called $C_G(u, v)$, analogous to $C_p(u, v)$, to address these problems. For defining the new MPCI, Chakraborty and Das (2007) have made the following realistic assumptions:

1. Underlying process distribution is multivariate normal with mean vector μ and dispersion matrix Σ.
2. The process has hyper-rectangular specification region.
3. For each process variable specification limits are symmetric about its mean.
4. $\boldsymbol{T} = \boldsymbol{M}$ as otherwise the specification region will become asymmetric with respect to the target.

Based on these assumptions, $C_G(u, v)$ can be defined as,

$$C_G(u, v) = \frac{1}{3}\sqrt{\frac{(\mathbf{d} - u\mathbf{D})'\Sigma^{-1}(\mathbf{d} - u\mathbf{D})}{1 + v(\mu - \mathbf{T})'\Sigma^{-1}(\mu - \mathbf{T})}} \tag{27}$$

where, $\mathbf{D} = (|\mu_1 - M_1|, |\mu_2 - M_2|, \ldots, |\mu_p - M_p|)'$;

$\mathbf{d} = (\frac{\mathrm{USL}_1 - \mathrm{LSL}_1}{2}, \frac{\mathrm{USL}_2 - \mathrm{LSL}_2}{2}, \ldots, \frac{\mathrm{USL}_p - \mathrm{LSL}_p}{2})'$;

$\mathbf{T} = (T_1, T_2, \ldots, T_p)'$;

$\mathbf{M} = (M_1, M_2, \ldots, M_p)'$; $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)'$.

Here, T_i is the target value, M_i is the nominal value for the ith characteristic of the item; 'p' denotes the number of characteristics under consideration and

Σ = Variance—covariance matrix of the variable X;

μ_i = Mean of the ith characteristic of the variable X, for $i = 1, 2, \ldots, p$;

μ = The mean vector of the variable X,

u and v are the scalar constants that can take any non-negative integer value.

Note that, the member indices of $C_G(u, v)$, viz., $C_G(0, 0)$, $C_G(1, 0)$, $C_G(0, 1)$ and $C_G(1, 1)$ are analogous to the four classical PCIs $C_p, C_{\mathrm{pk}}, C_{\mathrm{pm}}$ and C_{pmk} of univariate PCIs for symmetric specification limits.

Chatterjee and Chakraborty (2013) have observed that, for $u = 0, 1$ and $v = 0, 1$, the member indices of $C_G(u, v)$ are inter-related among themselves through the following relationships:

$$\left.\begin{array}{l} C_G(0,0) \geq C_G(1,0) \geq C_G(1,1) \\ C_G(0,0) \geq C_G(0,1) \geq C_G(1,1) \end{array}\right\} \tag{28}$$

and there exists no clear-cut relationship between $C_G(1,0)$ and $C_G(0,1)$. Note that such relationships are analogous to those between the member indices of $C_p(u,v)$.

Chatterjee and Chakraborty (2013) have also explored the relationship between the minimum attainable proportion of non-conformance (P_{NC}) and $C_G(0,0)$ and have observed that

$$P_{\mathrm{NC}} = 2\left\{1 - P[Y \le 9C_G^2(0,0) | Y \sim \chi_p^2]\right\} \tag{29}$$

Since by definition, $C_G(0,0)$ is always non-negative, Eq. (26) establishes a one-to-one relationship between $C_G(0,0)$ and P_{NC}. Chatterjee and Chakraborty (2013) have also made an extensive comparative study among the member indices of $C_G(u,v)$ and $C_p(u,v)$ to help these MPCIs gain higher amount of acceptability among the practitioners.

6.2 $C_M(u,v)$—a Multivariate Process Capability Index for Asymmetric Specification Region

Like in the univariate case, for processes with multiple quality characteristics also, it is common to encounter processes with asymmetric specification regions, i.e., where, $\boldsymbol{T} \neq \boldsymbol{M}$. Although Grau (2007) proposed some MPCIs to assess the capability of such processes, his formulations are complicated in nature and hence are of interest more for theoreticians than the shop-floor people who are ultimately going to use these PCIs.

As have been already discussed in Sect. 3, $C_p''(u,v)$, defined in Eq. (4), is more suitable for measuring capability of the processes with single quality characteristic and asymmetric specification limits with respect to T as compared to the other PCIs available in literature. Chatterjee and Chakraborty (2011) have defined an MPCI called $C_M(u,v)$, which generalizes $C_p''(u,v)$ for processes with multiple quality characteristics. Here, $C_M(u,v)$ is defined as

$$C_M(u,v) = \frac{1}{3}\sqrt{\frac{(\mathbf{d}^* - u\mathbf{G}^*)'\Sigma^{-1}(\mathbf{d}^* - u\mathbf{G}^*)}{1 + v\mathbf{G}'\Sigma^{-1}\mathbf{G}}}, \tag{30}$$

where, $\mathbf{d}^* = \begin{pmatrix} \min(D_{1L}, D_{1U}) \\ \min(D_{2L}, D_{2U}) \\ \vdots \\ \min(D_{pL}, D_{pU}) \end{pmatrix}$ i.e. $d_i^* = \min(D_{iL}, D_{iU}), \quad \text{for } i = 1(1)p$ with

$$D_U = \begin{pmatrix} D_{1U} \\ D_{2U} \\ \vdots \\ D_{pU} \end{pmatrix} \text{ and } D_L = \begin{pmatrix} D_{1L} \\ D_{2L} \\ \vdots \\ D_{pL} \end{pmatrix}.$$

Also $\mathbf{d} = \begin{pmatrix} \frac{\mathrm{USL}_1 - \mathrm{LSL}_1}{2} \\ \frac{\mathrm{USL}_2 - \mathrm{LSL}_2}{2} \\ \vdots \\ \frac{\mathrm{USL}_p - \mathrm{LSL}_p}{2} \end{pmatrix}$ i.e. $d_i = \frac{\mathrm{USL}_i - \mathrm{LSL}_i}{2}, \quad \text{for}\, i = 1(1)p$

For multivariate case '**G**' can be defined as

$\mathbf{G} = \begin{pmatrix} a_1 d_1 \\ a_2 d_2 \\ \vdots \\ a_p d_p \end{pmatrix}$, where, $a_i = [\max\{\frac{\mu_i - T_i}{D_{iU}}, \frac{T_i - \mu_i}{D_{iL}}\}], \forall\, i = 1(1)p.$

As such $\mathbf{G} = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & a_p \end{pmatrix} \mathbf{d} = A\mathbf{d}$, say and its univariate counterpart is given as 'F' in (4).

Similarly, 'F^*' can be generalized as $\mathbf{G}^* = A\mathbf{d}^*$ for the multivariate case. Also, for $p = 1$, $C_M(u, v) = C_p''(u, v)$. For $u = 0, 1$ and $v = 0, 1$, the authors have observed following relationship between the member indices of $C_M(u, v)$:

$$C_M(0,0) \geq C_M(1,0) \geq C_M(1,1)$$

$$C_M(0,0) \geq C_M(0,1) \geq C_M(1,1)$$

Also, no clear-cut relationship exists between $C_M(1,0)$ and $C_M(0,1)$ like in the case of $C_p''(u, v)$.

Note that here $C_M(0,0)$, which is independent of μ, measures the potential process capability and this is quite justified by the above relationships as all the other member indices of $C_M(u, v)$ can achieve at most the capability value projected by $C_M(0,0)$.

6.3 Example

To demonstrate the ability of $C_M(u, v)$, for $u = 0, 1$ and $v = 0, 1$, we consider the data set originally used by Sultan (1986). Here we have two correlated characteristics viz., brinell hardness (H) and tensile strength (S). The USL and LSL for 'H' are 241.3 and 112.7 respectively; while for 'S', these values are 73.30 and 32.70 respectively. Also, the target vector for the said process is $\boldsymbol{T} = (177, 53)'$. Thus, $\boldsymbol{T} = \boldsymbol{M}$ and hence $C_G(u, v)$ will be applicable here.

A random sample of size 25 is drawn from the process and the corresponding summary statistics are as follows:

$n = 25,\ \overline{\boldsymbol{X}} = \begin{pmatrix} 177.2 \\ 52.316 \end{pmatrix},\ \hat{\Sigma} = \begin{pmatrix} 338 & 88.8925 \\ 88.895 & 33.6247 \end{pmatrix}$ and hence the observed correlation coefficient between 'H' and 'S' is, $\hat{\rho} = 0.8338$ which is quite high. Thus, $\hat{C}_G(0,0) = 1.2181, \hat{C}_G(1,0) = 1.1971, \hat{C}_G(0,1) = 1.1870$ and $\hat{C}_G(1,1) = 1.1666$. Hence we conclude that the process is capable. Also, the computed MPCI values follow the interrelationship established in Eq. (28). These strongly suggest that the process is performing satisfactorily.

However, before assessing the capability of the process, we need to check the validity of the assumption of multivariate normality of the present data. The p value associated with Shapiro–Wilk test (refer Shapiro and Wilk 1965) is 0.006764 and that with Royston's test (refer Royston 1983) is 0.02586. Since both of these p values are less than 0.05, it is logical to expect that the underlying distribution of the present data set is not multivariate normal (refer Chatterjee and Chakraborty 2013).

In order to assess the capability of the process, the data is transformed using Box–Cox Power Transformation (refer Box and Cox 1964). For the transformed data, p value corresponding to the Shapiro–Wilk multivariate normality test is found to be 0.07627; while that using Royston's test is 0.1103. Therefore, it is logical to expect that the transformed data set indeed follow multivariate normal distribution.

Moreover, since the data set has been transformed to have multivariate normal distribution, it is now required to transform **USL**, **LSL** and **T**, by virtue of the same transformation. The transformed specification limits and targets for H_{new} and S_{new} are as follows:

$$\left.\begin{array}{l} \text{USL}_{H_{\text{new}}} = 240.3 \\ \text{LSL}_{H_{\text{new}}} = 111.7 \\ T_{H_{\text{new}}} = 176 \end{array}\right\} \qquad \left.\begin{array}{l} \text{USL}_{S_{\text{new}}} = 2685.945 \\ \text{LSL}_{S_{\text{new}}} = 534.145 \\ T_{S_{\text{new}}} = 1404.000 \end{array}\right\}$$

Thus, although, apparently the specification region was symmetric with respect to the target vector, the transformed specification region is asymmetric about the transformed target vector, viz., $T_{\text{new}} = (T_{H_{\text{new}}}, T_{S_{\text{new}}}) = (177, 1404)$.

For the transformed data, $\boldsymbol{d} = (64.3, 1075.9)'$, $\boldsymbol{d}^* = (64.3, 869.855)'$, $A = \begin{pmatrix} 0.0031 & 0 \\ 0 & 0.0228 \end{pmatrix}$, $G = A\boldsymbol{d} = (0.2, 24.5868)$ and $G^* = A\boldsymbol{d}^* = (0.2, 19.8782)$, $\widehat{\mu} = (176.20, 1384.122)'$ and $\hat{\Sigma} = \begin{pmatrix} 338 & 4435.277 \\ 4435.277 & 81311.074 \end{pmatrix}$.

Hence, $\widehat{C}_M(0,0) = 1.1672$, $\widehat{C}_M(1,0) = 1.1623$, $\widehat{C}_M(0,1) = 1.1551$ and $\widehat{C}_M(1,1) = 1.1503$. Also, the threshold value of $\widehat{C}_M(0,0)$ is computed as 1.1672. Thus, the process is potentially just capable as the threshold value coincides with the value of $\widehat{C}_M(0,0)$. However, all of $\widehat{C}_M(1,0)$, $\widehat{C}_M(0,1)$ and $\widehat{C}_M(1,1)$ have values lower than the threshold value. This indicates that the actual capability level of the process is not satisfactory.

Thus, assertion of the underlying distribution of the quality characteristic(s) is utmost necessary before assessing the capability of a process.

7 Process Capability Indices for Non-normal Statistical Distributions

As has been observed by Kotz and Johnson (2002), the assumption of normality of the underlying statistical distribution of the concerned quality characteristic, is one of the basic assumptions for defining process capability indices, irrespective of the nature of the specification limits. Despite of giving some computational advantage, such normality assumption is not valid in many practical situations.

For example, for quality characteristics of smaller the better type (like surface roughness, flatness and so on), for which only USL is available, some times, the quality characteristic has a skewed distribution with a long tail towards the larger values (refer Vannman and Albing 2007).

Clements (Clements 1989) first addressed this problem and suggested replacing 6σ by the length of the interval between the upper and lower 0.135 percentile points of the actual distribution. The author redefined estimators of C_p and C_{pk}, for quality characteristics with non-normal statistical distributions as follows:

$$C_p^{'} = \frac{U - L}{\xi_{1-\alpha} - \xi_\alpha} \tag{31}$$

$$C_{\text{pk}}^{'} = \frac{d - |\xi_{0.5} - M|}{(\xi_{1-\alpha} - \xi_\alpha)/2} \tag{32}$$

where, $\xi_{1-\alpha}$ and ξ_α are the upper and lower α percentiles of the distribution of the corresponding random variable X and $\xi_{0.5}$ is the corresponding median. Generally, $\alpha = 0.00135$ is considered for computational purposes.

Following Clements' (1989) approach, Pearn and Kotz (1994), redefined the estimators of C_{pm} and C_{pmk} for non-normal quality characteristics as,

$$C_{\text{pm}}^{'} = \frac{U - L}{6\sqrt{\left[\frac{\xi_{1-\alpha} - \xi_\alpha}{6}\right]^2 + (M - T)^2}} \tag{33}$$

$$C_{\text{pmk}}^{'} = \min\left\{\frac{U - M}{3\sqrt{\left[\frac{\xi_{1-\alpha} - M}{3}\right]^2 + (M - T)^2}}, \frac{M - L}{3\sqrt{\left[\frac{M - \xi_\alpha}{3}\right]^2 + (M - T)^2}}\right\} \tag{34}$$

Pearn et al. (1999) generalized these indices for asymmetric specification limits. Wright (1995) proposed the following PCI which is sensitive to skewness:

$$C_s = \frac{\frac{d}{\sigma} - \frac{|\mu - M|}{\sigma}}{3\sqrt{1 + \left(\frac{\mu - T}{\sigma}\right)^2 + |\sqrt{\beta_1}|}} \tag{35}$$

where, $\sqrt{\beta_1} = \frac{\mu_3}{\sigma^{3/2}}$ is a widely used measure of skewness and μ_3 is the third order raw moment of the corresponding random variate 'X'.

However, the quantile or percentile based approach of dealing with non-normality, while measuring capability of a process, suffers from a basic problem. Often these PCIs involve extreme percentiles viz., 99.73th or 0.27th percentiles. However, accurate estimation of these extreme percentiles require a huge amount of data, which is often difficult to obtain, especially for processes requiring destructive testing (refer Pearn et al. 1992). Wu et al. (1998) have observed that, PCIs based on Clements' approach fail to measure the capability of a process accurately, especially, when the underlying distribution of the concerned quality characteristic is skewed.

Another approach of dealing with non-normality is to transform the original non-normal data into a normal one through the use of appropriate transformations and then apply the PCIs defined for normal data. Some of the statistical transformations, which are available in literature, are

1. Johnson's (1949) transformation, based on the method of moments;
2. Box–Cox's (1964) power transformation;
3. Somerville and Montgomery's (1996) square-root based transformation for skewed distributions;
4. Hosseinifard et al.'s (2009) root transformation method

Farnum (1996) has extensively discussed the use of Johnson's transformation in the context of non-normal process data. Yang et al. (2010) have carried out a comparative study between the performances of Box–Cox transformation and Johnson's transformation in assessing capability of a process.

One can also choose a process distribution from a smaller family of distributions such as gamma, lognormal or weibull which in turn, simplifies the corresponding inferential problem. Rodriguez (1992) have enlisted the following advantages of using families of distributions for computing PCIs of non-normal processes:

1. Method of maximum likelihood can be used to have stable and straight forward estimation of the concerned parameters.
2. Since the method of maximum likelihood yields asymptotic variances for estimates of the parameters, it can be used to construct confidence intervals for the plug-in estimators of the PCIs.
3. For various families of distributions like gamma, lognormal and weibull, goodness-of-fit tests based on empirical distribution functions are also available.

4. For standard families of distributions, estimated values of the percentiles and proportion of non-conformance, related to the plug-in estimators of the PCIs, can be easily computed using standard results.

It is interesting to note that 'potential capability' means 'possibility of achieving' rather than 'actually achieving' (refer Kotz and Johnson 2002). Veevers (1998) has used the term 'viability' to represent 'capability potential' and has proposed a viability index from a more general perspective, as compared to C_p, in a sense that the viability index is neither restricted to normal distribution of 'X' nor even to univariate situations.

The univariate viability index is defined as

$$V_t = \frac{w}{2d} \tag{36}$$

where, 'w' is the 'window of opportunity' measured by the length of interval of θ for which the distribution of $(X + \theta)$ would generate an expected PNC not greater than the conventional $0.27\,\%$.

Under the assumption of normality of the quality characteristic under consideration,

$$M - (d - 3\sigma) \leq \mu \leq M + (d - 3\sigma) \tag{37}$$

i.e. the window of opportunity for μ can be defined as, $w = 2(d - 3\sigma)$ and the corresponding viability index will be

$$\begin{aligned} V_t &= \frac{2(d - 3\sigma)}{2d} \\ &= 1 - \frac{1}{C_p} \end{aligned} \tag{38}$$

Unlike most of the PCIs, V_t can assume negative values. If V_t is less than zero, there is no possibility of attaining a PNC value of $0.27\,\%$ or lower and hence, the process is considered to be 'non-viable'.

For processes with unilateral specification limits also, substantial research work has been done to assess the capability of a process when the underlying statistical distribution is non-normal. Vannman and Albing (2007) modified $C_{\mathrm{pvu}}(u, v)$ (see Eq. (16)) for the case, where the quality characteristic has a skewed distribution with a long tail towards large values and a 'USL' with target set at '0', i.e. the quality characteristic has a skewed zero-bound distribution. This superstructure is defined as

$$C_{\mathrm{MA}}(\tau, v) = \frac{\mathrm{USL}}{\sqrt{q_{1-\tau}^2 + v q_{0.5}^2}} \tag{39}$$

where, $v \geq 0$ and q_τ is the τth quantile of the quality characteristic. The parameter τ should be small and chosen in a suitable way, e.g. $\tau = 0.0027$.

However, Chatterjee and Chakraborty (2012) have observed the following drawbacks in this superstructure:

1. Vannman and Albing (2007) have modified only $C_{\mathrm{pvu}}(u, v)$. Neither $C_{\mathrm{pau}}(u, v)$ was modified nor any justification for omitting the same was given. However, as has been pointed out by Grau (2009), $C_{\mathrm{pvu}}(u, v)$ is not suitable for assessing capability of a process with unilateral specifications.
2. There is room for studying whether considering $\tau = 0.0027$ is justified even if the underlying distribution of the quality characteristic is not normal.
3. Some constants of $C_{\mathrm{pvu}}(u, v)$ were omitted just for simplicity without studying the impact of such omission.
4. $C_{\mathrm{MA}}(\tau, v)$ fails to perform if the target is other than '0'.
5. The ideal values of v have not been studied.

Albing (2009) has modified the superstructure $C_{\mathrm{MA}}(\tau, v)$ which is defined in Eq. (36) for the quality characteristic under Weibull distribution, as follows:

$$C_{\mathrm{MAW}}(\tau, v) = \frac{\mathrm{USL}}{a\sqrt{(ln(\frac{1}{\tau})^{\frac{2}{b}}) + v(ln2)^{\frac{2}{b}}}} \tag{40}$$

where, 'a' is the scale parameter and 'b' is the shape parameter of a two-parameter Weibull distribution. However, since this super-structure is an extension of $C_{\mathrm{MA}}(\tau, v)$, it inherits the drawbacks of $C_{\mathrm{MA}}(\tau, v)$ as listed above. Moreover, $C_{\mathrm{MAW}}(\tau, v)$ is valid only when the underlying distribution of the quality characteristic is Weibull. It fails to perform in case of all the other types of statistical distributions.

Rodriguez (1992) has also suggested other methods like goodness-of-fit, quantile-quantile plot, kernel density estimation and comparative histograms to assess capabilities of non-normal processes. For a thorough review of the PCIs for non-normal distributions, one can refer to Pearn and Kotz (2007); Tang and Than (1999) and the references there-in.

Finally, the capability assessments for multivariate processes with non-normal process distributions have been studied by Abbasi and Niaki (2010), Ahmad et al. (2009), Polansky (2001) and so on.

The example considered at the end of Sect. 6.2 can be considered here as well. Recall that, there we have transformed the multivariate non-normal data into a multivariate normal one and then applied $C_M(u, v)$. The MPCIs discussed in the present section can also be used for this purpose. In particular, Polansky (2001) used the same data and concluded that the performance of the process is not satisfactory, which supports the observations made by Chatterjee and Chakraborty (2013). Moreover, the approach of transforming the data to multivariate normality and then applying $C_M(u, v)$ is easier to execute as compared to using Polansky's (2001) MPCI.

8 How to Check Univariate and Multivariate Normality of Data

Asserting the underlying distribution of the quality characteristic under consideration plays a major role in capability assessment of a process. Often, in practice, PCIs for univariate and multivariate normal distributions are used to assess the capability of a process, without exploring the actual statistical distribution of the concerned quality characteristic. This may lead to wrong judgement of the actual capability of a process. Hence proper testing of the normality assumption of the available data is utmost solicited.

Now a days, such checking of normality is possible through almost all the statistical softwares available in market, viz., R, SPSS, STATISTICA, MINITAB, SAS, MATLAB and so on. Among these, R is a open source and hence can be freely downloaded from internet. We shall now discuss the procedure of testing normality through the statistical package **R**.

Following are some functions and packages in **R**, which deal with univariate and multivariate normality testing:

(i) Shapiro–Wilk test (Shapiro and Wilk 1965) for univariate normality can be done using the function **shapiro.test**.
(ii) qqnorm is a function that produces a normal quantile–quantile (QQ) plot of a data. The corresponding qqline adds a line to a 'theoretical', by default normal, quantile–quantile plot which passes through the first and third quartiles.
(iii) For testing multivariate normality of a data, one can use the library **MVN** which provides functions for Mardia's multivariate normality test (refer Mardia 1970, 1974) and Royston's multivariate normality test (refer Royston 1983).
(iv) Generalized Shapiro–Wilk test for multivariate normality (refer Royston 1983 and Villasenor-Alva and Gonzalez-Estrada 2009) can be carried out using libraries like **mvShapiroTest** and **mvnormtest**.

Also, to transform a non-normal data into a normal one, one can use the library **alr3** for Box–Cox transformation (refer Box and Cox 1964) of the data.

9 Concluding Remarks

This chapter deals with measurement of process capability analysis for different situations by mostly suggesting appropriate indices for a given situation. However, there are criticisms for making process capability index as the sole measure of the capability of the process. One can refer to Gunter (1989), Dovich (1991), Carr (1991), Herman (1989), Pignatiello and Ramberg (1993) and many others. Some even suggested that none of the so called PCIs adds any knowledge or understanding about the process beyond that contained in the equivalent basic parameters like μ, σ, target value and the specification limits.

The main problem seems to be that a PCI is taken as a one-time measure or a snap shot of the process and is highly dependent on the chosen sample. This leads to a fear of manipulation which is genuine. We suggest that for a PCI to be calculated, a necessary condition to be fulfilled is that the process should be stable. A sufficient condition could be that the PCI, calculated over a period of time should show stability. This requires an appropriate control charting technique for each PCI depending on the type of distribution a PCI would follow. The authors are now developing these control charts which will settle the issue.

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