Chapter 8
Kinematic Design

Sebastian Kassner

Abstract The kinematic design of haptic interfaces is a crucial step especially when designing interfaces with mainly kinaesthetic feedback (see Sect.12.2). This is often the case in the context of robotic applications. In these devices, a mechanical mechanism is used to link the user and the feedback generating actuators. Furthermore, the user's input commands are often given by moving a mechanical mechanism, e.g., a joystick.Accordingly, the kinematic design is a crucial aspect for a device with ergonomic design and good haptic transmission. This chapter gives an introduction to the classes of mechanisms and how they are designed.

8.1 Introduction and Classification

Figure 8.1 shows the basic elements of a mechanical mechanism: base platform, rods, joints, and the ↪ TCP. The base platform is that part of a mechanism that is static regarding its motion. This means that for all calculations to be performed to design a mechanism as well as for all calculations executed during the operation of the haptic interface, position, speed, and accelerations are given with respect to the base platform.

In common mechanisms, at least one joint is located in the base platform. Rods and further joints make up a kinematic chain linking the base platform and the ↪ TCP. Kinematic joints used in mechanisms are (Fig. 8.2): revolute (R) joints, prismatic (P) joints, helical (H) joints, universal (U) joints, cylindrical (C) joints, spherical (S) joints, and planar (E) joints.

S. Kassner (✉)
Institute of Electromechanical Design, Technische Universität Darmstadt,
Merckstr. 25, 64283 Darmstadt, Germany
e-mail: s.kassner@hapticdevices.eu

C. Hatzfeld and T.A. Kern (eds.), *Engineering Haptic Devices*,
Springer Series on Touch and Haptic Systems, DOI 10.1007/978-1-4471-6518-7_8

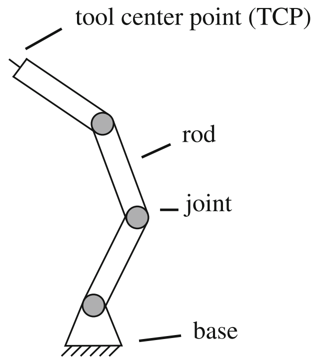


Fig. 8.1 Basic elements of a kinematic structure

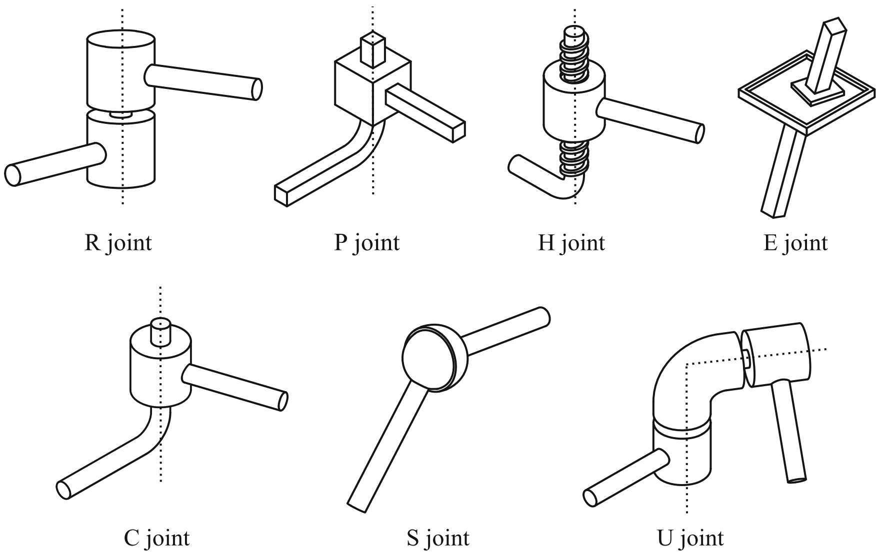


Fig. 8.2 Kinematic joints, figure based on [6]

The kinematic chain is a mathematical model to calculate the kinematics of a mechanical system. This is defined as:

Definition *Kinematics* Kinematics is the science of motion of points and bodies in space, characterized by their position, speed, and acceleration. Thereby the external causes of motion (forces and torques) are not taken into account. Motion with respect forces and torques is covered by the science of kinetics.

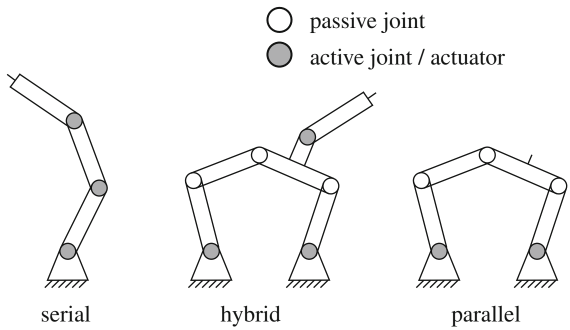


Fig. 8.3 Basic kinematic structures

The ↪ TCP is the point of interaction between mechanism and environment. It is able to move in space with a certain number of ↪ DoF. In the case of haptic interfaces, the haptic feedback and user interaction usually take place via this point.

8.1.1 Classification of Mechanisms

Depending on their kinematic chains mechanical mechanisms are classified as

- serial: open kinematic chains;
- parallel: closed kinematic chains, at least two paths from the base platform to the tool-center point;
- hybrid: combination of serial and parallel mechanisms.

Figure 8.3 shows the three fundamental configurations with typically passive and actively driven joints.

Serial Mechanisms

Serial mechanisms are widely used in all kinds of robotic applications. A classic example is serial assembly robots in an assembly line production of motor vehicles. Purely serial mechanisms include no passive joints. All actuators are in serial order within one single kinematic chain.

The advantages of serial mechanisms are their simple design and their relatively large workspace. They are furthermore easy to control, especially in positioning tasks. This is mainly due to the serial sequence of joints and rods allowing the application of mathematical step-by-step transformations. An established method is the DANAVIT-HARTENBERG transformation [4], which is not covered in this chapter.

The major drawback of a serial mechanism is its dynamic behavior. Since a load is carried by a single kinematic chain, serial mechanisms usually have lower structural stiffness with respect to their own weight. Additionally, the dynamic behavior is restricted by the comparatively high masses. One reason for this effect is the mass

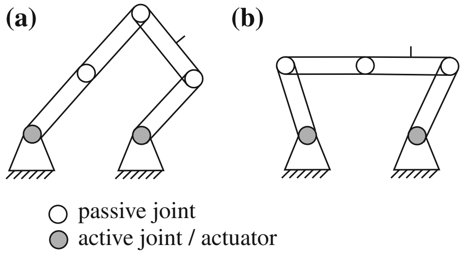


Fig. 8.4 Singular positions: **a** First kind. **b** Second kind

of the rods to gain a high structural stiffness. A second reason is the weight of the actuators within the mechanism. Every actuator has to accelerate all following actuators in the kinematic chain.

Parallel Mechanisms

This is the main advantage of parallel structures. They comprise actuators that are fixed to the base platform or that are only moved slightly in space. The load on a ↪ TCP is distributed into several kinematic chains. This allows the design of lightweight, yet stiff mechanical structures. This leads to a dynamic transmission behavior with high cutoff frequencies and thus a more transparent transmission of the haptic feedback. Because of these properties, parallel mechanisms are of high significance in the design of haptic interfaces.

On the other hand, parallel mechanism has a small workspace in comparison with serial structures. The parallel mechanism's kinematic is often mathematically more complex and usually nonlinear. Furthermore, the transmission behavior changes with respect to the mechanism's position. Thus, it is directional and anisotropic throughout the workspace.

Parallel mechanisms have special positions that have to be taken into account when designing a haptic interface: singular positions. These positions occur when at least two rods of the mechanism are aligned. One distinguishes two types of singular positions (Fig. 8.4):

Singularity of the first kind The actuator's motion is not transmitted to the ↪ TCP any longer. This position typically occurs at the edge of the mechanism's workspace.

Singularity of the second kind The actuator's force or torque is not transmitted to the ↪ TCP any longer and the ↪ TCP can carry no load. The mechanism is jammed regarding its actuators. This position typically occurs within the mechanism's workspace.

If a mechanism approaches a singular position its transmission or gear ratio changes quickly until the mechanism is locked in the singular position. In the singular position, the mechanism's degrees of freedom change in an undesirable way. The mechanism runs into danger of damage or cannot be controlled any longer. Thus, possible singular positions have to be analyzed thoroughly during the design of a parallel haptic device. During operation, they have to be avoided by all means. How singular positions can be identified mathematically is discussed in Sect. 8.3.

If a serial, parallel, or hybrid mechanism is suitable for the design of a haptic interface, it should be decided on a case-by-case basis. All are used in haptic applications.

8.2 Design Step 1: Topological Synthesis—Defining the Mechanism's Structure

The topological synthesis is the first step in designing a haptic interface. It leads to the basic configuration of joints, rods, and actuators. While the basic structure of the haptic interface is defined in this step, the topological synthesis has be carried out thoroughly.

Topological synthesis should be based on an analysis of the specific task and the following issues must be addressed:

- *degrees of freedom*: In how many ↪ DoFs should the user interact with the haptic interface? Which ↪ DoFs are required (e.g., one pure rotatory as in a jog wheel, three to mimic spatial interaction or even six to display three translations and three rotations)?
- *adaption of existing structures*: Should the device adapt the structure of the task (e.g., a controlled robot) or of the user (e.g., the user's finger or arm)?
- *workspace*: How large is the desired workspace, the ↪ TCP has to move in? Are there any restrictions (e.g., areas of the workspace that should not be accessed)?
- *mobility*: Is the haptic interface designed as a device standing on a fixed place, e.g., on a table or is designed as a portable device?

The analysis of these requirements lays the foundation for the design of an easy-to-use and ergonomic haptic interface, which will be accepted by the user.

8.2.1 Synthesis of Serial Mechanisms

A serial mechanism is not less nor more than a sequence of rods and actuators, whereas the actuators can be regarded as driven joints. Whether the actuators are linear or rotary is of no importance for the complexity of the kinematic problem. For the workspace and the orientation of the tool-center-point, however, this aspect is of highest importance. A spacial serial mechanism with three rotatory drives changes the orientation of its ↪ TCP all over its workspace. If it is not intended to generate a torque as output to the user, the handle attached to this serial mechanism has to be

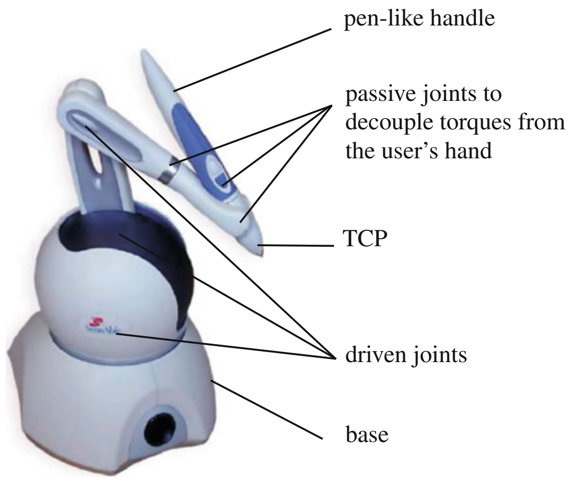


Fig. 8.5 The GEOMAGIC TOUCH haptic device is an example for a spacial working serial kinematic haptic device. The hand is decoupled from rotational movements by passive joints. Thus, no torques are induced to the hand

equipped with a passive universal joint. Such a realization as haptic device can be found in Fig. 8.5. Torques are decoupled from the hand. The handle does not have to be placed exactly in the ↪ TCP, as the moments are eliminated by the passive joints. Force vectors can be displaced arbitrarily within space. As a result, the hand experiences the same forces as the ↪ TCP.

8.2.2 Synthesis of Parallel Mechanisms

The synthesis of a parallel mechanism in general is a less intuitive process than the synthesis of a serial mechanism.

Since a parallel structure comprises several kinematic chains, the first step is to determine the required number of kinematic chains with respect to the desired degrees of freedom of the mechanism. This can be done using the ratio of the number of chains k and the degrees of freedom F of the mechanism leading to the degree of parallelism (see [4])

$$P_g = \frac{k}{F}. \tag{8.1}$$

A mechanism is partially parallel for $P_g < 1$, fully parallel for $P_g = 1$, and highly parallel for $P_g > 1$. Assuming the most common case of fully parallel mechanisms, this results in a number of kinematic chains—or "legs" of the mechanism—which is equal to the desired number of the mechanism's ↪ DoF.

The next step is to determine the number of joints ↪ DoF in the mechanism. This is done using the GRUEBLER-KUTZBACH-CHEBYCHEFF mobility criterion

$$F = \lambda \cdot (n - g - 1) + \sum_{i=1}^{g} f_i - f_{id} + s \tag{8.2}$$

with

F	mechanism's ↪ DoF
n	number of rods
g	number of joints
f_i	↪ DoF of the ith joint
f_{id}	number of identical links
s	number of constrains
λ	factor with $\lambda = 3$ for planar and $\lambda = 6$ for spatial mechanisms

An identical link is given for example when a rod has universal joints at both its ends. The rod will be able to rotate around its axis, without violating any constraints. Another example is two coaxial-oriented linear joints.

Constraints appear whenever conditions have to be fulfilled to enable the movement. If five joint axes have to be parallel to a 6th axis to enable a movement, then $s = 5$. Another example for a passive condition is two driving rods that have to be placed in parallel to enable a motion.

Applying Eq. (8.2) at this stage of design is usually not possible in the given form since the number of rods n and joints g is not known yet. However, it is possible to link $n - g$ with the already known number of kinematic chains k via

$$n = g - k + 2. \tag{8.3}$$

Assuming a spatial mechanism ($\lambda = 6$) then leads to

$$\sum_{i=1}^{g} f_i = F + 6 \cdot (k - 1). \tag{8.4}$$

for the sum of the joint ↪ DoFs that have to be distributed in the mechanism.

8.2.3 *Special Case: Parallel Mechanisms with Pure Translational Motion*

An important task of many haptic interfaces is the displaying of three-dimensional spatial sensation. An example is interaction with a pen-like tool where only forces in (x, y, z) should be displayed to the user.

A special class of 3-↪ DoF parallel mechanisms is used for these applications: ↪ Translational Parallel Machines (TPM). A ↪ TPM is a mechanism whose ↪ TCP can only move in three Cartesian coordinates (x, y, z). This is achieved by kinematic chains which block one or more rotatory ↪ DoFs of the ↪ TCP and are able to perform translational motion in all directions.

According to CARRICATO [1, 2], two restrictions have to be fulfilled to ensure a parallel kinematic mechanism with pure translational motion:

- ball joints shall not be used
- the rotatory axis of rotatory joints shall not be parallel to the axis of a degree of freedom which should be constrained

Neglecting overdetermined configurations, this results in so-called T_5-mechanisms, each comprising four or five rotatory joints. Each joint constrains the rotation of the ↪ TCP about one axis, defined by the unity vector $\mathbf{n}_i$ $(i = 1, 2, 3)$. To constrain a rotation about $\mathbf{n}_i$, all rotatory axes of a chain are orientated perpendicularly to $\mathbf{n}_i$.

There are two ways to design a T_5 mechanism. The first type is made of three T_5' chains, each having

- two rotatory joints following each other, with axis parallel to the unity vector $\mathbf{w}_{1i}$;
- two rotatory joints following each other, with axis parallel to the unity vector $\mathbf{w}_{2i}$ but not parallel to $\mathbf{w}_{1i}$;
- a prismatic joint at an arbitrary position in the chain or a fifth rotatory joint, parallel to one of its contiguous joints.

The second type is made of three T_5'' chains, having

- two rotatory joints with axis parallel to the unity vector $\mathbf{w}_{1i}$;
- two rotatory joints located between the first two rotatory joints with axis parallel to $\mathbf{w}_{2i}$ but not parallel to $\mathbf{w}_{1i}$;
- a prismatic joint at an arbitrary position in the chain or a fifth rotatory joint, parallel to one of its contiguous joints.

Figure 8.6 shows examples of these ↪ TPM chains with only 1-↪ DoF joints. This does not mean that ↪ TPMs are restricted to 1-↪ DoF. For instance can two adjoining and perpendicular rotatory joints be concentrated as a universal joint?

An important distinction between T_5' and T_5'' chains which is taken into account during design is the position of singular positions: Whereas in a T_5' mechanism, singular positions only occur at the edge of the workspace, T_5'' mechanisms can have singular positions within the workspace as well. Since a mechanism cannot pass through a singular position, this can lead to a split and therefore restricted usable workspace.

An exemplary topology synthesis is shown in the following example. However, one should keep in mind that topology synthesis is a process that requires some experience and cannot only be executed by application of straightforward design rules!

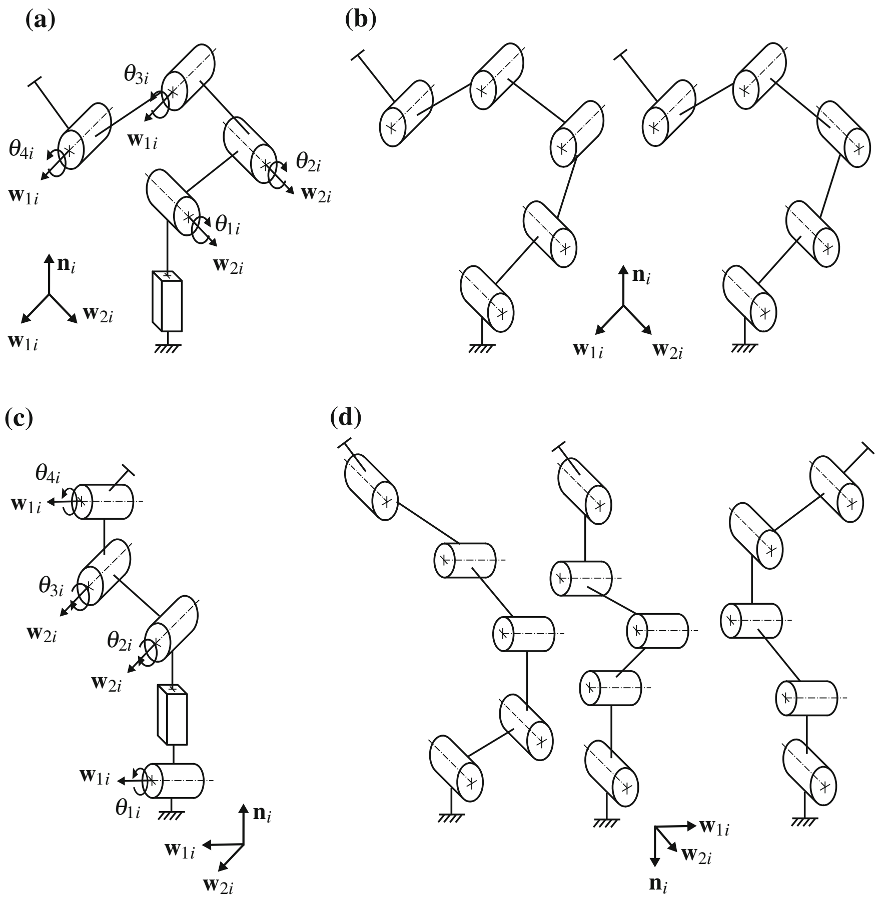


Fig. 8.6 Examples for ↪ TPM chains with five joint ↪ DoFs (based on [1])

8.2.4 Example: The DELTA Mechanism

One of the most common topologies to display spatial interaction is the parallel DELTA mechanism. Due to its relevance in the field of haptic interfaces, it is used as an example for topological synthesis.

Let us assume the design goal of a parallel kinematic haptic interface for spatial interaction in (x, y, z). Thus, a mechanism with three degrees of freedom is required. Using Eq. (8.1) for a fully parallel mechanism ($P_g = 1$) on $F = 3$ haptic degrees of freedom leads to a mechanism with $k = 3$ kinematic chains or legs.

In a second step, we have to determine the required joint degrees of freedom using GRUEBLER's formula (Eq. 8.4). This leads to the sum of $\sum_i f_i = 15$ joint degrees of freedom. Regarding an equal behavior in all spatial directions, it is self-evident to

Table 8.1 Topologies for 3-↪ DoF mechanisms with five ↪ DoF in each kinematic chain

Joints per chain	Topologies
1 × 1 DoF, 2 × 2 DoF	UUP, UPU, PUU, UUR, URU, RUU, CUP, CPU, CUR, CRU, RCU, UCP, UPC, PCU, UCR, URC, RUC, CCP, CPC, PCC, CCR, CRC, RCC
2 × 1 DoF, 1 × 3 DoF	SPP, SRR, SPR, SRP, PSP, RSP, PSR, RSR, PPS, RRS, RPS, PRS
3 × 1 DoF, 1 × 2 DoF	RRRU, RRUR, RURR, URRR, RRPU, RRUP, RURP, URRP, RPRU, RPUR, RUPR, URPR, PRRU, PRUR, PURR, UPRR, RPPU, RPUP, RUPP, URPP, PRPU, PRUP, PURP, UPRP, PPPU, PPUP, PUPP, UPPP, RRRC, RRCR, RCRR, CRRR, RRPC, RRCP, RCRP, CRRP, RPRC, RPCR, RCPR, CRPR, PRRC, PRCR, PCRR, CPRR, RPPC, RPCP, RCPP, PRPC, PCRP, PPPC, PPCP, PCPP, CPPP
5 × 1 DoF	32 iterations of P- and R-joints

distribute the 15 joint degrees of freedom with five degrees in each leg. This leads to the topologies in Table 8.1. The topologies are denominated according to the joints in one leg, e.g., a UUP mechanism comprises two universal and one prismatic joint.

The selection of an appropriate topology then can be carried on by a systematic reduction of the 3 ↪ DoF topologies in Table 8.1. The reduction is based on the following criteria:

- *Functionality as a* ↪ TPM: Criteria like the number of R-joints or the existence of an S-joint eliminate a large number of topologies.
- *Position of actors*: Rotatory, linear, or piston actors (e.g., in a hydraulic system) act as R-, P-, or C-joints. When having topologies with a U-joint attached to the base platform, this would lead to actors located within the kinematic chain. The required acceleration to move the actors' relatively high masses then would inhibit the dynamic advantages of a parallel mechanism to have the fullest effect.
- *Number of joints*: A concentration of two R-joints into one U-joint and an R- and P-joint into a C-joint, respectively, simplifies the mechanism's geometry and thereby its kinematic equations.

Table 8.2 shows the eliminated topologies. The remaining configurations are: UPU, PUU, CUR, CRU, RUU, and RUC.

Looking carefully at these topologies in Fig. 8.7 one recognizes that only RUU and RUC have rotatory joint attached to the base platform. Thus, these are the only two topologies that can be reasonably driven by a rotatory electrical motor. What makes the RUU- or DELTA mechanism special is that there are only joints with rotatory degrees of freedom within the kinematic chains. All forces and torques are converted into rotatory motion and there is no chance for the mechanism to cant. From a ↪ TPM point of view, the RUU/DELTA is a T_5'' mechanism having singular positions within the workspace. This has to be considered when dimensioning the mechanism.

The RUU/DELTA was introduced in 1988 by Clavel [3]. Besides acting as spatial haptic interface, the mechanism is widely used for robotic applications (e.g., pick-and-place tasks). Two popular examples are shown in Fig. 8.8.

Table 8.2 Eliminated topologies, sorted by the number of 1-, 2-, and 3-DoF joints in each leg

Elimination criterion	5 × 1 DoF	3 × 1 DoF, 1 × 2 DoF	2 × 1 DoF, 1 × 3 DoF	1 × 1 DoF, 2 × 2 DoF
No TPM	RRRPP, RRPRR, RRPPR, RRPPP, RPRRR, RPRRP, RPRPR, RPRPP, RPPRR, RPPRP, RPPPR, RPPPP, PRRRR, PRRRP, PRRPR, PRRPP, PRPRR, PRPRP, PRPPR, PPRRP, PPRPR, PPRPP, PPPRR, PPPRP, PPPPR, PPPPP, PRPPP	RPPU, RPUP, RUPP, URPU, PURP, UPRP, PPPU, PPUP, PUPP, UPPP, RRPC, RRCP, RCRP, CRRP, RPRC, RPCR, RCPR, CRPR, PRRC, PRCR, PCRR, CPRR, RPPC, RPCP, RCPP, CRPP, PRPC, PRCP, PCRP, CPRP, PPPC, PPCP, PCPP, CPPP	SPP, SRR, SPR, SRP, PSP, RSP, PSR, RSR, PPS, RRS, RPS, PRS	CUP, CPU, RCU, UCP, UPC, PCU, UCR, CCP, CPC, PCC, CCR, CRC, RCC
High number of joints	RRRRR, RRRRP, RRRPR, RRPRP, PPRRR	RRRU, RRUR, RURR, RRPU, RRUP, RURP, RPRU, RPUR, RUPR, PRRU, PRUR, PURR, UPRR, RRRC, RRCR RCRR, CRRR		
Base joint cannot be used as an actor		URRR, URRP, URPR		UUP, UUR, URU, URC

In these devices with mainly kinaesthetic feedback, a mechanical mechanism is used to link the user and the feedback generating actuators. Furthermore, the user's input commands are often given by moving a mechanical mechanism.

8.3 Design Step 2: Kinematic Equations

The kinematics of a mechanical mechanism describe its motion by means of position and orientation, speed, acceleration, and—of special importance for haptic interfaces—force and torques. The kinematic equations relate those measures at the input and output of a mechanism, typically at the base platform and the ↪ TCP. In other words: the kinematics represent the gearing properties of a mechanism.

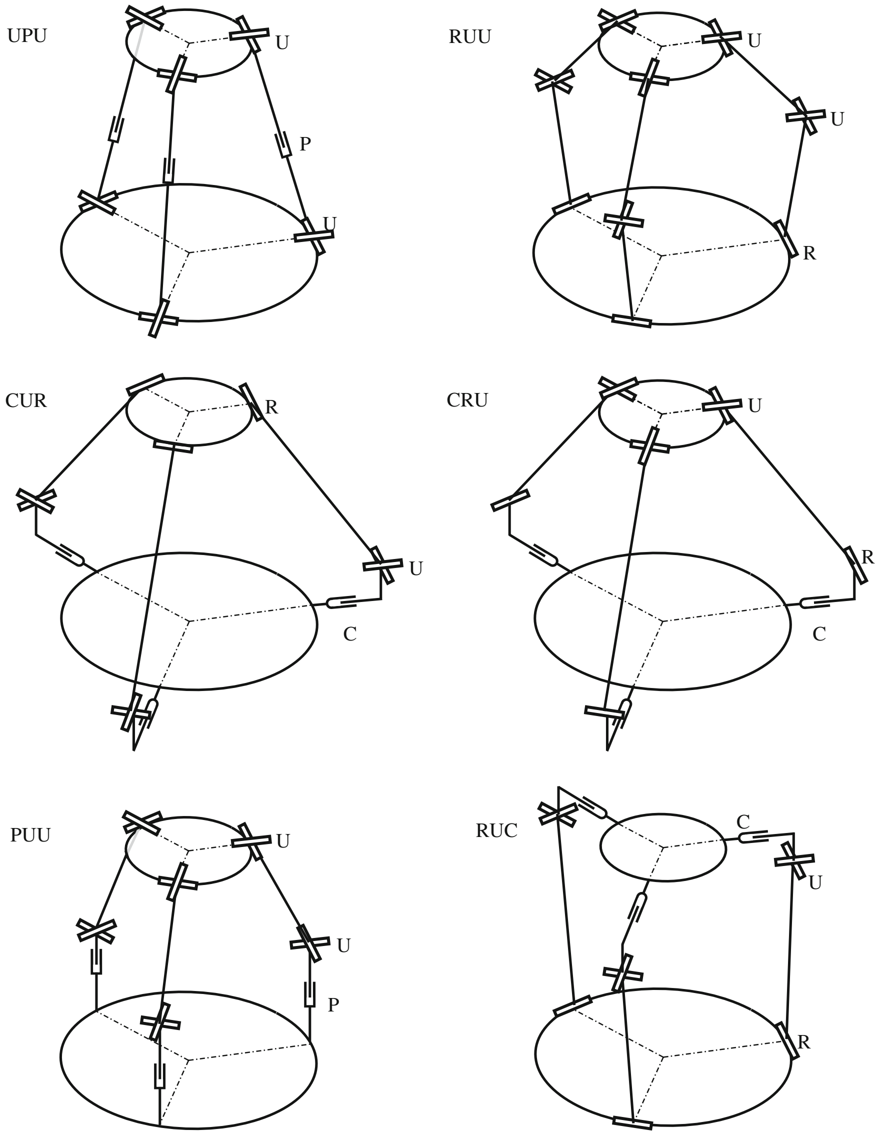


Fig. 8.7 Possible translational topologies with 3 DoF

They are of equal importance in the design and operation of a haptic device. This chapter gives an introduction to the basics of kinematic equations.

Fig. 8.8 *Left* FLEX PICKER for pick-and-place tasks (*source* ABB); *right* the FALCON as a 3-DoF haptic device (*source* Novint Technologies Inc.)

8.3.1 Kinematics: Basic Equations for Design and Operation

The transmission of motion in a mechanism can be described in two directions: from the actors to the ↪ TCP and vice versa. This leads to two basic kinds of kinematic equations: the *direct kinematic problem* or forward kinematics and the *inverse kinematic problem* or inverse kinematics.

Direct Kinematic Problem

The direct kinematic problems give a vector $\mathbf{x} = (x_1, x_2, ..., x_m)$ of ↪ TCP coordinates (position and orientation) with respect to a vector $\mathbf{q} = (q_1, q_2, ..., q_n)$ of actor coordinates by

$$\mathbf{x} = f(\mathbf{q}). \tag{8.5}$$

In contrast to serial mechanisms, for parallel mechanisms, the direct kinematic problem can only be solved numerically. However, there are exceptions as can bee seen later.

An important application of the direct kinematic problem is the calculation of an input command in impedance-controlled device. The users move the device, the mechanism's joint is detected and based; thereon the mechanism's ↪ TCP position is derived by Eq. (8.5).

Inverse Kinematic Problem

In the opposite direction, the transformation from ↪ TCP coordinates is given by the inverse kinematic problem

$$\mathbf{q} = f^{-1}(\mathbf{x}). \tag{8.6}$$

Equation (8.6) is used to determine required actor positions with respect to a desired ↪ TCP position. Thus, it is the essential equation in robotic positioning tasks. In admittance-controlled displays, it is used to calculate the required evasive movement in order to regulate a desired contact force between the user and the haptic interface.

The procedure of calculating the inverse kinematic problem can be split into the following three steps:

1. Formulation of closed vector chains for each leg, starting at the reference coordinate system enclosing the ↪ TCP coordinate system and going back to the reference coordinate system.
2. Splitting the vector chains in all—Cartesian—movement directions of the individual leg.
3. Solving the resulting system of equations according to the ↪ TCP coordinates.

In the design process, inverse kinematic problem the inverse kinematic problem can be used to derive the haptic interface's workspace, the space in which the user can operate the haptic device. This can be done using the fact that a point **x** is within the workspace if it yields a real solution for Eq. (8.6).

JACOBIan Matrix

In both the direct and inverse kinematic problems, the vectors **x** and **q** are linked via the mechanism's gearing properties. These properties are represented by the JACOBIan matrix **J**. For the mechanism's kinematics, the JACOBIan matrix represents the transmission matrix of the first order. It carries all information regarding dimensions and transmission properties. **J** is defined by the partial derivative of the direct kinematic problem Eq. (8.5) with respect to the actor or joint coordinates **q**.

From a mathematical perspective, the transformation is a mapping of the differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $n = 1 \ldots 6$, $m = 1 \ldots 6$ via a $n \times m$ matrix. It is

$$\mathbf{J}\left(\mathbf{q}\right) = \frac{\partial f}{\partial \mathbf{q}^{\mathrm{T}}} = \begin{pmatrix} \frac{\partial f_1}{\partial q_1} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \cdots & \frac{\partial f_m}{\partial q_n} \end{pmatrix}. \tag{8.7}$$

By its derivatives, the JACOBIan matrix gives correlation of speeds between actor and ↪ TCP coordinates.

Using the JACOBIan matrix the direct kinematic problem is expressed as

$$\mathrm{d}\mathbf{x} = \mathbf{J} \cdot \mathrm{d}\mathbf{q} \tag{8.8}$$

where the inverse kinematic problem is given by

$$\mathrm{d}\mathbf{q} = \mathbf{J}^{-1} \cdot \mathrm{d}\mathbf{x}. \tag{8.9}$$

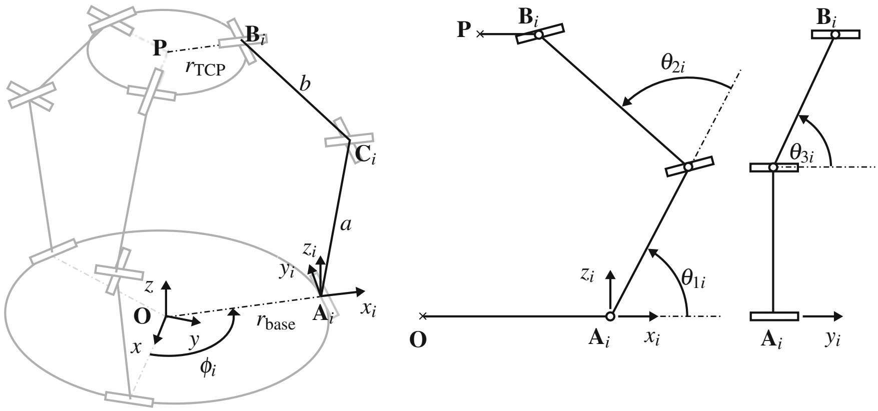


Fig. 8.9 Geometry of the DELTA mechanism (based on [9])

For the design and operation of haptic interfaces a third equation of high importance is the transformation of forces and torques by a mechanism. Again the JACOBIan matrix can be used. To display a force or torque vector **F** the required actor forces and torques $\boldsymbol{\tau}$ are given by

$$\boldsymbol{\tau} = \mathbf{J}^{\mathrm{T}} \cdot \mathbf{F}. \tag{8.10}$$

8.3.2 Example: The DELTA Mechanism

Figure 8.9 shows the dimensions and angles necessary to derive the mechanism's kinematic equations. It is desired to express all these equations with respect to the world coordinate system (WKS) in the middle of the base platform.

The x axis points toward the first leg. By means of simplification, we introduce a local coordinate system (x_i, y_i, z_i) in the start point $\mathbf{A}_i$ of the ith leg. The local coordinate system is rotated by $\phi_i = (i-1) \cdot 120$, $i = 1, 2, 3$ with respect to the WKS.

8.3.2.1 Direct Kinematic Problem

As mentioned above, the direct kinematic problem in general cannot be solved for parallel kinematic mechanisms. In case of the DELTA mechanism, it is different. Here the method of trilateration can be applied. This approach is based on the fact that—if looking at one leg—all points $\mathbf{B}_i$ are on the surface of a sphere with radius b and the center point $\mathbf{C}_i$. The surface is given by the sphere equation

$$\left(x - x_{C_i}\right)^2 + \left(y - y_{C_i}\right)^2 + \left(z - z_{C_i}\right)^2 = b^2 \tag{8.11}$$

with the center coordinates $\left(x_{C_i}, y_{C_i}, z_{C_i}\right)$ of the sphere. Assuming a leg's start point $\mathbf{A}_i'$ not in the distance r_{Basis} from the basis' origin but with the distance $(r_{\text{base}} - r_{\text{TCP}})$, all sphere surfaces of the three legs intersect in the point $\mathbf{P}$. With respect to the WKS, the assumed center point $\mathbf{C}_i'$ of the sphere is at

$$\mathbf{C}_i' = \begin{pmatrix} \cos(-\phi_i) & \sin(-\phi_i) & 0 \\ -\sin(-\phi_i) & \cos(-\phi_i) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \left[\begin{pmatrix} a \cdot \cos\theta_{1i} \\ 0 \\ a \cdot \sin\theta_{1i} \end{pmatrix} + \begin{pmatrix} r_{\text{base}} - r_{\text{TCP}} \\ 0 \\ 0 \end{pmatrix} \right] \tag{8.12}$$

The position of the ↪ TCP center point $\mathbf{P} = (x_P \; y_P \; z_P)^{\mathrm{T}}$, which is the solution to the direct kinematic problem, can be derived by the solution of the three sphere equations with the center points $\mathbf{C}_i'$

$$\left(x_P - x_{C_i'}\right)^2 + \left(y_P - y_{C_i'}\right)^2 + \left(z_P - z_{C_i'}\right)^2 = b^2 \tag{8.13}$$

Since the lower rod in point $\mathbf{A}_i'$, respectively, $\mathbf{A}_i$ rotates solely around the axis x_i axis, point $\mathbf{C}_i'$ is always within the x_i/z_i plane and for the y coordinate of $\mathbf{C}_i'$ one can write

$$y_{C_i'} = 0. \tag{8.14}$$

With respect to the WKS, the assumed center of the assumed sphere $\mathbf{C}_i'$ hence is given as

$$\mathbf{C}_i' = \underbrace{\begin{pmatrix} \cos(-\phi_i) & \sin(-\phi_i) & 0 \\ -\sin(-\phi_i) & \cos(-\phi_i) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotational matrix}} \cdot \left[\begin{pmatrix} a \cdot \cos\theta_{1i} \\ 0 \\ a \cdot \sin\theta_{1i} \end{pmatrix} + \begin{pmatrix} r_{\text{base}} - r_{\text{TCP}} \\ 0 \\ 0 \end{pmatrix} \right] \tag{8.15}$$

The rotational matrix maps the coordinates of the local coordinate system in $\mathbf{A}_i$ respectively $\mathbf{A}_i'$ into the WKS. The rotation is done clockwise by $-\phi_i$.

This means that the equation system of the three sphere equations has to be solved:

$$\left(x_P - x_{C_1'}\right)^2 + \left(y_P - y_{C_1'}\right)^2 + \left(z_P - z_{C_1'}\right)^2 = b^2 \tag{8.16}$$

$$\left(x_P - x_{C_2'}\right)^2 + \left(y_P - y_{C_2'}\right)^2 + \left(z_P - z_{C_2'}\right)^2 = b^2 \tag{8.17}$$

$$\left(x_P - x_{C_3'}\right)^2 + \left(y_P - y_{C_3'}\right)^2 + \left(z_P - z_{C_3'}\right)^2 = b^2 \tag{8.18}$$

from (8.16)–(8.17) we get

$$x_P\left(-2x_{C_1'}+2x_{C_2'}\right)+y_P\cdot 2y_{C_2'}+z_P\left(-2z_{C_1'}+2z_{C_2'}\right)$$
$$=-x_{C_1'}^2+x_{C_2'}^2+y_{C_2'}^2-z_{C_1'}^2+z_{C_2'}^2, \tag{8.19}$$

from (8.16)–(8.18) we get

$$x_P\left(-2x_{C_1'}+2x_{C_3'}\right)+y_P\cdot 2y_{C_3'}+z_P\left(-2z_{C_1'}+2z_{C_3'}\right)$$
$$=-x_{C_1'}^2+x_{C_3'}^2+y_{C_3'}^2-z_{C_1'}^2+z_{C_3'}^2 \tag{8.20}$$

and from (8.17)–(8.18) we get

$$x_P\left(-2x_{C_2'}+2x_{C_3'}\right)+y_P\left(-2y_{C_2'}+2y_{C_3'}\right)+z_P\left(-2z_{C_2'}+2z_{C_3'}\right)$$
$$=-x_{C_2'}^2+x_{C_3'}^2-y_{C_2'}^2+y_{C_3'}^2-z_{C_2'}^2+z_{C_3'}^2 \tag{8.21}$$

The solution yields to two points of intersection of the spheres, whereas only one solution is geometrically meaningful. The calculation of $\mathbf{P}=(x_P\ y_P\ z_P)^{\mathrm{T}}$ should be computer-assisted , e.g., by using Mathematica®.

Inverse Kinematic Problem

The DELTA mechanism is especially known from impedance-controlled devices. In the mode of operation, the inverse kinematic problem is not needed. However, it is a useful tool in the design process to determine the available workspace which is shown later. Furthermore, it provides an effective way to determine the JACOBIan matrix of DELTA mechanism.

As mentioned above a standard approach to determine a mechanism's inverse kinematic is using closed vector chains. In the case on hand, this can be done via

$$\overrightarrow{\mathbf{OP}}+\overrightarrow{\mathbf{PB}_i}=\overrightarrow{\mathbf{OA}_i}+\overrightarrow{\mathbf{A}_i\mathbf{C}_i}+\overrightarrow{\mathbf{C}_i\mathbf{B}_i}. \tag{8.22}$$

This leads to the coordinates of the point $\mathbf{B}_i$ with

$$\begin{pmatrix} x_{B_i}\\ y_{B_i}\\ z_{B_i}\end{pmatrix}=\begin{pmatrix} a\cdot\cos\theta_{1i}+b\cdot\sin\theta_{3i}\cdot\cos\left(\theta_{1i}+\theta_{2i}\right)\\ b\cdot\cos\theta_{3i}\\ a\cdot\sin\theta_{1i}+b\cdot\sin\theta_{3i}\cdot\sin\left(\theta_{1i}+\theta_{2i}\right)\end{pmatrix} \tag{8.23}$$

Since we want to derive the mechanism's base angles with respect to the ↪ TCP position in point $\mathbf{P}$, we can use the relation between $\mathbf{P}$ and $\mathbf{B}_i$ which is given by

$$\begin{pmatrix} x_{B_i}\\ y_{B_i}\\ z_{B_i}\end{pmatrix}=\begin{pmatrix} \cos\phi_i & \sin\phi_i & 0\\ -\sin\phi_i & \cos\phi_i & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix} x_P\\ y_P\\ z_P\end{pmatrix}+\begin{pmatrix} r_{\mathrm{TCP}}-r_{\mathrm{base}}\\ 0\\ 0\end{pmatrix}. \tag{8.24}$$

By solving Eq. (8.23), we can determine the solution for the inverse kinmatic equations as given by TSAI [9] as

$$\theta_{3i} = \arccos \frac{y_{B_i}}{b} \tag{8.25}$$
$$\theta_{2i} = \arccos \frac{x_{B_i}^2 + y_{B_i}^2 + z_{B_i}^2 - a^2 - b^2}{2ab \sin \theta_{3i}} \tag{8.26}$$
$$\theta_{1i} = \arctan \frac{x_{B_i} - b \sin \theta_{3i} \cos(\theta_{1i} + \theta_{2i})}{z_{B_i} - b \sin \theta_{3i} \sin(\theta_{1i} + \theta_{2i})} \tag{8.27}$$

Especially Eq. (8.25) gives the angles of joints attached to the base platform with respect to the ↪ TCP position, and therefore, the solution for the inverse kinematic problem. Furthermore, the inverse JACOBIan matrix

$$\mathbf{J}^{-1} = \begin{pmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{pmatrix} \tag{8.28}$$

comprises the matrix elements [9]

$$j_{i1} = \frac{\cos(\theta_{1i} + \theta_{2i}) \sin \theta_{3i} \cos \phi_i - \cos \theta_{3i} \sin \phi_i}{a \sin \theta_{2i} \sin \theta_{3i}} \tag{8.29}$$
$$j_{i2} = \frac{\cos(\theta_{1i} + \theta_{2i}) \sin \theta_{3i} \sin \phi_i + \cos \theta_{3i} \sin \phi_i}{a \sin \theta_{2i} \sin \theta_{3i}} \tag{8.30}$$
$$j_{i3} = \frac{\sin(\theta_{1i} + \theta_{2i})}{a \sin \theta_{2i}}. \tag{8.31}$$

This closed-form solution provides an effective way to calculate the DELTA mechanism's JACOBIan matrix during operation.

8.4 Design Step 3: Dimensioning

The design step of dimensioning covers the optimization and determination of all designable lengths and angles within a topology that has been defined in step 1. Since especially the dimensioning of parallel kinematic mechanisms is a rather complex procedure, the following section focuses on this class of mechanisms.

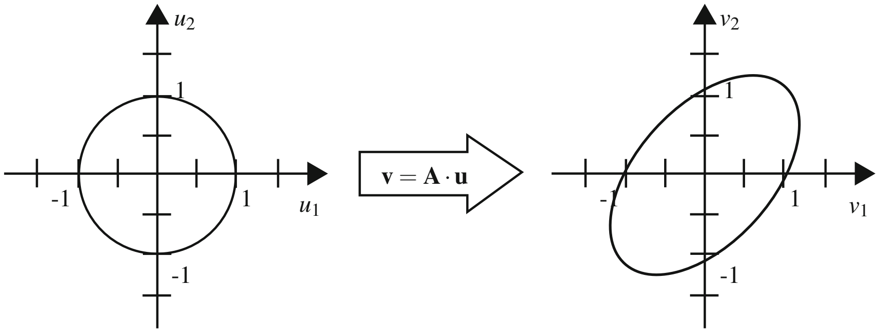


Fig. 8.10 Linear mapping of a vector, Example based on [5]

8.4.1 Isotropy and Singular Positions

As discussed in Sect. 8.1, parallel kinematic has a rather complex transmission behavior. Especially two effects have to be taken into account: isotropy and singular positions.

The key to analyze these effects in the design process is again based on properties of the JACOBIan matrix. A key performance index which is derived from the JACOBIan matrix properties is the condition number κ. It is introduced in the following section.

The Conditioning Number

The kinematic transmission behavior is rated by the singular values σ_i of the inverse JACOBIan matrix $\mathbf{J}^{-1}$. In general the singular values of a matrix $\mathbf{A}$ are defined as

$$\sigma_i\left(\mathbf{A}\right) = \sqrt{\lambda_i\left(\mathbf{A}^\mathrm{T}\mathbf{A}\right)}. \tag{8.32}$$

They are a measure for the distortion of the general linear projection of the vector $\mathbf{u}$ to $\mathbf{v}$ via

$$\mathbf{v} = \mathbf{A} \cdot \mathbf{u} \tag{8.33}$$

Figure 8.10 shows an example for this kind of distortion in a two-dimensional projection.

The role of the singular values can be shown by GOLUB's method of singular value decomposition. It is based on the fact that each complex $m \times n$-matrix $\mathbf{A}$ with the rank r can be fractioned in the product

$$\mathbf{A} = \mathbf{U} \cdot \mathbf{\Sigma} \cdot \mathbf{V}^* \tag{8.34}$$

of the unitary $m \times m$-matrix $\mathbf{U}$ and the adjoint matrix $\mathbf{V}^*$ of the $n \times n$-matrix $\mathbf{V}$. $\mathbf{\Sigma}$ is a $m \times n$- diagonal matrix with

$$\boldsymbol{\Sigma} = \left(\begin{array}{ccc|c} \sigma_1 & & & \vdots \\ & \ddots & & \cdots 0 \cdots \\ & & \sigma_r & \vdots \\ \hline & \vdots & & \\ \cdots & 0 & \cdots & \cdots 0 \cdots \\ & \vdots & & \vdots \end{array}\right) \tag{8.35}$$

and $\sigma_1 \geq \cdots \geq \sigma_r > 0$. In the linear projection, **U** and **V*** act as rotations and $\boldsymbol{\Sigma}$ as elongation and compression of the ellipse from Fig. 8.10. $\sigma_{\min}$ and $\sigma_{\max}$ quantity the minimal and maximal amplifications of the vector **u**.

A measure to rate the distortion is the conditioning number

$$\kappa = \frac{\sigma_{\max}\left(\mathbf{J}^{-1}\right)}{\sigma_{\min}\left(\mathbf{J}^{-1}\right)} \tag{8.36}$$

as the ratio of the two maximal singular values $\sigma_{\max}$ and $\sigma_{\min}$ of $\mathbf{J}^{-1}$. Thus, the conditioning number κ is a measure for the equal amplification **u** in all spatial directions. As a function of the JACOBIan matrix κ changes with respect to the mechanism's position. The conditioning number can reach values from $\frac{1}{\kappa} = 0 \ldots 1$.

Isotropic Transmission and Singular Positions

The goal of kinematic design is a highly isotropic transmission. From the distortion properties of singular values, the design target of

$$\frac{1}{\kappa} = 1 \tag{8.37}$$

for an isotropic transmission can be derived. On the other hand, one has to avoid singular behavior with

$$\frac{1}{\kappa} = 0. \tag{8.38}$$

In singular positions, the rank of the JACOBIan matrix decreases. This means that the transformation equations are no longer independent of each other. Practically, this leads to the loss of one or several controllable degrees of freedom.

For the two introduced kinds of singular positions (see Fig. 8.4), the loss of rank is characterized as

1. Singularity of the first kind: $\det(\mathbf{J}) = 0$
2. Singularity of the second kind: $\det\left(\mathbf{J}^{-1}\right) = 0$

For all conclusions drawn from the JACOBIan matrix, one has to take care of the used definition. In this book, we use the definition as in Eq. (8.7), but also the inverse definition

$$\mathbf{J}_{\text{alternative}}(\mathbf{x}) = \begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \cdots & \frac{\partial q_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial q_m}{\partial x_1} & \cdots & \frac{\partial q_m}{\partial x_n} \end{pmatrix} = \mathbf{J}^{-1} \tag{8.39}$$

is possible and used in the literature. However, given the fact that for singular values of a matrix $\mathbf{A}$ applies

$$\sigma(\mathbf{A}) = \frac{1}{\sigma(\mathbf{A}^{-1})} \tag{8.40}$$

the derived conditions from the JACOBIan matrix can be transferred into each other.

Transmission of Force and Speed

Besides the analysis of isotropy the second aspect one has to take care of in the design process is the transmission of force and speed.

1. *Transmission of force*: To limit the maximal required force and torque and thereby limit also the size of the used actuators, it is important reach a good transmission of forces and torques even in cases of a disadvantageous scaling σ_i. From the transposed Eq. (8.10)

$$\mathbf{F} = \mathbf{J}^{\text{-T}} \cdot \boldsymbol{\tau} \tag{8.41}$$

we can derive the criteria

$$\sigma_{\min}\left(\mathbf{J}^{\text{-T}}\right) \rightarrow \max \tag{8.42}$$

and with $\sigma(\mathbf{A}) = \sigma\left(\mathbf{A}^{\text{T}}\right)$ we get

$$\sigma_{\min}\left(\mathbf{J}^{-1}\right) \rightarrow \max \tag{8.43}$$

as a design criterion for the force transmission.

Therefore, we have to maximize the speed transmission for the most disadvantageous spatial direction. With the JACOBIan matrix, a measure for speed transmission

$$\dot{\mathbf{x}} = \mathbf{J} \cdot \dot{\mathbf{q}} \tag{8.44}$$

we accordingly get

$$\sigma_{\min}(\mathbf{J}) \rightarrow \max \tag{8.45}$$

Table 8.3 Summarization of the experiments

Design aspect	Criterion
Force transmission	$\sigma_{\min}(\mathbf{J}) \to \max$
No singular positions	$\sigma_{\min}(\mathbf{J}) \to \max$
High stiffness	$\sigma_{\min}(\mathbf{J}) \to \max$
Speed transmission	$\sigma_{\max}(\mathbf{J}) \to \min$
Isotropy	$\frac{\sigma_{\min}(\mathbf{J})}{\sigma_{\max}(\mathbf{J})} \to \max$

as a design goal. Analogous to the design goal for the force transmission with Eq. (8.40) from

$$\begin{aligned}\sigma_{\min}(\mathbf{J}) &= \min\{\sigma_1(\mathbf{J}), \ldots, \sigma_r(\mathbf{J})\}\\ &= \frac{1}{\max\left\{\frac{1}{\sigma_1(\mathbf{J})}, \ldots, \frac{1}{\sigma_r(\mathbf{J})}\right\}}\\ &= \frac{1}{\max\left\{\sigma_1\left(\mathbf{J}^{-1}\right), \ldots, \sigma_r\left(\mathbf{J}^{-1}\right)\right\}}\\ &= \frac{1}{\sigma_{\max}\left(\mathbf{J}^{-1}\right)}\end{aligned} \tag{8.46}$$

we can derive the criterion

$$\sigma_{\max}\left(\mathbf{J}^{-1}\right) \to \min. \tag{8.47}$$

Table 8.3 sums up the various design aspects.

Looking at the definition of the conditioning number in Eq. (8.36), we see that the requirements for $\sigma_{\min}\left(\mathbf{J}^{-1}\right) \to \max$ are $\sigma_{\max}\left(\mathbf{J}^{-1}\right) \to \min$ are covered by one single value. Therein both requirements are weighted equally. Thus, the conditioning number $1/\kappa$ covers the evaluation of isotropy and of force and torque, respectively, speed at the same time.

One major drawback of Eq. (8.36) is that it rates the mechanism for JACOBIan matrix or position. The pure optimization of $1/\kappa$ would in fact lead to one single position where the mechanism reaches high isotropy. However, one cannot draw the conclusion that the whole workspace in total has an optimized transmission behavior.

What is needed is a measure to rate $1/\kappa$ of a whole workspace. This measure is provided by the global conditioning index (e.g., MERLET [7])

$$\nu = \frac{\int_W \frac{1}{\kappa} \mathrm{d}W}{\int_W \mathrm{d}W}. \tag{8.48}$$

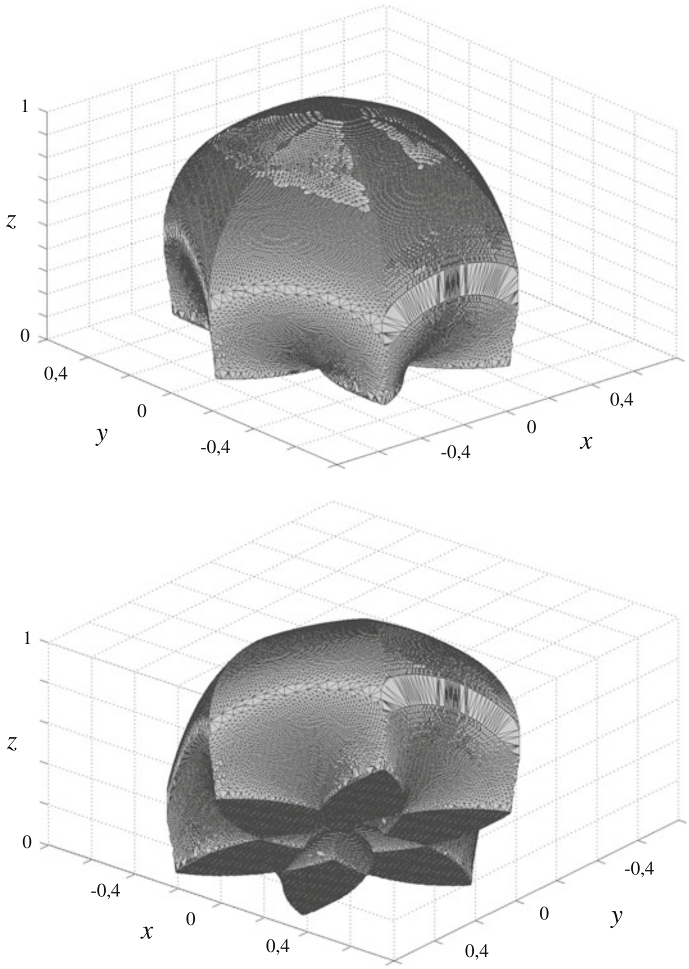


Fig. 8.11 Hull of DELTA's workspace from two angles

In a computer-assisted algorithm, it can be programmed in discrete form as

$$\nu = \frac{\mathrm{d}W \cdot \sum_n \frac{1}{\kappa}}{n \cdot \mathrm{d}W} \tag{8.49}$$

with n as the number of sampling points in the workspace and dW the size of a discretized voxel.

8.4.2 Example: The DELTA Mechanism

The parameters to be designed are the length of the rods a and b and the radii of the $\hookrightarrow$ TCP platform r_{TCP} and of the base platform r_{base}. The design goal is a mechanism with a workspace of suitable size and isotropic transmission behavior.

This is done by an algorithm which is executed in two steps for each set of $a, b, r_{\mathrm{TCP}}, r_{\mathrm{base}}$

1. determine the workspace,
2. evaluate the global conditioning index.

The workspace is defined by the set of possible $\hookrightarrow$ TCP positions. As mentioned above, a point is within this set if a real solution for the inverse kinematic problem exists. Using this criterion, the workspace can be determined using an algorithm that solves the inverse kinematic equations (8.25), (8.26), (8.27) pointwise in space. Simultaneously, the size of resulting workspace is determined by the number of points or voxels which fulfills the inverse kinematic equation. The shape of the DELTA's workspace obtained by this method is shown in Fig. 8.11.

To rate the mechanism's isotropy in each workspace, we use the global conditioning index as in Eq. (8.49). Figure 8.12 shows ν, calculated for rod lengths $a = 0.2 \ldots 0.8$. Cubic spaces have been discretized with voxels of $\mathrm{d}W = 0.001$. The result shows a global maximum at $a = 0.46$ and $b = 0.54$, respectively. This ratio yields a mechanism with a maximal global conditioning index.

A similar analysis shows that the variations of r_{TCP} and r_{base} have rather small influence on the global conditioning index and regarding the size of the workspace $r_{\mathrm{TCP}} = r_{\mathrm{base}} = 1$ turns out to be a good choice.

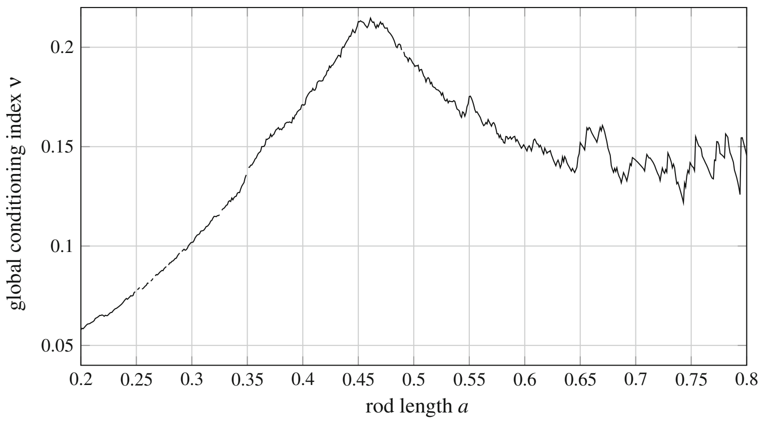
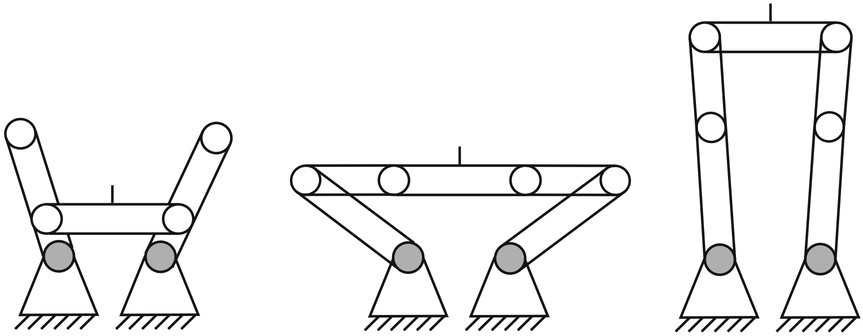


Fig. 8.12 Global conditioning index ν for varied rod lengths a and $b = 1 - a$ with $r_{\mathrm{TCP}} = r_{\mathrm{base}} = 1$ (all lengths without dimensions)



1. singular posittion of the first kind

singular position of the second kind

2. singular position of the first kind

Fig. 8.13 Singular positions

As mentioned in the first part of the example in Sect. 8.2.4, we have to keep in mind that the RUU/DELTA mechanism is a T_5'' mechanism having singular positions within the workspace. Moving the ↪ TCP through its workspace this leads to the singular positions as shown in Fig. 8.13. Since a mechanism cannot cross singular positions—or in the case of a haptic device provide a feedback in singular positions—the workspace is divided by the second kind of singular position. The device is only operated in one part of the theoretically available workspace. In the shown case, this would be in the part above the second kind of singular position.

Recommended Background Reading

[6] Kong, X. & Gosselin, C.: **Synthesis of Parallel Mechanisms**. Springer, Germany, 2007.
Comprehensive description of the design of parallel kinematic mechanisms.

[7] Merlet, J.P.: **Parallel robots**. Springer, Netherlands, 2009.
Standard textbook on parallel mechanisms with a great variety of examples.

[8] Tsai, L.: **Robot analysis: the mechanics of serial and parallel manipulators**. John Wiley & Sons, Netherlands, 1999.
Standard textbook on parallel and serial mechanisms with deep discussion of mathematical backgrounds.

References

1. Carricato M, Parenti-Castelli V (2002) Singularity-free fully-isotropic translational parallel mechanisms. Int J Robot Res 21(2):161–174. doi:10.1177/027836402760475360

2. Carricato M, Parenti-Castelli V (2003) A family of 3-DOF translational parallel manipulators. J Mech Des 125:302. doi:10.1115/1.1563635
3. Clavel R (1988) DELTA, a fast robot with parallel geometry. In: 18th international symposium on industrial robots. pp 91–100
4. Husty M et al (1997) Kinematik und Robotik. Springer, Berlin. ISBN: 978-3-642-59029-0
5. Kirchner J (2001) Mehrkriterielle Optimierung von Parallelkinematiken. Dissertation. Technische Universität Chemnitz. ISBN: 3928921673
6. Kong X, Gosselin C (2007) Type synthesis of parallel mechanisms, vol 33. Springer Tracts in Advanced Robotics, Berlin
7. Merlet J (2006) Parallel robots. Springer, New York. ISBN 978-1402003851
8. Tsai L-W (1999) Robot analysis: the mechanics of serial and parallel manipulators. Wiley, New York. ISBN: 978-0471325932
9. Tsai L, Joshi S (2002) Kinematic analysis of 3-DOF position mechanisms for use in hybrid kinematic machines. J Mech Des 124:245–253. doi:10.1115/1.1468860