Chapter 2
Stochastic Calculus

Abstract We review important results of stochastic calculus. We introduce a Brownian motion, a random measure and a compensated random measure. Examples of Lévy processes, step processes and their jump measures are given. We investigate stochastic integrals with respect to Brownian motion and compensated random measures and we recall their properties. We discuss the weak property of predictable representation for local martingales. Equivalent probability measures are defined, and Girsanov's theorem for Brownian motion and random measures is stated. We give differentiation rules of the Malliavin calculus.

We review important results of stochastic calculus which we use in this book. This chapter is written in the spirit of a rèsume and we collect facts needed to investigate BSDEs driven by Brownian motions and compensated random measures.

2.1 Brownian Motion and Random Measures

Let us consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a filtration $\mathscr{F} = (\mathscr{F}_t)_{0\leq t\leq T}$ and a finite time horizon $T < \infty$. We assume that the filtration $\mathscr{F}$ satisfies the usual hypotheses of completeness ($\mathscr{F}_0$ contains all sets of $\mathbb{P}$-measure zero) and right continuity ($\mathscr{F}_t = \mathscr{F}_{t+}$).

A stochastic process $V(\omega, t)$ is a real function defined on $\Omega \times [0, T]$ such that $\omega \mapsto V(\omega, t)$ is $\mathscr{F}$-measurable for any $t \in [0, T]$. A stochastic process V is called $\mathscr{F}$-adapted if $\omega \mapsto V(\omega, t)$ is $\mathscr{F}_t$-measurable for any $t \in [0, T]$. The natural filtration generated by a process V is denoted by $\mathscr{F}^V$. We always assume that the natural filtration is completed with sets of measure zero. By $\mathscr{B}(A)$ we denote the Borel sets of $A \subset \mathbb{R}$, by $\mathscr{P}$ we denote the σ-field on $\Omega \times [0, T]$ generated by all left-continuous and adapted processes. The field $\mathscr{P}$ is called the predictable σ-field. A process $V : \Omega \times [0, T] \to \mathbb{R}$, or $V : \Omega \times [0, T] \times \mathscr{E} \to \mathbb{R}$, is called $\mathscr{F}$-predictable if it is $\mathscr{F}$-adapted and $\mathscr{P}$-measurable, or $\mathscr{P} \otimes \mathscr{B}(\mathscr{E})$-measurable. Clearly, the limit of a converging sequence of predictable processes is a predictable process. If there is no confusion, the reference to the filtration $\mathscr{F}$ is omitted. A process is called càdlàg if its trajectories are right-continuous and have left limits. By K we denote

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constants, which are allowed to vary from line to line. The term *a.s.* means *almost surely* with respect to the probability measure, and, unless specified, the term *a.e.* means *almost everywhere* with respect to the Lebesgue measure. All statements for random variables and stochastic processes should be understood a.s.

We introduce a Brownian motion and a random measure. Brownian motion and random measures are used to develop financial and actuarial stochastic models.

Definition 2.1.1 An $\mathscr{F}$-adapted process $W := (W(t), 0 \le t \le T)$ with $W(0) = 0$ is called a Brownian motion if

(i) for $0 \le s < t \le T$, $W(t) - W(s)$ is independent of $\mathscr{F}_s$,
(ii) for $0 \le s < t \le T$, $W(t) - W(s)$ is a Gaussian random variable with mean zero and variance $t - s$.

There exists a modification of a Brownian motion which has continuous paths.

Definition 2.1.2 A function N defined on $\Omega \times [0, T] \times \mathbb{R}$ is called a random measure if

(i) for any $\omega \in \Omega$, $N(\omega, .)$ is a σ-finite measure on $\mathscr{B}([0, T]) \otimes \mathscr{B}(\mathbb{R})$,
(ii) for any $A \in \mathscr{B}([0, T]) \otimes \mathscr{B}(\mathbb{R})$, $N(., A)$ is a random variable on $(\Omega, \mathscr{F}, \mathbb{P})$.

We remark that $N(\omega, [0, t], A)$ may be equal to infinity (see Example 2.3 and the case of Lévy processes).

Example 2.1 Let $(T_n)_{n\ge 1}$ denote the sequence of jump times of a Poisson process. The function

$$N\big(\omega, [s, t]\big) = \sharp\big\{n \ge 1, T_n \in [s, t]\big\}, \quad 0 \le s < t \le T,$$

which counts the number of jumps of the Poisson process in the time interval $[s, t]$, defines a random measure. If we fix ω, then the sequence of jump times $(T_n)_{n\ge 1}$ of the Poisson process is given on the time axis, and N as a function of $[s, t]$ is a finite measure which counts the number of $(T_n)_{n\ge 1}$ which are in the interval $[s, t]$. If we fix $[s, t]$, then N is a Poisson distributed random variable which counts the number of random jump times $(T_n)_{n\ge 1}$ of the Poisson process which are in the interval $[s, t]$.

Next, we introduce a predictable compensator of a random measure.

Definition 2.1.3 A random measure N is called $\mathscr{F}$-predictable if for any $\mathscr{F}$-predictable process V such that the integral $\int_0^T \int_{\mathbb{R}} |V(s, z)| N(ds, dz)$ exists, the process $(\int_0^t \int_{\mathbb{R}} V(s, z) N(ds, dz), 0 \le t \le T)$ is $\mathscr{F}$-predictable.

Definition 2.1.4 For a random measure N we define

$$E_N(A) = \mathbb{E}\bigg[\int_{[0,T]\times\mathbb{R}} \mathbf{1}_A(\omega, t, z) N(\omega, dt, dz)\bigg], \quad A \in \mathscr{F} \otimes \mathscr{B}\big([0, T]\big) \otimes \mathscr{B}(\mathbb{R}).$$

If there exists an $\mathscr{F}$-predictable random measure ϑ such that

(i) E_ϑ is a σ-finite measure on $\mathscr{P} \otimes \mathscr{B}(\mathbb{R})$,
(ii) the measures E_N and E_ϑ are identical on $\mathscr{P} \otimes \mathscr{B}(\mathbb{R})$,

then we say that the random measure N has a compensator ϑ.

We remark that the compensator is uniquely determined, see Theorem 11.6 in He et al. (1992).

Given the compensator ϑ of a random measure N, we can define the compensated random measure

$$\tilde{N}(\omega, dt, dz) = N(\omega, dt, dz) - \vartheta(\omega, dt, dz).$$

Random measures are usually related to jumps of discontinuous processes. We state the following result, see Theorem 11.15 He et al. (1992).

Proposition 2.1.1 *Let $J := (J(t), 0 \le t \le T)$ be an $\mathscr{F}$-adapted, càdlàg process, and set $D = \{\Delta J \neq 0\}$. Then*

$$N(dt, dz) = \sum_{s \in (0,T]} \mathbf{1}_{(s, \Delta J(s))}(dt, dz) \mathbf{1}_{\{\Delta J(s) \neq 0\}}(s) \mathbf{1}_D\{s\}, \tag{2.1}$$

is an integer-valued random measure which has a unique $\mathscr{F}$-predictable compensator.

The measure N defined in Proposition 2.1.1 is called the jump measure of the process J. The measure $N([0, T], A)$ counts the number of jumps of the process J of size specified in the set A in the time interval $[0, T]$.

Two important families of discontinuous processes should be pointed out. In financial and actuarial applications we usually deal with Lévy processes and step processes.

Definition 2.1.5 An $\mathscr{F}$-adapted process $J := (J(t), 0 \le t \le T)$ with $J(0) = 0$ is called a Lévy process if

(i) for $0 \le s < t \le T$, $J(t) - J(s)$ is independent of $\mathscr{F}_s$,
(ii) for $0 \le s < t \le T$, $J(t) - J(s)$ has the same distribution as $J(t - s)$,
(iii) the process J is continuous in probability, for any $t \in [0, T]$ and $\varepsilon > 0$ we have $\lim_{s \to t} \mathbb{P}(|L(t) - L(s)| > \varepsilon) = 0$.

There exists a modification of a Lévy process which has càdlàg paths.

Example 2.2 The Poisson process and the compound Poisson process are the prime examples of discontinuous Lévy processes. It is easy to conclude that the jump measure of a compound Poisson process with intensity λ and jump distribution q has the compensator $\vartheta(dt, dz) = \lambda q(dz)dt$. The jump measure of a compound Poisson process is a finite random measure.

Example 2.3 The family of Lévy processes contains Variance Gamma, Normal Inverse Gaussian and stable processes, see Chap. 4 in Cont and Tankov (2004). In general, the jump measure of a Lévy process has the compensator $\vartheta(dt,dz)=\nu(dz)dt$ where ν is a σ-finite measure (called a Lévy measure) satisfying $\int_{|z|<1} z^2\nu(dz) < \infty$, see Proposition 3.7 in Cont and Tankov (2004). The measure ν determines properties of the Lévy process (we can have a finite variation or an infinite variation process with an infinite number of small jumps in every finite time interval), see Chaps. 3, 4 in Cont and Tankov (2004). For all Lévy processes except the compound Poisson process, the jump measure of a Lévy process is a σ-finite random measure with $N([0,T],\mathbb{R})=+\infty$.

If the random measure (2.1) is generated by a Lévy process, then it is called a Poisson random measure.

Definition 2.1.6 A process J is called a step process if its trajectories are càdlàg step functions having a finite number of jumps in every finite time interval. An $\mathscr{F}$-adapted step process J with $J(0)=0$ has the representation

$$J(t)=\sum_{n=1}^{\infty}\xi_n\mathbf{1}\{T_n\le t\}, \tag{2.2}$$

where

(i) $(T_n)_{n\ge 1}$ is a sequence of $\mathscr{F}$-stopping times such that $0\le T_1\le T_2\le\cdots\le T_n\uparrow\infty$, $n\to\infty$,
(ii) $\xi_n\in\mathscr{F}_{T_n}$, $n\ge 1$,
(iii) for each $n\ge 1$, $T_n<\infty\Rightarrow T_n<T_{n+1}$,
(iv) for each $n\ge 1$, $\xi_n\ne 0\Leftrightarrow T_n<\infty$.

In representation (2.2), T_n denotes the nth jump time of J and ξ_n denotes the jump size of J at time T_n. The sequence $(T_n)_{n\ge 1}$ defines a non-explosive point process. The jump measure of a step process is a finite random measure.

Example 2.4 The compound Poisson process is a step process.

Example 2.5 The compound Cox process is a second example of a step process. The compound Cox process J can be defined by $J(t)=j(\int_0^t\lambda(s)ds)$ where λ is a stochastic intensity process and j is an independent compound Poisson process with intensity 1 and jump size distribution q, see Theorem 12.2.3 in Rolski et al. (1999). We can deduce that the compensator of the corresponding jump measure is of the form $\vartheta(dt,dz)=\lambda(t)q(dz)$.

Example 2.6 Take a continuous process $\lambda:\Omega\times[0,T]\to(0,\infty)$ and define the hazard process $\Psi(t)=\int_0^t\lambda(s)ds$. We introduce a random time τ which has the conditional distribution

$$\mathbb{P}\big(\tau>t|\mathscr{F}_t^{\lambda}\big)=e^{-\Psi(t)}=e^{-\int_0^t\lambda(s)ds},\quad 0\le t\le T,$$

and we define the step process

$$J(t) = \mathbf{1}\{t \geq \tau\}, \quad 0 \leq t \leq T.$$

The compensated process $J(t) - \Psi(t \wedge \tau)$ is an $\mathscr{F}^{\lambda} \vee \mathscr{F}^{J}$-martingale, see Proposition 2.13 in Jeanblanc and Rutkowski (2000), and the jump measure of J has the compensator $\vartheta(dt, \{1\}) = (1 - J(t-))\lambda(t)dt$.

We remark that given the conditional distribution of (T_{n+1}, ξ_{n+1}) with respect to $\mathscr{F}_{T_n}$, it is possible to derive the compensator of the step process, see Theorem 11.49 in He et al. (1992).

We need some assumptions concerning the random measure and its compensator. We always assume that

(RM) the random measure N is an integer-valued random measure with the compensator

$$\vartheta(dt, dz) = Q(t, dz)\eta(t)dt, \tag{2.3}$$

where $\eta : \Omega \times [0, T] \to [0, \infty)$ is a predictable process, and Q is a kernel from $(\Omega \times [0, T], \mathscr{P})$ into $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ satisfying

$$\int_0^T \int_{\mathbb{R}} z^2 Q(t, dz)\eta(t)dt < \infty. \tag{2.4}$$

We also set $N(\{0\}, \mathbb{R}) = N((0, T], \{0\}) = \vartheta((0, T], \{0\}) = 0$.

This is our standing assumption and any random measure N considered in this book satisfies (RM). From the definition of a kernel we recall that for $(\omega, t) \in \Omega \times [0, T]$, $Q(t, .)$ is a measure on $\mathscr{B}(\mathbb{R})$, and for $A \in \mathscr{B}(\mathbb{R})$, $Q(., A)$ is a predictable process. Notice that the compensators considered in our examples satisfy assumption (2.3). In fact, the representation of the compensator (2.3) holds in most practical cases, we refer to Theorem II.1.8 in Jacod and Shiryaev (2003) for a general representation of the compensator of a random measure. In (2.3) we assume that the compensator is absolutely continuous with respect to the Lebesgue measure dt. The absolute continuity of the compensator with respect to the Lebesgue measure dt can be motivated by financial and actuarial applications in which we investigate jump measures of quasi-left continuous, càdlàg, adapted processes. Let us recall that a càdlàg, adapted process is called quasi-left continuous if a sequence of totally inaccessible stopping times exhausts its jump times, see Proposition I.2.26 and Corollary II.1.19 in Jacod and Shiryaev (2003). In other words, quasi-left continuous processes and absolutely continuous compensators of jump measures arise if we model jumps that arrive in an unpredictable way. Indeed, this is the right probabilistic framework for discontinuous processes used in insurance and finance. Assumption (2.4) implies that the quasi-left continuous process related to the jump measure is locally square integrable, see Theorem 11.31 in He et al. (1992) (in applications we deal with square integrable processes). The measure zero of the set $\{0\}$ indicates that N is indeed a jump measure, see Theorem 11.25 in He et al. (1992). In many actuarial applications, in which we deal with step processes, η can be interpreted as a claim intensity and Q as a claim distribution. If we consider a Lévy process, then we simply set $\eta(t) = 1$ and $Q(t, dz) = \nu(dz)$ where ν is a σ-finite Lévy measure.

2.2 Classes of Functions, Random Variables and Stochastic Processes

We start with defining spaces of functions, random variables and stochastic processes which we use in this book.

- Let $L^2_\nu(\mathbb{R})$ denote the space of measurable functions $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying

$$\int_{\mathbb{R}} \left|\varphi(z)\right|^2 \nu(dz) < \infty,$$

  where ν is a σ-finite measure,
- Let $\mathscr{C}^{1,2}([0, T], \mathbb{R})$ denote the space of continuous functions $\varphi : [0, T] \times \mathbb{R} \to \mathbb{R}$ which have continuous partial derivatives $\frac{\partial}{\partial t}\varphi(t, x)$, $\frac{\partial}{\partial x}\varphi(t, x)$ and $\frac{\partial^2}{\partial x^2}\varphi(t, x)$. Partial derivatives are denoted by $\phi_t, \phi_x, \phi_{xx}$. If there is no confusion, first derivative is denoted by ϕ'.
- Let $\mathbb{L}^2(\mathbb{R})$ denote the space of random variables $\xi : \Omega \to \mathbb{R}$ satisfying

$$\mathbb{E}\big[|\xi|^2\big] < \infty.$$

- Let $\mathbb{H}^2(\mathbb{R})$ denote the space of predictable processes $Z : \Omega \times [0, T] \to \mathbb{R}$ satisfying

$$\mathbb{E}\left[\int_0^T \left|Z(t)\right|^2 dt\right] < \infty.$$

- Let $\mathbb{H}^2_N(\mathbb{R})$ denote the space of predictable processes $U : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfying

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \left|U(t, z)\right|^2 Q(t, dz)\eta(t)dt\right] < \infty,$$

  where we integrate with respect to the predictable compensator of the random measure N.
- Let $\mathbb{S}^2(\mathbb{R})$ denote the space of adapted, càdlàg processes $Y : \Omega \times [0, T] \to \mathbb{R}$ satisfying

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|Y(t)\right|^2\right] < \infty.$$

- Let $\mathbb{S}^2_{inc}(\mathbb{R})$ denote the subspace of $\mathbb{S}^2(\mathbb{R})$ which contains processes with non-decreasing trajectories, and let $\mathbb{S}^\infty(\mathbb{R})$ denote the subspace of $\mathbb{S}^2(\mathbb{R})$ which contains bounded processes.

The spaces $\mathbb{H}^2(\mathbb{R})$, $\mathbb{H}^2_N(\mathbb{R})$ and $\mathbb{S}^2(\mathbb{R})$ are endowed with the norms:

$$\|Z\|^2_{\mathbb{H}^2} = \mathbb{E}\Big[\int_0^T e^{\rho t}\big|Z(t)\big|^2 dt\Big],$$
$$\|U\|^2_{\mathbb{H}^2_N} = \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}} e^{\rho t}\big|U(t,z)\big|^2 Q(t,dz)\eta(t)dt\Big],$$
$$\|Y\|^2_{\mathbb{S}^2} = \mathbb{E}\Big[\sup_{t\in[0,T]} e^{\rho t}\big|Y(t)\big|^2\Big],$$

with some $\rho \geq 0$.

We also define classes of processes which are differentiable in the Malliavin sense. First, we present the idea behind the Malliavin derivative.

If we investigate Malliavin differentiability, then we deal with the completed filtration generated by a Lévy process. We work with the product of two canonical spaces $(\Omega_W \times \Omega_N, \mathscr{F}_W \otimes \mathscr{F}_N, \mathbb{P}_W \otimes \mathbb{P}_N)$ completed with sets of measure zero. The space $(\Omega_W, \mathscr{F}_W, \mathbb{P}_W)$ is the usual canonical space for a one-dimensional Brownian motion (the space of continuous functions on $[0,T]$ with the σ-algebra generated by the topology of uniform convergence and Wiener measure). The space $(\Omega_N, \mathscr{F}_N, \mathbb{P}_N)$ is a canonical space for a pure jump Lévy process, and for its proper definition we refer to Solé et al. (2007). In the product space $(\Omega_W \times \Omega_N, \mathscr{F}_W \otimes \mathscr{F}_N, \mathbb{P}_W \otimes \mathbb{P}_N)$ we can study a two-parameter Malliavin derivative.

We follow the exposition from Solé et al. (2007). Let ν be a Lévy measure such that $\int_{\mathbb{R}} |z|^2\nu(dz) < \infty$. Consider the finite measure υ

$$\upsilon(A) = \int_{A(0)} \sigma^2 dt + \int_{A'} z^2\nu(dz)dt, \quad A \in \mathscr{B}\big([0,T]\big) \otimes \mathscr{B}(\mathbb{R}),$$

where $A(0) = \{t \in [0,T]; (t,0) \in A\}$ and $A' = A \setminus A(0)$. We define the martingale-valued random measure Υ

$$\Upsilon(A) = \int_{A(0)} \sigma dW(t) + \int_{A'} z\tilde{N}(dt,dz), \quad A \in \mathscr{B}\big([0,T]\big) \otimes \mathscr{B}(\mathbb{R}),$$

and its continuous and discontinuous parts

$$\Upsilon^c(t) = \int_0^t \sigma dW(s), \quad \Upsilon^d(t,A) = \int_0^t \int_A z\tilde{N}(ds,dz), \quad A \in \mathscr{B}(\mathbb{R}).$$

We introduce the multiple two-parameter integral with respect to Υ

$$I_n(\varphi_n) = \int_{([0,T]\times\mathbb{R})^n} \varphi\big((t_1,z_1),\ldots(t_n,z_n)\big)\Upsilon(dt_1,dz_1)\cdot\ldots\cdot\Upsilon(dt_n,dz_n),$$

for functions $\varphi \in L^2_\upsilon(([0,T]\times\mathbb{R})^n)$ satisfying

$$\|\varphi_n\|^2_{L^2_\upsilon} = \int_{([0,T]\times\mathbb{R})^n} \big|\varphi_n\big((t_1,z_1),\ldots,(t_n,z_n)\big)\big|^2 \upsilon(dt_1,dz_1)\cdot\ldots\cdot\upsilon(dt_n,dz_n) < \infty.$$

We finally recall the chaotic decomposition property which states that any square integrable random variable ξ measurable with respect to the completed natural filtration generated by a Lévy process has the unique representation

$$\xi = \sum_{n=0}^{\infty} I_n(\varphi_n), \tag{2.5}$$

where $\varphi_n \in L^2_\upsilon(([0,T]\times\mathbb{R})^n)$ are symmetric in the n pairs (t_i, z_i), $1 \le i \le n$. The Malliavin derivative uses the chaotic decomposition property (2.5).

We consider the following spaces:

- Let $\mathbb{D}^{1,2}(\mathbb{R})$ denote the space of random variables $\xi \in \mathbb{L}^2(\mathbb{R})$ which are measurable with respect to the natural filtration generated by a Lévy process and have the representation $\xi = \sum_{n=0}^{\infty} I_n(\varphi_n)$ such that

$$\sum_{n=1}^{\infty} n n! \|\varphi_n\|^2_{L^2_\upsilon} < \infty.$$

  For a random variable $\xi \in \mathbb{D}^{1,2}(\mathbb{R})$ we define the Malliavin derivative $D\xi : \Omega \times [0,T]\times\mathbb{R} \to \mathbb{R}$ to be a stochastic process of the form

$$D_{t,z}\xi = \sum_{n=1}^{\infty} n I_{n-1}\big(\varphi_n\big((t,z),\cdot\big)\big). \tag{2.6}$$

- Let $\mathbb{L}^{1,2}(\mathbb{R})$ denote the space of adapted and product measurable processes $V : \Omega\times[0,T]\times\mathbb{R} \to \mathbb{R}$ satisfying

$$\mathbb{E}\Big[\int_{[0,T]\times\mathbb{R}} \big|V(s,y)\big|^2 \upsilon(ds,dy)\Big] < \infty,$$

$$V(s,y) \in \mathbb{D}^{1,2}(\mathbb{R}), \quad \upsilon\text{-a.e. } (s,y) \in [0,T]\times\mathbb{R},$$

$$\mathbb{E}\Big[\int_{([0,T]\times\mathbb{R})^2} \big|D_{t,z}V(s,y)\big|^2 \upsilon(ds,dy)\upsilon(dt,dz)\Big] < \infty.$$

Let us define a stopping time, (local) martingale, quadratic variation and *BMO* martingale.

Definition 2.2.1 A random variable $\tau : \Omega \to [0,T]$ is called an $\mathscr{F}$-stopping time if $\{\tau \le t\} \in \mathscr{F}_t$ for every $t \in [0,T]$.

Definition 2.2.2 An $\mathscr{F}$-adapted process $M := (M(t), 0 \le t \le T)$ is called an $\mathscr{F}$-martingale (supermartingale/submartingale) if

(i) $\mathbb{E}[|M(t)|] < \infty$, $0 \le t \le T$,
(ii) $\mathbb{E}[M(t)|\mathscr{F}_s] = M(s)$, $0 \le s < t \le T$, $(\mathbb{E}[M(t)|\mathscr{F}_s] \le M(s) / \mathbb{E}[M(t)|\mathscr{F}_s] \ge M(s))$.

Definition 2.2.3 An $\mathscr{F}$-adapted process $M := (M(t), 0 \le t \le T)$ is called an $\mathscr{F}$-local martingale if there exists a sequence of $\mathscr{F}$-stopping times $(\tau_n, n \in \mathbb{N})$ such that $\tau_n \to T$, $n \to \infty$, and $(M(t \wedge \tau_n), 0 \le t \le T)$ is an $\mathscr{F}$-martingale.

Definition 2.2.4 The quadratic variation process of a càdlàg semimartingale V is defined by

$$[V, V](t) = \lim_{n\to\infty} \sum_{i=1}^{n} \big(V\big(t_{i+1}^n \wedge t\big) - V\big(t_i^n \wedge t\big)\big)^2, \quad 0 \le t \le T,$$

where $\lim_{n\to\infty} \sup_{i=1,\dots,n} |t_{i+1}^n - t_i^n| = 0$, and the convergence is uniform in probability.

Example 2.7 The quadratic variation of a Brownian motion is given by $[W, W](t) = t$, and the quadratic variation of a quadratic pure jump process J (a purely discontinuous Lévy process or a step process) is given by $[J, J](t) = \sum_{s\le t} |\Delta J(s)|^2$, see Theorems II.28 and II.39 in Protter (2004).

Definition 2.2.5 Let $M := (M(t), 0 \le t \le T)$ be an $\mathscr{F}$-local martingale. The process M is called a BMO (bounded mean oscillation) martingale if there exists a constant K such that

$$\mathbb{E}\big[[M, M](T) - [M, M](\tau)|\mathscr{F}_\tau\big] \le K,$$
$$\big|\Delta M(\tau)\big| \le K,$$

for any $\mathscr{F}$-stopping time $\tau \in [0, T]$.

We end this chapter with two important martingale inequalities, which are often applied in this book. We state the Burkholder-Davis-Gundy inequalities, see Theorem IV.48 in Protter (2004).

Theorem 2.2.1 *Let M be a local martingale. For any $p \ge 1$ there exist constants K_1, K_2, depending on p but independent from M, such that*

$$\mathbb{E}\Big[\sup_{0\le t\le T} \big|M(t)\big|^p\Big] \le K_1 \mathbb{E}\big[\big|[M, M](T)\big|^{p/2}\big] \le K_2 \mathbb{E}\Big[\sup_{0\le t\le T} \big|M(t)\big|^p\Big]. \quad (2.7)$$

We also recall the Doob's inequality, see Theorem I.20 in Protter (2004).

Theorem 2.2.2 *Let M be a positive submartingale. For any $p > 1$ we have*

$$\mathbb{E}\Big[\sup_{0\le t\le T} \big|M(t)\big|^p\Big] \le K \sup_{0\le t\le T} \mathbb{E}\big[\big|M(t)\big|^p\big]. \quad (2.8)$$

As a corollary, we can conclude that the martingale $M(t) = \mathbb{E}[\xi|\mathscr{F}_t]$, $0 \le t \le T$, $\xi \in \mathbb{L}^2(\mathbb{R})$, satisfies the inequality

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|M(t)\big|^2\Big] \le K\mathbb{E}\big[|\xi|^2\big].$$

2.3 Stochastic Integration

We state main properties of stochastic integrals with respect to Brownian motion and compensated random measures.

Theorem 2.3.1

(a) *Let* $V : \Omega \times [0,T] \to \mathbb{R}$ *be a predictable process satisfying*

$$\int_0^T \big|V(t)\big|^2 dt < \infty,$$

Then $(\int_0^t V(s)dW(s), 0 \le t \le T)$ *is a continuous local martingale with the quadratic variation process*

$$\Big[\int_0^\cdot V(s)dW(s), \int_0^\cdot V(s)dW(s)\Big](t) = \int_0^t \big|V(s)\big|^2 ds, \quad 0 \le t \le T.$$

(b) *Let* $V : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a predictable process satisfying*

$$\int_0^T \int_{\mathbb{R}} \big|V(t,z)\big|^2 Q(t,dz)\eta(t)dt < \infty,$$

where we integrate with respect to the compensator of a random measure N. *Then* $(\int_0^t \int_{\mathbb{R}} V(s,z)\tilde{N}(ds,dz), 0 \le t \le T)$ *is a càdlàg local martingale with the quadratic variation process*

$$\Big[\int_0^\cdot \int_{\mathbb{R}} V(s,z)\tilde{N}(ds,dz), \int_0^\cdot \int_{\mathbb{R}} V(s,z)\tilde{N}(ds,dz)\Big](t)$$
$$= \int_0^t \int_{\mathbb{R}} \big|V(s,z)\big|^2 N(ds,dz), \quad 0 \le t \le T.$$

Proof Case (a) follows from Theorems IV.22 and IV.28 in Protter (2004). Case (b) follows from Definition 11.16 and Theorem 11.21 in He et al. (1992). □

We also use the following result, see Theorem 11.21 in He et al. (1992).

Theorem 2.3.2 *Let* $V : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a predictable process satisfying*

$$\int_0^T \int_{\mathbb{R}} \big|V(t,z)\big| Q(t,dz)\eta(t)dt < \infty,$$

where we integrate with respect to the compensator of a random measure N. Then $(\int_0^t\int_{\mathbb{R}} V(s,z)\tilde{N}(ds,dz), 0 \le t \le T)$ is a càdlàg local martingale and $(\int_0^t\int_{\mathbb{R}} V(s,z)N(ds,dz), 0 \le t \le T)$ is a càdlàg process. Let N be the jump measure of a càdlàg process J. We also have the property

$$\int_0^t\int_{\mathbb{R}} V(s,z)N(ds,dz) = \sum_{s\in(0,t]} V\big(s, \Delta J(s)\big)\mathbf{1}_{\Delta J(s)\neq 0}(s), \quad 0 \le t \le T.$$

Notice that if V is a non-negative predictable process satisfying $\mathbb{E}[\int_0^T\int_{\mathbb{R}} V(t,z)Q(t,dz)\eta(t)dt] < \infty$, then

$$\mathbb{E}\bigg[\int_0^T\int_{\mathbb{R}} V(t,z)N(dt,dz)\bigg] = \mathbb{E}\bigg[\int_0^T\int_{\mathbb{R}} V(t,z)Q(t,dz)\eta(t)dt\bigg]. \tag{2.9}$$

To prove (2.9), from Theorem 2.3.2 we first deduce

$$\mathbb{E}\bigg[\int_0^{\tau_n}\int_{\mathbb{R}} V(t,z)N(dt,dz)\bigg] = \mathbb{E}\bigg[\int_0^{\tau_n}\int_{\mathbb{R}} V(t,z)Q(t,dz)\eta(t)dt\bigg],$$

where $(\tau_n)_{n\ge 1}$ is a sequence of stopping times, and we next apply the monotone convergence theorem.

We need a stronger version of Theorem 2.3.1.

Theorem 2.3.3

(a) *Let $V \in \mathbb{H}^2(\mathbb{R})$. Then $(\int_0^t V(s)dW(s), 0 \le t \le T)$ is a continuous, square integrable martingale which satisfies*

$$\mathbb{E}\bigg[\bigg|\int_0^T V(s)dW(s)\bigg|^2\bigg] = \mathbb{E}\bigg[\int_0^T \big|V(s)\big|^2 ds\bigg].$$

(b) *Let $V \in \mathbb{H}_N^2(\mathbb{R})$. Then $(\int_0^t\int_{\mathbb{R}} V(s,z)\tilde{N}(ds,dz), 0 \le t \le T)$ is a càdlàg, square integrable martingale which satisfies*

$$\mathbb{E}\bigg[\bigg|\int_0^T\int_{\mathbb{R}} V(s,z)\tilde{N}(ds,dz)\bigg|^2\bigg] = \mathbb{E}\bigg[\int_0^T\int_{\mathbb{R}} \big|V(s,z)\big|^2 Q(s,dz)\eta(s)ds\bigg].$$

Proof Case (a) follows from Lemma IV.27 and Theorem IV.22 in Protter (2004). We prove case (b). By Theorem 2.3.1 the process $\int_0^t\int_{\mathbb{R}} V(s,z)\tilde{N}(ds,dz)$ is a càdlàg local martingale. By Theorem 2.3.2 and property (2.9) we obtain

$$\mathbb{E}\bigg[\int_0^T\int_{\mathbb{R}} \big|V(s,z)\big|^2 N(ds,dz)\bigg] = \mathbb{E}\bigg[\int_0^T\int_{\mathbb{R}} \big|V(s,z)\big|^2 Q(s,dz)\eta(s)ds\bigg] < \infty.$$

Since $\int_0^t\int_{\mathbb{R}} V(s,z)\tilde{N}(ds,dz)$ is a local martingale with integrable quadratic variation, it is a square integrable martingale, see Corollary II.26.3 in Protter (2004). By Corollary II.26.3 in Protter (2004) we also derive

$$\mathbb{E}\Big[\Big|\int_0^T\int_{\mathbb{R}}V(s,z)\tilde{N}(ds,dz)\Big|^2\Big]$$
$$=\mathbb{E}\Big[\Big[\int_0^{\cdot}\int_{\mathbb{R}}V(s,z)\tilde{N}(ds,dz),\int_0^{\cdot}\int_{\mathbb{R}}V(s,z)\tilde{N}(ds,dz)\Big](T)\Big]$$
$$=\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}}\big|V(s,z)\big|^2N(ds,dz)\Big],$$

and the proof is complete. □

From Sect. II.6 in Protter (2004) we also recall that

$$\Big[\int_0^{\cdot}V_1(s)dW(s),\int_0^{\cdot}\int_{\mathbb{R}}V_2(s,z)\tilde{N}(ds,dz)\Big](T)=0.$$

Finally, let us present the Itô's formula, see Theorem II.32 in Protter (2004).

Theorem 2.3.4 *Consider a process $\mathscr{X}:=(\mathscr{X}(t),0\le t\le T)$ which satisfies the dynamics*

$$\mathscr{X}(t)=\mathscr{X}(0)+\int_0^t\mu(s)ds+\int_0^t\sigma(s)dW(s)$$
$$+\int_0^t\int_{\mathbb{R}}\gamma(s,z)\tilde{N}(ds,dz),\quad 0\le t\le T,$$

where μ, σ and γ are predictable processes such that $\int_0^T|\mu(s)|ds<\infty$, $\int_0^T|\sigma(s)|^2ds<\infty$, $\int_0^T\int_{\mathbb{R}}|\gamma(s,z)|^2Q(s,dz)\eta(s)ds<\infty$. Let $\varphi\in\mathscr{C}^{1,2}([0,T]\times\mathbb{R})$. Then

$$\varphi\big(\tau,\mathscr{X}(\tau)\big)=\varphi\big(0,\mathscr{X}(0)\big)+\int_0^\tau\varphi_t\big(s,\mathscr{X}(s-)\big)ds+\int_0^\tau\varphi_x\big(s,\mathscr{X}(s-)\big)d\mathscr{X}(s)$$
$$+\int_0^\tau\frac{1}{2}\varphi_{xx}\big(s,\mathscr{X}(s-)\big)\sigma^2(s)ds+\int_0^\tau\int_{\mathbb{R}}\Big(\varphi\big(s,\mathscr{X}(s-)+\gamma(s,z)\big)$$
$$-\varphi\big(s,\mathscr{X}(s-)\big)-\varphi_x\big(s,\mathscr{X}(s-)\big)\gamma(s,z)\Big)N(ds,dz),$$

for any stopping time $0\le\tau\le T$.

Example 2.8 Let $M(t)=e^{W(t)-\frac{1}{2}t}$, $0\le t\le T$. Then

$$M(t)=1+\int_0^tM(s)dW(s),\quad 0\le t\le T.$$

Let $M(t)=e^{\int_0^t\int_{\mathbb{R}}z\tilde{N}(ds,dz)-\int_0^t\int_{\mathbb{R}}(e^z-z-1)Q(s,dz)\eta(s)ds}$, $0\le t\le T$, where N is a random measure with a compensator satisfying $\int_0^T\int_{\mathbb{R}}z^2Q(s,dz)\eta(s)ds<\infty$ and

$\int_0^T \int_{\mathbb{R}} (e^z - 1)^2 Q(ds, dz)\eta(s)ds < \infty$. Then

$$M(t) = 1 + \int_0^t \int_{\mathbb{R}} M(s-)\big(e^z - 1\big)\tilde{N}(ds, dz), \quad 0 \le t \le T.$$

We also use the following result, which is a special case of the multidimensional Itô's formula, see Theorem II.33 in Protter (2004).

Proposition 2.3.1 *Consider the processes* $\mathscr{X}_i := (\mathscr{X}_i(t), 0 \le t \le T)$, $i = 1, 2$, *which satisfy the dynamics*

$$\mathscr{X}_i(t) = \mathscr{X}_i(0) + \int_0^t \mu_i(s)ds + \int_0^t \sigma_i(s)dW(s) + \int_0^t \int_{\mathbb{R}} \gamma_i(s, z)\tilde{N}(ds, dz), \quad 0 \le t \le T,\ i = 1, 2,$$

where μ_i, σ_i *and* γ_i *are predictable processes such that* $\int_0^T |\mu_i(s)|ds < \infty$, $\int_0^T |\sigma_i(s)|^2 ds < \infty$, $\int_0^T \int_{\mathbb{R}} |\gamma_i(s, z)|^2 Q(s, dz)\eta(s)ds < \infty$, *for* $i = 1, 2$. *Then*

$$\mathscr{X}_1(\tau)\mathscr{X}_2(\tau) = \mathscr{X}_1(0)\mathscr{X}_2(0) + \int_0^\tau \mathscr{X}_1(s-)d\mathscr{X}_2(s) + \int_0^\tau \mathscr{X}_2(s-)d\mathscr{X}_1(s) + \int_0^\tau \sigma_1(s)\sigma_2(s)ds + \int_0^\tau \int_{\mathbb{R}} \gamma_1(s, z)\gamma_2(s, z)N(ds, dz),$$

for any stopping time $0 \le \tau \le T$.

2.4 The Property of Predictable Representation

We now introduce the property of predictable representation, see Sect. XIII.2 in He et al. (1992) and Sect. III.4 in Jacod and Shiryaev (2003). The predictable representation property is the key concept in the theory of BSDEs which allows us to construct a solution to a BSDE. From the practical point of view, the predictable representation yields hedging strategies for financial claims.

Let us consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a filtration $\mathscr{F} = (\mathscr{F}_t)_{0 \le t \le T}$. In this book we always assume that the weak property of predictable representation holds, that is

(PR) any $\mathscr{F}$-local martingale M has the representation

$$M(t) = M(0) + \int_0^t Z(s)dW(s) + \int_0^t \int_{\mathbb{R}} U(s, z)\tilde{N}(ds, dz) \quad 0 \le t \le T, \tag{2.10}$$

where Z and U are $\mathscr{F}$-predictable processes integrable with respect to W and $\tilde{N}$.

This is our second standing assumption, next to (RM) from Sect. 2.1. If M is a locally square integrable local martingale, then the processes Z and U are locally square integrable in the sense of the assumptions from Theorem 2.3.1, see Definition III.4.2 in Jacod and Shiryaev (2003) and Theorem 11.31 in He et al. (1992). By Theorems 2.3.1–2.3.2 we also get

$$\mathbb{E}\big[[M,M](\tau_n)\big] \\ = M^2(0) + \mathbb{E}\Big[\int_0^{\tau_n} \big|Z(s)\big|^2 ds\Big] + \mathbb{E}\Big[\int_0^{\tau_n}\int_{\mathbb{R}} \big|U(s,z)\big|^2 Q(s,dz)\eta(s)ds\Big], \quad (2.11)$$

where $(\tau_n)_{n\geq 1}$ is a sequence of stopping times. If we now assume that M is a square integrable martingale, then $\mathbb{E}[[M,M](T)] < \infty$, see Corollary II.26.3 in Protter (2004), and applying the monotone convergence theorem and Fatou's lemma to (2.11) we can conclude that $Z \in \mathbb{H}^2(\mathbb{R})$ and $U \in \mathbb{H}^2_N(\mathbb{R})$. Moreover, we can easily deduce that the representation of a square integrable martingale M is unique in $\mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$. Consequently, in this book we assume that any square integrable $\mathscr{F}$-martingale M has the unique representation

$$M(t) = M(0) + \int_0^t Z(s)dW(s) + \int_0^t\int_{\mathbb{R}} U(s,z)\tilde{N}(ds,dz), \quad 0 \leq t \leq T, \quad (2.12)$$

where $(Z,U) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$. We can also assume that any square integrable $\mathscr{F}_T$-measurable random variable ξ has the unique representation

$$\xi = \mathbb{E}[\xi] + \int_0^T Z(s)dW(s) + \int_0^T\int_{\mathbb{R}} U(s,z)\tilde{N}(ds,dz), \quad (2.13)$$

where $(Z,U) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$. Representation (2.13) follows immediately from (2.12) by taking the martingale $M(t) = \mathbb{E}[\xi|\mathscr{F}_t]$, $0 \leq t \leq T$.

We point out that we introduce the predictable representation property (PR) as an assumption. In general, the predictable representation property does not have to hold. However, in our case it is possible to construct a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ in such a way that any $\mathscr{F}$-local martingale has the predictable representation. It is known that the weak property of predictable representation holds for a Brownian motion, a Lévy process, a step process and the corresponding completed natural filtration, see Theorems 13.19 and 13.49 in He et al. (1992). Moreover, given a Brownian motion W and an independent jump process J (a Lévy process or a step process), the weak property of predictable representation holds for (W,J) and the product of their completed natural filtrations. Finally, the weak property of predictable representation holds for (W,J) under any equivalent probability measure, see Theorem 13.22 in He et al. (1992). Hence, by the change of measure we can establish the predictable representation for a Brownian motion and a jump process with a random compensator (depending on W and J), see Sect. 2.5. For such a construction we refer to Becherer (2006) and Chap. 7 in Crépey (2011). We comment on the predictable representation in our financial and insurance model in Sect. 7.2.

2.5 Equivalent Probability Measures

Let us recall that for a semimartingale V such that $V(0) = 0$ there exists a unique càdlàg solution $\mathscr{E}$ to the forward stochastic differential equation

$$d\mathscr{E}(t) = \mathscr{E}(t-)dV(t), \quad \mathscr{E}(0) = 1,$$

given by

$$\mathscr{E}(t) = e^{V(t)-\frac{1}{2}[V,V](t)} \prod_{0<u\leq t} \big(1 + \Delta V(u)\big)e^{-\Delta V(u)+\frac{1}{2}|\Delta V(u)|^2}, \quad 0 \leq t \leq T. \quad (2.14)$$

The process $\mathscr{E}$ is called the stochastic exponential of V, see Theorem II.37 in Protter (2004). If $\Delta V(t) > -1, 0 \leq t \leq T$, then the stochastic exponential $\mathscr{E}$ is positive.

Let $\mathbb{P}$ and $\mathbb{Q}$ be two equivalent probability measures, $\mathbb{Q} \sim \mathbb{P}$. There exists a positive martingale $M := (M(t), 0 \leq t \leq T)$ such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathscr{F}_t = M(t), \quad 0 \leq t \leq T, \quad (2.15)$$

see Definition III.8.1 in Protter (2004) and Theorem 12.4 in He et al. (1992). In the view of the predictable representation property, we define

$$\frac{dM(t)}{M(t-)} = \phi(t)dW(t) + \int_{\mathbb{R}} \kappa(t,z)\tilde{N}(dt,dz), \quad M(0) = 1, \quad (2.16)$$

where $\phi := (\phi(t), 0 \leq t \leq T)$ and $\kappa := (\kappa(t,z), 0 \leq t \leq T, z \in \mathbb{R})$ are $\mathscr{F}$-predictable processes satisfying

$$\int_0^T \big|\phi(t)\big|^2 dt < \infty, \quad \int_0^T \int_{\mathbb{R}} \big|\kappa(t,z)\big|^2 Q(t,dz)\eta(t)dt < \infty,$$
$$\kappa(t,z) > -1, \quad 0 \leq t \leq T, \ z \in \mathbb{R}. \quad (2.17)$$

The process M defined by (2.16) under assumptions (2.17) is only a local martingale, see Theorem 2.3.1. We have to impose stronger assumptions on (ϕ, κ) so that the local martingale M is a true martingale. In this book we use the following proposition.

Proposition 2.5.1 *Let $M := (M(t), 0 \leq t \leq T)$ be the stochastic exponential defined by*

$$\frac{dM(t)}{M(t-)} = \phi(t)dW(t) + \int_{\mathbb{R}} \kappa(t,z)\tilde{N}(dt,dz), \quad M(0) = 1,$$

where ϕ and κ are predictable processes such that

$$\left|\phi(t)\right| \le K, \quad \int_{\mathbb{R}} \left|\kappa(t,z)\right|^2 Q(t,dz)\eta(t) \le K, \quad 0 \le t \le T,$$
$$\kappa(t,z) > -1, \quad 0 \le t \le T, \ z \in \mathbb{R}.$$

The process M is a square integrable, positive martingale.

Proof From Theorem 2.3.1 and (2.14) we conclude that M is a positive local martingale. We define the sequence of stopping times $\tau_n = \inf\{t : |M(t)| \ge n\} \wedge T$. We can derive the inequality

$$\begin{aligned}
\mathbb{E}\big[\left|M(t)\right|^2 \mathbf{1}\{t \le \tau_n\}\big] &\le \mathbb{E}\big[\left|M(\tau_n \wedge t)\right|^2\big] \\
&\le K\mathbb{E}\bigg[1 + \bigg|\int_0^{\tau_n \wedge t} M(s-)\phi(s)dW(s)\bigg|^2 \\
&\quad + \bigg|\int_0^{\tau_n \wedge t}\int_{\mathbb{R}} M(s-)\kappa(s,z)\tilde{N}(ds,dz)\bigg|^2\bigg] \\
&= K\bigg(1 + \mathbb{E}\bigg[\int_0^{\tau_n \wedge t} \left|M(s-)\phi(s)\right|^2 ds\bigg] \\
&\quad + \mathbb{E}\bigg[\int_0^{\tau_n \wedge t}\int_{\mathbb{R}} \left|M(s-)\kappa(s,z)\right|^2 Q(s,dz)\eta(s)ds\bigg]\bigg) \\
&\le K\bigg(1 + \int_0^t \mathbb{E}\big[\left|M(s)\right|^2 \mathbf{1}\{s \le \tau_n\}\big]ds\bigg), \quad 0 \le t \le T,
\end{aligned}$$

where we use Theorem 2.3.3. By the Gronwall's inequality, see Theorem V.68 in Protter (2004), we obtain

$$\mathbb{E}\big[\left|M(t)\right|^2 \mathbf{1}\{t \le \tau_n\}\big] \le K, \quad 0 \le t \le T.$$

We let $n \to \infty$, apply Fatous' lemma and we can deduce that M is uniformly square integrable. The uniform integrability yields that the local martingale M is a true martingale, see Theorem I.51 in Protter (2004). □

We state Girsanov's theorem which plays an important role in stochastic calculus and financial mathematics.

Theorem 2.5.1 *Let W and N be a $(\mathbb{P}, \mathscr{F})$-Brownian motion and a $(\mathbb{P}, \mathscr{F})$-random measure with compensator $\vartheta(ds,dz) = Q(s,dz)\eta(s)ds$. We define an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ with a positive $\mathscr{F}$-martingale (2.16). The processes*

$$\begin{aligned}
W^{\mathbb{Q}}(t) &= W(t) - \int_0^t \phi(s)ds, \quad 0 \le t \le T, \\
\tilde{N}^{\mathbb{Q}}(t,A) &= N(t,A) \\
&\quad - \int_0^t \int_{\mathbb{R}} \big(1 + \kappa(s,z)\big)Q(s,dz)\eta(s)ds, \quad 0 \le t \le T, \ A \in \mathscr{B}(\mathbb{R}),
\end{aligned} \tag{2.18}$$

are a $(\mathbb{Q}, \mathscr{F})$-Brownian motion and a $(\mathbb{Q}, \mathscr{F})$-compensated random measure.

Proof Let M denote the martingale (2.16) which changes the measure. The result of our theorem follows from the Girsanov-Meyer theorem, see Theorem III.40 in Protter (2004), which states that if for a $\mathbb{P}$-local martingale V the sharp bracket process $\langle V, M\rangle$ exists under $\mathbb{P}$, then

$$V(t) - \int_0^t \frac{1}{M(s-)} d\langle V, M\rangle(s), \quad 0 \le t \le T,$$

is a $\mathbb{Q}$-local martingale. The first assertion for the Brownian motion can be deduced from Theorem III.46 in Protter (2004). We prove the second assertion for the compensated random measure. The measure $\vartheta(dt, dz) = (1 + \kappa(t, z))Q(t, dz)\eta(t)dt$ is an $\mathscr{F}$-predictable random measure, see Definition 2.1.3. We choose a nonnegative, predictable function V such that $\int_0^t \int_{\mathbb{R}} V(s, z)N(ds, dz)$ is locally integrable under $\mathbb{Q}$. We set $V^m(s, z) = V(s, z) \wedge (m|z|)$. We can now deal with the $\mathbb{P}$-local martingale $\int_0^t \int_{\mathbb{R}} V^m(s, z)\tilde{N}(ds, dz)$, see Theorem 2.3.1. We define the quadratic covariation process

$$\begin{aligned}\Big[\int_0^{\cdot} \int_{\mathbb{R}} V^m(s, z)\tilde{N}(ds, dz), M\Big](t) \\ = \int_0^t \int_{\mathbb{R}} M(s-)\kappa(s, z)V^m(s, z)N(ds, dz), \quad 0 \le t \le T. \end{aligned} \tag{2.19}$$

Since the martingale M is càdlàg, we get

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \big|M(s-)\kappa(s, z)V^m(s, z)\big|Q(s, dz)\eta(s)ds \\ &\le K\sqrt{\int_0^T \int_{\mathbb{R}} \big|\kappa(s, z)\big|^2 Q(s, dz)\eta(s)ds \int_0^{\tau_n} \int_{\mathbb{R}} m|z|^2 Q(s, dz)\eta(s)ds} < \infty,\end{aligned}$$

and from Theorem 2.3.2 we deduce that the process $\int_0^t \int_{\mathbb{R}} M(s-)\kappa(s, z)V^m(s, z) \times \tilde{N}(ds, dz)$ is a $\mathbb{P}$-local martingale and the quadratic covariation process (2.19) is locally integrable under $\mathbb{P}$. Hence, the compensator of the covariation process (2.19) (the sharp bracket) exists under $\mathbb{P}$, see Sect. III.5 in Protter (2004), and it takes the form

$$\Big\langle \int_0^{\cdot} \int_{\mathbb{R}} V^m(s, z)\tilde{N}(ds, dz), M\Big\rangle(t) = \int_0^t \int_{\mathbb{R}} M(s-)\kappa(s, z)V^m(s, z)Q(s, dz)\eta(s)ds.$$

The Girsanov-Meyer theorem now yields that

$$\int_0^t \int_{\mathbb{R}} V^m(s, z)\big(N(ds, dz) - \big(1 + \kappa(s, z)\big)Q(s, dz)\eta(s)ds\big), \quad 0 \le t \le T,$$

is a $\mathbb{Q}$-local martingale. Let $(\tau_k)_{k\ge1}$ be a localizing sequence of stopping times for $\int_0^t \int_{\mathbb{R}} V^m(s, z)\tilde{N}^{\mathbb{Q}}(ds, dz)$, let $(\tau_n)_{n\ge1}$ be a localizing sequence of stopping times for $\int_0^t \int_{\mathbb{R}} V(s, z)N(ds, dz)$, and let τ be a stopping time. We have

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau_k\wedge\tau_n\wedge\tau}\int_{\mathbb{R}} V^m(s,z)N(ds,dz)\right]$$
$$=\mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau_k\wedge\tau_n\wedge\tau}\int_{\mathbb{R}} V^m(s,z)\big(1+\kappa(s,z)\big)Q(s,dz)\eta(s)ds\right].$$

Taking the limit $k\to\infty$, $m\to\infty$ and applying the Lebesgue monotone convergence theorem, we show

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau_n\wedge\tau}\int_{\mathbb{R}} V(s,z)N(ds,dz)\right]$$
$$=\mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau_n\wedge\tau}\int_{\mathbb{R}} V(s,z)\big(1+\kappa(s,z)\big)Q(s,dz)\eta(s)ds\right].$$

Hence, by Lemma I.1.44 in Jacod and Shiryaev (2003) the process $\int_0^t\int_{\mathbb{R}} V(s,z)\times\tilde{N}^{\mathbb{Q}}(ds,dz)$ is a $\mathbb{Q}$-local martingale. We now choose a predictable function V such that $\int_0^t\int_{\mathbb{R}}|V(s,z)|N(ds,dz)$ is locally integrable under $\mathbb{Q}$. Following the same reasoning, we show that $\int_0^t\int_{\mathbb{R}} V^+(s,z)\tilde{N}^{\mathbb{Q}}(ds,dz)$ and $\int_0^t\int_{\mathbb{R}} V^-(s,z)\tilde{N}^{\mathbb{Q}}(ds,dz)$ are $\mathbb{Q}$-local martingales, and $\int_0^t\int_{\mathbb{R}} V(s,z)\tilde{N}^{\mathbb{Q}}(ds,dz)$ is a $\mathbb{Q}$-local martingale. The proof is complete by Theorem II.1.8 in Jacod and Shiryaev (2003) and Definition 2.1.4. □

We give two examples which illustrate the change of measure.

Example 2.9 Consider the dynamics

$$\frac{dS(t)}{S(t)}=\mu(t)dt+\sigma(t)dW(t),\qquad S(0)=s,$$

where μ, σ are predictable, bounded processes. Let r be a predictable, nonnegative, bounded process. Define the stochastic exponential

$$\frac{dM(t)}{M(t)}=-\frac{\mu(t)-r(t)}{\sigma(t)}dW(t),\qquad M(0)=1,\tag{2.20}$$

and assume that $t\mapsto\frac{\mu(t)-r(t)}{\sigma(t)}$ is a.s bounded. By Proposition 2.5.1 the stochastic exponential M is a square integrable martingale. Hence, we can define an equivalent probability measure $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathscr{F}_T=M(T)$. From Theorem 2.5.1 we deduce that the dynamics of S under the new measure $\mathbb{Q}$ is given by

$$\frac{dS(t)}{S(t)}=r(t)dt+\sigma(t)dW^{\mathbb{Q}}(t).$$

The Itô's formula and Proposition 2.5.1 yield that $e^{-\int_0^t r(s)ds}S(t)$ is a $\mathbb{Q}$-martingale.

Example 2.10 Consider a compound Poisson process J with intensity λ and jump size distribution q. Let N denote the corresponding jump measure. Choose a predictable process κ such that $|\kappa(t,z)| < 1$, $(t,z) \in [0,T] \times \mathbb{R}$. We define the stochastic exponential

$$\frac{dM(t)}{M(t-)} = \int_{\mathbb{R}} \kappa(t,z)\tilde{N}(dt,dz), \quad M(0) = 1. \tag{2.21}$$

By Proposition 2.5.1 the stochastic exponential M is a square integrable martingale. Hence, we can define an equivalent probability measure $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathscr{F}_T = M(T)$. From Theorem 2.5.1 we deduce that

$$N(dt,dz) - \big(1 + \kappa(t,z)\big)\lambda q(dz)dt,$$

is the compensated random measure of the process J under the equivalent probability measure $\mathbb{Q}$. Consequently, under the equivalent probability measure $\mathbb{Q}$ the process J has the jump size distribution and the intensity

$$q^{\mathbb{Q}}(t,dz) = \frac{1+\kappa(t,z)}{\int_{\mathbb{R}}(1+\kappa(t,z))q(dz)} q(dz), \quad 0 \le t \le T,\ z \in \mathbb{R},$$

$$\lambda^{\mathbb{Q}}(t) = \int_{\mathbb{R}} \big(1+\kappa(t,z)\big)q(dz)\lambda, \quad 0 \le t \le T.$$

In general, since κ is a stochastic process then the new distribution $q^{\mathbb{Q}}$ and the new intensity $\lambda^{\mathbb{Q}}$ are stochastic processes as well. The set of equivalent probability measures determined by the martingales (2.21) with processes κ such that $|\kappa(t,z)| < 1$, $(t,z) \in [0,T] \times \mathbb{R}$ defines the set of equivalent scenarios for the compound Poisson process J, see Example 1.3.

2.6 The Malliavin Calculus

The Malliavin calculus plays an important role in the theory of BSDEs. It allows us to characterize a solution to a BSDE, prove path regularities of a solution and develop numerical schemes for finding a solution. Since Definition 2.6 of the Malliavin derivative is not very useful in calculations, we present some practical differentiation rules.

Consider the canonical Lévy space $(\Omega_W \times \Omega_N, \mathscr{F}_W \otimes \mathscr{F}_N, \mathbb{P}_W \otimes \mathbb{P}_N)$ and recall the Malliavin derivatives $D_{t,0}$, $D_{t,z}$ and the measures ν, υ, Υ from Sect. 2.2. The derivative $D_{t,0}$ is derivative with respect to the continuous component of a Lévy process (the Brownian motion) and we can apply the classical Malliavin calculus for Hilbert space-valued random variables, see Nualart (1995). By D_t we denote the classical Malliavin derivative on the Wiener space $(\Omega_W, \mathscr{F}_W, \mathbb{P}_W)$. We state the first result, see Proposition 3.5 in Solé et al. (2007).

Proposition 2.6.1 *If for $\mathbb{P}^N$-a.e. $\omega_N \in \Omega_N$ a random variable $\xi(., \omega_N)$ on $(\Omega_W, \mathscr{F}_W, \mathbb{P}_W)$ is Malliavin differentiable, then*

$$D_{t,0}\xi(\omega_W, \omega_N) = \frac{1}{\sigma} D_t\xi(., \omega_N)(\omega_W), \quad a.s., a.e., (\omega, t) \in \Omega \times [0, T], \tag{2.22}$$

where D_t denotes the Malliavin derivative on the Wiener space.

The derivative $D_{t,z}$, for $z \neq 0$, is derivative with respect to the pure jump component of a Lévy process. In order to calculate this derivative, we use the following increment quotient operator

$$\mathscr{I}_{t,z}\xi(\omega_W, \omega_N) = \frac{\xi(\omega_W, \omega_N^{t,z}) - \xi(\omega_W, \omega_N)}{z}, \tag{2.23}$$

where $\omega_N^{t,z}$ transforms a family $\omega_N = ((t_1, z_1), (t_2, z_2), \ldots) \in \Omega_N$ into a new family $\omega_N^{t,z} = ((t, z), (t_1, z_1), (t_2, z_2), \ldots) \in \Omega_N$ by adding a jump of size z at time t into the trajectory of the Lévy process. We can state the second result, see Propositions 5.4 and 5.5 in Solé et al. (2007).

Proposition 2.6.2 *Consider $\xi \in \mathbb{L}^2(\mathbb{R})$ which is measurable with respect to the natural filtration generated by a Lévy process. If $\mathbb{E}[\int_0^T \int_{\mathbb{R}\setminus\{0\}} |\mathscr{I}_{t,z}\xi|^2 z^2 \nu(dz)dt] < \infty$, then*

$$D_{t,z}\xi = \mathscr{I}_{t,z}\xi, \quad a.s., \upsilon\text{-}a.e.\ (\omega, t, z) \in \Omega \times [0, T] \times \big(\mathbb{R} \setminus \{0\}\big). \tag{2.24}$$

Let us now present some differentiation rules.

Proposition 2.6.3 *Consider the natural filtration $\mathscr{F}$ generated by a Lévy process and let $\xi \in \mathbb{D}^{1,2}(\mathbb{R})$. For $0 \leq s \leq T$ we have $\mathbb{E}[\xi|\mathscr{F}_s] \in \mathbb{D}^{1,2}(\mathbb{R})$, and*

$$D_{t,z}\mathbb{E}[\xi|\mathscr{F}_s] = \mathbb{E}[D_{t,z}\xi|\mathscr{F}_s]\mathbf{1}\{t \leq s\}, \quad a.s., \upsilon\text{-}a.e.\ (\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}.$$

Proof The result follows by adapting the proof of Proposition 1.2.8 from Nualart (1995) into our setting. □

It follows from Proposition 2.6.3 that if ξ is $\mathscr{F}_s$-measurable then $D_{t,z}\xi = 0$ a.s., υ-a.e. $(\omega, t, z) \in \Omega \times (s, T] \times \mathbb{R}$, see Corollary 1.2.1 in Nualart (1995).

We state the chain rule.

Proposition 2.6.4 *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function. Under the assumptions of Propositions 2.6.1 and 2.6.2 we have $\varphi(\xi) \in \mathbb{D}^{1,2}(\mathbb{R})$. Moreover:*

(a) *There exists an a.s. bounded random variable ζ such that*

$$D_{t,0}\varphi(\xi) = \zeta D_{t,0}\xi, \quad a.s., a.e.(\omega, t) \in \Omega \times [0, T].$$

If the law of ξ is absolutely continuous with respect to the Lebesgue measure or φ is continuously differentiable, then $\zeta = \varphi'(\xi)$.

(b) *We have the relation*

$$D_{t,z}\varphi(\xi) = \frac{\varphi(\xi + zD_{t,z}\xi) - \varphi(\xi)}{z},$$
$$a.s., \upsilon\text{-}a.e.\ (\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R} \setminus \{0\}.$$

Proof Case (a) follows from Proposition 1.2.4 in Nualart (1995) and Proposition 2.6.1. Case (b) follows from Proposition 2.6.2 and the definition of the operator (2.23). □

The next two results are taken from Delong and Imkeller (2010b).

Proposition 2.6.5 *Consider a finite measure q on $\mathbb{R}$. Let $\varphi : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be a product measurable, adapted process which satisfies*

$$\begin{aligned}
&\mathbb{E}\Big[\int_{[0,T]\times\mathbb{R}} \big|\varphi(s, y)\big|^2 q(dy)ds\Big] < \infty,\\
&\varphi(s, y) \in \mathbb{D}^{1,2}(\mathbb{R}), \quad a.e.\ (s, y) \in [0, T] \times \mathbb{R},\\
&\mathbb{E}\Big[\int_{([0,T]\times\mathbb{R})^2} \big|D_{t,z}\varphi(s, y)\big|^2 q(dy)ds\upsilon(dt, dz)\Big] < \infty.
\end{aligned} \tag{2.25}$$

Then $\int_{[0,T]\times\mathbb{R}} \varphi(s, y)q(dy)ds \in \mathbb{D}^{1,2}(\mathbb{R})$ and we have the differentiation rule

$$D_{t,z}\int_0^T \int_{\mathbb{R}} \varphi(s, y)q(dy)ds = \int_t^T \int_{\mathbb{R}} D_{t,z}\varphi(s, y)q(dy)ds,$$

a.s., υ-a.e. $(\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}$.

Proposition 2.6.6 *Let $\varphi : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be a predictable process which satisfies $\mathbb{E}[\int_{[0,T]\times\mathbb{R}} |\varphi(s, y)|^2\upsilon(ds, dy)] < \infty$. Then*

$$\varphi \in \mathbb{L}^{1,2}(\mathbb{R}) \quad \textit{if and only if} \quad \int_{[0,T]\times\mathbb{R}} \varphi(s, y)\Upsilon(ds, dy) \in \mathbb{D}^{1,2}(\mathbb{R}).$$

Moreover, if $\int_{[0,T]\times\mathbb{R}} \varphi(s, y)\Upsilon(ds, dy) \in \mathbb{D}^{1,2}(\mathbb{R})$, then

$$D_{t,z}\int_0^T \int_{\mathbb{R}} \varphi(s, y)\Upsilon(ds, dy) = \varphi(t, z) + \int_t^T \int_{\mathbb{R}} D_{t,z}\varphi(s, y)\Upsilon(ds, dy),$$

a.s., υ-a.e. $(\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}$, and $\int_{[0,T]\times\mathbb{R}} D_{t,z}\varphi(s, y)\Upsilon(ds, dy)$ is a stochastic integral in the Itô sense.

Notice that we can also establish the following relation

$$D_{t,z}\int_s^T\int_{\mathbb{R}}\varphi(r,y)\Upsilon(dr,dy)$$
$$=D_{t,z}\Big(\int_0^T\int_{\mathbb{R}}\varphi(r,y)\Upsilon(dr,dy)-\int_0^s\int_{\mathbb{R}}\varphi(r,y)\Upsilon(dr,dy)\Big)$$
$$=\int_s^T\int_{\mathbb{R}}D_{t,z}\varphi(r,y)\Upsilon(dr,dy),\quad 0\le t\le s\le T,\ s>0.$$

We now give examples which illustrate the differentiation rules.

Example 2.11 Consider a square integrable function $V:\mathbb{R}\to\mathbb{R}$ and a Lipschitz continuous function $\varphi:\mathbb{R}\to\mathbb{R}$. Let

$$\xi=\varphi\Big(\int_0^T V(s)dW(s)\Big).$$

We can write $\xi=\varphi(\int_0^T\frac{V(s)}{\sigma}d\Upsilon^c(s))$. It is know that the random variable $\int_0^T V(s)dW(s)$ is normally distributed, see Lemma 4.3.11 in Applebaum (2004). By Propositions 2.6.4 and 2.6.6 we obtain

$$D_{t,0}\xi=\varphi'\Big(\int_0^T V(s)dW(s)\Big)\frac{V(t)}{\sigma},$$

a.s., a.e. $(\omega,t)\in\Omega\times[0,T]$.

Example 2.12 Consider the put option $\xi=(K-V(T))^+$ where $V(T)=e^{\sigma W(T)-\frac{1}{2}\sigma^2T}$ models the terminal value of a stock. In applications we would like to use the Malliavin derivative of ξ. Unfortunately, we cannot use the result from Example 2.11 since the exponential function is not Lipschitz continuous. We follow a different approach. First, we find the Malliavin derivative of $V(T)$. Let us define the process $V(t)=e^{\sigma W(t)-\frac{1}{2}\sigma^2t}$, $0\le t\le T$, and by the Itô's formula we get

$$V(t)=1+\int_0^t V(s)\sigma dW(s),\quad 0\le t\le T.$$

In Sect. 4.1 we show that the process V, which solves a linear forward stochastic differential equation, is Malliavin differentiable, see Theorem 4.1.2. We can now apply Proposition 2.6.6 and we derive the equation

$$D_{u,0}V(t)=V(u)+\int_u^t D_{u,0}V(s)\sigma dW(s),\quad 0\le u\le t\le T.$$

Since $D_{u,0}V$ turns out to be a stochastic exponential of the Brownian motion W, we conclude that $D_{u,0}V(t)=V(u)e^{\int_u^t\sigma dW(s)-\frac{1}{2}\int_u^t\sigma^2ds}=V(t)$, $0\le u\le t\le T$. By Proposition 2.6.4 we now get the Malliavin derivative

$$D_{t,0}\xi = -e^{\sigma W(T)-\frac{1}{2}\sigma^2 T}\mathbf{1}\big\{e^{\sigma W(T)-\frac{1}{2}\sigma^2 T} < K\big\},$$

a.s., a.e. $(\omega, t) \in \Omega \times [0, T]$.

Example 2.13 Let N be the jump measure of a compound Poisson process with jump size distribution q. Consider a function $V : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that $\int_0^T \int_{\mathbb{R}} |V(s, y)|^2 q(dy)ds < \infty$ and a Lipschitz continuous function $\varphi : \mathbb{R} \to \mathbb{R}$. Let

$$\xi = \varphi\bigg(\int_0^T \int_{\mathbb{R}} V(s, y)\tilde{N}(ds, dy)\bigg).$$

We can write $\xi = \varphi(\int_0^T \int_{\mathbb{R}} \frac{V(s,y)}{y} \Upsilon^d(ds, dy))$. By Propositions 2.6.6 and 2.6.4 we obtain

$$D_{t,z}\xi = \frac{\varphi(\int_0^T \int_{\mathbb{R}} V(s, y)\tilde{N}(ds, dy) + V(t, z)) - \varphi(\int_0^T \int_{\mathbb{R}} V(s, y)\tilde{N}(ds, dy))}{z},$$

a.s., υ-a.e. $(\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R} \setminus \{0\}$.

Example 2.14 Let N be the jump measure of a compound Poisson process with jump size distribution q. Consider the stop-loss contract $\xi = (J(T) - K)^+$ where $J(t) = \int_0^t \int_0^\infty yN(ds, dy)$, $0 \le t \le T$, is the compound Poisson process used for modelling insurer's claims. We assume that the claim size distribution q is supported on $(0, \infty)$ and satisfies $\int_0^\infty y^2 q(dy) < \infty$. In applications we would like to use the Malliavin derivative of ξ. From Example 2.13 we immediately deduce that

$$D_{t,z} = \frac{(J(T) + z - K)^+ - (J(T) - K)^+}{z},$$

a.s., υ-a.e. $(\omega, t, z) \in \Omega \times [0, T] \times (0, \infty)$.

Bibliographical Notes Definitions are taken from He et al. (1992) and Protter (2004). Propositions and theorems are taken from the sources cited in the text. Stochastic integration and the theory of semimartingales are studied in Applebaum (2004) (see for Lévy processes), Brémaud (1981) (see for point processes), Karatzas and Shreve (1988) (see for Brownian motions), Nualart (1995) (see for the Malliavin calculus) and He et al. (1992), Jacod and Shiryaev (2003), Protter (2004) (see for the general theory). Financial and actuarial applications of Brownian motions, Lévy processes and step processes are investigated in Cont and Tankov (2004), Mikosch (2009), Øksendal and Sulem (2004), Pham (2009), Rolski et al. (1999), Schmidli (2007) and Shreve (2004).