Chapter 2
Simple Switching Transients

2.1 Laplace Transform

The Laplace transformation is among the most useful tools used by electrical engineers. The purpose of this book is not to provide a mathematical treatment of the Laplace transform. Instead, the next pages focus on the practical application of the method and its use on different types of circuits.

The Laplace transformation is a method that can be used for solving ordinary differential equations by reducing a differential equation to an algebraic equation which can then be solved by the more common and easy algebraic operations. Moreover, the Laplace transformation is a linear operation, i.e., $L(af(t) + bg(t)) = aL(f(t)) + bL(g(t))$ and it can be applied to piecewise continuous functions, meaning that the function may have finite "jumps".

The Laplace transform of a function $f(t)$ is defined by (2.1). However, it is normally unnecessary to solve the equation as tables of transforms exist for the more common expressions. Table 2.1 shows some of the main transformations used by electrical engineers.

$$F(s) = L(f(t)) = \int_0^{\infty} e^{-st} f(t)\mathrm{d}t \tag{2.1}$$

One question that may arise is: can Laplace transform applied to any function. The theory states that the transform can be applied for any $s > \gamma$ as long as the condition (2.2) is fulfilled for all $t \geq 0$ and for some constant M and γ [1]. In other words, the transform exists if $e^{-st}f(t)$ goes to zero when $t \to \infty$, something which is true for any physical system, since that *for any real physical stimulus there will be a real physical response* [2].

$$|f(t)| \leq Me^{\gamma t} \tag{2.2}$$

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Table 2.1 Some Laplace transforms of the more common functions

	$f(t)$	$F(s)$		$f(t)$	$F(s)$		$f(t)$	$F(s)$
I	1	$\frac{1}{s}$	IV	e^{at}	$\frac{1}{s-a}$	VII	$e^{-a\|t\|}$	$\frac{2a}{a^2-s^2}$
II	t	$\frac{1}{s^2}$	V	$1-e^{-at}$	$\frac{a}{s(s+a)}$	VIII	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
III	t^n	$\frac{n!}{s^{n+1}}$	VI	$\frac{t^n}{n!}e^{-at}$	$\frac{1}{(s+a)^{n+1}}$	IX	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

As mentioned the Laplace transform is normally used to solve differential equations. These equations contain a combination of derivative and integration functions and in the following, we will see how to work with these functions.

The Laplace transform of a differentiation corresponds to the multiplication of the transform $F(s)$ by the complex Laplace variable s, whereas the Laplace transform of an integration corresponds to a division. In other words, the differential equation becomes a polynomial equation.

Equation (2.3) shows how to apply the Laplace transform to a first order derivate. The variable s multiplies the transformation and the initial value of the function f is subtracted. It is important to keep in mind that the solution to a differential equation has two parts, the *general* solution and the *particular* solution. The general solution is a characteristic of the system being studied, and it is independent of the system condition [left term in (2.3)], whereas the particular solution depends on the system condition, normally the system initial conditions [right term in (2.3)].

$$\mathrm{L}\left(f'(t)\right) = s\mathrm{L}(f(t)) - f(0) \tag{2.3}$$

The same method can be applied to higher order derivate functions, with minor changes. Equation (2.4) shows how to apply the Laplace transform to a second order derivate. The deduction of the formula is rather straightforward and shown in (2.5).

$$\mathrm{L}\left(f''(t)\right) = s^2\mathrm{L}(f(t)) - sf(0) - f'(0) \tag{2.4}$$

$$\begin{aligned} \mathrm{L}(f''(t)) &= s\mathrm{L}(f'(t)) - f'(0) \\ &= s(s\mathrm{L}(f) - f(0)) - f'(0) \\ &= s^2\mathrm{L}(f(t)) - sf(0) - f'(0) \end{aligned} \tag{2.5}$$

The reasoning applied in (2.5) can be applied to any higher order derivate function and the following general expression is obtained (2.6).

$$\mathrm{L}(f^n(t)) = s^n\mathrm{L}(f(t)) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0) \tag{2.6}$$

The integration is the opposite operation of derivation, and thus, there is a division instead of a multiplication. Equation (2.7) shows how to apply the Laplace transform to an integral defined between 0 and a random time t.

$$\mathrm{L}\left(\int_0^t f(t)\mathrm{d}t\right) = \frac{1}{s}\mathrm{L}(f(t)) \tag{2.7}$$

Do not worry if it seems complicated. The use of the Laplace transformation of the different proprieties will become clearer via the examples shown on the next pages.

2.2 Switching of RL Circuits (or Shunt Reactors)

The first circuit that we are going to analyse is a basic RL circuit. An RL circuit is a first-order circuit and due to its simplicity, it is a good choice for an introduction to the world of transients. The circuit also resembles the behaviour of a shunt reactor which is an important piece of equipment in any high voltage cable-based network. Figure 2.1 shows the single-line of an RL circuit consisting of a resistance and inductance in series connected to a voltage source through a switch (CB).

2.2.1 DC Source

The simplest example is the energisation of the RL load by a DC voltage source. Such a system is mathematically described by (2.8). The application of the Laplace transform to (2.8) leads to (2.9).

$$V = RI + L\frac{\mathrm{d}I}{\mathrm{d}t} \tag{2.8}$$

$$V = RI(s) + L(sI(s) - I(0)) \tag{2.9}$$

The circuit is considered as being discharged previous to the switching and thus $I(0) = 0$, because of the inductor, and the solution to the equation is the one shown in (2.10).

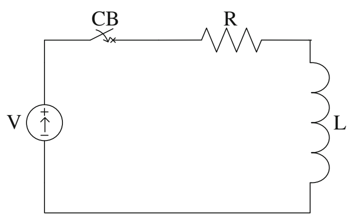


Fig. 2.1 Switching of an RL load

$$\frac{V}{s} = RI(s) + sLI(s) \Leftrightarrow I(s) = \frac{V}{s(R + sL)} \Leftrightarrow I(s) = \frac{V}{L}\frac{1}{s(R/L + s)} \tag{2.10}$$

Applying the inverse Laplace transform (V in Table 2.1) to (2.10) allows to obtain the time domain expression of the current (2.11).

$$\begin{aligned} I(t) &= L^{-1}(I(s)) \Leftrightarrow I(t) = L^{-1}\left(\frac{V}{L}\frac{L}{R}\frac{R/L}{s(R/L + s)}\right) \\ &\Leftrightarrow I(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right) \end{aligned} \tag{2.11}$$

We can see that the current starts at zero and increases up to *V*/*R* with a time constant (τ) equal *L*/*R* and with the behaviour of an inverse exponential decay function. It is common to consider the current in steady-state after approximately 5τ.

It is important to keep in mind that the current cannot change instantaneously due to the conservation of the moment in the magnetic flux associated to the inductor. In other words, the magnetic flux has to be continuous, i.e., an instant change in the current would require an infinite voltage which is obviously impossible in a real system.

Example:
Figure 2.2 shows the current during the switching of an RL load for the following parameters: $V = 100$ V, $R = 1\ \Omega$ and $L = 0.1$ H.

We know from (2.11) that the time constant of the circuit is $L/R = 0.1$ s and the steady-state value is equal to $V/R = 100$ A. Figure 2.2 confirms the results showing a current with the shape of an inverse exponential decaying function with an approximate steady-state value of 100 A after 0.5 s (5×0.1).

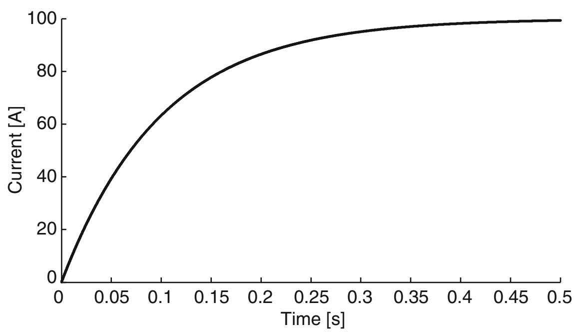


Fig. 2.2 Current in the load during the first 0.5 s when using a DC voltage source

2.2.2 AC Source

The previous example was for a DC circuit, however, the majority of the power systems are AC. The change of the voltage source from DC to AC leads to radical changes in both the equation and the waveform.

The first step is to solve the equation describing the circuit (2.12) and apply the Laplace transform (2.13). For simplification, it is considered that the circuit is switched at zero voltage and that it was unloaded prior to the energisation, $I(0) = 0$ A.

$$V_P \sin(\omega t) = RI + L\frac{\mathrm{d}I}{\mathrm{d}t} \tag{2.12}$$

$$V_P \frac{\omega}{s^2+\omega^2} = I(R+sL) - I(0) \tag{2.13}$$

$$I = V_P \frac{\omega}{s^2+\omega^2}\frac{1}{R+sL} \Leftrightarrow I = V_P \frac{\omega}{L}\frac{1}{s^2+\omega^2}\frac{1}{s+\frac{R}{L}} \tag{2.14}$$

The resolution of (2.14) requires the use of the partial fraction method as done in (2.15) and (2.16), where $a = R/L$.

$$\frac{1}{s^2+\omega^2}\frac{1}{s+a} = \frac{As+B}{s^2+\omega^2} + \frac{C}{s+a} \tag{2.15}$$

$$\begin{cases} s^2 : A + C = 0 \\ s^1 : Aa + B = 0 \\ s^0 : Ba + C\omega^2 = 1 \end{cases} \tag{2.16}$$

By solving (2.16), (2.17) is obtained. Using (2.17) in (2.15) and later in (2.14), (2.18) is obtained.

$$A = \frac{-1}{a^2+\omega^2} \wedge B = \frac{a}{a^2+\omega^2} \wedge C = \frac{1}{a^2+\omega^2} \tag{2.17}$$

$$\begin{aligned} I =& V_P \frac{\omega}{L}\frac{1}{a^2+\omega^2}\left(\frac{-s+a}{\omega^2+s^2} + \frac{1}{s+a}\right) \\ \Leftrightarrow I =& V_P \frac{\omega}{L}\frac{1}{a^2+\omega^2}\left(\frac{-s}{s^2+\omega^2} + \frac{a}{s^2+\omega^2} + \frac{1}{s+a}\right) \end{aligned} \tag{2.18}$$

The inverse Laplace transform can be applied to (2.18) leading to (2.19). Notice how the linearity of the Laplace transform act as an advantage in this case.

$$I(t) = V_P \frac{\omega}{L} \frac{1}{a^2+\omega^2}\left(-\cos(\omega t) + \frac{a}{\omega}\sin(\omega t) + e^{-at}\right)$$
$$\Leftrightarrow I(t) = \frac{V_P}{L}\frac{1}{\sqrt{a^2+\omega^2}}\left(-\frac{\omega}{\sqrt{a^2+\omega^2}}\cos(\omega t) + \frac{a}{\sqrt{a^2+\omega^2}}\sin(\omega t) + \frac{\omega}{\sqrt{a^2+\omega^2}}e^{-at}\right) \tag{2.19}$$

It is known that the power factor [$\cos(\phi)$] of an RL load is given by (2.20) which is equivalent to (2.21).

$$\cos(\phi) = \frac{R}{\sqrt{R^2 + (\omega L)^2}} \tag{2.20}$$

$$\cos(\phi) = \frac{R}{L\sqrt{\left(\frac{R}{L}\right)^2 + \omega^2}} \Leftrightarrow \cos(\phi) = \frac{a}{\sqrt{a^2+\omega^2}} \tag{2.21}$$

Similar relations can be obtained for $\sin(\phi)$ (2.22) and $\tan(\phi)$ (2.23).

$$\sin^2(\phi) + \cos^2(\phi) = 1 \Leftrightarrow \sin(\phi) = \sqrt{1 - \frac{a^2}{a^2+\omega^2}} \Leftrightarrow \sin(\phi) = \frac{\omega}{a^2+\omega^2} \tag{2.22}$$

$$\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)} \Leftrightarrow \tan(\phi) = \frac{\omega}{a} \tag{2.23}$$

Replacing (2.21) and (2.22) in (2.19), (2.24) is obtained.

$$I(t) = \frac{V_P}{L}\frac{1}{\sqrt{a^2+\omega^2}}\left(-\sin(\phi)\cos(\omega t) + \cos(\phi)\sin(\omega t) + \sin(\phi)e^{-at}\right) \tag{2.24}$$

Equation (2.24) can be further simplified using the trigonometric relation (2.25), leading to (2.26).

$$\cos(\phi)\sin(\omega t) - \sin(\phi)\cos(\omega t) = \sin(\omega t - \phi) \tag{2.25}$$

$$I(t) = \frac{V_P}{L}\frac{1}{\sqrt{a^2+\omega^2}}\left(\sin(\omega t - \phi) + \sin(\phi)e^{-at}\right)$$
$$\Leftrightarrow I(t) = \frac{V_P}{L}\frac{1}{\sqrt{a^2+\omega^2}}\left(\sin\left(\omega t - \tan^{-1}\left(\frac{\omega}{a}\right)\right) + \sin\left(\tan^{-1}\left(\frac{\omega}{a}\right)\right)e^{-at}\right)$$
$$\Leftrightarrow I(t) = \frac{V_P}{\sqrt{R^2+(\omega L)^2}}\left(\sin\left(\omega t - \tan^{-1}\left(\omega\frac{L}{R}\right)\right) - \sin\left(-\tan^{-1}\left(\omega\frac{L}{R}\right)\right)e^{-\frac{R}{L}t}\right) \tag{2.26}$$

The equation were developed assuming that the circuit was switched on for zero voltage. However, the circuit can be energised at any chosen instant. A process

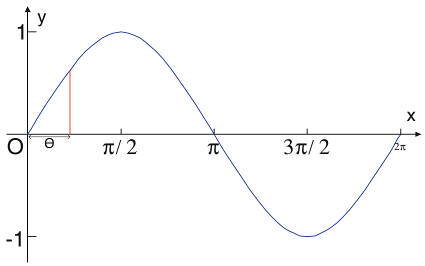


Fig. 2.3 Switching angle in a sinusoidal wave

similar to the one just done in the last pages can be done, which would result in (2.27), where θ is the switching angle (Fig. 2.3).

$$I(t) = \frac{V_P}{\sqrt{R^2 + (\omega L)^2}} \left(\sin\left(\omega t + \theta - \tan^{-1}\left(\omega \frac{L}{R} \right) \right) - \sin\left(\theta - \tan^{-1}\left(\omega \frac{L}{R} \right) \right) e^{-\frac{R}{L}t} \right) \tag{2.27}$$

As previously stated, a shunt reactor is basically an RL circuit and it is described by (2.27). However, the high inductance of a shunt reactor which is typically hundreds of times larger than the resistance allows simplification of (2.27) to the more compact (2.28).

$$I(t) \simeq \frac{V_P}{\sqrt{R^2 + (\omega L)^2}} \left(\sin\left(\omega t + \theta - \frac{\pi}{2} \right) - \sin\left(\theta - \frac{\pi}{2} \right) e^{-\frac{R}{L}t} \right) \tag{2.28}$$

We obtained an equation describing a general RL circuit, but we have not yet analysed and explained it. The current in (2.28) is the summation of two parts:

- The steady-state component (also known as forced regime): $\sin\left(\omega t + \theta - \frac{\pi}{2}\right)$;
- The transient component (also known as homogeneous regime): $\sin\left(\theta - \frac{\pi}{2}\right)e^{-\frac{R}{L}t}$.

The steady-state component of the current is basically a sinusoidal wave oscillating at power frequency with a phase difference of approximately 90° to the voltage, and it is independent of the switching instant or conditions.

The transient component is a decaying DC current whose amplitude depends on the initial conditions, i.e., switching instant and the energy stored in the inductance. The energy stored is typically zero ($I(0) = 0$ A), but the switching instant can be any, depending on the application and type of CB.

To better understand the high importance of the switching instant, let's see two examples.

Energisation at peak voltage:
The energisation of the RL load for peak voltage is equivalent to having $\theta = \pm 90°$. As a consequence, the transient component is zero and the current is only the steady-state component.

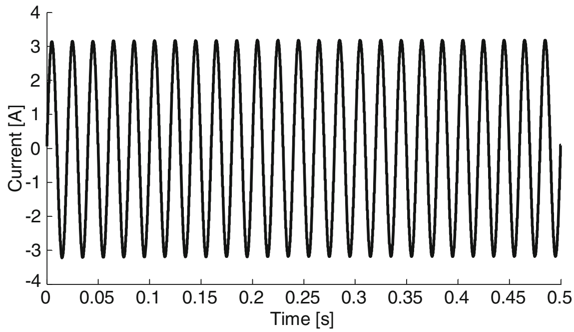


Fig. 2.4 Current in the load during the first 0.5 s when using an AC voltage source and energising at peak voltage

Figure 2.4 shows the energisation of the RL circuit for peak voltage where only the steady-state component is present.

Energisation at zero voltage:
The energisation of the RL load for zero voltage is equivalent to having $\theta = 0°$. The steady-state component is the same for the previous example, as it would be for any example that we can come up with for this circuit. However, the transient component is radically different.

The transient component is a result of the need of maintaining the continuity of the current at the inductor. The steady-state current component has a phase difference of almost 90° to the voltage. Due to energy conservation, the current in the inductor must be continuous; therefore, the transient component has an initial value equal to that of the steady-state current in the connection moment with an opposite sign. Thus, if the RL load is energised when the voltage is zero, the DC component will be at its maximum with a value which is in theory equal to the peak value of the steady-state component.

Figure 2.5 shows the energisation of the RL circuit for zero voltage. Notice the decaying DC component whose initial amplitude is equal to the peak value of the steady-state component, ~ 3.2 A. The decaying rate of the DC component depends

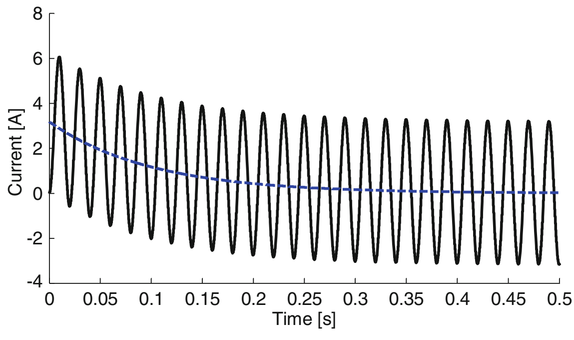


Fig. 2.5 Current during the first 0.5 s when using an AC voltage and energising at zero voltage. *Solid line* current in the RL load, *dashed line* transient component

on the time constant *R/L*, and the smaller the time constant, the longer it takes to damp the transient.

The DC component in the previous example is positive, but it can also be negative. A voltage passes twice by zero during a cycle, and the DC component is maxima for both zeros. However, the signal of the DC component depends on derivate of the voltage. If the voltage is going from negative values to positive values (positive derivate) the DC component has a positive signal; if the voltage is going from positive values to negative values (negative derivate) the DC component has a negative signal.

2.2.3 Summary

In this section, we took contact with our first transient, the energisation of an RL load. We saw how to use the Laplace transform to solve the system and how the switching instant influences the transient. In the case of a RL circuit, the peak current doubles if the circuit is energised at zero-voltage when compared with an energisation at peak voltage. Thus, contrary to what many believe, in this specific case, it would be advantageous to energise at peak voltage instead of zero voltage.

The influence of the switching instant on the transient waveform is common to many transients and it is very often the difference between a smooth transient and a highly undesired transient, as we shall see in Chap. 4.

2.3 Switching of RC Circuits (or Capacitor Banks)

After analysing an RL circuit, we are ready for the next step which is the RC circuit. The RC circuit is also a first order circuit and it resembles the behaviour of a capacitor bank. Figure 2.6 shows the single-line of an RC circuit consisting in a resistance and capacitor in series, connected to a voltage source through a switch.

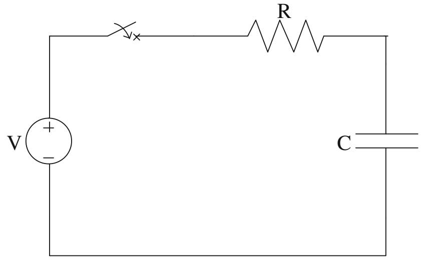


Fig. 2.6 Switching of an RC load

2.3.1 AC Source

An RC load behaves like an open circuit for a DC current, and its transient is identical to the one obtained when the load is connected to an AC source at peak voltage. Thus, we do an analysis for an AC source. Like before, the first thing to do is to write the equation describing the system (2.29) and to apply the Laplace transform (2.30). In order to be able to generalise and become even more acquainted with the Laplace transform, we now consider that the switch can be at any given instant.

$$V_P \sin(\omega t + \theta) = RI + \frac{1}{C}\int I\mathrm{d}t \tag{2.29}$$

$$\begin{aligned} &V_P\left(\frac{\omega\cos(\theta)}{s^2+\omega^2} + \frac{s\sin(\theta)}{s^2+\omega^2}\right) = RI + \frac{1}{C}\frac{I}{s} + \frac{0}{s} \\ &\Leftrightarrow I = V_P\frac{saC}{s+a}\left(\frac{\omega\cos(\theta)}{s^2+\omega^2} + \frac{s\sin(\theta)}{s^2+\omega^2}\right), \text{ where } a = \frac{1}{RC} \end{aligned} \tag{2.30}$$

The easiest way to solve the equation is to divide the second term of (2.31) into two parts (2.32).

$$I = V_P aC\left(\frac{s\omega\cos(\theta)}{(s^2+\omega^2)(s+a)} + \frac{s^2\sin(\theta)}{(s^2+\omega^2)(s+a)}\right) \tag{2.31}$$

$$\begin{aligned} (1) &: \tfrac{s\omega\cos(\theta)}{(s^2+\omega^2)(s+a)} \\ (2) &: \tfrac{s^2\sin(\theta)}{(s^2+\omega^2)(s+a)} \end{aligned} \tag{2.32}$$

By applying the partial fraction method to (2.32), (2.33) is obtained. Replacing (2.33) in (2.31), (2.34) is obtained.

$$\begin{aligned} (1) &: \tfrac{\omega\cos(\theta)}{\omega^2+a^2}\left(\tfrac{sa+\omega^2}{s^2+\omega^2} - \tfrac{a}{s+a}\right) \\ (2) &: \tfrac{\sin(\theta)}{\omega^2+a^2}\left(\tfrac{s\omega^2-a\omega^2}{s^2+\omega^2} + \tfrac{a^2}{s+a}\right) \end{aligned} \tag{2.33}$$

$$I = V_P\frac{aC}{a^2+\omega^2}\left[\omega\cos(\theta)\left(\frac{sa+\omega^2}{s^2+\omega^2} - \frac{a}{s+a}\right) + \sin(\theta)\left(\frac{s\omega^2-a\omega^2}{s^2+\omega^2} + \frac{a^2}{s+a}\right)\right] \tag{2.34}$$

The inverse Laplace transform (VI, VIII and IX in Table 2.1) is applied to (2.34) and (2.35) is obtained.

$$\begin{aligned} I(t) = &V_P\frac{aC}{a^2+\omega^2}\left[\omega\cos(\theta)(a\cos(\omega t) + \omega\sin(\omega t) - ae^{-at})\right] \\ &+ \sin(\theta)\left(\omega^2\cos(\omega t) - a\omega\sin(\omega t) + a^2e^{-at}\right) \end{aligned} \tag{2.35}$$

The power factor of an RC load is given by (2.36), which is equivalent to (2.37).

$$\cos(\phi) = \frac{R}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \tag{2.36}$$

$$\begin{aligned} \cos(\phi) &= \frac{1}{\frac{1}{R}\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \Leftrightarrow \cos(\phi) = \frac{1}{\sqrt{1 + \left(\frac{1}{\omega RC}\right)^2}} \Leftrightarrow \cos(\phi) \\ &= \frac{1}{\frac{1}{\omega}\sqrt{\omega^2 + \left(\frac{1}{RC}\right)^2}} \Leftrightarrow \cos(\phi) = \frac{\omega}{\sqrt{\omega^2 + a^2}} \end{aligned} \tag{2.37}$$

Similar relations can be obtained for $\sin(\phi)$ (2.38) and $\tan(\phi)$ (2.39).

$$\sin^2(\phi) + \cos^2(\phi) = 1 \Leftrightarrow \sin(\phi) = \sqrt{1 - \frac{\omega^2}{a^2 + \omega^2}} \Leftrightarrow \sin(\phi) = \frac{a}{a^2 + \omega^2} \tag{2.38}$$

$$\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)} \Leftrightarrow \tan(\phi) = \frac{a}{\omega} \tag{2.39}$$

Substituting (2.37)–(2.39), (2.40) is obtained.

$$\begin{aligned} I(t) &= V_P \frac{aC}{\sqrt{a^2 + \omega^2}} \omega\cos(\theta)(\sin(\phi)\cos(\omega t) + \cos(\phi)\sin(\omega t) - \sin(\phi)e^{-at}) \\ &\quad + V_P \frac{aC}{\sqrt{a^2 + \omega^2}} \omega\sin(\theta)\left(\cos(\phi)\cos(\omega t) - \sin(\phi)\sin(\omega t) + \frac{a}{\omega}\sin(\phi)e^{-at}\right) \end{aligned} \tag{2.40}$$

Equation (2.40) can be further simplified using the trigonometric relation (2.42) and (2.42), leading to (2.43).

$$\sin(\phi)\cos(\omega t) + \cos(\phi)\sin(\omega t) = \sin(\omega t + \phi) \tag{2.41}$$

$$\cos(\phi)\cos(\omega t) - \sin(\phi)\sin(\omega t) = \cos(\omega t + \phi) \tag{2.42}$$

$$\begin{aligned} I(t) &= V_P \frac{aC}{\sqrt{a^2 + \omega^2}} \\ &\left[\omega\cos(\theta)(\sin(\omega t + \phi) - \sin(\phi)e^{-at}) + \omega\sin(\theta)\left(\cos(\phi + \omega t) + \frac{a}{\omega}\sin(\phi)e^{-at}\right)\right] \end{aligned} \tag{2.43}$$

Equation (2.43) can be written as (2.44).

$$I(t) = \frac{V_P}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \cos(\theta)\left(\sin\left(\omega t + \tan^{-1}\left(\frac{1}{\omega RC}\right)\right) - \sin\left(\tan^{-1}\left(\frac{1}{\omega RC}\right)\right)e^{-\frac{1}{RC}t}\right) + \\ + \frac{V_P}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \sin(\theta)\left(\cos\left(\omega t + \tan^{-1}\left(\frac{1}{\omega RC}\right)\right) + \frac{1}{\omega RC}\sin\left(\tan^{-1}\left(\frac{1}{\omega RC}\right)\right)e^{-\frac{1}{RC}t}\right) \tag{2.44}$$

A capacitance of either a cable or a capacitor bank is in the order of micro-farad. Thus, (2.44) can be simplified into (2.45).

$$I(t) = \frac{V_P}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \left[\cos(\theta)\left(\sin\left(\omega t + \frac{\pi}{2}\right) - e^{-\frac{1}{RC}t}\right) + \sin(\theta)\left(\cos\left(\omega t + \frac{\pi}{2}\right) + \frac{1}{\omega RC}e^{-\frac{1}{RC}t}\right)\right] \tag{2.45}$$

Another consequence of having typically a capacitance of micro-farad is the low value of the time constant $\tau = RC$. Meaning that an energisation transient is damped in some micro-seconds; something that must be taken into account when choosing the simulation time-step.

Energisation at zero voltage:

Like before, we are going to separate the analysis of the energisation of the load at zero voltage and the energisation of the load at peak voltage into two cases, starting with the former.

The energisation of the RC load for zero voltage is equivalent to having $\theta = 0°$. Thus, (2.45) can be simplified into (2.46).

$$I(t) = \frac{V_P}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \left(\sin\left(\omega t + \frac{\pi}{2}\right) - e^{-\frac{1}{RC}t}\right) \tag{2.46}$$

The equation shows us that the current starts at zero and rises while the transient component is damped. It was previously stated that the time constant is typically very high and that the transient component is damped in just some micro-seconds. As a result, the current rises to a value approximately equal to the peak in a matter

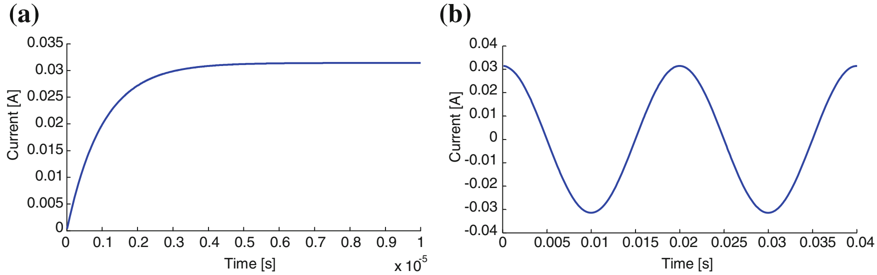


Fig. 2.7 Current in the load when energising at zero voltage. **a** First 10 μs. **b** First 40 ms

of micro-seconds (remember that the variation of a sinus function magnitude around the maximum is very slow).

Figure 2.7 shows the energisation of a RC load with a resistance of 1 Ω and a capacitance of 1 μF connected to a 100 V-peak voltage source. Figure 2.7a zooms the first 10 μs of the energisation and shows that the current reaches the peak value in just 5 μs and that after this instant, only the steady-state current is present.

Energisation at peak voltage:
The energisation of the RC load for zero voltage is equivalent to having $\theta = \pm 90°$. Thus, (2.45) can be simplified into (2.47).

$$I(t) = \pm \frac{V_P}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \left(\cos\left(\omega t + \frac{\pi}{2}\right) + \frac{1}{\omega RC} e^{-\frac{1}{RC}t} \right) \tag{2.47}$$

The analysis indicated that the current jumps from 0 A to current approximately equal to V_P/R instantaneously. Some readers may now be a little confused as they were taught that an instantaneous current change is impossible.

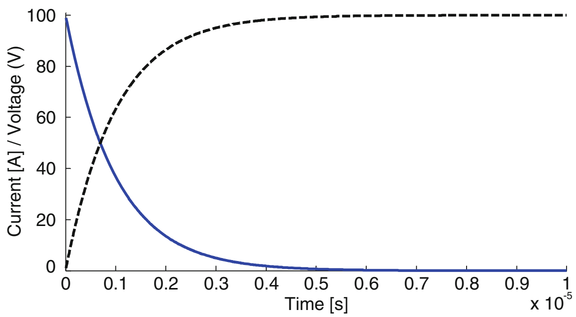
At this point, it is a good idea to distinguish between the electrical and the magnetic fields. A capacitor is an element that stores energy in the electric field, whereas an inductor is an element that stores energy in the magnetic field.

A change in the electric field requires a change in the voltage or charge which is opposed by a current. Thus, an instantaneous change of the voltage would require infinite current, something which is impossible because it would require infinite power. In other words, there must be conservation of charge and the voltage may be continuous at the capacitor.

As an example, think about the voltage-current relation of a capacitor (2.48). If the voltage changes suddenly, like in the connection of the capacitor to an ideal voltage source, the value of dV/dt would be infinite and so also would be the current.

$$I(t) = C\frac{\mathrm{d}V}{\mathrm{d}t} \tag{2.48}$$

Fig. 2.8 Current (*solid line*) and voltage in the capacitor (*dashed line*) during the first 10 μs of the energisation of an RC load



A similar situation happens in a magnetic field, where a change in the current is opposed by an electromotive force (emf), i.e., a voltage. Thus, an instantaneous change of the current would require infinite voltage.

Going back to our RC example, we see that there is an instantaneous change of the current, but not to infinite. This difference is a consequence of the law of conservation of charge preciously stated and the presence of the resistance.

The capacitor is initially uncharged, but it starts to charge when the switch is closed. The voltage at the capacitor has to be continuous, meaning that at $t(0^+)$ all the voltage is dropped at the resistor. As a result, the current at $t(0^+)$ is equal to V_P/R. As the capacitor charges, the voltage at the resistor drops, as does the current, until the system reaches steady-state conditions.

Figure 2.8 shows the current and voltage in the capacitor for an energisation at peak voltage of the RC load used in the previous example. Notice how the voltage at the capacitor raises from zero to approximately 100 V, while the current decreases to the steady-state peak value.

In this particular case, the resistance is 1 Ω and the voltage at the resistor is equal to the current. Therefore, the summation of the two curves at any given instant is equal to the voltage in the source for this particular case.

A capacitor bank strongly resembles a RC circuit and a large current will be present in the energisation of a capacitor bank, if no extra precautions are taken. This current is called inrush current and it has both a high magnitude and a high frequency.

However, a real circuit also has some inductance, which reduces the current magnitude and frequency and more importantly assures the continuity of the current, i.e., there is no longer the current jump that is present in a RC circuit.

2.3.2 *The Importance of the Time-Step*

We saw that the energisation transient of an RC load is damped in just some micro-seconds. Thus, special care regarding the time-step is necessary when simulating this phenomenon in an EMTP-type software.

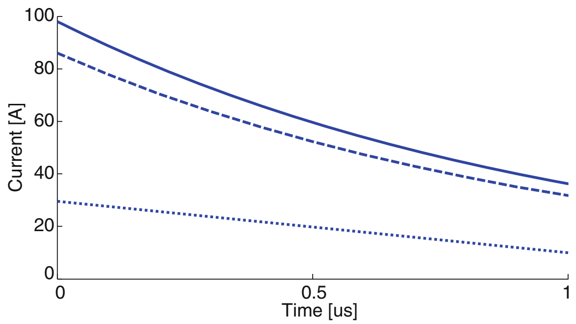


Fig. 2.9 Current in the in the capacitor during the first 10 µs for different time steps. *Solid line* 0.01 µs, *dashed line* 0.1 µs, *dotted line* 1 µs

Figure 2.9 shows the current in the RC load during the energisation for different simulation time-steps in EMTDC/PSCAD. The peak current is different for all the three cases and the smaller the time-step, the higher the peak current.

The time constant of this particular example is 1 μs. Thus, the peak current of the 1 μs is approximately 30 A, remembering that the time constant is the time necessary for the current to decay 63.2%.

This is a simple example of the importance of choosing a good time-step and how sometimes it is necessary to use very small time-steps. However, we want to have the time-step as large as possible in order to reduce the simulation running time. Thus, it is necessary to learn how to choose the right time-step in function of the expected phenomenon.

2.3.3 Summary

In this section, we analysed the energisation of an RC load. We saw another example of the use of the Laplace transform, how the transient changes in function of the switching instant and how the transient current and the frequency may be very high when energising the load at peak voltage.

Contrary to the RL load, the transient is "smoother" when energising at zero voltage and more problematic when energising at peak voltage.

Finally, we saw how the time-step influences the results and how it is important to choose the appropriate time-step.

2.4 Switching of RLC Circuits

The final example of circuits with lumped-parameters is the RLC circuit. The RLC circuit is a second-order circuit whose behaviour is more complex than in the previous two examples, as both electric and magnetic fields are present.

The conjugation of these three elements can describe, up to some degree, many of the electrical equipment that exists in real life. As an example, a capacitor bank can be described as just a resistor and a capacitor, but it would never be connected to an ideal voltage source. In reality there is always some inductance between the two elements, making the circuit similar to an RLC circuit. Another example is the pi-model used to represent overhead lines or cables, which we will use in future chapters.

2.4.1 DC Source

A series RLC circuit connected to a DC voltage source is described by (2.49), which can be written in the frequency domain (2.50). Note: For a parallel RLC circuit, see the exercises.

$$V = RI + L\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C}\int I\mathrm{d}t \tag{2.49}$$

$$\frac{\mathrm{d}(V_P)}{\mathrm{d}t} = R\frac{\mathrm{d}I}{\mathrm{d}t} + L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{I}{C} \Leftrightarrow 0 = \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{I}{LC} \tag{2.50}$$

A series RLC circuit connected to a DC source has only the transient component of the current, whereas the steady-state current is zero; remember that the capacitor is like an open circuit when in the presence of a DC current. The transient component is a result of an exchange of energy between the capacitor and the inductor that it is eventually damped by the resistance.

Equation (2.50) is a homogeneous differential equation and it can easily be solved without using the Laplace Transform. The derivatives are replaced by λ and (2.50) is replaced by (2.51). The roots of (2.51) are calculated (2.52) and replaced in the general solution of a homogeneous differential equation (2.53).

$$0 = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} \tag{2.51}$$

$$\lambda_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{2.52}$$

$$I(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{2.53}$$

The roots solution ($\lambda_{1,2}$) can be one of three types, each with a different type of solution.

Two distinct real roots: $(R/2L)^2 > 1/(LC)$ –> Overdamped circuit

Two complex conjugate roots: $(R/2L)^2 < 1/(LC)$ –> Underdamped circuit (Oscillating)

A double root: $(R/2L)^2 = 1/(LC)$ –> Critically damped circuit.

It is still necessary to calculate the values of C_1 and C_2, which depend on the system initial conditions. There are two variables, meaning that two equations are needed (2.54).

$$\begin{cases} I(0) = C_1 + C_2 \\ \dot{I}(0) = \lambda_1 C_1 + \lambda_2 C_2 \end{cases} \tag{2.54}$$

The circuit is de-energised prior to the switching and the value of $I(0)$ is simply zero.

For calculating the value of $\dot{I}(0)$, we must do some deductions. An RLC circuit has both capacitance and inductance and it is not possible to have a sudden change of the voltage because of the capacitor, or of the current because of the inductor. As the current cannot change instantaneously, its value is 0 A at $t = 0^+$ and there is no voltage drop at the resistor. Therefore, all the voltage is dropped in the inductor (2.55).

$$V(0^+) = L\frac{\mathrm{d}I}{\mathrm{d}t}\bigg|_{t=0^+} \Rightarrow \dot{I}(0) = \frac{V}{L} \tag{2.55}$$

Substituting in (2.54), (2.56) is obtained.

$$\begin{cases} 0 = C_1 + C_2 \\ \frac{V}{L} = \lambda_1 C_1 + \lambda_2 C_2 \end{cases} \tag{2.56}$$

Example:
In the following example, the following three possible scenarios are considered:

1. Overdamped circuit: $V = 100$ V; $L = 0.1$ H; $C = 1$ μF; $R = 1{,}000$ Ω
2. Underdamped circuit: $V = 100$ V; $L = 0.1$ H; $C = 1$ μF; $R = 100$ Ω
3. Critically damped circuit; $V = 100$ V; $L = 0.1$ H; $C = 1$ μF; $R = 633$ Ω.

The first step is to calculate the roots' values for each case:

1. $\lambda_1 = -1273$; $\lambda_2 = -8873$;
2. $\lambda_{1,2} = -500 \pm \mathrm{j}3123$
3. $\lambda_{1,2} \sim 3161$ (in reality, there is a small difference between the roots, as R is an irrational number).

The next step is to calculate the value of the constants C_1 and C_2. The initial current is zero, therefore, $C_1 = -C_2$ for all the cases.

1, $C_1 = -C_2 = 0.1291$
2. $C_1 = -C_2 = -\mathrm{j}0.1601$
3. $C_1 = -C_2 = 341$.

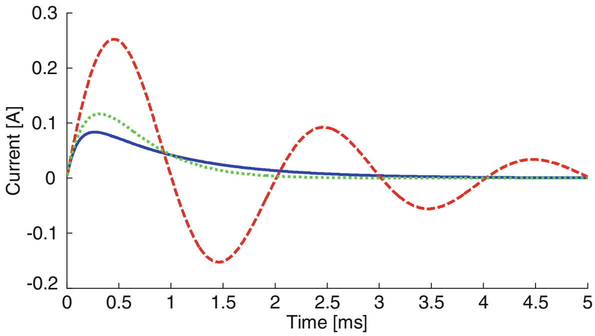


Fig. 2.10 Current during the energisation of an RLC circuit. *Solid line* overdamped circuit, *dashed line* underdamped circuit, *dotted line* critically damped circuit

Figure 2.10 shows the transient current for all the three scenarios:

1. The current increases up to a peak value and it damps to zero;
2. The current oscillates around zero, decaying as the time advances. In an LC circuit, the current continues to oscillate at approximately the same frequency, but it is not damped;
3. Identical to scenario 1, but it is the limit case. If the resistance was smaller, and the inductance and capacitance were the same, there would be an oscillation.

2.4.2 AC Source

One would think that the transient obtained when connecting an RLC load to an AC source is completely different from connecting the load to an equivalent DC source (i.e., the same peak amplitude). However, we will see that this is not necessarily true.

We start by writing the equation describing the system on the time domain (2.57) and simplify it to (2.58) and apply the Laplace Transform (2.59).

$$V_P \sin(\omega t + \theta) = RI + L\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C}\int I \mathrm{d}t \tag{2.57}$$

$$\frac{\mathrm{d}(V_P \sin(\omega t + \theta))}{\mathrm{d}t} = R\frac{\mathrm{d}I}{\mathrm{d}t} + L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \frac{I}{C} \Leftrightarrow V_P\omega \cos(\omega t + \theta) = L\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{I}{C} \tag{2.58}$$

$$V_P\omega\left(\frac{s\cos(\theta)}{s^2+\omega^2} - \frac{\omega\sin(\theta)}{s^2+\omega^2}\right) = I\left(s^2L + sR + \frac{1}{C}\right) - sLI(0) - L\dot{I}(0) - RI(0) \tag{2.59}$$

We already know from the previous sections that the current at t_0 is zero because of the continuity of the current, and that the derivative of the current at the same instant is given by $V_P(0)/L$. Thus, (2.59) becomes (2.60).

$$V_P\omega\left(\frac{s\cos(\theta)}{s^2+\omega^2} - \frac{\omega\sin(\theta)}{s^2+\omega^2}\right) = I\left(s^2L + sR + \frac{1}{C}\right) - L\frac{V_P(0)}{L} \tag{2.60}$$

At this point, we may solve the system using the Laplace transform as described in the previous sections, but this is unnecessary. The current is a summation of the forced and homogeneous components (2.61).

$$I(t) = I_f(t) + I_h(t) \tag{2.61}$$

The forced component is the steady-state component and it is easy to obtain by using phasors (2.62).

$$I_f(t) = \frac{V_P}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right) \tag{2.62}$$

The homogeneous component is similar to the one obtained when using a DC source, with the difference that the value of $V_P(0)$ depends on the switching instant and that the derivative of the voltage is no longer zero. The relation (2.63) continues to be valid, but the values of $I_h(0)$ and $\dot{I}_h(0)$ need to be calculated. The premises used for the DC source example continue to be valid; the initial current has to be zero, thus (2.64), and the value of the derivative of the current at $t = 0$ is still given by having all the voltage dropping in the inductor. Thus, the derivative of the homogeneous current component is given by (2.65). The variables λ_1 and λ_2 continue to be calculated by (2.52).

$$\begin{cases} I_h(0) = C_1 + C_2 \\ \dot{I}_h(0) = \lambda_1 C_1 + \lambda_2 C_2 \end{cases} \tag{2.63}$$

$$I(0) = 0 \Leftrightarrow I_h(0) = -I_f(0) \tag{2.64}$$

$$\dot{I}(0) = \dot{I}_h(0) + \dot{I}_f(0) \Rightarrow \dot{I}_h(0) = \left(\frac{V_P\omega}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos\left(\theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right)\right) - \left(\frac{V_P}{L}\sin(\theta)\right) \tag{2.65}$$

Replacing (2.64) and (2.65) in (2.63), (2.66) is obtained for C_1 and C_2. Replacing the results in (2.61), the general expression of the current in the RLC load when connected to an AC source is obtained (2.67).

$$\begin{cases} C_1 = \frac{\dot{I}_h(0) + I_f(0)\lambda_2}{\lambda_1 - \lambda_2} \\ C_2 = \frac{\dot{I}_h(0) + I_f(0)\lambda_1}{\lambda_2 - \lambda_1} \end{cases} \tag{2.66}$$

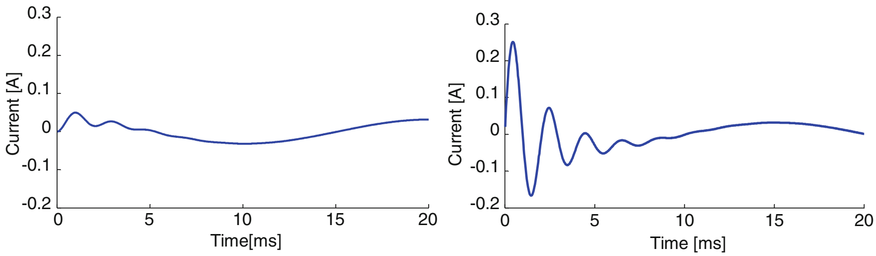


Fig. 2.11 Current during the energisation of the RLC load. *Left* energisation at $\theta = 0°$, *right* energisation at $\theta = 90°$

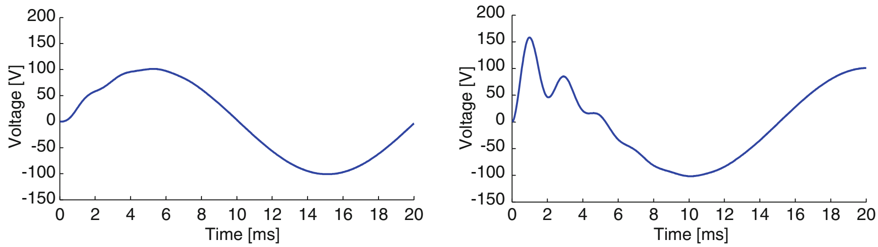


Fig. 2.12 Voltage in the capacitor (*VC*) during the energisation of the RLC load. *Left* energisation at $\theta = 0°$, *right* energisation at $\theta = 90°$

$$I(t) = \frac{V_P}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left(\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right) + \frac{\dot{I}_h(0) + I_f(0)\lambda_2}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + \frac{\dot{I}_h(0) + I_f(0)\lambda_1}{\lambda_2 - \lambda_1} e^{\lambda_2 t} \tag{2.67}$$

Figure 2.11 shows the current during an energisation at zero voltage and peak voltage respectively. The simulation parameters are $V_P = 100$ V; $L = 0.1$ H; $C = 1$ μF; $R = 100$ Ω.

Notice that the waveform for the energisation at peak voltage is very similar, regarding both shape and magnitude, to the one obtained when energising the same load using a DC source of equal magnitude. The homogeneous component depends on the voltage value at the energisation instant and the load initial conditions.

The voltage initial conditions and the load initial conditions are the same in both cases. The small difference between the two cases is present because the forced component is not zero at $t = 0$. However, as this component is close to zero, the difference between the two waveforms is very small.

It should also be noticed that the homogeneous waveform is influenced by the load parameters in the same way as for the energisation with a DC source. As an example, an overdamped oscillation would be present if the resistor has of 1,000 Ω.

Until now, we have been focusing on the current behaviour and not considering the voltage much. Yet, in the same way that there is a transient waveform for the current, there is also a transient waveform for the voltage.

The voltage at the terminals of each element is a function of the current: linear function if a resistor, derivative if an inductor and integrative if a capacitor. Thus, the voltage at the element's terminal is expected to be larger when energising at peak voltage, as both the magnitude and the current variation are larger in this situation.

Figure 2.12 shows the voltage at the capacitor terminals for an energisation at zero and peak voltage, where it can be seen that the voltage magnitude is substantially larger for the second case.

We will return to this topic in the next chapters and understand better how the switching instant may be a determining parameter when switching a cable.

2.4.3 Summary

In this section, we analysed an RLC circuit and saw how its behaviour is strongly influenced by both the switching instant and the load parameters. We saw how the current or voltage can be divided into two types, the forced and homogeneous regime. The forced regime consists of the system in steady-state condition, whereas the homogenous regime is a transient condition, the total current/voltage is the summation of both regimes.

The equations were solved without the use of the Laplace transform in order to show the reader other possibilities. However, the system can still be solved using the Laplace transform. As a matter of fact, this is one of the exercises proposed next, and the solution may be found online.

2.5 Exercises

1. Obtain the current expression for the RLC circuit of Sect. 2.4 (DC source) using the Laplace Transform.

   A: $I = \frac{V}{L}\left(\frac{\frac{1}{2\sqrt{\left(\frac{R}{2L}\right)^2-\left(\frac{1}{\sqrt{LC}}\right)^2}}}{s-\left(-\frac{R}{2L}-\sqrt{\left(\frac{R}{2L}\right)^2-\left(\frac{1}{\sqrt{LC}}\right)^2}\right)} - \frac{\frac{1}{2\sqrt{\left(\frac{R}{2L}\right)^2-\left(\frac{1}{\sqrt{LC}}\right)^2}}}{s-\left(-\frac{R}{2L}+\sqrt{\left(\frac{R}{2L}\right)^2-\left(\frac{1}{\sqrt{LC}}\right)^2}\right)}\right)$

2. Repeat exercise 1, but for a parallel RLC circuit, for both a DC and AC voltage sources.

   A: $I(t) = \frac{V}{R} + \frac{V}{L}t + \mathrm{VC}\delta(t)$; $I = \frac{V\sin(\omega t)}{R} + \frac{V}{\omega L} - \frac{V}{\omega L}\cos(\omega t) + \omega CV\cos(\omega t)$

3. Obtain the expression for the transient of the voltage at the receiving end of a line connected to an AC source using a pi-model. The energisation is made at peak voltage and the values are the following: $R = 0.62\ \Omega$; $L = 44.7$ mH; $C = 3.9\ \mu$F; $V_P = 100$ kV.

   A: $V_2 = \frac{1}{1.95\times10^{-6}}\left(0.2\cos(\omega t) + e^{-6.94t}(4\times10^{-4}\sin(3387t) + 0.2\cos(3387t))\right)$

4. For the same pi-model obtain the expression of the current at the sending end of the line.

   A: $I(t) = -61.26\sin(\omega t) - 61.75\sin(\omega t) - e^{-6.94t}(666\sin(3387t))$

References and Further Reading

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