Chapter 6
Nonlinear Expectations and g-Expectations

Abstract We investigate nonlinear expectations. We briefly discuss Choquet expectations and we focus on g-expectations defined by BSDEs. The connection between filtration-consistent nonlinear expectations and g-expectations is presented. We study the properties of translation invariance, positive homogeneity, convexity and sub-linearity of g-expectations and we show that these properties are determined by the generator of the BSDE defining the g-expectation.

The original motivation for studying nonlinear expectations comes from the theory of decision making. The Allais paradox proved that the linear expectation (the expected value operator) might fail in an attempt to describe choices made by decision makers and the Ellsberg's paradox disqualified the notion of linear probability in representing beliefs of decision makers. It was shown that decisions made in the real world contradicted optimal decisions based on additive probabilities and the expected utility theory. Consequently, economists and mathematicians begun to look for a new notion of expectation.

A nonlinear expectation is an operator which preserves all essential properties of the standard expected value operator except linearity. In this chapter we focus on nonlinear expectations called g-expectations which are defined by BSDEs. In Chap. 13 we use g-expectations to define dynamic risk measures which can be used for actuarial and financial valuation.

6.1 Choquet Expectations

Before we study g-expectations, we briefly discuss Choquet expectations. It is well-known that the expected value can be calculated by the formula

$$\mathbb{E}[\xi] = \int_{-\infty}^{0} \big(\Pr(\xi \geq x) - 1\big)dx + \int_{0}^{\infty} \Pr(\xi \geq x)dx. \qquad (6.1)$$

The idea by Choquet (1953) was to replace an additive probability measure $\Pr(\cdot)$ with a non-additive capacity measure $V(\cdot)$. We can define a nonlinear operator in

the following way

$$C[\xi] = \int_{-\infty}^{0} \big(V(\xi \geq x) - 1\big)dx + \int_{0}^{\infty} V(\xi \geq x)dx. \qquad (6.2)$$

The nonlinear operator (6.2) is called the Choquet expectation or the Choquet integral.

The key example of a non-additive capacity measure, often applied in insurance and finance, is a distorted probability. We can define a non-additive capacity measure by distorting the original probability

$$V(\xi \geq x) = \Psi\big(\mathrm{Pr}(\xi \geq x)\big), \qquad (6.3)$$

where we choose a nonlinear function $\Psi : [0, 1] \rightarrow [0, 1]$ such that $\Psi(0) = 0$, $\Psi(1) = 1$ and $x \mapsto \Psi(x)$ is non-decreasing. The function Ψ is called a distortion. The Wang transform is an important distortion function used for actuarial and financial applications, see Wang (2000). We remark that Value-at-Risk and Tail-Value-at-Risk are examples of the Choquet expectations derived under distorted probabilities.

The idea behind the Choquet expectation and distorted probabilities is clear. Consequently, Choquet expectations have found numerous applications in insurance and financial mathematics. Unfortunately, it is very difficult to define a dynamic version of a Choquet expectation. It turns out that BSDEs can be very useful for defining dynamic nonlinear expectations.

6.2 Filtration-Consistent Nonlinear Expectations and g-Expectations

We define a nonlinear expectation and a filtration-consistent nonlinear expectation, see Coquet et al. (2002).

Definition 6.2.1 A functional $\mathscr{E} : \mathbb{L}^2(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R}) \rightarrow \mathbb{R}$ is called a nonlinear expectation if it satisfies

 (i) the property of strict monotonicity:

$$\xi_1 \geq \xi_2 \quad \Rightarrow \quad \mathscr{E}[\xi_1] \geq \mathscr{E}[\xi_2],$$
$$\xi_1 \geq \xi_2 \quad \text{and} \quad \mathscr{E}[\xi_1] = \mathscr{E}[\xi_2] \quad \Leftrightarrow \quad \xi_1 = \xi_2,$$

for all $\xi_1, \xi_2 \in \mathbb{L}^2(\mathbb{R})$,
(ii) the invariance property:

$$\mathscr{E}[c] = c, \quad \text{for all } c \in \mathbb{R}.$$

Definition 6.2.2 A nonlinear expectation $\mathscr{E}$ is called an $\mathscr{F}$-consistent nonlinear expectation if for each $\xi \in \mathbb{L}^2(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R})$ and $t \in [0, T]$ there exists a random variable $\zeta \in \mathbb{L}^2(\Omega, \mathscr{F}_t, \mathbb{P}; \mathbb{R})$ such that

$$\mathscr{E}[\xi \mathbf{1}_A] = \mathscr{E}[\zeta \mathbf{1}_A], \quad A \in \mathscr{F}_t.$$

We remark that ζ is uniquely determined, see Lemma 3.1 in Coquet et al. (2002). The random variable ζ is denoted by $\mathscr{E}[\xi | \mathscr{F}_t]$. Notice that the definition of an $\mathscr{F}$-consistent nonlinear expectation is analogous to the definition of the standard linear conditional expectation but in Definition 6.2.2 the expectation is taken under a nonlinear operator.

We state properties of $\mathscr{F}$-consistent nonlinear expectations, see Coquet et al. (2002).

Proposition 6.2.1 *Let $\mathscr{E}$ be an $\mathscr{F}$-consistent nonlinear expectation. The following properties hold*:

(a) $\mathscr{E}[\mathscr{E}[\xi | \mathscr{F}_t] | \mathscr{F}_s] = \mathscr{E}[\xi | \mathscr{F}_s]$ *for all* $0 \le s \le t \le T$.
(b) $\mathscr{E}[\xi \mathbf{1}_A | \mathscr{F}_t] = \mathbf{1}_A \mathscr{E}[\xi | \mathscr{F}_t]$ *for all* $0 \le t \le T$ *and* $A \in \mathscr{F}_t$.
(c) *If* $\xi_1 \ge \xi_2$, *then* $\mathscr{E}[\xi_1 | \mathscr{F}_t] \ge \mathscr{E}[\xi_2 | \mathscr{F}_t]$ *for all* $0 \le t \le T$. *In addition, if* $\mathscr{E}[\xi_1 | \mathscr{F}_t] = \mathscr{E}[\xi_2 | \mathscr{F}_t]$ *a.s. for some* $t \in [0, T]$, *then* $\xi_1 = \xi_2$.

Proposition 6.2.1 shows that all essential properties of the standard linear conditional expectation, except linearity, are preserved under the notion of an $\mathscr{F}$-consistent nonlinear expectation.

From the modelling point of view, we should be able to generate $\mathscr{F}$-consistent nonlinear expectations in a feasible way. The next example shows one possible way of generating $\mathscr{F}$-consistent nonlinear expectations.

Example 6.1 Choose a continuous, strictly increasing function $\varphi : \mathbb{R} \to \mathbb{R}$. The operator

$$\mathscr{E}[\xi | \mathscr{F}_t] = \varphi^{-1}\big(\mathbb{E}\big[\varphi(\xi) | \mathscr{F}_t\big]\big), \quad 0 \le t \le T, \tag{6.4}$$

is an $\mathscr{F}$-consistent nonlinear expectation. The nonlinear expectation (6.4) can be interpreted as the indifference price of ξ determined by an agent with utility φ, see Royer (2006).

It turns out that $\mathscr{F}$-consistent nonlinear expectations can be defined by nonlinear BSDEs. In this chapter we study the BSDEs

$$Y(t) = \xi + \int_t^T g\big(s, Y(s), Z(s), U(s, .)\big) ds$$

$$- \int_t^T Z(s) dW(s) - \int_t^T \int_{\mathbb{R}} U(s, z) \tilde{N}(ds, dz), \quad 0 \le t \le T. \tag{6.5}$$

By a nonlinear BSDE we mean a BSDE with a nonlinear generator g.

Definition 6.2.3 Consider $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times L_Q^2(\mathbb{R}) \to \mathbb{R}$ such that

(i) g satisfies (A2) from Chap. 3,
(ii) g satisfies the inequality

$$g(\omega, t, y, z, u) - g(\omega, t, y, z, u')$$

$$\leq \int_{\mathbb{R}} \delta^{y,z,u,u'}(t, x)\big(u(x) - u'(x)\big)Q(t, dx)\eta(t),$$

a.s., a.e. $(\omega, t) \in \Omega \times [0, T]$, for all $(y, z, u), (y, z, u') \in \mathbb{R} \times \mathbb{R} \times L_Q^2(\mathbb{R})$, where $\delta^{y,z,u,u'} : \Omega \times [0, T] \times \mathbb{R} \to (-1, \infty)$ is a predictable process such that the mapping $t \mapsto \int_{\mathbb{R}} |\delta^{y,z,u,u'}(t, x)|^2 Q(t, dx)\eta(t)$ is uniformly bounded in (y, z, u, u'),

(iii) $g(t, y, 0, 0) = 0$ for all $(t, y) \in [0, T] \times \mathbb{R}$.

(a) We define the g-expectation $\mathscr{E}_g : \mathbb{L}^2(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R}) \mapsto \mathbb{R}$ by

$$\mathscr{E}_g[\xi] = Y(0),$$

where $Y(0)$ is the unique solution to the BSDE (6.5) with the generator g satisfying (i)–(iii) and the terminal condition $\xi \in \mathbb{L}^2(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R})$.

(b) We define the conditional g-expectation $\mathscr{E}_g : \mathbb{L}^2(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R}) \to \mathbb{L}^2(\Omega, \mathscr{F}_t, \mathbb{P}; \mathbb{R})$ by

$$\mathscr{E}_g[\xi|\mathscr{F}_t] = Y(t), \quad 0 \leq t \leq T,$$

where $Y(t)$ is the unique solution to the BSDE (6.5) with the generator g satisfying (i)–(iii) and the terminal condition $\xi \in \mathbb{L}^2(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R})$.

Notice that for the g-expectation its dynamic version is naturally defined. We state the first key result of this chapter.

Theorem 6.2.1 *The g-expectation $\mathscr{E}_g$ is an $\mathscr{F}$-consistent nonlinear expectation.*

Proof The strict monotonicity of $\mathscr{E}_g$ follows from the comparison principle established in Theorem 3.2.2. Since $g(t, y, 0, 0) = 0$, we can choose $Y = c, Z = U = 0$ as the unique solution to the BSDE (6.5) with $\xi = c$. Hence, the invariance property of $\mathscr{E}_g$ holds. We now prove the $\mathscr{F}$-consistency of $\mathscr{E}_g$. Choose $t \in [0, T]$ and $A \in \mathscr{F}_t$. We investigate

$$Y(0) = \mathscr{E}_g[\xi \mathbf{1}_A], \qquad Y'(0) = \mathscr{E}_g\big[Y(t)\mathbf{1}_A\big],$$

where Y and Y' denote the unique solutions to the BSDEs

$$Y(u) = \xi \mathbf{1}_A + \int_u^T g\big(s, Y(s), Z(s), U(s)\big)ds$$

$$- \int_u^T Z(s)dW(s) - \int_u^T \int_{\mathbb{R}} U(s, z)\tilde{N}(ds, dz), \quad 0 \leq u \leq T,$$

$$Y'(u) = Y(t)\mathbf{1}_A + \int_u^T g\big(s, Y'(s), Z'(s), U'(s)\big)ds$$

$$- \int_u^T Z'(s)dW(s) - \int_u^T \int_{\mathbb{R}} U'(s, z)\tilde{N}(ds, dz), \quad 0 \le u \le T.$$

Since $g(t, y, 0, 0) = 0$, we can put $Y'(s) = Y(t)\mathbf{1}_A$, $Z'(s) = U'(s, z) = 0$, $(s, z) \in [t, T] \times \mathbb{R}$. Consequently, we obtain the equations

$$Y(u) = Y(t) + \int_u^t g\big(s, Y(s), Z(s), U(s)\big)ds$$

$$- \int_u^t Z(s)dW(s) - \int_u^t \int_{\mathbb{R}} U(s, z)\tilde{N}(ds, dz), \quad 0 \le u \le t,$$

$$Y'(u) = Y(t)\mathbf{1}_A + \int_u^t g\big(s, Y'(s), Z'(s), U'(s)\big)ds$$

$$- \int_u^t Z'(s)dW(s) - \int_u^t \int_{\mathbb{R}} U'(s, z)\tilde{N}(ds, dz), \quad 0 \le u \le t.$$

Consider the BSDE

$$Y''(u) = \xi + \int_u^T g\big(s, Y''(s), Z''(s), U''(s)\big)ds$$

$$- \int_u^T Z''(s)dW(s) - \int_u^T \int_{\mathbb{R}} U''(s, z)\tilde{N}(ds, dz), \quad 0 \le u \le T.$$

Since $g(t, y, 0, 0) = 0$ we can also put $Y(s) = Y''(s)\mathbf{1}_A$, $Z(s) = Z''(s)\mathbf{1}_A$, $U(s, z) = U''(s, z)\mathbf{1}_A$ $(s, z) \in [t, T] \times \mathbb{R}$. Hence, we end up with the equations

$$Y(u) = Y''(t)\mathbf{1}_A + \int_u^t g\big(s, Y(s), Z(s), U(s)\big)ds$$

$$- \int_u^t Z(s)dW(s) - \int_u^t \int_{\mathbb{R}} U(s, z)\tilde{N}(ds, dz), \quad 0 \le u \le t,$$

$$Y'(u) = Y''(t)\mathbf{1}_A + \int_u^t g\big(s, Y'(s), Z'(s), U'(s)\big)ds$$

$$- \int_u^t Z'(s)dW(s) - \int_u^t \int_{\mathbb{R}} U'(s, z)\tilde{N}(ds, dz), \quad 0 \le u \le t.$$

By uniqueness of solutions we finally conclude that $Y(s) = Y'(s)$, $Z(s) = Z'(s)$, $U(s, z) = U'(s, z)$, $(s, z) \in [0, t] \times \mathbb{R}$. Hence, $\mathscr{E}_g$ is a filtration-consistent expectation with the conditional expectation $\mathscr{E}_g[\xi|\mathscr{F}_t] = Y(t)$. $\qquad\square$

Any g-expectation clearly satisfies the properties from Proposition 6.2.1, which can now be derived from properties of BSDEs.

Example 6.2 If we consider a BSDE with zero generator, then the *g*-expectation coincides with the linear conditional expectation. If we consider the BSDE from Proposition 3.3.2 or 3.4.2, then the *g*-expectation is a filtration-consistent nonlinear expectation.

We are now interested in a converse of Theorem 6.2.1. The first results in this field were proved by Coquet et al. (2002) and Rosazza Gianin (2006) for the Brownian filtration. We present the result proved by Royer (2006) for the filtration generated by a Lévy process. First, we introduce two particular types of *g*-expectations.

Proposition 6.2.2 *Consider the natural filtration $\mathscr{F}$ generated by a Lévy process with a Lévy measure ν. For $\alpha > 0$ and $-1 < \beta \leq 0$ we define the generators*

$$g^*_{\alpha,\beta}(t,z,u) = \alpha|z| + \alpha \int_{\mathbb{R}} \big(1 \wedge |x|\big)u^+(x)\nu(dx) - \beta \int_{\mathbb{R}} \big(1 \wedge |x|\big)u^-(x)\nu(dx),$$

$$g^{**}_{\alpha,\beta}(t,z,u) = -\alpha|z| - \alpha \int_{\mathbb{R}} \big(1 \wedge |x|\big)u^-(x)\nu(dx) + \beta \int_{\mathbb{R}} \big(1 \wedge |x|\big)u^+(x)\nu(dx).$$

The corresponding g-expectations have the representations

$$\mathscr{E}_{g^*_{\alpha,\beta}}[\xi|\mathscr{F}_t] = \sup_{\mathbb{Q} \in \mathscr{Q}} \mathbb{E}^{\mathbb{Q}}[\xi|\mathscr{F}_t], \quad 0 \leq t \leq T,$$

$$\mathscr{E}_{g^{**}_{\alpha,\beta}}[\xi|\mathscr{F}_t] = \inf_{\mathbb{Q} \in \mathscr{Q}} \mathbb{E}^{\mathbb{Q}}[\xi|\mathscr{F}_t], \quad 0 \leq t \leq T,$$

where

$$\mathscr{Q} = \left\{ \mathbb{Q} \sim \mathbb{P}, \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathscr{F}_t} = M^{\phi,\kappa}(t), \ 0 \leq t \leq T \right\},$$

$$\frac{dM^{\phi,\kappa}(t)}{M^{\phi,\kappa}(t-)} = \phi(t)dW(t) + \int_{\mathbb{R}} \kappa(t,x)\tilde{N}(dx,dt), \quad M^{\phi,\kappa}(0) = 1,$$

and (ϕ,κ) are $\mathscr{F}$-predictable processes satisfying

$$\big|\phi(t)\big| \leq \alpha, \quad \kappa(t,x) > -1,$$

$$\kappa^+(t,x) \leq \alpha\big(1 \wedge |x|\big), \quad \kappa^-(t,x) \leq -\beta\big(1 \wedge |x|\big), \quad (t,x) \in [0,T] \times \mathbb{R}.$$

Proof The result can be derived by following the arguments from Propositions 3.3.2 and 3.4.2, see also Proposition 3.6 in Royer (2006).   □

We now state the second key result of this chapter, see Theorem 4.6 in Royer (2006).

Theorem 6.2.2 *Consider the natural filtration $\mathscr{F}$ generated by a Lévy process with a Lévy measure ν. Let $\mathscr{E}$ be an $\mathscr{F}$-consistent nonlinear expectation such that*

(i) *for all* $\xi_1, \xi_2 \in \mathbb{L}^2(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R})$

$$\mathscr{E}[\xi_1 + \xi_2] - \mathscr{E}[\xi_1] \leq \mathscr{E}_{g^*_{\alpha,\beta}}[\xi_2], \quad \text{with some } \alpha \geq 0, \ -1 < \beta \leq 0,$$

   where $\mathscr{E}_{g^*_{\alpha,\beta}}$ is the g-expectation defined in Proposition 6.2.2,

(ii) *for all* $\xi_1 \in \mathbb{L}^2(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R})$ *and* $\xi_2 \in \mathbb{L}^2(\Omega, \mathscr{F}_t, \mathbb{P}; \mathbb{R})$

$$\mathscr{E}[\xi_1 + \xi_2 | \mathscr{F}_t] = \mathscr{E}[\xi_1 | \mathscr{F}_t] + \xi_2, \quad 0 \leq t \leq T.$$

Then, there exists a function $g : \Omega \times [0, T] \times \mathbb{R} \times L^2_Q \to \mathbb{R}$ and the g-expectation $\mathscr{E}_g$ such that

$$\mathscr{E}[\xi | \mathscr{F}_t] = \mathscr{E}_g[\xi | \mathscr{F}_t], \quad \xi \in \mathbb{L}^2(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R}), \quad 0 \leq t \leq T.$$

Moreover, the following properties hold:

 (i) g *satisfies* (A2) *from Chap.* 3,
(ii) g *satisfies the inequality*

$$g(\omega, t, z, u) - g(\omega, t, z, u') \leq \int_{\mathbb{R}} \delta^{z,u,u'}(t, x)\big(u(x) - u'(x)\big)\nu(dx),$$

   a.s., a.e. $(\omega, t) \in \Omega \times [0, T]$, for all $(z, u), (z, u') \in \mathbb{R} \times L^2_Q(\mathbb{R})$, where $\delta^{z,u,u'}$: $\Omega \times [0, T] \times \mathbb{R} \to (-1, \infty)$ is a predictable process such that $\delta^{z,u,u'}(t, x) > -1$ and $|\delta^{z,u,u'}(t, x)| \leq K(1 \wedge |x|)$ for all $(t, x, z, u, u') \in [0, T] \times \mathbb{R} \times \mathbb{R} \times L^2_Q \times L^2_Q$,
(iii) $g(t, 0, 0) = 0$ *for all* $t \in [0, T]$,
(iv) g *satisfies the growth conditions*

$$g^{**}_{\alpha,\beta}(t, z, u) \leq g(t, z, u) \leq g^*_{\alpha,\beta}(t, z, u),$$

   for $(t, z, u) \in [0, T] \times \mathbb{R} \times L^2_Q(\mathbb{R})$.

   The first condition of Theorem 6.2.2 is called the domination condition. We remark that a large class of nonlinear expectations satisfies the domination condition, see Rosazza Gianin (2006) and Royer (2006). The second condition requires translation invariance of the nonlinear expectation with respect to "known" pay-offs, which is a reasonable assumption provided that discounting of pay-offs is not allowed in the valuation, see Sect. 13.1.

   The importance of Theorem 6.2.2 is obvious. Theorem 6.2.2 shows that all filtration-consistent nonlinear expectations which satisfy the domination condition and the translation invariance property can be derived from BSDEs. Consequently, when we study "regular" filtration-consistent nonlinear expectations we can focus on g-expectations. Notice that the generator derived under the assumptions of Theorem 6.2.2 depends only on the control processes (Z, U) and is independent of Y. This is the consequence of the assumed translation invariance property for the nonlinear expectation.

It is clear that the generator g of a BSDE plays a crucial role in defining a g-expectation. Some important properties of g-expectations can be related to properties of generators g.

Proposition 6.2.3 *Let $\mathscr{E}_g$ be a g-expectation.*

(a) *If g is independent of y, then $\mathscr{E}_g$ is translation invariant*

$$\mathscr{E}_g[\xi + c|\mathscr{F}_t] = \mathscr{E}_g[\xi|\mathscr{F}_t] + c, \quad c \in \mathbb{R}, \ 0 \le t \le T.$$

(b) *If g is positively homogenous, then $\mathscr{E}_g$ is positively homogenous*

$$\mathscr{E}_g[c\xi|\mathscr{F}_t] = c\mathscr{E}_g[\xi|\mathscr{F}_t], \quad c > 0, \ 0 \le t \le T.$$

(c) *If g is convex*

$$g\big(t, cy_1 + (1-c)y_2, cz_1 + (1-c)z_2, cu_1 + (1-c)u_2\big)$$
$$\le cg(t, y_1, z_1, u_1) + (1-c)g(t, y_2, z_2, u_2),$$
$$c \in (0, 1), \ (t, y_1, z_1, u_1), (t, y_2, z_2, u_2) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times L_Q^2,$$

then $\mathscr{E}_g$ is convex

$$\mathscr{E}_g\big[c\xi_1 + (1-c)\xi_2|\mathscr{F}_t\big]$$
$$\le c\mathscr{E}_g[\xi_1|\mathscr{F}_t] + (1-c)\mathscr{E}_g[\xi_2|\mathscr{F}_t], \quad c \in (0, 1), \ 0 \le t \le T.$$

(d) *If g is sub-linear: sub-additive*

$$g(t, y_1 + y_2, z_1 + z_2, u_1 + u_2)$$
$$\le g(t, y_1, z_1, u_1) + g(t, y_2, z_2, u_2),$$
$$(t, y_1, z_1, u_1), (t, y_2, z_2, u_2) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times L_Q^2,$$

and positively homogenous, then $\mathscr{E}_g$ is sub-linear: sub-additive

$$\mathscr{E}_g[\xi_1 + \xi_2|\mathscr{F}_t] \le \mathscr{E}_g[\xi_1|\mathscr{F}_t] + \mathscr{E}_g[\xi_2|\mathscr{F}_t], \quad 0 \le t \le T,$$

positively homogenous.

Proof (a) We deal with two BSDEs

$$Y^{\xi+c}(t) = \xi + c + \int_t^T g\big(s, Z^{\xi+c}(s), U^{\xi+c}(s)\big)ds$$

$$- \int_t^T Z^{\xi+c}(s)dW(s) - \int_t^T \int_{\mathbb{R}} U^{\xi+c}(s, z)\tilde{N}(ds, dz), \quad 0 \le t \le T,$$

$$Y^\xi(t) = \xi + \int_t^T g\big(s, Z^\xi(s), U^\xi(s)\big)ds$$

$$- \int_t^T Z^\xi(s)dW(s) - \int_t^T \int_{\mathbb{R}} U^\xi(s, z)\tilde{N}(ds, dz), \quad 0 \le t \le T.$$

We can easily conclude that $Y^{\xi+c}(t) = Y^\xi(t) + c$, $Z^{\xi+c}(t) = Z^\xi(t)$, $U^{\xi+c}(t, z) = U^\xi(t, z)$, $(t, z) \in [0, T] \times \mathbb{R}$.

(b) We deal with two BSDEs

$$Y^{c\xi}(t) = c\xi + \int_t^T g\big(s, Y^{c\xi}(s), Z^{c\xi}(s), U^{c\xi}(s)\big)ds$$

$$- \int_t^T Z^{c\xi}(s)dW(s) - \int_t^T \int_{\mathbb{R}} U^{c\xi}(s, z)\tilde{N}(ds, dz), \quad 0 \le t \le T,$$

$$Y^\xi(t) = \xi + \int_t^T g\big(s, Y^\xi(s), Z^\xi(s), U^\xi(s)\big)ds$$

$$- \int_t^T Z^\xi(s)dW(s) - \int_t^T \int_{\mathbb{R}} U^\xi(s, z)\tilde{N}(ds, dz), \quad 0 \le t \le T.$$

We can easily conclude that $Y^{c\xi}(t) = cY^\xi(t)$, $Z^{c\xi}(t) = cZ^\xi(t)$, $U^{c\xi}(t, z) = cU^\xi(t, z)$, $(t, z) \in [0, T] \times \mathbb{R}$.

(c) We deal with three BSDEs

$$Y^{c\xi_1+(1-c)\xi_2}(t) = c\xi_1 + (1 - c)\xi_2$$

$$+ \int_t^T g\big(s, Y^{c\xi_1+(1-c)\xi_2}(s), Z^{c\xi_1+(1-c)\xi_2}(s), U^{c\xi_1+(1-c)\xi_2}(s)\big)ds$$

$$- \int_t^T Z^{c\xi_1+(1-c)\xi_2}(s)dW(s)$$

$$- \int_t^T \int_{\mathbb{R}} U^{c\xi_1+(1-c)\xi_2}(s, z)\tilde{N}(ds, dz), \quad 0 \le t \le T,$$

$$Y^{\xi_i}(t) = \xi_i + \int_t^T g\big(s, Y^{\xi_i}(s), Z^{\xi_i}(s), U^{\xi_i}(s)\big)ds$$

$$- \int_t^T Z^{\xi_i}(s)dW(s) - \int_t^T \int_{\mathbb{R}} U^{\xi_i}(s, z)\tilde{N}(ds, dz), \quad i = 1, 2, \ 0 \le t \le T.$$

We introduce the processes $Y(t) = cY^{\xi_1}(t) + (1 - c)Y^{\xi_2}(t)$, $Z(t) = cZ^{\xi_1}(t) + (1 - c)Z^{\xi_2}(t)$, $U(t, z) = U^{\xi_1}(t, z) + (1 - c)U^{\xi_2}(t, z)$. It is straightforward to notice that (Y, Z, U) satisfies the BSDE

$$Y(t) = c\xi_1 + (1 - c)\xi_2$$

$$+ \int_t^T \big(cg\big(s, Y^{\xi_1}(s), Z^{\xi_1}(s), U^{\xi_1}(s)\big)$$

$$+ (1 - c)g\big(s, Y^{\xi_2}(s), Z^{\xi_2}(s), U^{\xi_2}(s)\big)\big)ds$$

$$- \int_t^T Z(s)dW(s) - \int_t^T \int_{\mathbb{R}} U(s, z)\tilde{N}(ds, dz), \quad 0 \le t \le T. \quad (6.6)$$

Since g satisfies

$$cg\big(s, Y^{\xi_1}(s), Z^{\xi_1}(s), U^{\xi_1}(s)\big) + (1 - c)g\big(s, Y^{\xi_2}(s), Z^{\xi_2}(s), U^{\xi_2}(s)\big)$$

$$\ge g\big(s, Y(s), Z(s), U(s)\big), \quad 0 \le s \le T,$$

the BSDE (6.6) can be written as

$$Y(t) = c\xi_1 + (1 - c)\xi_2$$

$$+ \int_t^T \Big(g\big(s, Y(s), Z(s), U(s)\big)$$

$$+ h\big(s, Y^{\xi_1}(s), Z^{\xi_1}(s), U^{\xi_1}(s), Y^{\xi_2}(s), Z^{\xi_2}(s), U^{\xi_2}(s)\big)\Big)ds$$

$$- \int_t^T Z(s)dW(s) - \int_t^T \int_{\mathbb{R}} U(s, z)\tilde{N}(ds, dz), \quad 0 \le t \le T,$$

with a nonnegative function h. By the comparison principle we get $Y^{c\xi_1 + (1-c)x_2}(t) \le Y(t) = cY^{\xi_1}(t) + (1 - c)Y^{\xi_2}(t), 0 \le t \le T$.

(d) Adapting the arguments from (b) and (c), we can prove the assertion. $\square$

The properties from Proposition 6.2.3 are used in Chap. 13 where we deal with dynamic risk measures.

Bibliographical Notes   The Choquet expectation was introduced by Choquet (1953). Properties of Choquet expectations and the Wang transform together with their failures in non-Gaussian financial models are discussed by Nguyen et al. (2012). The g-expectations was introduced by Peng (1997). For the connection between the Choquet expectation and the g-expectation we refer to Chen et al. (2005) and Chen and Kulperger (2006). In the proof of Proposition 6.2.3 we follow the arguments from Rosazza Gianin (2006) and Jiang (2008). We refer to Rosazza Gianin (2006) and Jiang (2008) for stronger relations between static and dynamic properties of g-expectations and generators of BSDEs defining the g-expectations. For a representation of a filtration-consistent nonlinear expectation in a general separable space we refer to Cohen (2011). We remark that g-expectations allow for introducing nonlinear versions of some well-known probabilistic results, see Coquet et al. (2002), Peng (1997), Rosazza Gianin (2006) and Royer (2006). For g-martingales, g-submartingales, g-supermartingales and nonlinear Doob-Meyer decomposition we refer to Coquet et al. (2002) and Royer (2006).