Chapter 9
Shocks as Burn-in

As described in the previous chapters, in conventional burn-in, the main parameter of the burn-in procedure is its duration. However, in order to shorten the length of this procedure, burn-in is most often performed in an accelerated environment. This indicates that high environmental stress can be more effective in eliminating weak items from a population. In this case, obviously, the larger values of stress should correspond to the shorter duration of burn-in. By letting the stress to increase, we can end up (as some limit) with very short (negligible) durations, in other words, *shocks*. In practice, the most common types of shocks as a method of burn-in are "thermal shock" and "physical drop". In these cases, the item is subjected to a very rapid cold-to-hot, or hot-to-cold, instantaneous thermal change or the item is dropped by a "drop tester" which is specifically designed to drop it without any rotational motion, to ensure the most rigorous impact. In this case, the stress level (to be called shock's *severity*) can be a controllable parameter for the corresponding optimization, which in a loose sense is an analogue of the burn-in duration in accelerated burn-in (see e.g., [1, 9].

This general reasoning suggests that 'electrical' (e.g., the increased voltage for a short period of time for some electronic items), thermal and mechanical shocks can be used for burn-in in heterogeneous populations of items. If the initial population is not 'sufficiently reliable', then the items that have survived a shock might be more suitable for field usage, as their *predicted* reliability characteristics could be better. Therefore, in this chapter, we consider shocks as a method of burn-in and develop the corresponding optimization model. It should be noted that several approaches (such as Environmental Stress Screening to be considered in the next chapter) that exhibit a similar initial reasoning were already implemented in industry as a practical tool (see, for example, [13, 16, 17].

As in the previous chapters, we will also assume that the population is the mixture of stochastically ordered subpopulations. As before, we will consider both discrete and continuous mixture models. Under this and some other natural assumptions, we consider the problem of determining the optimal severity level of a stress. Furthermore, we develop approaches that minimize the risks of selecting items with large levels of individual failure rates for missions of high importance,

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where failures can result, e.g., in substantial economic losses. We consider some new measures for describing the corresponding optimal burn-in, which boils up in obtaining the optimal severity of shocks. For instance, the losses that are monotonically increasing with the value of the failure rate of items after burn-in are introduced. Furthermore, focusing on the quality of relatively poor (with large failure rates) items in the mixed population, some conservative measures for the population quality are defined and the corresponding optimal burn-in with respect to these measures is also investigated.

We will also consider burn-in for items that will operate (after burn-in) in the environment with shocks. We assume that there are two competing risk causes of failure—the ‘usual’ one (in accordance with aging processes in a system) and environmental shocks. A new type of burn-in via the controlled (laboratory) test shocks is considered and the problem of obtaining the optimal level (severity) of these shocks is investigated.

9.1 Discrete Mixtures

9.1.1 General Setting

We assume in this section that a population is a mixture of two ordered subpopulations—the strong subpopulation and the weak subpopulation. Let the lifetime of a component from the strong subpopulation be denoted by X_S and its absolutely continuous cumulative distribution function (Cdf), probability density function (pdf) and the failure rate function be $F_1(t), f_1(t)$ and $\lambda_1(t)$, respectively. Similarly, the lifetime, the Cdf, pdf, and the failure rate function of a weak component are denoted by X_W, $F_2(t), f_2(t)$ and $\lambda_2(t)$, respectively. Let the lifetimes in these subpopulations be ordered either in the sense of the failure rate ordering:

$$\lambda_1(t) \leq \lambda_2(t), \text{for all } t \geq 0$$

or in the sense of the usual stochastic ordering

$$\overline{F}_1(t) \geq \overline{F}_2(t), \text{for all } t \geq 0,$$

where $\overline{F}_i(t) = 1 - F_i(t)$, $i = 1, 2$. Assume that the mixing proportion (distribution) is given by

$$\pi(z) = \begin{cases} \pi, & z = z_1 \\ 1 - \pi, & z = z_2 \end{cases},$$

where z_1 and z_2, $z_1 < z_2$, are variables that represent the strong and the weak subpopulations, respectively. Therefore, $Z = (z_1, z_2)$ can be considered as the discrete frailty in this case. Then the corresponding mixture distribution and the density functions are defined as in the previous chapters:

$$F_m(t) = \pi F_1(t) + (1-\pi)F_2(t),$$
$$f_m(t) = \pi f_1(t) + (1-\pi)f_2(t),$$

respectively, and the mixture failure rate is

$$\lambda_m(t) = \frac{\pi f_1(t) + (1-\pi)f_2(t)}{\pi \bar{F}_1(t) + (1-\pi)\bar{F}_2(t)} = \pi(z_1|t)\lambda_1(t) + \pi(z_2|t)\lambda_2(t),$$

where the time-dependent probabilities are

$$\pi(z_1|t) = \frac{\pi \bar{F}_1(t)}{\pi \bar{F}_1(t) + (1-\pi)\bar{F}_2(t)},$$
$$\pi(z_2|t) = 1 - \pi(z_1|t) = \frac{(1-\pi)\bar{F}_2(t)}{\pi \bar{F}_1(t) + (1-\pi)\bar{F}_2(t)}.$$

Assume that at time $t = 0$ an instantaneous shock has occurred and with complementary probabilities it either 'kills' an item (i.e., a failure occurs), or 'leaves it unchanged'. The following is the basic assumption in our reasoning:

Basic Assumption
The more frail (e.g., with the larger failure rate) the items are, the more susceptible they are to be 'killed' by a shock.

Let $\pi_s(z)$ denote the frailty distribution *after a shock* and let T_s and $\lambda_{ms}(t)$ be the corresponding lifetime and the mixture (observed) failure rate, respectively. Denote the probabilities of failures caused by each shock for two subpopulations as:

$$p(z) = \begin{cases} p_1, & z = z_1, \\ p_2, & z = z_2. \end{cases} \tag{9.1}$$

Here, in accordance with our Basic Assumption, $p_1 \le p_2$. It is easy to show that

$$\pi_s(z) = \begin{cases} \frac{(1-p_1)\pi}{(1-p_1)\pi + (1-p_2)(1-\pi)} \equiv \pi_s, & z = z_1, \\ \frac{(1-p_2)(1-\pi)}{(1-p_1)\pi + (1-p_2)(1-\pi)} \equiv 1 - \pi_s, & z = z_2. \end{cases}$$

and

$$\lambda_{ms}(t) = \frac{\pi_s f_1(t) + (1-\pi_s)f_2(t)}{\pi_s \bar{F}_1(t) + (1-\pi_s)\bar{F}_2(t)} = \pi_s(z_1|t)\lambda_1(t) + \pi_s(z_2|t)\lambda_2(t), \tag{9.2}$$

where

$$\pi_s(z_1|t) = \frac{\pi_s \bar{F}_1(t)}{\pi_s \bar{F}_1(t) + (1-\pi_s)\bar{F}_2(t)},$$
$$\pi_s(z_2|t) = 1 - \pi_s(z_1|t) = \frac{(1-\pi_s)\bar{F}_2(t)}{\pi_s \bar{F}_1(t) + (1-\pi_s)\bar{F}_2(t)}.$$

The corresponding survival function is given by

$$\overline{F}_{ms}(t) = \pi_s \overline{F}_1(t) + (1 - \pi_s)\overline{F}_2(t).$$

The following initial result justifies the fact that a shock can be considered as the burn-in procedure.

Theorem 9.1 *Let* $p_1 \le p_2$.

(i) *If* $\lambda_1(t) \le \lambda_2(t)$, *for all* $t \ge 0$, *then* $\lambda_{ms}(t) \le \lambda_m(t), \forall t \in [0, \infty)$.
(ii) *If* $\overline{F}_1(t) \ge \overline{F}_2(t)$, *for all* $t \ge 0$, *then* $\overline{F}_{ms}(t) \ge \overline{F}_m(t), \forall t \in [0, \infty)$.

Proof Observe that $\lambda_m(t)$ and $\lambda_{ms}(t)$ are weighted averages of $\lambda_1(t)$ and $\lambda_2(t)$. Then it is sufficient to show that $\pi_s(z_1|t) \ge \pi(z_1|t)$. Note that

$$\pi(z_1|t) = \frac{\pi\bar{F}_1(t)}{\pi\bar{F}_1(t) + (1 - \pi)\bar{F}_2(t)} = \frac{\bar{F}_1(t)}{\bar{F}_1(t) + (1/\pi - 1)\bar{F}_2(t)}$$

is increasing in π, and

$$\pi_s - \pi = \frac{(1 - p_1)\pi}{(1 - p_1)\pi + (1 - p_2)(1 - \pi)} - \pi = \frac{\pi(1 - \pi)(p_2 - p_1)}{(1 - p_1)\pi + (1 - p_2)(1 - \pi)} \ge 0.$$

Therefore, $\pi_s(z_1|t) \ge \pi(z_1|t)$ and we can conclude that $\lambda_{ms}(t) \le \lambda_m(t), \forall t \in [0, \infty)$.

On the other hand, $\overline{F}_m(t)$ and $\overline{F}_{ms}(t)$ are also weighted averages of $\overline{F}_1(t)$ and $\overline{F}_2(t)$. Then the second result is obvious from the fact that $\pi_s \ge \pi$. □

Remark 9.1 The above result implies that reliability characteristics of a population of items that have survived a shock have improved. This justifies the described burn-in procedure as a measure of improving the 'quality' of a heterogeneous population. Depending on assumptions, Theorem 9.1 states that the population lifetime random variable after a shock is larger than that before the shock either in the sense of the failure rate ordering, or in the sense of the usual stochastic ordering. Note that individual characteristics of an item that has survived a shock, due to our assumption, are same as before.

9.1.2 Optimal Severity for Population Quality Measures

The optimal burn-in time is the main characteristic of interest in conventional burn-in procedures. In our model, the 'severity' of a shock in a way corresponds to this burn-in time. Therefore, we will suggest now an approach for determining an optimal magnitude of a shock that maximizes the 'quality' of our population after burn-in.

Denote the magnitude of a shock by $s \in [0, \infty]$. Assume that the 'strength' of the component in a strong subpopulation is a continuous random variable, which is denoted by U. By 'strength' we understand here the corresponding measure of resistance to a single shock, i.e., if $s > U$, then the failure occurs. Let the Cdf, the survival function, and the failure rate function of U are denoted by $G(s)$, $\overline{G}(s)$, and $r(s)$, respectively. Similarly, let the strength of the component in a weak subpopulation be denoted by U_w. Then, in accordance with our Basic Assumption, let

$$U \geq_{st} U_w.$$

It is easy to see that this inequality is equivalent to

$$G_w(s) = G(\rho(s)), \text{for all } s \geq 0, \tag{9.3}$$

where $G_w(s)$ is the Cdf of U_w, $\rho(s)$ is an increasing function, $\rho(s) \geq s$ for all $s \geq 0$, and $\rho(0) = 0$. It follows from (9.1) that the probabilities of failure for this case are given by

$$p(z, s) = \begin{cases} p_1 = G(s), & z = z_1, \\ p_2 = G(\rho(s)), & z = z_2. \end{cases} \tag{9.4}$$

Then $p_1 \leq p_2$ holds for all $s \in [0, \infty)$. Under the above setting, $\lambda_{ms}(t)$ is also a function of s and therefore will be denoted as $\lambda_{ms}(t; s)$:

$$\lambda_{ms}(t; s) = \frac{\pi_s \bar{F}_1(t)}{\pi_s \bar{F}_1(t) + (1 - \pi_s)\bar{F}_2(t)} \cdot \lambda_1(t) + \frac{(1 - \pi_s)\bar{F}_2(t)}{\pi_s \bar{F}_1(t) + (1 - \pi_s)\bar{F}_2(t)} \cdot \lambda_2(t),$$

where

$$\pi_s = \frac{(1 - G(s))\pi}{(1 - G(s))\pi + (1 - G(\rho(s)))(1 - \pi)},$$
$$1 - \pi_s = \frac{(1 - G(\rho(s)))(1 - \pi)}{(1 - G(s))\pi + (1 - G(\rho(s)))(1 - \pi)}.$$

Denote the expected lifetime (as a function of s) of an item that has survived a shock by $m(s)$ and, by $P(\tau, s)$, the probability of success (survival probability) for a mission time τ. We are interested in 'pure' maximization of these functions without considering any costs or gains. Thus we want to maximize (with respect to s) the following functions:

$$m(s) = \int_0^\infty \exp\left\{ - \int_0^t \lambda_{ms}(u; s) \mathrm{d}u \right\} \mathrm{d}t, \tag{9.5}$$

$$P(\tau, s) = \exp\left\{ - \int_0^\tau \lambda_{ms}(u; s) \mathrm{d}u \right\}. \tag{9.6}$$

Intuitively, the first guess would be: the larger is the level of severity s, the larger are the functions of interest, which means that *formally* $s^* = \infty$ and we understand this notation here and in the rest of the chapter only in the described sense. However, as the strength of the item is given by distributions in (9.3), there can be the other non-intuitively evident possibility.

In order to investigate the maximizations of (9.5) and (9.6), consider a more general problem—the uniform minimization of $\lambda_{ms}(t;s)$, for all fixed $t \geq 0$, with respect to $s \in [0, \infty]$. That is, find s^* which satisfies

$$s^* = \arg \inf_{s \in [0,\infty]} \lambda_{ms}(t;s), \text{for all fixed } t \geq 0.$$

Denote by $R(s) \equiv \int_0^s r(u)\mathrm{d}u$ the cumulative failure rate that corresponds to the Cdf $G(s)$. Then the following result describes the optimal severity s^*.

Theorem 9.2 *Let $\lambda_1(t) \leq \lambda_2(t)$, for all $t \geq 0$. Then the optimal s^* is the value which maximizes $R(\rho(s)) - R(s)$. In particular,*

(i) *If $r(s)$ is increasing and $\rho'(s) > 1$, then $s^* = \infty$.*
(ii) *If $\frac{\rho'(s)r(\rho(s))}{r(s)} > 1$, for $s < s_0$, and $\frac{\rho'(s)r(\rho(s))}{r(s)} < 1$, for $s > s_0$, then $s^* = s_0$.*

Proof Note again that in accordance with (9.2), $\lambda_{ms}(t;s)$ is the weighted average of $\lambda_1(t)$ and $\lambda_2(t)$ with the corresponding weights $\pi_s(z_1|t)$ and $\pi_s(z_2|t) = 1 - \pi_s(z_1|t)$, respectively, and

$$\pi_s(z_1|t) = \frac{\pi_s\bar{F}_1(t)}{\pi_s\bar{F}_1(t) + (1-\pi_s)\bar{F}_2(t)} = \frac{\bar{F}_1(t)}{\bar{F}_1(t) + (1/\pi_s - 1)\bar{F}_2(t)}$$

is increasing in π_s. Thus, for each fixed $t \geq 0$, as $\lambda_1(t) \leq \lambda_2(t)$, the minimum of $\lambda_{ms}(t;s)$ is obtained by maximizing

$$\pi_s = \frac{(1-G(s))\pi}{(1-G(s))\pi + (1-G(\rho(s)))(1-\pi)}. \tag{9.7}$$

This problem is equivalent to minimizing

$$\frac{1-G(\rho(s))}{1-G(s)} = \exp\{-[R(\rho(s)) - R(s)]\}.$$

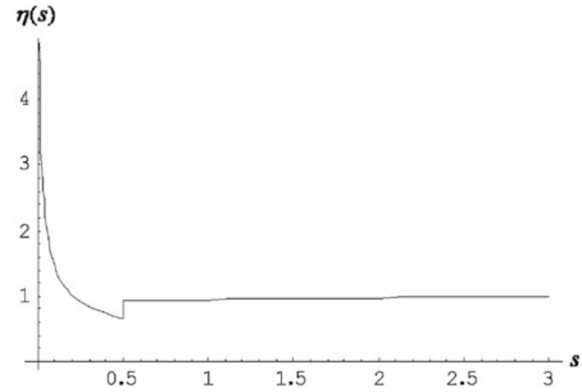
Therefore, the minimum can now be attained by maximizing $R(\rho(s)) - R(s)$.

(i) Denote $\phi(s) \equiv R(\rho(s)) - R(s)$. Then $\phi'(s) \equiv \rho'(s)r(\rho(s)) - r(s)$. As $\rho'(s) > 1$ and $r(x)$ is increasing,

$$\phi'(s) = \rho'(s)r(\rho(s)) - r(s) > r(\rho(s)) - r(s) \geq 0,$$

where assumption $\rho(s) \geq s$ is used. Thus, in this case, $s^* = \infty$.

Fig. 9.1 Graph for $\eta(s)$



(ii) Assume now that $\frac{\rho'(s)r(\rho(s))}{r(s)} > 1$, for $s < s_0$, and $\frac{\rho'(s)r(\rho(s))}{r(s)} < 1$, for $s > s_0$. Then $\phi'(s) > 0$, for $s < s_0$, and $\phi'(s) < 0$, for $s > s_0$, which implies $s^* = s_0$.

□

Example 9.1 Let $r(s) = e^{-s} + 1, s \geq 0,$ and $\rho(s) = \sqrt{s}, 0 \leq s \leq 1/2;$ $\rho(s) = s + \left(1/\sqrt{2} - 1/2,\right)s \geq 1/2$. The graph for $\eta(s) \equiv \rho'(s)r(\rho(s))/r(s)$ is given in Fig. 9.1. Then it can be seen that there exists some $0 < s_0 < \infty$ which satisfies

$$\frac{\rho'(s)r(\rho(s))}{r(s)} > 1, \text{for } s < s_0 \text{ and } \frac{\rho'(s)r(\rho(s))}{r(s)} < 1, \text{for } s > s_0.$$

Thus, obtaining this value numerically: $s^* = s_0 = 0.204$.

Remark 9.2 In practice, obviously, there exists a maximum level of stress $s_a < \infty$ that can be applied to items without destroying the whole population or without the non-negligible damage in the survived items. In this case, the first part of Theorem 9.2 is modified to $s^* = s_a$, whereas, for the second part of Theorem 9.2, if $s_0 \leq s_a$ then $s^* = s_0$; otherwise $s^* = s_a$.

Let s^* be the optimal severity level which satisfies

$$s^* = \arg \sup_{s \in [0,\infty]} \overline{F}_{ms}(t; s), \text{for all fixed } t \geq 0.$$

Corollary 9.1 *Suppose that* $\overline{F}_1(t) \geq \overline{F}_2(t)$, *for all* $t \geq 0$. *Then the optimal* s^* *is the same as the value which minimizes* $\lambda_{ms}(t; s)$, *for all fixed* $t \geq 0$.

Proof Observe that $\overline{F}_{ms}(t; s)$ is the weighted average of $\overline{F}_1(t)$ and $\overline{F}_2(t)$:

$$\overline{F}_{ms}(t; s) = \pi_s \overline{F}_1(t) + (1 - \pi_s)\overline{F}_2(t).$$

As $\overline{F}_1(t) \geq \overline{F}_2(t)$ and s^*, in accordance with Theorem 9.2, maximizes π_s, the result follows immediately. □

Note that maximizations of $m(s)$ and $P(\tau; s)$, which can be expressed as [see Eqs. (9.5) and (9.6)]

$$m(s) = \int_0^\infty \overline{F}_{ms}(t; s)\mathrm{d}t,$$

$$P(\tau; s) = \overline{F}_{ms}(\tau; s),$$

is equivalent to uniform maximization of $\overline{F}_{ms}(t; s)$. Therefore, optimal s^* is the same as given in Corollary 9.1.

In the framework of our burn-in model, consider now the corresponding gains and penalties defined for four mutually exclusive events. Denote:

- g_1: gain due to the survival of a strong component
- c_1: penalty incurred by the elimination of a strong component
- g_2: gain due to the elimination of a weak component
- c_2: penalty incurred by the survival of a weak component.

In accordance with this notation and relationship (9.4), the expected gain resulting from the burn-in procedure performed by a shock is given by the following function of severity s:

$$\begin{aligned}\varphi(s) &= g_1\pi\overline{G}(s) + g_2(1-\pi)G(\rho(s)) - c_1\pi G(s) - c_2(1-\pi)\overline{G}(\rho(s)) \\ &= -(\pi g_1 + \pi c_1)G(s) + ((1-\pi)g_2 + (1-\pi)c_2)G(\rho(s)) + g_1\pi - c_2(1-\pi).\end{aligned} \tag{9.8}$$

It is clear that maximization of $\varphi(s)$ is equivalent to minimization of

$$\pi(g_1 + c_1)G(s) + (1-\pi)(g_2 + c_2)(1 - G(\rho(s)))$$

or to minimization of

$$\psi(s) \equiv w_1 G(s) + w_2(1 - G(\rho(s))), \tag{9.9}$$

where the weights w_1 and w_2 are

$$w_1 = \frac{\pi(g_1 + c_1)}{\pi(g_1 + c_1) + (1-\pi)(g_2 + c_2)}, \; w_2 = 1 - w_1.$$

Note that the probability of failure of a strong component $G(s)$ can be interpreted as the risk that the strong component will be eliminated by a shock. On the other hand, $(1 - G(\rho(s))$ can be regarded as the risk that a weak component will survive a shock. Expressions (9.8) and (9.9) imply that maximization of expected gain is equivalent to minimization of the weighted risk. Observe that when $s = 0$, $\psi(0) = w_2$ and when $s \to \infty$, $\psi(\infty) = w_1$.

The optimal severity s^* should be obtained numerically, however, we can define an upper bound for s^* under some additional conditions.

Theorem 9.3 *Let* $w_1 > w_2$, $\rho'(s) < w_1/w_2$, *for all* $s > s_0$, *and* $r(s)$ *is decreasing for* $s > s_1$. *Then the upper bound for optimal severity level* s^* *is given by* $\max\{s_0, s_1\}$, *that is,* $s^* \le \max\{s_0, s_1\}$.

Proof Observe that

$$\psi'(s) \equiv w_1 r(s) \exp\{-R(s)\} - w_2 \rho'(s) r(\rho(s)) \exp\{-R(\rho(s))\},$$

where $R(s) \equiv \int_0^s r(u)\mathrm{d}u$. If $\rho'(s) < w_1/w_2$, for all $s > s_0$, and $r(s)$ is decreasing for $s > s_1$, then $\psi'(s) > 0$, for all $s > \max\{s_0, s_1\}$. This implies that $\psi(s)$ is strictly increasing for $s > \max\{s_0, s_1\}$. Thus the upper bound for s^* is given by $\max\{s_0, s_1\}$. □

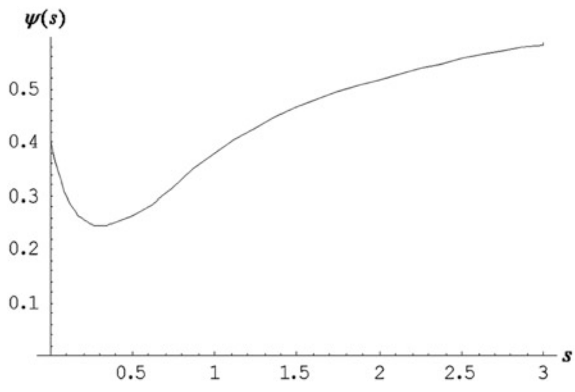
Example 9.2 Suppose that $w_1 = 0.6, w_2 = 0.4, r(s) = 1, 0 \le s < 2;\ r(s) = e^{s-2}, s \ge 2$, and $\rho(s) = 5s, 0 \le s < 1;\ \rho(s) = s + 4, s \ge 1$. Then, in this case, $s_0 = 1.0$ and $s_1 = 2.0$. Therefore, $s^* \le \max\{s_0, s_1\} = 2.0$. The graph for $\psi(s)$ is given in Fig. 9.2.

It can be obtained numerically that $s^* = 0.302$.

9.1.3 Optimal Severity for Minimizing Expected Costs

In this section, we consider two models of determining the optimal severity minimizing the expected cost function, which takes into account burn-in and field operation.

Fig. 9.2 Graph for $\psi(s)$



9.1.3.1 Model 1: Minimization Without Replacement During Field Operation

An item is chosen at random from our heterogeneous population and is exposed to a shock. If it survives, then it is considered to be ready for usage, otherwise the failed item is discarded and the new one is chosen from the population, etc. This procedure is repeated until the first survived item is obtained.

Let c_{sr} be the shop replacement cost and c_s be the cost for conducting a single shock. Let $c_1(s)$, as a function of s, be the expected cost for eventually obtaining a component which has survived a shock. Conditioning on the event that the component survives (or fails) a shock, the following equation can be obtained:

$$c_1(s) = (1-P)c_s + ((c_s + c_{sr}) + c_1(s))P, \tag{9.10}$$

where $P = G(s)\pi + G(\rho(s))(1-\pi)$ is the probability that an item from the mixture population does not survive the shock. Then, from Eq. (9.10):

$$c_1(s) = \frac{c_s + c_{sr}P}{1-P} = -c_{sr} + \frac{c_s + c_{sr}}{1-P}. \tag{9.11}$$

Let:

The cost c_m is incurred by the event $\{T_s \leq \tau\}$ (Failure of the Mission);
The gain g_m results from the event $\{T_s > \tau\}$ (Success of the Mission).

Then the expected costs during field operation, $c_2(s)$, is given by

$$c_2(s) = -g_m\left(\pi_s\overline{F}_1(\tau) + (1-\pi_s)\overline{F}_2(\tau)\right) + c_m(\pi_s F_1(\tau) + (1-\pi_s)F_2(\tau)),$$

where π_s is defined by Eq. (9.7). Then the total expected cost $c(s)$ is

$$\begin{aligned} c(s) &= c_1(s) + c_2(s) = -c_{sr} + \frac{c_s + c_{sr}}{\overline{G}(s)\pi + \overline{G}(\rho(s))(1-\pi)} \\ &\quad - (g_m + c_m)\left(\frac{\overline{G}(s)\pi}{\overline{G}(s)\pi + \overline{G}(\rho(s))(1-\pi)}\overline{F}_1(\tau) + \frac{\overline{G}(\rho(s))(1-\pi)}{\overline{G}(s)\pi + \overline{G}(\rho(s))(1-\pi)}\overline{F}_2(\tau)\right) + c_m. \end{aligned}$$

Let s^* be the optimal severity level that satisfies

$$s^* = \arg \inf_{s\in[0,\infty]} c(s).$$

The following theorem defines properties of optimal s^*.

Theorem 9.4 *Let $\overline{F}_1(t) \geq \overline{F}_2(t)$, for all $t \geq 0$. If $R(\rho(s)) - R(s)$ strictly decreases for $s > s_0$, then $s^* \leq s_0$. In particular,*

(i) *If $\rho'(s) > 1$ and $r(x)$ is increasing, then $s^* < \infty$.*
(ii) *If $\frac{\rho'(s)r(\rho(s))}{r(s)} < 1$, for $s > s_0$, then $s^* \leq s_0$.*

Proof Note that $c_1(s)$ strictly increases from $c_1(0) = c_s$ to $c_1(\infty) = \infty$. Also observe that $c_2(s) = -(g_m + c_m)\overline{F}_{ms}(\tau; s) + c_m$, where $\overline{F}_{ms}(t; s)$ is the weighted average of $\overline{F}_1(t)$ and $\overline{F}_2(t)$ with the corresponding weights π_s and $1 - \pi_s$, respectively. If $R(\rho(s)) - R(s)$ strictly decreases for $s > s_0$, then, by similar arguments as those described in the proof of Theorem 9.2, $c_2(s)$ strictly increases for $s > s_0$. This imply that $c(s)$ strictly increases for $s > s_0$ and thus we can conclude that optimal $s^* \leq s_0$.

(i) From the proof of Theorem 9.2, it can be seen that if $\rho'(s) > 1$ and $r(x)$ is increasing, then $c_2(s)$ strictly decreases for $s > 0$. But $c(\infty) = \infty$ and thus $s^* < \infty$.
(ii) If $\rho'(s) r(\rho(s))/r(s) < 1$, for $s > s_0$ then, from the proof of Theorem 9.2, it is easy to see that $c_2(s)$ strictly increases for $s > s_0$, and thus $s^* \leq s_0$. □

Assume now that the expected gain during field operation is proportional to the mean lifetime. Then the expected cost (i.e., the negative gain) during field operation is

$$c_2(s) = -k\left(\pi_s \int_0^\infty \overline{F}_1(t)\mathrm{d}t + (1 - \pi_s)\int_0^\infty \overline{F}_2(t)\mathrm{d}t\right),$$

and the total expected cost is given by

$$\begin{aligned} c(s) = &- c_{sr} + \frac{c_s + c_{sr}}{\overline{G}(s)\pi + \overline{G}(\rho(s))(1-\pi)} \\ &- k\left(\frac{\overline{G}(s)\pi}{\overline{G}(s)\pi + \overline{G}(\rho(s))(1-\pi)}\int_0^\infty \overline{F}_1(t)\mathrm{d}t + \frac{\overline{G}(\rho(s))(1-\pi)}{\overline{G}(s)\pi + \overline{G}(\rho(s))(1-\pi)}\int_0^\infty \overline{F}_2(t)\mathrm{d}t\right), \end{aligned} \tag{9.12}$$

where k is a constant of proportionality. Then the following corollary holds:

Corollary 9.2 *Let $\overline{F}_1(t) \geq \overline{F}_2(t)$, for all $t \geq 0$. Then the properties of optimal s^* for the total expected cost function (9.12) are the same as those described in Theorem 9.4.*

The *proof* is similar to that of Theorem 9.4. □

9.1.3.2 Model 2: Minimization with Replacement During Field Operation

Assume that if an item fails during field operation, it is replaced by another item which has survived a shock at a cost $c_f > c_{sr}$. The time intervals between two consecutive replacements constitute a renewal process. Therefore, in accordance with $\overline{F}_{ms}(t) = \pi_s\overline{F}_1(t) + (1-\pi_s)\overline{F}_2(t)$ and Eq. (9.7), the mean time between two consecutive replacements is equal to

$$\frac{\overline{G}(s)\pi}{\overline{G}(s)\pi + \overline{G}(\rho(s))(1-\pi)}\int_0^\infty \overline{F}_1(t)\mathrm{d}t + \frac{\overline{G}(\rho(s))(1-\pi)}{\overline{G}(s)\pi + \overline{G}(\rho(s))(1-\pi)}\int_0^\infty \overline{F}_2(t)\mathrm{d}t. \tag{9.13}$$

Then, by the renewal reward theory argument, the expected cost rate $\tilde{c}(s)$ is given by

$$\begin{aligned}\tilde{c}(s) = {} & \frac{1}{\frac{\overline{G}(s)\pi}{\overline{G}(s)\pi+\overline{G}(\rho(s))(1-\pi)}\int_0^\infty \overline{F}_1(t)\mathrm{d}t + \frac{\overline{G}(\rho(s))(1-\pi)}{\overline{G}(s)\pi+\overline{G}(\rho(s))(1-\pi)}\int_0^\infty \overline{F}_2(t)\mathrm{d}t} \\ & \times\left(\frac{c_s + c_{sr}}{\overline{G}(s)\pi + \overline{G}(\rho(s))(1-\pi)} + (c_f - c_{sr})\right),\end{aligned} \tag{9.14}$$

where the denominator is just an expected duration of a renewal cycle given by Eq. (9.13) and the numerator defines the expected cost incurred during this cycle (taking into account that the expected cost during burn-in is given by (9.11) and the replacement cost during field operation is given by c_f).

Let s^* denote the optimal severity which satisfies

$$s^* = \arg \inf_{s\in[0,\infty]} \tilde{c}(s).$$

Then, similar to Theorem 9.4, the following result is also true:

Theorem 9.5 *Let $\overline{F}_1(t) \geq \overline{F}_2(t)$, for all $t \geq 0$. If $R(\rho(s)) - R(s)$ strictly decreases for $s > s_0$, then optimal $s^* \leq s_0$. In particular,*

(i) *If $\rho'(s) > 1$ and $r(s)$ is increasing, then $s^* < \infty$.*
(ii) *If $\frac{\rho'(s)r(\rho(s))}{r(s)} < 1$, for $s > s_0$, then the optimal $s^* \leq s_0$.*

Proof Rearranging terms in (9.14):

$$\begin{aligned}\tilde{c}(s) = {} & \frac{c_s + c_{sr}}{\overline{G}(s)\pi\int_0^\infty \overline{F}_1(t)\mathrm{d}t + \overline{G}(\rho(s))(1-\pi)\int_0^\infty \overline{F}_2(t)\mathrm{d}t} \\ & + \frac{c_f - c_{sr}}{\frac{\overline{G}(s)\pi}{\overline{G}(s)\pi+\overline{G}(\rho(s))(1-\pi)}\int_0^\infty \overline{F}_1(t)\mathrm{d}t + \frac{\overline{G}(\rho(s))(1-\pi)}{\overline{G}(s)\pi+\overline{G}(\rho(s))(1-\pi)}\int_0^\infty \overline{F}_2(t)\mathrm{d}t}.\end{aligned}$$

The first term in the right-hand side strictly increases for $s > 0$. Note that the denominator of the second term is the weighted average of $\int_0^\infty \overline{F}_1(t)\mathrm{d}t$ and $\int_0^\infty \overline{F}_2(t)\mathrm{d}t$ $\left(\int_0^\infty \overline{F}_1(t)\mathrm{d}t \geq \int_0^\infty \overline{F}_2(t)\mathrm{d}t\right)$ with the corresponding weights π_s and

$1 - \pi_s$, respectively. Then, following the procedures described in the proof of Theorem 9.4, we can obtain the desired result. □

Remark 9.3 In 'ordinary' burn-in, as discussed in the previous chapters, when the lifetimes of items are described by the distributions with the bathtub-shaped failure rate, the following property holds: the optimal burn-in time should be smaller than the first change point (see, e.g., [5, 12]). In our reasoning, optimal stress levels, in accordance with Theorems 9.2, 9.4, and 9.5, in a similar way also depend on the properties of the distribution of strength.

Remark 9.4 In practice, the cost parameters $\left(c_s, c_{sr}, c_f, c_m, g_m\right)$ might not be estimated precisely, which could make the optimization procedure difficult. In this case, the Receiver Operating Characteristic (ROC) analysis can be adopted and effectively used to determine the optimal burn-in time which minimizes the corresponding cost functions. A reference for this approach can be found in Wu and Xie [15], where the application of ROC analysis is used to remove the weak subpopulation in burn-in problems.

9.2 Continuous Mixtures

9.2.1 *The Impact of Shocks on Mixed Populations*

Consider a population of identically distributed items with lifetimes $T_i, i = 1, 2, \ldots$. Each item 'is affected' by a non-observable univariate frailty parameter Z_i and the lifetimes T_i are conditionally independent given the values of parameters $Z_i = z_i$. Assume that these parameters are i.i.d with a common pdf $\pi(z)$ and with support in $[0, \infty)$. (The general support $[a, b)$, $0 \le a < b \le \infty$ can be considered as well.) Then, obviously $T_i, i = 1, 2, \ldots$ are also i.i.d. For convenience, the sub index "i" will be omitted and, therefore, the lifetimes and frailties for all items will be denoted by T and Z, respectively. Thus, obviously, T is described by the *mixture* Cdf and pdf

$$F_m(t) = \int_0^\infty F(t, z)\pi(z)\mathrm{d}z,$$

$$f_m(t) = \int_0^\infty f(t, z)\pi(z)\mathrm{d}z,$$

respectively, where $F(t, z) \equiv F(t|z) = \Pr[T \le t|Z = z]$, $f(t, z) = F'(t, z)$ are the corresponding conditional characteristics for realization $Z = z$.

Then the mixture (*observed*) failure rate $\lambda_m(t)$, similar to (5.11, 5.12) is

$$\lambda_m(t) = \frac{f_m(t)}{\bar{F}_m(t)} = \frac{\int_0^\infty f(t,z)\pi(z)\mathrm{d}z}{\int_0^\infty \bar{F}(t,z)\pi(z)\mathrm{d}z} = \int_0^\infty \lambda(t,z)\pi(z|t)\mathrm{d}z, \tag{9.15}$$

where

$$\pi(z|t) \equiv \pi(z)\frac{\bar{F}(t,z)}{\int_0^\infty \bar{F}(t,z)\pi(z)\mathrm{d}z}. \tag{9.16}$$

In the framework of the model described above, we will consider mixed populations of stochastically ordered subpopulations.

Remark 9.5 The foregoing definitions and properties describe a standard statistical mixture (or frailty) model for an item and for the collection of items (population) as well. However, the following *interpretation* can be also useful, as frailty models were initially developed in demographic and actuarial studies as a method of describing heterogeneity in large populations (see, e.g., [3, 11, 14]; and references therein). Thus, we assume that heterogeneity, described by the unobserved frailty, is a property of an infinite population. It usually means that, due to different environments, conditions, different manufacturers, etc., the population consists of subpopulations of items with different statistical properties. Pooling at random items from this population results in the described mixture model.

Assume that an item is put into operation for the mission time τ with the required survival probability $P_r(\tau)$. If

$$\exp\left\{-\int_0^\tau \lambda_m(u)\mathrm{d}u\right\} \geq P_r(\tau), \tag{9.17}$$

then everything is fine and we do not need to improve the performance of our items. On the contrary, if this inequality does not hold, the burn-in procedure can be performed. There are different types of these procedures and we will consider here the burn-in that is performed via shocks that can eliminate the weak items.

Throughout this section, the impact of a shock is described by the following *general assumption:*

Assumption An instantaneous shock either 'kills' an item with a given probability or does not change its stochastic properties with the complementary probability. The more 'frail' (e.g., with larger failure rate or with smaller survival function) an item is, the larger is the probability that a shock will 'kill' it.

The following burn-in procedure is employed:

- *Burn-in procedure by means of shocks.* An item is exposed to a shock. If it survives, it is considered to be ready for usage, otherwise the failed item is discarded and a new one is exposed to a shock, etc.

This setting can be defined probabilistically in the following way: Let $\pi_s(z)$ denote the pdf of the frailty Z_s (with support in $[0,\infty)$) after a shock and let $\lambda_{ms}(t)$ be the corresponding mixture failure rate. In accordance with (9.15):

$$\lambda_{ms}(t) = \int_0^\infty \lambda(t,z)\pi_s(z|t)\mathrm{d}z,$$

where, similar to (9.16), $\pi_s(z|t)$ is defined by the right-hand side of (9.16) with $\pi(z)$ substituted by $\pi_s(z)$.

First, assume formally that population frailties before and after a shock are ordered in the sense of the likelihood ratio (see Sect. 2.8):

$$Z \geq_{LR} Z_s, \tag{9.18}$$

which in our terms is defined as

$$\pi_s(z) = \frac{g(z)\pi(z)}{\int_0^\infty g(z)\pi(z)\mathrm{d}z}, \tag{9.19}$$

where $g(z)$ is a decreasing function and therefore $\pi_s(z)/\pi(z)$ is decreasing. As it will be discussed in the next subsection, the function $g(z)$ can be interpreted as the survival probability of an item with frailty z after the shock. Therefore, the assumption that $g(z)$ is a decreasing function of z corresponds to our general "Assumption". Note that the 'likelihood ratio ordering' for mixing (frailty) distributions was used by Block et al. [4] for ordering optimal burn-in times in 'ordinary' burn-in settings (without shocks): the larger frailty corresponds to the larger optimal burn-in time for some specified cost functions. It seems that this ordering is natural for stochastic modeling in heterogeneous populations. The following important theorem shows that depending on assumptions, the likelihood ratio ordering of frailties leads either to the failure rate or to the usual stochastic ordering of population lifetimes.

Theorem 9.6 *Let relationship* (9.19)*, defining the mixing density after a shock, where* $g(z)$ *is a decreasing function, hold.*

(i) *Assume that*

$$\lambda(t,z_1) \leq \lambda(t,z_2), \quad z_1 < z_2, \forall z_1, z_2 \in [0,\infty], t \geq 0. \tag{9.20}$$

Then

$$\lambda_{ms}(t) \leq \lambda_m(t);\ \forall t \geq 0. \tag{9.21}$$

(ii) *Assume that*

$$\overline{F}(t, z_1) \geq \overline{F}(t, z_2), \quad z_1 < z_2, \forall z_1, z_2 \in [0, \infty], t \geq 0. \tag{9.22}$$

Then

$$\overline{F}_{ms}(t) \geq \overline{F}_m(t), \forall t \geq 0, \tag{9.23}$$

where $\lambda_{ms}(t), \bar{F}_{ms}(t)$ are the population (mixture) failure rate and the survival function after a shock, respectively.

Proof Note that, inequalities (9.20) and (9.22) define two types of stochastic orderings for subpopulations, i.e., the failure rate ordering and the usual stochastic ordering, respectively.

(i) It can be shown [10: p. 164] that:

$$\begin{aligned} &sign[\lambda_{ms}(t) - \lambda_m(t)] \\ &= sign \int\limits_{\substack{0 \\ u>s}}^{\infty} \int\limits_0^{\infty} \bar{F}(t,u)\bar{F}(t,s)(\lambda(t,u) - \lambda(t,s))(\pi_s(u)\pi(s) - \pi_s(s)\pi(u))duds, \end{aligned} \tag{9.24}$$

which is negative due to definition (9.19) and assumptions of this theorem.

(ii) As $g(z)$ is a decreasing function, and the survival function $\bar{F}(t, z)$ is also decreasing in z, it can be easily shown using the mean value theorem that

$$\overline{F}_{ms}(t) - \overline{F}_m(t) = \frac{\int_0^{\infty} \overline{F}(t, z)g(z)\pi(z)\mathrm{d}z}{\int_0^{\infty} g(z)\pi(z)\mathrm{d}u} - \int\limits_0^{\infty} \overline{F}(t, z)\pi(z)\mathrm{d}z \geq 0. \tag{9.25}$$

Indeed

$$\int\limits_0^{\infty} g(z)\pi(z)\mathrm{d}z = g(z^*)$$

and

$$\int\limits_0^{\infty} \overline{F}(t, z)g(z)\pi(z)\mathrm{d}z = g(z^{**}) \int\limits_0^{\infty} \overline{F}(t, z)\pi(z)\mathrm{d}z,$$

where $g(z^*)$ and $g(z^{**})$ are the corresponding mean values. As $\bar{F}(t, z)$ is decreasing in z, $z^{**} \leq z^*$. Therefore, taking into account that $g(z)$ is a decreasing function, (9.25) follows. Note that the usage of the mean value theorem relies on the continuity of $g(z)$. Alternatively, the general case (without this assumption) can be proved similar to the proof in (i) (see also Theorem 9.7). □

Remark 9.6 Inequality (9.20) is a natural ordering in the family of failure rates $\lambda(t,z), z \in [0,\infty)$ and trivially holds, e.g., for the specific multiplicative model:

$$\lambda(t,z) = z\lambda(t). \tag{9.26}$$

Remark 9.7 Theorem 9.6 means that the *population quality* (in terms of the failure rate or the survival function) has improved after a shock. Thus, in accordance with our statistical 'frequentistic' *interpretation* (see Remark 9.5) when 'the whole population' is exposed to a shock, the items that have passed this test form a new population with better stochastic characteristics. On the other hand, following our formal initial setting, it turns out that the benefit of a non-destructive shock is of 'informational' type, i.e., surviving a shock has the 'Bayesian' effect of modifying the posterior distribution of Z, which is Z_s in our notation.

Remark 9.8 In accordance with (9.21) and (9.23), inequality (9.17) can be already achieved after one shock, otherwise new shocks should be applied or the "severity" of a single shock (see later) should be increased. It is also worth noting that the replacement of condition (9.18) by the usual stochastic ordering: $Z \geq_{st} Z_s$ will not guarantee orderings (9.21) and (9.23) for all t.

9.2.2 *The Impact of Shocks on an Item*

Now we must consider a more specific mechanism of a shock's impact on an item. Let each item fail with probability $p(z)$ and survive (as good as new) with probability $q(z) = 1 - p(z)$. Here, the condition that corresponds to the general "Assumption" in Sect. 9.2.1 is that $p(z)$ is an increasing function of $z\,(0 \leq p(z) \leq 1)$. This assumption makes sense as, in accordance with (9.20), larger values of frailty correspond to larger values of the failure rate. Therefore, items with larger values of frailty are more susceptible to failures. Equation (9.19) reads now

$$\pi_s(z) = \frac{q(z)\pi(z)}{\int_0^\infty q(z)\pi(z)\mathrm{d}z}, \tag{9.27}$$

where $\pi_s(z)$ is the pdf of Z_s (*predictive*, or *posterior* pdf, as it has been called in Bayesian terminology). As $q(z)$ is decreasing with z, it follows from Theorem 9.6 that the failure rate ordering (9.21) and the usual stochastic ordering (9.23) hold.

If we are not concerned about the costs (e.g., when the mission is very important) and inequality

$$\exp\left\{-\int_0^\tau \lambda_{ms}(u)\mathrm{d}u\right\} \geq P_r(\tau) \tag{9.28}$$

holds, then the burn-in is over and the item that has survived a shock can be put into field operation. Otherwise, a shock with the higher level of severity or several shocks should be performed for each item in order to achieve this inequality.

On the other hand, in most practical situations the costs are involved. In order to consider the corresponding optimization, we must define the costs and probabilities of interest. A convenient and useful model for $p(z)$ (although oversimplified for practical usage) is the step function:

$$p(z) = \begin{cases} 0, & 0 \leq z \leq z_b \\ 1, & z > z_b \end{cases}. \tag{9.29}$$

It means that all 'weak' items with $z > z_b$ will be eliminated and only 'strong' items will remain in the population. In accordance with (9.29), the probability of not surviving the shock in this case is

$$P_{z_b} \equiv \bar{\Pi}(z_b) = \int_{z_b}^{\infty} \pi(z)\mathrm{d}z, \tag{9.30}$$

where $\Pi(z)$ is the Cdf that corresponds to the pdf $\pi(z)$. Obviously, for a general form of $p(z)$, this probability is defined by the following mixture

$$P = \int_0^{\infty} p(z)\pi(z)\mathrm{d}z. \tag{9.31}$$

9.2.3 *Shock's Severity*

It is clear that the parameter z_b in the specific model (9.29) can be considered as a parameter of severity: the larger values of z_b correspond to a smaller severity. Now we can deal with the issue of severity in a more general context, that is, when $p(z)$ is not a simple step function but a continuous function of z.

For this discussion, define the functions $p(z)$ and $q(z)$ as functions of the frailty variable z and the severity parameter $s \in [0, \infty)$, $p(z, s)$ and $q(z, s)$. Assume that $q(z, s)$ is decreasing in z for each fixed s and is decreasing in s for each z. The assumption that $q(z, s)$ is decreasing in z for each fixed s is just what was assumed in our general "Assumption" in Sect. 9.2.1. The assumption that $q(z, s)$ is decreasing in s for each fixed z is also quite natural and implies that items characterized by the same value of frailty have larger failure probabilities under larger severity levels.

Denote the corresponding failure rate and the survival function by $\lambda_{ms}(t; s)$ and $\overline{F}_{ms}(t; s)$, respectively. Similar to (9.19) and (9.16):

$$\pi_s(z,s) = \frac{q(z,s)\pi(z)}{\int_0^\infty q(u,s)\pi(u)\mathrm{d}u}, \pi_s(z,s|t) \equiv \pi_s(z,s)\frac{\bar{F}(t,z)}{\int_0^\infty \bar{F}(t,u)\pi_s(u,s)\mathrm{d}u}.$$

In order to compare two severity levels, we need the following definition.

Definition 9.1

(i) The severity (stress) level s is said to be *dominated* under the failure rate criterion if there exists another level s' such that

$$\lambda_{ms}(t;s) \geq \lambda_{ms}(t;s'), \text{for all } t \geq 0.$$

(ii) The severity (stress) level s is said to be *dominated* under the survival probability criterion if there exists another level s' such that

$$\overline{F}_{ms}(t;s') \geq \overline{F}_{ms}(t;s), \text{for all } t \geq 0.$$

Otherwise, the severity (stress) level s is called *non-dominated.*

Theorem 9.7 *Assume that $q(z,s)$ is decreasing in z for each fixed s and is decreasing in s for each z. Consider two stress levels s and s'. Let*

$$q(u,s')q(v,s) - q(v,s')q(u,s) \leq 0, \text{for all } u > v, \tag{9.32}$$

which means that $q(z,s')/q(z,s)$ is decreasing in z.

(i) *If $\lambda(t,z_1) \leq \lambda(t,z_2), \quad z_1 < z_2, \forall z_1, z_2 \in [0,\infty], t \geq 0,$ then the severity level s is dominated under the failure rate criterion.*

(ii) *If $\overline{F}(t,z_1) \geq \overline{F}(t,z_2), \quad z_1 < z_2, \forall z_1, z_2 \in [0,\infty], t \geq 0,$ then the severity level s is dominated under the survival probability criterion.*

Proof

(i) Similar to (9.24):

$$sign[\lambda_{ms}(t;s') - \lambda_{ms}(t;s)]$$
$$= sign \int\limits_{\substack{0 \\ u>v}}^{\infty} \int\limits_0^{\infty} \bar{F}(t,u)\bar{F}(t,v)(\lambda(t,u) - \lambda(t,v))(\pi_s(u,s')\pi_s(v,s) - \pi_s(v,s')\pi_s(u,s))dudv.$$

Thus, if (9.32) holds, then

$$\pi_s(u,s')\pi_s(v,s) - \pi_s(v,s')\pi_s(u,s) \leq 0,$$

which implies the result in (i).

(ii)

$$\overline{F}_{ms}(t;s') - \overline{F}_{ms}(t;s) = \frac{\int_0^\infty \overline{F}(t,z)q(z,s')\pi(z)\mathrm{d}z}{\int_0^\infty q(u,s')\pi(u)\mathrm{d}u} - \frac{\int_0^\infty \overline{F}(t,z)q(z,s)\pi(z)\mathrm{d}z}{\int_0^\infty q(u,s)\pi(u)\mathrm{d}u},$$

and the corresponding numerator can be transformed to

$$\int\limits_{\substack{0\\u>v}}^{\infty}\int\limits_{0}^{\infty}\pi(u)\pi(v)(\overline{F}(t,u)-\overline{F}(t,v))(q(u,s')q(v,s)-q(v,s')q(u,s))\mathrm{d}u\mathrm{d}v.$$

Therefore, if (9.32) holds, then

$$\overline{F}_{ms}(t;s')-\overline{F}_{ms}(t;s)\geq 0,\text{for all } t\geq 0.$$

□

Remark 9.9 Note that although the assumption that $q(z,s)$ is decreasing in z for each fixed s and is decreasing in s for each z is not used formally in the foregoing proof, it represents some basic 'physical properties' of the model and should be checked in applications.

Remark 9.10 In accordance with Remark 9.7, Theorem 9.7 means that the *population quality* (in terms of the failure rate or the survival function) is better after the shock with severity s' than after the shock with severity s.

Example 9.3 Consider the following illustrative discrete example. Suppose that there are only three stress levels: s_1, s_2, and $s_3(s_1<s_2<s_3)$. Let $q(z,s_1)=0.2e^{-z}+0.6, q(z,s_2)=0.6e^{-z}+0.2$, and $q(z,s_3)=0.2e^{-z}+0.2$. Then $q(z,s_i)$ is decreasing in z, for each $i=1,2,3$. Furthermore, for each fixed z, $q(z,s_1)\geq q(z,s_2)\geq q(z,s_3)$ and in this way the condition for ordering the stress levels $(s_1<s_2<s_3)$ is justified. Observe that

$$\frac{q(z,s_2)}{q(z,s_1)} \text{ and } \frac{q(z,s_2)}{q(z,s_3)}$$

strictly decrease in z. Therefore, as follows from Theorem 9.7, the stress levels s_1 and s_3 are dominated and, in this case, the stress level s_2 minimizes the failure rate and maximizes the survival function after a shock. Thus s_2 is the optimal level.

Remark 9.11 Intuitively, it can be believed that a higher level of severity results in a 'better population' but it is not always true as shown in this example. A similar observation is true for the conventional burn-in in homogeneous populations when the larger time of burn-in does not necessarily lead to a 'better population'. In this case, the shape of the failure rate (e.g., bathtub) plays a crucial role in the corresponding analysis.

Consider again the specific case (9.29). For convenience, and in accordance with our reasoning, let us change the notation in the following way:

$$q(z,s)=\begin{cases}1, & 0\leq z\leq z_s\\ 0, & z>z_s\end{cases}, \tag{9.33}$$

where $z_s > z_{s'}$ if $s' > s,\ s, s' \in [0, \infty)$. Then we have the following corollary.

Corollary 9.3 *Let the model* (9.33) *hold and fix* $s' > 0$.

(i) *If* $\lambda(t, z_1) \leq \lambda(t, z_2), \quad z_1 < z_2, \forall z_1, z_2 \in [0, \infty], t \geq 0$, *then the severity level* s *for* $\forall s \leq s'$ *is dominated under the failure rate criterion. That is,*

$$\lambda_{ms}(t; s) \geq \lambda_{ms}(t; s'), \text{for all } t \geq 0, \text{for all } s \leq s'.$$

(ii) *If* $\overline{F}(t, z_1) \geq \overline{F}(t, z_2), \quad z_1 < z_2, \forall z_1, z_2 \in [0, \infty], t \geq 0$, *then the severity level* s *for* $\forall s \leq s'$ *is dominated under the survival probability criterion. That is,*

$$\overline{F}_{ms}(t; s') \geq \overline{F}_{ms}(t; s), \text{for all } t \geq 0, \text{for all } s \leq s'.$$

Proof It is easy to check that condition

$$q(u, s')q(v, s) - q(v, s')q(u, s) \leq 0, \text{for all } u > v,$$

is always satisfied for $q(z, s)$ given by Relationship (9.33) for all $s' > s$. □

It follows from this corollary that the better population quality (see Remark 9.7) can be obtained by increasing s (formally, $s \to \infty$, but the level of severity is always bounded in practice).

Remark 9.12 In Theorem 9.7, considering general form of $q(z, s)$, it was assumed that $q(z, s')/q(z, s)$ decreases in z for some fixed s' and s. If we now assume that this quotient decreases in z for all $s' > s$, then, similar to the specific case of Corollary 9.3, the better population quality can be obtained by increasing s $(s \to \infty)$.

Remark 9.13 It should be noted that there is a certain analogy between describing the usual burn-in for heterogeneous populations during a given time period and the burn-in via shocks. It was shown in Finkelstein [10] that, if two different frailty distributions are ordered in the sense of the likelihood ratio and inequality (9.20) holds, then the smaller frailty implies the smaller mixture failure rate (the better population quality after burn-in). In the case under consideration, Inequality (9.32) can be also interpreted as the corresponding likelihood ordering of frailties after the shocks with two stress levels s and s', respectively.

9.2.4 The Cost of Burn-in and Optimal Problem

In field operation, items are frequently required to survive a pre-specified time period, which is called the mission time, τ. In this subsection, optimal severity of a shock, which minimizes the average cost incurred during the burn-in and the operation phase will be considered.

As previously, a new component randomly selected from the heterogeneous population is burned-in by means of a shock. If the first one did not survive then we take another one from infinite heterogeneous population and burn-in again. This procedure is repeated until we obtain the first component which survives burn-in. Then this component is put into the field operation. Assume, first, for simplicity, that the cost of conducting a single shock $c_s = 0$. Denote by c_1 the expected cost of the burn-in until obtaining the first item that has survived shocks. It is clear that

$$\begin{aligned} c_1 &= 0 \times (1-P) + c_{sr}P(1-P) + 2c_{sr}P^2(1-P) + 3c_{sr}P^3(1-P) + \cdots \\ &= c_{sr}P(1-P)(1+2P+3P^2+\cdots) = \frac{c_{sr}P}{1-P}, \end{aligned} \tag{9.34}$$

where c_{sr} is the shop replacement cost. Similarly, when $c_s \neq 0$

$$c_1 = \frac{c_{sr}P + c_s}{1-P}. \tag{9.35}$$

Obviously, this function increases when P increases in $[0, 1)$. Note that P is now a function of the stress level s, that is, $P(s)$ [see definition (9.31), where $p(z)$ should be substituted by $p(z, s)$] and thus, in the following, c_1 in (9.34) and (9.35) should be also understood as a function of $s, c_1(s)$.

Let:

The cost c_m is incurred by the event $\{T_s \le \tau\}$ (Failure of the Mission);
The gain g_m results from the event $\{T_s > \tau\}$ (Success of the Mission).

Obviously, the expected cost during field operation is:

$$\begin{aligned} c_2(s) &= -g_m\overline{F}_{ms}(\tau; s) + c_m(1 - \overline{F}_{ms}(\tau; s)) \\ &= -(g_m + c_m)\overline{F}_{ms}(\tau; s) + c_m. \end{aligned}$$

Therefore, the total expected cost function (as a function of the stress level s) for the burn-in and the field operation phases is given by

$$c(s) = c_1(s) + c_2(s), \tag{9.36}$$

where $c_1(s)$ is defined in (9.35). The values c_{sr}, c_s, g_m, c_m are assumed to be known. Thus the corresponding optimization problem can be formalized as

$$s^* = \arg\ \min c(s). \tag{9.37}$$

It is worth noting that condition (9.28) can also be imposed as an additional requirement for obtaining minimum of the total costs function.

Theorem 9.8 *Suppose that*

$$\overline{F}(t, z_1) \ge \overline{F}(t, z_2), \quad z_1 < z_2, \forall z_1, z_2 \in [0, \infty], t \ge 0.$$

(i) *If, for any* $s_2 > s_1$, $q(u,s_2)q(v,s_1) - q(v,s_2)q(u,s_1) \leq 0$, *for all* $u > v$, i.e., $q(z,s_2)/q(z,s_1)$ *decreases in* z *for all* $s_2 > s_1$, *then there exists the finite optimal level* $s^* < \infty$ *for the optimization problem* (9.37).

(ii) *If there exists* $s_0 < \infty$ *such that for all levels* $s > s_0$, *the level* s *is dominated by* s_0 *under the survival probability criterion, then* $s^* < s_0$.

Proof

(i) Observe that $c_1(s)$ strictly increases in s with $c_1(0) = c_s$ to $c_1(\infty) = \infty$ and $c_2(s)$ can be minimized by maximizing $\overline{F}_{ms}(\tau; s)$. If $q(z,s_2)/q(z,s_1)$ decreases in z for all $s_2 > s_1$, then $c_2(s)$ strictly decreases for $s > 0$ since $\overline{F}_{ms}(\tau; s)$. strictly increases for $s > 0$ by Theorem 9.7. But $c(\infty) = \infty$ and thus, $s^* < \infty$.

(ii) If there exists $s_0 < \infty$ such that for all stress levels $s > s_0$, the level s is dominated by s_0 then it is obvious that $c(s_0) \leq c(s)$, for all $s > s_0$. Therefore, $s^* < s_0$. □

Assume now that the expected gain during field operation is proportional to the mean lifetime of an item, which is also a reasonable assumption that is often used in practice. Then the expected cost during the field operation, $c_2(s)$, is given by

$$c_2(s) = -k\int_0^\infty \overline{F}_{ms}(t;s)\mathrm{d}t = -k\frac{\int_0^\infty \left\{\int_0^\infty \overline{F}(t,z)\mathrm{d}t\right\}q(z;s)\pi(z)\mathrm{d}z}{\int_0^\infty q(u;s)\pi(u)\mathrm{d}u},$$

where k is the proportionality constant. It is obvious that if

$$\overline{F}(t,z_1) \geq \overline{F}(t,z_2), \quad z_1 < z_2, \forall z_1, z_2 \in [0,\infty], t \geq 0,$$

then

$$\int_0^\infty \overline{F}(t,z_1)\mathrm{d}t \geq \int_0^\infty \overline{F}(t,z_2)\mathrm{d}t, \quad z_1 < z_2, \forall z_1, z_2 \in [0,\infty]$$

and, as in Theorem 9.8, the same result for optimal severity level s^* can be obtained (See also the proof of Theorem 9.7-(ii)).

If our goal is only to achieve minimum of $c(s)$ and a shock can be made as severe as we wish, then no further shocks are needed. However, if the shock's severity beyond certain level (that is usually defined by the physical processes in the item subject to a shock) results in a non-negligible damage in the 'survived' item, then we cannot go above this level of severity and should consider an option of performing additional shocks. Note that additional shocks in the framework of the specific model (9.29) do not improve the quality of a population. This can be easily seen by deriving $P_{z_b}^{(2)}$- the probability of not surviving the second shock with the same level of z_b. Using (9.27) and (9.30),

$$P_{z_b}^{(2)} = \int_{z_b}^{\infty} \pi_s(z)\mathrm{d}z = \frac{\int_{z_b}^{\infty} q(z)\pi(z)\mathrm{d}z}{\int_0^{\infty} q(z)\pi(z)\mathrm{d}z} = 0.$$

On the other hand, the general model (9.31) gives a *positive probability* of not surviving the second shock (with the same level of severity $p(z,s)$) after an item had survived the first shock:

$$P^{(2)}(s) = \int_0^{\infty} p(z,s)\pi_s(z)\mathrm{d}z = \frac{\int_0^{\infty} p(z,s)q(z,s)\pi(z)\mathrm{d}z}{\int_0^{\infty} q(z,s)\pi(z)\mathrm{d}z} > 0.$$

Therefore, when the high level of stress can negatively affect even those items that had formally passed it (did not fail), we can perform a more 'friendly' burn-in with a lower level of stress by increasing the number of shocks as opposed to the option of one shock.

Denote the posterior density after the nth shock by $\pi_s^{(n)}(z)$, where $\pi_s^{(1)}(z) = \pi_s(z)$. Then, (9.27) is generalized to:

$$\pi_s^{(n)}(z) = \frac{q^n(z,s)\pi(z)}{\int_0^{\infty} q^n(z,s)\pi(z)\mathrm{d}z}, \tag{9.38}$$

meaning that for the given $q(z,s)$, this density tends (in the sense of generalized functions) to the 'one-sided' δ-function (in the positive neighborhood of 0). Therefore, if we assume that there is no penalty (cost) for additional shocks, then obviously, we can reach the desired level of severity (the same as with one 'unfriendly' shock) with a finite number of shocks. This 'multi-shock reasoning' can be generalized to an extended model considering the relevant costs and the corresponding optimal problem. In essence, as all shocks are applied in a relatively short period of time, we are treating the sequence of shocks as one 'aggregated' shock.

In this case, the number of shocks can be considered as a measure of severity. Let s_i denote the level of severity with i shocks, $i = 1, 2, \ldots$, that is, for example, at level s_1 only one shock with severity level s is applied; at level s_2 two consecutive shocks with severity level s are applied, and so on. Let $\tilde{q}(z,s_i)(\tilde{q}(z,s_1) \equiv q(z,s))$ be the item's survival probability for this 'multi-shock structure'. Obviously, from (9.38), we have $\tilde{q}(z,s_i) = q^i(z,s)$. As

$$\frac{\tilde{q}(z,s_{i+1})}{\tilde{q}(z,s_i)} = q(z,s)$$

is decreasing in z, by Remark 9.12, we can conclude that the better quality of a population can be obtained by monotonically increasing the number of shocks. Using this property, similar results as in Theorem 9.8 can be obtained when the corresponding cost structure is considered.

Example 9.4 Consider the multiplicative model (9.26) with the constant baseline failure rate $\lambda(t,z) = z\lambda$. This is a real-life example as, e.g., many electronic components have a constant failure rate which is varying from component to component due to production instability, etc. Note that 'traditional' burn-in (i.e., for the specified time) for these heterogeneous populations was usually executed by the manufacturers especially when the items had to meet high reliability requirements (e.g., for military field usage).

Assume for simplicity that Z is also exponentially distributed (it can easily be generalized to the gamma distribution): $\Pr(Z \le z) = 1 - \exp\{-\alpha z\}$. It is well known that the mixture failure rate in this case is

$$\lambda_m(t) = \frac{\int_0^\infty z\lambda \exp\{-z\lambda t\}\pi(z)\mathrm{d}z}{\int_0^\infty \exp\{-z\lambda t\}\pi(z)\mathrm{d}z} = \frac{\lambda}{\lambda t + \alpha}. \tag{9.39}$$

Consider a single shock defined by the specific $p(z)$ given by Eq. (9.29) [it is just more convenient for this particular example to use this parameterization rather than the equivalent parameterization (9.33)]. In accordance with (9.27):

$$\begin{aligned}\pi_s(z) &= \frac{q(z)\pi(z)}{\int_0^\infty q(z)\pi(z)\mathrm{d}z} = \frac{1}{\int_0^{z_b} \pi(z)\mathrm{d}z}\begin{cases}\pi(z), & 0 \le z \le z_b\\ 0, & z > z_b\end{cases}\\ &= \frac{1}{\Pi(z_b)}\begin{cases}\pi(z), & 0 \le z_b\\ 0, & z > z_b\end{cases}.\end{aligned}$$

Therefore, simple integration results in

$$\begin{aligned}\lambda_{ms}(t, z_b) &= \frac{\int_0^{z_b} z\lambda \exp\{-z\lambda t\}\pi(z)\mathrm{d}z}{\int_0^{z_b} \exp\{-z\lambda t\}\pi(z)\mathrm{d}z}\\ &= \frac{\lambda}{\lambda t + \alpha}\left(1 - \frac{z_b(\lambda t + \alpha)}{\exp\{z_b(\lambda t + \alpha)\} - 1}\right).\end{aligned} \tag{9.40}$$

It can be easily seen that $1 - z_b(\lambda t + \alpha)/(\exp\{z_b(\lambda t + \alpha)\} - 1)$ is increasing in z_b from 0 at $z_b = 0$ to 1 at $z_b = \infty$, for all fixed $t > 0$. Note that the value at $z_b = 0$ should be considered only like a limit (which obviously does not belong to admissible failure rates). Thus, when $z_b \to \infty$, (9.40) tends to the value defined by Eq. (9.39). It is also clear that the general inequality (9.21) holds in this specific case. It follows from (9.30) that the probability of not surviving a shock in this specific case is:

$$P(z_b) = \int_{z_b}^{\infty} \pi(z)\mathrm{d}z = \exp\{-\alpha z_b\}.$$

In accordance with (9.36), the corresponding total expected cost function is

$$c(z_b) = c_1(z_b) + c_2(z_b),$$

where

$$c_1(z_b) = \frac{c_{sr}\exp\{-\alpha z_b\} + c_s}{1 - \exp\{-\alpha z_b\}},$$

and

$$c_2(z_b) = -(g_m + c_m)\exp\left\{-\int_0^\tau \frac{\lambda}{\lambda u + \alpha}\left(1 - \frac{z_b(\lambda u + \alpha)}{\exp\{z_b(\lambda u + \alpha)\} - 1}\right)\mathrm{d}u\right\} + c_m.$$

It is obvious that $c_1(z_b)$ is decreasing in z_b and its limits are ∞ and c_s at $z_b = 0$ and $z_b = \infty$, respectively. On the other hand, as $1 - z_b(\lambda t + \alpha)/(\exp\{z_b(\lambda t + \alpha)\} - 1)$ is increasing in z_b from 0 at $z_b = 0$ to 1 at $z_b = \infty$ (for all fixed $t > 0$), $c_2(z_b)$ is increasing in z_b and its limits are $-g_m$ and $-(g_m + c_m)\exp\{-\int_0^\tau \lambda/(\lambda u + \alpha)\mathrm{d}u\} + c_m$, at $z_b = 0$ and $z_b = \infty$, respectively.

Thus, in this case, $c(z_b)$ has its limit

$$c_s - (g_m + c_m)\exp\left\{-\int_0^\tau \lambda/(\lambda u + \alpha)\mathrm{d}u\right\} + c_m.$$

Consider the following illustrative numerical values: $\lambda = 1.0, \alpha = 0.1, c_{sr} = 1.0, c_s = 1.0, g_m = 300, c_m = 200$, and $\tau = 5.0$. The corresponding graph is given in Fig. 9.3.

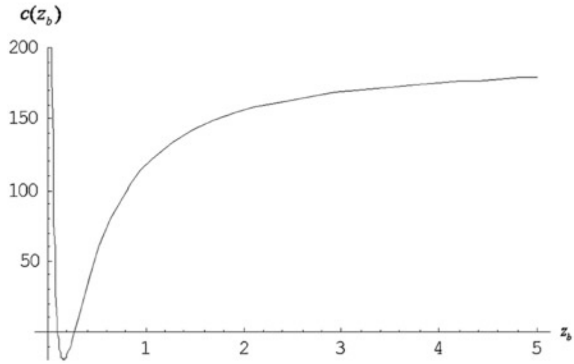
It follows from Theorem 9.8 that there exists a finite optimal stress level $s^* < \infty$, which implies that in our example there exists a positive optimal $z_b{}^* > 0$. For the chosen numerical values, we have: $z_b{}^* = 0.165$ and $c(z_b{}^*) \approx -19.63$. This result shows that for the given values of parameters the optimal stress level is relatively large ($z_b{}^*$is small).

9.3 Burn-in for Minimizing Risks

9.3.1 Discrete Mixtures

In the previous sections, it was shown that under reasonable assumptions, shocks will eliminate weaker items with larger probabilities than strong items, and in this way the burn-in can be performed. The optimal severity of shocks for some population quality measures was also studied. In this section, we will apply this methodology to the shock burn-in that minimizes the risks of selecting items (from heterogeneous populations) with poor reliability characteristics for important missions or missions, where failures can result, e.g., in a substantial economic loss. This type of burn-in can be beneficial when the 'ordinary' time burn-in does not

Fig. 9.3 The function $c(z_b)$



make sense (e.g., when the population failure rate is increasing), which will be illustrated by relevant examples (see also [8]). In what follows, we implicitly assume that shocks randomly occurring during 'normal' operation constitute one of the main causes of failure. Therefore, a single shock of a larger magnitude under the assumptions to be discussed can act as a method of burn-in.

Consider now the case of $n = 2$ subpopulations. For convenience, we repeat the initial setting of Sect. 9.1. First, we describe the composition of our population. Denote the lifetime of a component from the 'strong subpopulation' by T_s and its absolutely continuous Cdf, pdf, and the failure rate function by $F_1(t), f_1(t)$ and $\lambda_1(t)$, respectively. Similarly, the lifetime, the Cdf, pdf, and the failure rate function of a 'weak' component are $T_W, F_2(t), f_2(t)$ and $\lambda_2(t)$, accordingly. We define strong and weak subpopulations in the sense of the following failure rate ordering:

$$\lambda_2(t) \geq \lambda_1(t), \quad t \geq 0. \tag{9.41}$$

The initial $(t = 0)$ composition of our mixed population is as follows: the proportion of the strong items is π, whereas the proportion of the weak items is $1 - \pi$, which means that the distribution of the discrete *frailty* Z with realizations z_1 and z_2 in this case is

$$\pi(z) = \begin{cases} \pi, & z = z_1 \\ 1 - \pi, & z = z_2 \end{cases}$$

and z_1, z_2 $(z_1 < z_2)$, correspond to the strong and the weak subpopulations, respectively. The mixture (population) survival function is

$$\overline{F}_m(t) = \pi\overline{F}_1(t) + (1 - \pi)\overline{F}_2(t).$$

Then the mixture (the *observed* or the *population*) failure rate is

$$\lambda_m(t) = \frac{\pi f_1(t) + (1 - \pi) f_2(t)}{\pi\bar{F}_1(t) + (1 - \pi)\bar{F}_2(t)} = \pi_1(t)\lambda_1(t) + \pi_2(t)\lambda_2(t), \tag{9.42}$$

where the time-dependent probabilities are

$$\pi_1(t) = \frac{\pi \bar{F}_1(t)}{\pi \bar{F}_1(t) + (1-\pi)\bar{F}_2(t)}, \quad \pi_2(t) = \frac{(1-\pi)\bar{F}_2(t)}{\pi \bar{F}_1(t) + (1-\pi)\bar{F}_2(t)}. \tag{9.43}$$

We adopt the same assumption as in Sect. 9.1:

Basic Assumption 1 The more frail (e.g., with the larger failure rate during 'normal' operation) the items are, the more susceptible they are to be 'killed' by a single shock of a larger magnitude (burn-in).

Burn-in is applied in the following way:

- *Burn-in procedure by means of shocks.* An item from our heterogeneous population is exposed to a shock. If it survives, it is considered to be ready for usage, otherwise the failed item is discarded and a new one is exposed to a shock, etc.

Let $\pi_s(z)$ denote the frailty distribution *after the (burn-in) shock* and let $\lambda_{ms}(t)$ be the corresponding mixture (observed) failure rate. Denote the probabilities of failures caused by each shock for two subpopulations as:

$$p(z) = \begin{cases} p_1, & z = z_1, \\ p_2, & z = z_2. \end{cases} \tag{9.44}$$

Then $\pi_s(z), \lambda_{ms}(t)$ and $\overline{F}_{ms}(t)$ are defined as in Sect. 9.1 [see, e.g., Eq. (9.2)].

Consider now a simple motivating example, where the shock burn-in can be effective, whereas the ordinary time burn-in will only decrease reliability characteristics of items.

Example 9.5 Let $\lambda_1(t) = 0.3t + 1$, $\lambda_2(t) = 0.6t + 2$ and $\pi = 0.60$. Then, obviously, $\lambda_2(t) \geq \lambda_1(t), \quad t \geq 0$, and the mixture failure rate $\lambda_m(t)$ given in Fig. 9.4 is strictly increasing. Therefore, the time burn-in *should not be applied* for this heterogeneous population.

Let $p_1 = 0.1$ and $p_2 = 0.8$ [see Eq. (9.44)]. Then the mixture failure rate functions before and after (lower) the shock burn-in are given in Fig. 9.5.

Therefore, the shock burn-in improves the quality (reliability) characteristics of this population.

In the following, we consider the problem of determining the optimal severity of the shock burn-in for suitable measures of risk in operation. Denote the magnitude of a shock by $s \in [0, \infty]$. Assume that the 'strength' of the component in a strong subpopulation is a continuous random variable, which is denoted by U. By 'strength' we understand here the corresponding measure of resistance to a single shock, i.e., if $s > U$, then the failure occurs. Let the Cdf, the survival function, and the failure rate function of U are denoted by $G(s), \overline{G}(s)$, and $r(s)$, respectively. Similarly, let the strength of the component in a weak subpopulation be denoted by U_W. Then, in accordance with our Basic Assumption 1, let

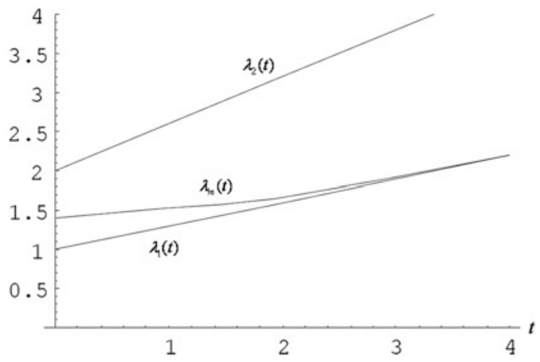


Fig. 9.4 Mixture failure rate

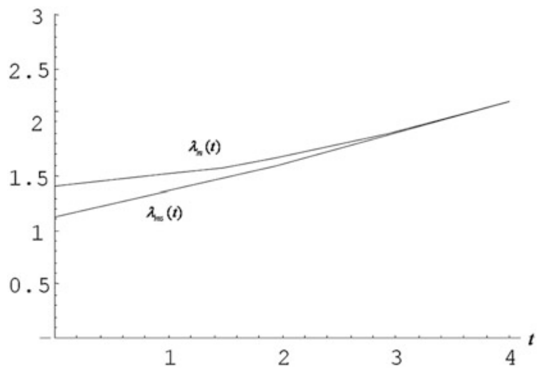


Fig. 9.5 The mixture failure rate functions before and after Shock Burn-in

$$U \geq_{st} U_W. \tag{9.45}$$

Then Eqs. (9.3) and (9.4) and the corresponding reasoning employed while deriving these equations hold.

Let an item from our population be operable at time $t > 0$ (in field operation). Then, if this is a weak item, the 'risk of instantaneous failure' is larger than that for a strong one. Therefore, a larger penalty (loss) should be imposed to the item with a larger risk. This allows us to define the following "point loss" at time t for the subpopulation i:

$$L_i(t) = g((\lambda_i(t)),\ i = 1, 2, \tag{9.46}$$

where $g(\cdot)$ is a strictly increasing function of its argument.

The following criterion of optimization of shock's severity level stems from definition (9.46):

Criterion 1 Find s^* which minimizes

$$\overline{L}(t|s) = \sum_{i=1}^{2} L(\lambda_i(t), 0)\pi_s(z_i|0) = \sum_{i=1}^{2} g(\lambda_i(t))\pi_s(z_i|0), \text{for all } t \geq 0. \tag{9.47}$$

Observe that $\overline{L}(t|s)$ in (9.47) corresponds to the mean loss at time t of an item which has experienced the shock burn-in with the corresponding magnitude s. Suppose that the subpopulations are ordered as in (9.41). Then, it is easy to see that maximization of the proportion of the strong components, $\pi_s(z_1|0) \equiv \pi_s$ minimizes (9.47) for all $t \geq 0$. Therefore, as follows from (9.45), the problem is the same as maximizing

$$\pi_s = \frac{(1 - G(s))\pi}{(1 - G(s))\pi + (1 - G(\rho(s)))(1 - \pi)},$$

which is the same as finding s^* that satisfies

$$s^* = \arg \inf_{s \in [0,\infty]} \lambda_{ms}(t; s), \text{for all fixed } t \geq 0.$$

The corresponding result can be found in Cha and Finkelstein [6]:

Theorem 9.9 [6] *Let $\lambda_1(t) \leq \lambda_2(t)$, for all $t \geq 0$. Then the optimal s^* is the value which maximizes $R(\rho(s)) - R(s)$, where $R(s) \equiv \int_0^s r(u)\mathrm{d}u$. In particular,*

(i) *If $r(s)$ is increasing and $\rho'(s) > 1$, then $s^* = \infty$.*
(ii) *If $\frac{\rho'(s)r(\rho(s))}{r(s)} > 1$, for $s < s_0$, and $\frac{\rho'(s)r(\rho(s))}{r(s)} < 1$, for $s > s_0$, then $s^* = s_0$.*

Consider now the second criterion. Let τ be the usage (mission) time for our components. Then, as the point loss varies during mission time, it should be averaged, i.e., it should be integrated for the mission interval (and then divided by the length of the interval) to measure the 'overall risk' during the mission. Thus, the average loss during the operational interval for subpopulation i can be defined as

$$\frac{\int_0^\tau L_i(t)\mathrm{d}t}{\tau} = \frac{\int_0^\tau g(\lambda_i(t))\mathrm{d}t}{\tau}, i = 1, 2.$$

As the selection of a component from a heterogeneous population is made just after the shock burn-in and the corresponding proportions after the burn-in are given by $\pi_s(z_i|0), i = 1, 2$, the mean loss for our mixture population (after burn-in) is

$$\Psi(s) = \sum_{i=1}^{2} \frac{\int_0^\tau L(\lambda_i(t), 0)\mathrm{d}t}{\tau} \cdot \pi_s(z_i|0) = \sum_{i=1}^{2} \frac{\int_0^\tau g(\lambda_i(t))\mathrm{d}t}{\tau} \cdot \pi_s(z_i|0). \tag{9.48}$$

Criterion 2 Find s^* which minimizes $\Psi(s)$.

Similar to the optimization based on **Criterion 1**, as the subpopulations are ordered in the sense of failure rate ordering, Theorem 9.9 could be also applied, which is illustrated by the following example.

Example 9 Let $\lambda_1(t) = 1.2 - \exp\{-1.2t\} + 0.01t$, $\lambda_2(t) = 1.4\exp\{-0.08t\} + 1.2 + 0.01t$, with $\pi = \pi_1(0) = 0.80$. Then $\lambda_2(t) \geq \lambda_1(t)$, $t \geq 0$ and the

Fig. 9.6 Mixture failure rate

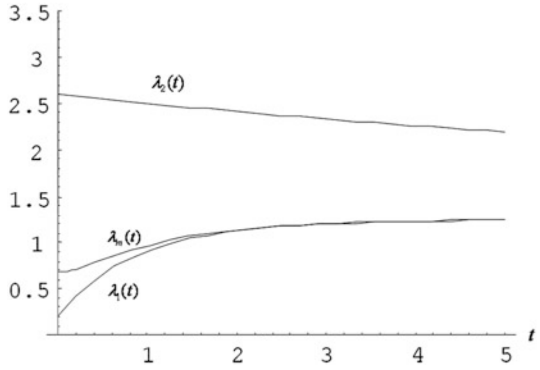
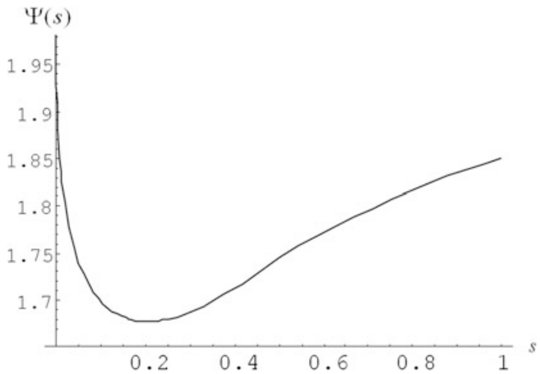


Fig. 9.7 $\Psi(s)$



corresponding strictly increasing mixture failure rate $\lambda_m(t)$ is given in Fig. 9.6. Let the failure rate of $G(s)$ be $r(s) = \exp\{-s\} + 1, \rho(s) = \sqrt{s}, 0 \le s \le 1/2; \rho(s) = s + (1/\sqrt{2} - 1/2)\exp(0.5 - s), s \ge 1/2.$ and $\tau = 3$. Then for $g(x) = x^2$, $\Psi(s)$ is given in Fig. 9.7. It can be numerically shown that there exists s_0 such that $\frac{\rho'(s)r(\rho(s))}{r(s)} > 1$, for $s < s_0$, and $\frac{\rho'(s)r(\rho(s))}{r(s)} < 1$, for $s > s_0$, and, as illustrated by Fig. 9.7, there exists the finite optimal severity level $(s^* \approx 0.20)$. Note that, as the failure rates are ordered, minimization of $\Psi(s)$ in (9.48) is equivalent to maximization of the proportion of the strong components, $\pi_s(z_1|0) \equiv \pi_s$. Therefore, the optimal severity in this case does not depend on the value of τ and this is also the optimal severity level for **Criterion 1**.

Note that the proportion of the strong subpopulation after the shock burn-in is $\pi_s \approx 0.86$. (compare with 0.80 before burn-in). In addition, it can be shown graphically that the mixture failure rate in this case has also been decreased for all $t \ge 0$, as in Fig. 9.5.

9.3.2 Continuous Mixtures

As in the previous parts of this chapter, consider now the case of the 'continuous' mixing model for a heterogeneous population, i.e.,

$$F_m(t) = \int_0^\infty F(t,z)\pi(z)\mathrm{d}z,\ f_m(t) = \int_0^\infty f(t,z)\pi(z)\mathrm{d}z, \tag{9.49}$$

where $F(t,z) \equiv F(t|z), f(t,z) \equiv f(t|z)$ are the Cdf and the pdf of subpopulations indexed (conditioned) by the frailty parameter Z and $\pi(z)$ is the pdf of Z with support in $[0,\infty)$ Then the mixture failure rate $\lambda_m(t)$ is defined as in (9.15), (9.16).

As in the discrete case, let our subpopulations be ordered in the sense of the failure (hazard) rate ordering:

$$\lambda(t,z_1) \leq \lambda(t,z_2), \quad z_1 < z_2, \forall z_1, z_2 \in [0,\infty),\ t \geq 0. \tag{9.50}$$

We choose an item from a heterogeneous population at random (or alternatively, our item is described by the unobserved frailty parameter Z). Thus, the mixture (population) failure rate of this item is $\lambda_m(t)$. Throughout this subsection, similar to the Basic Assumption 1, the impact of a shock is described by the following *general assumption* [6].

Basic Assumption 2 A shock either 'kills' an item with a given probability or does not change its stochastic properties with the complementary probability. The more 'frail' (e.g., with the larger failure rate during normal operation) an item is, the larger is the probability that a single burn-in shock will 'kill' it.

As we implicitly assume that shocks during normal operation constitute one of the main causes of failure, the above assumption can be justified. Note that, clearly, the burn-in procedure is the same as in the discrete case. The described setting can be defined probabilistically in the following way: Let $\pi_s(z)$ denote the pdf of the frailty Z_s (with support in $[0,\infty)$) after a shock and let $\lambda_{ms}(t)$ be the corresponding mixture failure rate. In accordance with (9.49):

$$\lambda_{ms}(t) = \int_0^\infty \lambda(t,z)\pi_s(z|t)\mathrm{d}z, \tag{9.51}$$

where, similar to (9.50), $\pi_s(z|t)$ is defined by the right-hand side of (9.50) with $\pi(z)$ substituted by $\pi_s(z)$.

Let $q(z)$ be "the survival probability" of an item with frailty z after the shock. Then $\pi_s(z)$ is [10]:

$$\pi_s(z) = \frac{q(z)\pi(z)}{\int_0^\infty q(z)\pi(z)\mathrm{d}z}, \tag{9.52}$$

where, in accordance with Basic Assumption 2, $q(z)$ is a decreasing function of z and therefore, $\pi_s(z)/\pi(z)$ is decreasing [the denominator of (9.52) is just a normalizing constant for the density]. That is, population frailties before $(\pi(z))$ and after $(\pi_s(z))$ a shock are ordered in the sense of the likelihood ratio (Sect. 2.8)

$$Z \geq_{LR} Z_s.$$

Define the functions $p(z)$ and $q(z)$ as functions of the frailty variable z and the severity parameter $s \in [0, \infty)$, $p(z, s)$, and $q(z, s)$. Assume that $q(z, s)$ is decreasing in z for each fixed s and is decreasing in s for each z. Denote the corresponding failure rate and survival functions by $\lambda_{ms}(t; s)$, and $\overline{F}_{ms}(t; s)$, respectively. Similar to (9.52) and (9.50):

$$\pi_s(z,s) = \frac{q(z,s)\pi(z)}{\int_0^\infty q(u,s)\pi(u)\mathrm{d}u}, \pi_s(z,s|t) \equiv \pi_s(z,s)\frac{\bar{F}(t,z)}{\int_0^\infty \bar{F}(t,u)\pi_s(u,s)\mathrm{d}u}. \quad (9.53)$$

For this continuous mixture case, the criteria defined for the discrete case can obviously be generalized as follows:

Criterion 1C Find s^* which minimizes

$$\overline{L}(t|s) = \int_0^\infty g(\lambda(t,z))\pi_s(z,s)\mathrm{d}z, \text{for all } t \geq 0.$$

Criterion 2C Find s^* which minimizes

$$\Psi(s) = \int_0^\infty \frac{\int_0^\tau g(\lambda(t,z))\mathrm{d}t}{\tau} \cdot \pi_s(z,s)\mathrm{d}z.$$

The following example illustrates the application of **Criterion 2C**.

Example 9.7 Suppose that $\lambda(t, z) = 0.1z\exp\{0.1t\} + 0.02t + 1$, and let Z be exponentially distributed with parameter $\theta = 0.5$. For brevity, we omit the graph showing that the mixture failure rate is strictly increasing in this case. Let $q(z, s) = 0.95e^{-zs} + 0.05$, $\tau = 3.0$, and $g(x) = x^2$. Then $\Psi(s)$ is given in Fig. 9.8.

Thus the optimal shock severity is $s^* \approx 2.03$. As in Example 9.6, the shock burn-in in this case has decreased the mixture failure rate (we omit the corresponding figure for brevity), which obviously cannot be attained by the ordinary time burn-in, as the mixture failure rate of our population is increasing. The frailty distributions before and after burn-in are given in Fig. 9.9.

It can be seen that the frailty density before the shock is much flatter allowing larger proportions of items with higher failure rates (weaker).

Fig. 9.8 $\Psi(s)$

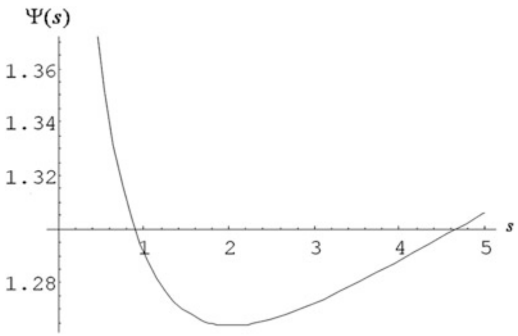
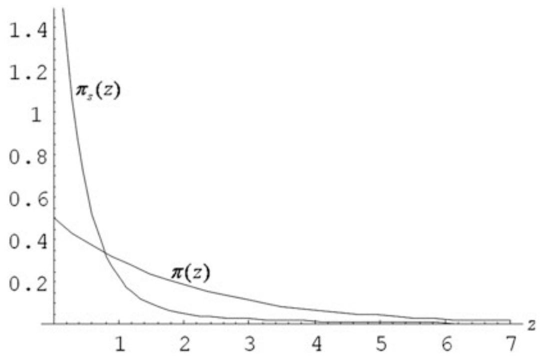


Fig. 9.9 Frailty densities before and after Burn-in



9.3.3 *Optimal Shock Burn-in Based on Conservative Measures*

Sometimes, failures of items may result in catastrophic or disastrous events. For example, failures in jet engines of aircrafts or those in gas safety valves may cause fatal consequences. Similarly, failures during important missions can cause huge economic loss. In these cases, we need to define some 'marginal quality' of the population that describes in some sense the "worst scenario". That is, if this worst scenario quality is still acceptable then the quality of our population as a whole is considered to be satisfactory. Thus, the marginal quality can be used as a conservative (safe) measure (or bound) for the quality of a population in such cases.

In this subsection, we consider the optimal burn-in procedure which optimizes the conservative measures and modify the approach that was developed in Cha and Finkelstein [7] (see also Sect. 8.3) for the time burn-in with respect to the shock burn-in. Obviously, this refers only to the continuous mixtures case.

Denote by $\Pi_s(z, s)$, the conditional distribution function which corresponds to $\pi_s(z, s)$, defined in (9.53). Define the following measure:

$$\lambda_\alpha(t|s) = \lambda(t, z(\alpha|s)),\ t \geq 0, \tag{9.54}$$

where $z(\alpha|s) \equiv \inf\{z : \Pi_s(z,s) \geq \alpha\}$ and α is usually close to 1 (e.g., 0.9 or 0.95). Thus, $\lambda_\alpha(t|s)$ is the (residual) failure rate of an item after a shock with magnitude s, which corresponds to the αth percentile $z(\alpha|s)$ of the conditional distribution of frailty $\Pi_s(z,s)$. When α is close to 1, this operation describes the αth worst scenario, which is the 'αth worst subpopulation' in the defined way. Based on the above setting, we can define the αth worst mean remaining lifetime (MRL) of the population after the shock burn-in with severity s:

$$M_\alpha(s) \equiv \int_0^\infty \exp\{-\int_0^t \lambda_\alpha(u|s)du\}dt.$$

Therefore, the following criterion can be applied:

Criterion 3 Determine the optimal severity s^* as the minimal severity s such that $M_\alpha(s) \geq m_r$, where m_r is the MRL that corresponds to the αth worst scenario.

Implementation of this approach can be clearly seen while considering the following meaningful example.

Example 9.8 Let the conditional failure rate and the mixing distribution be $\lambda(t,z) = z$ and $\pi(z) = \theta\exp\{-\theta z\}$, respectively. It is well known (see e.g., [2] that the mixture failure rate strictly decreases in this case. Let $q(z,s) = e^{-za(s)}$, where $a(s)$ is nonnegative strictly increasing function with $a(0) = 0$ and $\lim_{s\to\infty} a(s) = \infty$. In accordance with (9.53):

$$\Pi_s(z,s) = 1 - \exp\{-(\theta + a(s))z\}.$$

Then

$$z(\alpha|s) = -\frac{\ln(1-\alpha)}{\theta + a(s)},$$

and [see (9.54)]:

$$\lambda_\alpha(t|s) = -\frac{\ln(1-\alpha)}{\theta + a(s)},\ t \geq 0.$$

The criterion for the shock burn-in is as follows: Find the minimum shock severity such that, after burn-in, the mean (residual) lifetime of the lower $(1-\alpha)\,\%$ quality of items is, at least, m. As the lifetimes are exponential (for the fixed frailty), this MRL is, obviously,

$$M_\alpha(s) = 1/z(\alpha|s) = -(\theta + a(s))/\ln(1-\alpha).$$

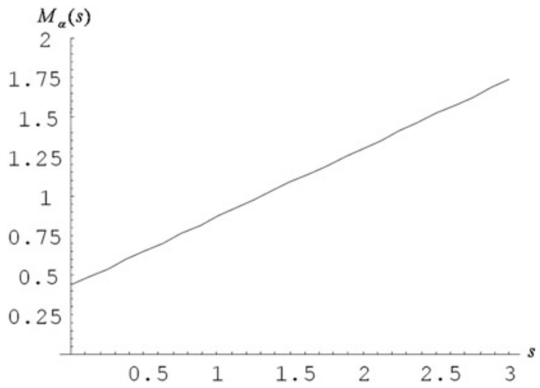


Fig. 9.10 $M_\alpha(s)$ for $\alpha = 9$, $\theta = 1.0$, $a(s) = s$

Let $\alpha = 9, \theta = 1.0$ and $a(s) = s$. Then the corresponding linear function is given in Fig. 9.10.

If, for instance, $m = 1.25$, then the corresponding minimum shock severity: $s^* \approx 1.88$.

The conservative measure (9.54) can be modified (generalized) to account for the *average* of the lower $(1 - \alpha)\,\%$ quality of items in the survived population after the shock with severity s. Then, after the shock with severity s, the *initial conditional frailty distribution* [which corresponds to $\pi(z)$ in (9.50)] for the items whose quality is lower than $(1 - \alpha)\,\%$ is given by

$$\frac{\pi_s(z,s)}{1-\alpha}, z(\alpha|s) \le z \le \infty,$$

where, as previously, $z(\alpha|s) \equiv \inf\{z : \prod_s(z,s) \ge \alpha\}$. Thus the conditional density after time t (in usage), which corresponds to $\pi(z|t)$ in (9.51) is

$$\pi_\alpha(z,s|t) \equiv \frac{\pi_s(z,s)}{1-\alpha}\frac{\bar{F}(t,z)}{\int_{z(\alpha|s)}^{\infty}\bar{F}(t,z)\frac{\pi_s(z,s)}{1-\alpha}\mathrm{d}z}, z(\alpha|s) \le z \le \infty.$$

Therefore, the mixture failure rate for the items in the survived population whose quality is lower than $(1 - \alpha)\,\%$ after the shock with severity s is obtained by

$$\lambda_m(t|s,\alpha) = \int_{z(\alpha|s)}^{\infty} \lambda(t,z)\pi_\alpha(z,s|t)\mathrm{d}z.$$

Example 9.9 (Example 9.8 Continued) As $z(\alpha|s) = -\ln(1-\alpha)/(\theta + a(s))$ and

$$\int_{z(\alpha|s)}^{\infty} \bar{F}(t,z)\frac{\pi_s(z,s)}{1-\alpha}\mathrm{d}z = \frac{1}{(1-\alpha)} \cdot \frac{\theta + a(s)}{\theta + a(s) + t} \cdot (1-\alpha)^{\frac{\theta+a(s)+t}{\theta+a(s)}},$$

we have,

$$\pi_\alpha(z, s|t) \equiv \frac{\pi_s(z,s)}{1-\alpha} \frac{\bar{F}(t,z)}{\int_{z(\alpha|s)}^{\infty} \bar{F}(t,z) \frac{\pi_s(z,s)}{1-\alpha} \mathrm{d}z}$$
$$= (\theta + a(s) + t) \cdot (1-\alpha)^{-\frac{\theta+a(s)+t}{\theta+a(s)}} \cdot \exp\{-(\theta + a(s) + t)z\}.$$

Thus

$$\lambda_m(t|s,\alpha) = \int\limits_{z(\alpha|s)}^{\infty} \lambda(t,z)\pi_\alpha(z,s|t)\mathrm{d}z = -\frac{\ln(1-\alpha)}{\theta + a(s)} + \frac{1}{\theta + a(s) + t}, t \geq 0.$$

The criterion for the shock burn-in is as follows: Find the minimum shock severity such that, after burn-in, the mean (residual) lifetime of the items whose quality is lower than $(1-\alpha)\,\%$ is at least m. Then we have to obtain the MRL of the items whose quality is lower than $(1-\alpha)\,\%$ after the shock burn-in at each severity level s, which is given by

$$\int\limits_0^{\infty} \exp\left\{ -\int\limits_0^x \lambda_m(t|s,\alpha)\mathrm{d}t \right\} \mathrm{d}x,$$

Let $\alpha = 9, \theta = 1.0$ and $a(s) = s$ and $m = 1.25$. Then it can be easily found numerically that the optimal shock severity is $s^* \approx 2.47$.

9.4 Burn-in for Systems in Environment with Shocks

Burn-in procedures are usually applied to items with large initial failure rate which operate under static operational environment. Similar to previous sections, we consider shocks as a method of burn-in, but in this section we assume that there are two competing risks causes of failure—the 'usual' one (in accordance with aging processes in a system) and environmental shocks. We also suggest a new type of burn-in via the *controlled* (laboratory) test shocks and consider the problem of obtaining the optimal level (severity) of these shocks that minimizes the overall expected cost (burn-in + field use). Furthermore, also to minimize these costs, we combine the conventional burn-in procedure with burn-in via shocks in one unified model. We start with the general description of the basic stress-strength model. In Sect. 4.7 and Sect. 4.10.3 we have already used some specific cases of this model for discussing the operation of thinning of point processes and processes with delay and cure.

9.4.1 Strength–Stress Shock Model

In this subsection, we consider a rather general stress-strength shock model, which will be used as an important supplementary result for considering burn-in problems of the subsequent subsections.

As in Chap. 4, consider a system subject to the nonhomogeneous Poisson process (NHPP) of shocks $N(t), t \geq 0$, with rate $\lambda(t)$ and arrival (waiting) times $T_i, i = 1, 2, \ldots$. Let S_i denote the magnitude (stress) of the ith shock. Assume that $S_i, i = 1, 2, \ldots$ are i.i.d. random variables with the common Cdf $M_f(s) = \Pr(S_i \leq s)$ $\left(\overline{M}_f(s) \equiv 1 - M_f(s)\right)$ and the corresponding pdf $m_f(s)$. Let U be a random strength of the system with the corresponding Cdf, Sf, pdf, and FR $G_U(u)$, $\overline{G}_U(u)$, $g_U(u)$ and $r_U(u)$, respectively. For each $i = 1, 2, \ldots$, the operable system survives if $S_i \leq U$ and fails if $S_i > U$, '*independently of everything else*'.

Let T be the lifetime of the system described above and $r(t)$ be the corresponding failure rate function, which will be derived in the rest of this subsection. Then the following theorem presents the formal and a more detailed proof of Eq. (4.50):

Theorem 9.10 *The failure rate function of the system lifetime $r(t)$ is given by*

$$r(t) = p(t)\lambda(t), \tag{9.55}$$

where

$$p(t) \equiv \frac{\int_0^\infty \int_0^v \exp\left\{-\overline{M}_f(r) \int_0^t \lambda(x)\mathrm{d}x\right\} \cdot g_U(r)\mathrm{d}r\, m_f(v)\mathrm{d}v}{\int_0^\infty \exp\left\{-\overline{M}_f(r) \int_0^t \lambda(x)\mathrm{d}x\right\} g_U(r)\mathrm{d}r}.$$

Proof Observe that

$$\begin{aligned}
&P(T > t | N(s), 0 \leq s \leq t, S_1, S_2, \ldots, S_{N(t)}) \\
&\quad = P(U > \max\{S_1, S_2, \ldots, S_{N(t)}\}) \\
&\quad = \int_0^\infty \left(M_f(r)\right)^{N(t)} g_U(r)\mathrm{d}r.
\end{aligned}$$

Thus,

$$\begin{aligned}
P(T > t) &= \int_0^\infty \left(\sum_{n=0}^\infty \left(M_f(r)\right)^n \frac{(\Lambda(t))^n}{n!} \exp\{-\Lambda(t)\}\right) g_U(r)\mathrm{d}r \\
&= \int_0^\infty \exp\{-(1 - M_f(r))\Lambda(t)\} g_U(r)\mathrm{d}r \\
&= \int_0^\infty \exp\{-\overline{M}_f(r)\Lambda(t)\} g_U(r)\mathrm{d}r,
\end{aligned}$$

where $\Lambda(t) \equiv \int_0^t \lambda(u)\mathrm{d}u$. The corresponding failure rate is

$$\begin{aligned}
r(t) &= -\frac{\mathrm{d}}{\mathrm{d}t}\ln P(T > t) \\
&= \frac{\int_0^\infty \overline{M}_f(r)\exp\{-\overline{M}_f(r)\Lambda(t)\}g_U(r)\mathrm{d}r \cdot \lambda(t)}{\int_0^\infty \exp\{-\overline{M}_f(r)\Lambda(t)\}g_U(r)\mathrm{d}r} \\
&= \frac{\int_0^\infty \int_r^\infty m_f(v)\mathrm{d}v \exp\{-\overline{M}_f(r)\Lambda(t)\}g_U(r)\mathrm{d}r \cdot \lambda(t)}{\int_0^\infty \exp\{-\overline{M}_f(r)\Lambda(t)\}g_U(r)\mathrm{d}r} \\
&= \frac{\int_0^\infty \int_0^v \exp\{-\overline{M}_f(r)\int_0^t \lambda(x)\mathrm{d}x\} \cdot g_U(r)\mathrm{d}r\, m_f(v)\mathrm{d}v}{\int_0^\infty \exp\left\{-\overline{M}_f(r)\int\limits_0^t \lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r}\lambda(t).
\end{aligned}$$

□

The expression for $p(t)$ is formally rather cumbersome, but it has a simple and meaningful probabilistic meaning, which is shown in the following remark.

Remark 9.14 Observe that

$$P(T > t|N(t) = n, U = u) = P(u \geq \max\{S_1, S_2, \ldots, S_n\}) = (M_f(u))^n.$$

Thus,

$$\begin{aligned}
P(T > t, U > u) &= \int\limits_u^\infty \sum_{n=0}^\infty (M_f(r))^n \frac{\left(\int_0^t \lambda(x)\mathrm{d}x\right)^n}{n!}\exp\left\{-\int\limits_0^t \lambda(x)\mathrm{d}x\right\} \cdot g_U(r)\mathrm{d}r \\
&= \int\limits_u^\infty \exp\left\{-\overline{M}_f(r)\int\limits_0^t \lambda(x)\mathrm{d}x\right\} \cdot g_U(r)\mathrm{d}r,
\end{aligned}$$

and

$$P(U > u|T > t) = \frac{\int_u^\infty \exp\{-\overline{M}_f(r)\int_0^t \lambda(x)dx\} \cdot g_U(r)\mathrm{d}r}{\int_0^\infty \exp\{-\overline{M}_f(r)\int_0^t \lambda(x)\mathrm{d}x\}g_U(r)\mathrm{d}r}.$$

Therefore, it can be seen that

$$p(t) = \int_0^\infty P(U < v | T > t) m_f(v) \mathrm{d}v. \tag{9.56}$$

As U is a random strength of our system and $m_f(v)$ is the pdf of the magnitude of any shock, $p(t)$ can be interpreted as the probability of a failure under a shock that had occurred at time t given that it did not occur before. The important feature of (9.56) is conditioning on the event $T > t$, which obviously has the Bayesian interpretation via the updating of the distribution of the system's strength. That is, even though random strength does not actually change, its distribution (on the condition that $T > t$) is updated as t increases, which eventually yields a time-dependent $p(t)$. This conditioning was overlooked in Cha and Finkelstein [7], which resulted in $p = \int_0^\infty P(U < v) m_f(v) \mathrm{d}v$. Relationship (9.56) will be very useful for our further discussion.

9.4.2 Optimal Level of Shock's Severity

We consider a system (a component, an item) that operates in an environment with shocks. Assume that in the absence of shocks, it can fail in accordance with the baseline distribution $F_0(t)$ with the corresponding failure rate function $r_0(t)$. In addition to this type of the 'baseline' failure, the environmental shocks can also cause system's failure. Assume that each shock, with probability $p(t)$ results in immediate system's failure and with probability $q(t) = 1 - p(t)$ it does not cause any change in the system. We use the same notation, as in (9.56), because $p(t)$ in (9.56) as an 'overall characteristic' can be also obviously interpreted in this way. If the shocks follow the NHPP with intensity $\lambda(t)$, then it is well known that the survival function of the system for this setting is given by

$$\begin{aligned} P(T > t) &= \exp\left(-\int_0^t r_0(u)\mathrm{d}u\right)\exp\left(-\int_0^t p(u)\lambda(u)\mathrm{d}u\right) \\ &= \exp\left(-\int_0^t r_0(u) + p(u)\lambda(u)\mathrm{d}u\right), t \geq 0, \end{aligned}$$

and thus the resulting failure rate is

$$r(t) = r_0(t) + p(t)\lambda(t). \tag{9.57}$$

Coming back to the burn-in setting, as in Sect. 9.4.1, we now further assume that the magnitude (stress) of the ith shock $S_i, i = 1, 2, \ldots$ are i.i.d. random variables with the common Cdf $M_f(s) = \Pr(S_i \leq s)$ $\left(\overline{M}_f(s) \equiv 1 - M_f(s)\right)$ and the corresponding pdf $m_f(s)$. For each $i = 1, 2, \ldots$, the operable system survives if $S_i \leq U$ and fails if $S_i > U$, *independently of everything else,* where U is the random strength of the system. When we apply the shock of the controlled magnitude s during burn-in, this means that the strength of the component that had passed it is larger than s, and the distribution of the *remaining strength* U_s (given that the strength is larger than s) is

$$G_U(u|s) \equiv \Pr[U \leq u | U > s] = 1 - \overline{G}(u)/\overline{G}(s), u > s.$$

Let T_s be the lifetime of the system that has survived the shock burn-in with the controlled magnitude s. Then, in accordance with the discussion in Sect. 9.4.1 and the result given by (9.55), the failure rate in (9.57) should now be modified to

$$r(t, s) = r_0(t) + p(s, t)\lambda(t), \tag{9.58}$$

where

$$\begin{aligned} p(s,t) &= \frac{\int_0^\infty \int_0^v \exp\left\{-\overline{M}_f(r) \int_0^t \lambda(x)\mathrm{d}x\right\} \cdot g_U(r|s)\mathrm{d}r\, m_f(v)\mathrm{d}v}{\int_0^\infty \exp\left\{-\overline{M}_f(r) \int_0^t \lambda(x)\mathrm{d}x\right\} g_U(r|s)\mathrm{d}r} \\ &= \frac{\int_s^\infty \int_s^v \exp\{-\overline{M}_f(r) \int\limits_0^t \lambda(x)dx\} \cdot g_U(r)dr\, m_f(v)dv}{\int_s^\infty \exp\left\{-\overline{M}_f(r) \int_0^t \lambda(x)\mathrm{d}x\right\} g_U(r)\mathrm{d}r}, \end{aligned} \tag{9.59}$$

and $g_U(u|s)$ is the corresponding pdf of $G_U(u|s)$, which is given by

$$g_U(u|s) = \begin{cases} 0, & \text{if } u \leq s \\ \frac{g_U(u)}{\overline{G}_U(s)}, & \text{if } u > s \end{cases}.$$

Therefore, similar to (9.56), Eq. (9.59) can be written in a compact and a meaningful way (via the corresponding mixture) as

$$p(s, t) = \int\limits_0^\infty I(v \in [s, \infty))P(U_s < v | T > t)m_f(v)\mathrm{d}v, \tag{9.60}$$

where

$$P(U_s < v | T > t) = \frac{\int_s^v \exp\left\{-\overline{M}_f(r) \int_0^t \lambda(x)\mathrm{d}x\right\} \cdot g_U(r)\mathrm{d}r}{\int_s^\infty \exp\left\{-\overline{M}_f(r) \int_0^t \lambda(x)\mathrm{d}x\right\} g_U(r)\mathrm{d}r}.$$

and the indicator $I(v \in [s, \infty))$ accounts for the fact that after the shock burn-in with magnitude s, the system's strength with probability 1 is larger than s.

In order to justify the shock burn-in, we must show that $p(s,t)$ in (9.59) is decreasing in s for each fixed t. Thus, by increasing the magnitude of the burn-in shock, we decrease the corresponding failure rate in (9.58). This property, which is important for our reasoning, is proved by the following simple theorem:

Theorem 9.11 *The function $p(s,t)$ is strictly decreasing in s for each fixed t.*

Proof Observe that

$$\begin{aligned}\frac{\partial}{\partial s}P(U_s<v|T>t) &= \frac{1}{\left(\int_s^\infty \exp\{-\overline{M}_f(r)\int_0^t \lambda(x)\mathrm{d}x\}g_U(r)\mathrm{d}r\right)^2}\\ &\quad\times\left[-\exp\left\{-\overline{M}_f(s)\int_0^t \lambda(x)\mathrm{d}x\right\}g_U(s)\cdot\int_s^\infty \exp\left\{-\overline{M}_f(r)\int_0^t \lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\right.\\ &\quad\left.+\exp\left\{-\overline{M}_f(s)\int_0^t \lambda(x)\mathrm{d}x\right\}g_U(s)\cdot\int_s^v \exp\left\{-\overline{M}_f(r)\int_0^t \lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\right]<0.\end{aligned}$$

This implies that $P(U_s<v|T>t)$ is strictly decreasing in s for all fixed v and t. Observe that the indicator in (9.60) is also strictly decreasing in s for all fixed v. Therefore, it can be concluded that $p(s,t)$ is strictly decreasing in s for each fixed t.□

Based on the new results obtained above, we now reconsider some of the previous burn-in models.

An item is chosen at random from our population and is exposed to a shock of magnitude s. If it survives, it is considered to be ready for usage, otherwise the failed item is discarded and the new one is chosen from the population, etc. This procedure is repeated until the first survived item is obtained. Let c_{sr} be the shop replacement cost and c_s be the cost for conducting a single shock. Let $c_1(s)$, as a function of s, be the expected cost for eventually obtaining a component which has survived a shock. Then

$$c_1(s)=\frac{c_s+c_{sr}G(s)}{\overline{G}(s)}=-c_{sr}+\frac{c_s+c_{sr}}{\overline{G}(s)}, \tag{9.61}$$

where $1/\overline{G}(s)$ is the total number of trials until the first 'success'.

Let K be the gain for the unit of time during the mission time. Then the expected gain during field operation (until failure) is given by

$$c_2(s)=-K\left(\int_0^\infty \exp\left\{-\int_0^t (r_0(u)+p(s,u)\lambda(u))\mathrm{d}u\right\}\mathrm{d}t\right)$$

and the total expected cost $c(s)$ is

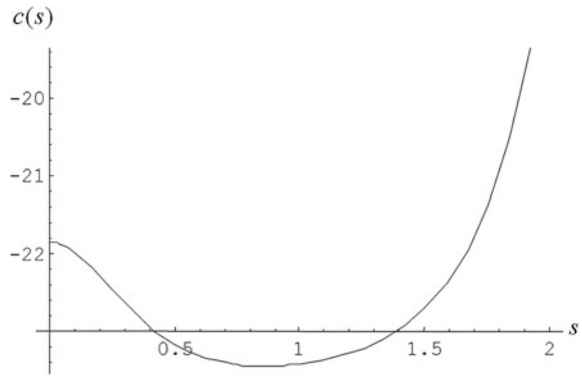


Fig. 9.11 Graph for $c(s)$ (Weibull $G(s)$)

$$
\begin{aligned}
c(s) &= c_1(s) + c_2(s)\\
&= -c_{sr} + \frac{c_s + c_{sr}}{\overline{G}(s)} - K\left(\int_0^{\infty} \exp\left\{-\int_0^t (r_0(u) + p(s,u)\lambda(u))\mathrm{d}u\right\}\mathrm{d}t\right),
\end{aligned}
\tag{9.62}
$$

where $p(s,u)$ is given by (9.59). The function $c_1(s)$ is strictly increasing to infinity and $c_2(s)$ is strictly decreasing (Theorem 9.11) to $-K\mu_0$, where μ_0 is the mean time to failure, which corresponds to the distribution with the failure rate $r_0(t)$. Therefore, there should be a *finite* optimal severity. Then, based on (9.62), the optimal severity level s^* that satisfies

$$s^* = \arg \min_{s \in [0,\infty]} c(s)$$

can be obtained.

In the following example, the strength of a system is described by the Weibull distribution.

Example 9.10 Assume that $\overline{G}_U(u) = \exp\{-u^2\}, u \geq 0, \overline{M}_f(s) = \exp\{-6s\}$, $s \geq 0, \lambda(t) = 1, t \geq 0$, and $r_0(t) = 0.06t + 0.2$, $t \geq 0$. Let $c_{sr} = 0.1, c_s = 0.01$, and $K = 8.0$.

Optimal severity in this case is given by $s^* = 0.86$ and the corresponding minimum cost is $c(s^*) = -23.46$ (Fig. 9.11).

Similar reasoning holds when our gain is defined by the success of the mission during the fixed interval of time τ. Let:

- The cost c_m is incurred by the event $\{T_s \leq \tau\}$ (Failure of the Mission);
- The gain g_m results from the event $\{T_s > \tau\}$ (Success of the Mission).

Then the burn-in costs are the same as in (9.61), whereas the expected cost during field operation, $c_2(s)$, is given by

$$c_2(s) = -g_m\left(\exp\left\{-\int_0^\tau (r_0(u)+p(s,u)\lambda(u))\mathrm{d}u\right\}\right) + c_m\left(1-\exp\left\{-\int_0^\tau (r_0(u)+p(s,u)\lambda(u))\mathrm{d}u\right\}\right)$$
$$= -(g_m+c_m)\left(\exp\left\{-\int_0^\tau (r_0(u)+p(s,u)\lambda(u))\mathrm{d}u\right\}\right) + c_m.$$

It is clear that $c_2(s)$ is strictly decreasing to

$$-(g_m+c_m)\left(\exp\left\{-\int_0^\tau r_0(u)\mathrm{d}u\right\}\right) + c_m,$$

and all further considerations are similar to those when the gain is proportional to the mean time to failure.

9.4.3 Burn-in Procedure Combining Shock and Conventional Burn-in

In this subsection, we will deal with the combined burn-in procedures considered in Cha and Finkelstein [7] using the results of the previous subsections. We have two possibilities: $B(b,s)$, the strategy when the systems are burned-in for time b (we will call it the 'time burn-in') and then the shock burn-in with severity s is applied to the systems, which survived the burn-in time b, whereas the strategy $B(s,b)$ applies shock first and then the survived systems are burned-in for time b. Unless otherwise specified, we assume that, during the time burn-in, the system is also subject to environmental shocks (as in field usage). In Cha and Finkelstein [7], the simple case of the homogeneous Poisson process of environmental shocks with intensity λ was considered, whereas in the current setting we are able to deal with the general NHPP case. In fact, the shock intensity during time burn-in and that during the field operation can be different. Let $\lambda_b(t)$ be the shock intensity at time t from the starting point of the burn-in and $\lambda_f(t)$ be the shock intensity at time t from the starting point of the field operation. Then the overall intensity function is

$$\lambda(t) = \begin{cases} \lambda_b(t), & \text{if } t \le b \\ \lambda_f(t-b), & \text{if } t > b, \end{cases}$$

where b is the burn-in time.

Let the assumptions and notation for the burn-in strategies under consideration be the same as before. As for the conventional burn-in procedure, assume additionally that the burn-in cost is proportional to the total burn-in time with proportionality constant c_0.

Consider first, the strategy $B(s, b)$. Let $h_1(s, b)$ be the expected burn-in cost for $B(s, b)$ and T_s be the lifetime of the system that has survived the shock burn-in. As our shock is of the fixed magnitude s, the corresponding survival function after the shock, in accordance with (9.58), is

$$\overline{F_s}(t) = \exp\left(-\int_0^t (r_0(u) + p(s,u)\lambda(u))\mathrm{d}u\right),$$

where $p(s, t)$ is defined in (9.59). Then, by similar arguments as those described in Cha and Finkelstein [7], we have:

$$h_1(s,b) = c_0 \frac{\int_0^b \overline{F_s}(t)\mathrm{d}t}{\overline{F_s}(b)} + \frac{c_s + c_{sr}}{\overline{F_s}(b)\overline{G}(s)} - c_{sr}. \tag{9.63}$$

On the other hand, when our system is not exposed to environmental shocks during the time burn-in, (9.63) changes to

$$h_1(s,b) = c_0 \frac{\int_0^b \overline{F_0}(t)\mathrm{d}t}{\overline{F_0}(b)} + \frac{c_s + c_{sr}}{\overline{F_0}(b)\overline{G}(s)} - c_{sr},$$

where $\overline{F_0}(t) = \exp\left(-\int_0^t r_0(u)\, du\right)$.

Consider a gain proportional to the mean time to failure in field usage, as in (9.62). Then the total expected cost $c_1(s, b)$ is

$$\begin{aligned} c_1(s,b) = {} & c_0 \frac{\int_0^b \overline{F_s}(t)\mathrm{d}t}{\overline{F_s}(b)} + \frac{c_s + c_{sr}}{\overline{F_s}(b)\overline{G}(s)} - c_{sr} \\ & - K\left(\int_0^\infty \exp\left\{-\int_0^t (r_0(b+u) + p(s,b+u)\lambda(b+u))\mathrm{d}u\right\}\mathrm{d}t\right), \end{aligned} \tag{9.64}$$

whereas the substitution of $\overline{F_s}(t)$ by $\overline{F_0}(t)$ and assuming that $\lambda_b(t) = 0$ corresponds to the case when there are no environmental shocks during the time burn-in.

As Cha and Finkelstein [7] did not take into account the existing dependence of the distribution of strength on time, the failure rate that corresponds to (9.58) was erroneously obtained as $r(t, s) = r_0(t) + p(s)\lambda$ for $\lambda(t) = \lambda$. In accordance with this equation it was stated that "the failures due to shocks during the time burn-in do not contribute to improvement of reliability characteristics in field use, but increase only the cost of burn-in" as time burn-in does not decrease the second term "$p(s)\lambda$". However, the following theorem shows that shocks during time burn-in *do contribute* to improvement of reliability characteristics in field use.

Theorem 9.12 *The function $p(s,t)$ is strictly decreasing in t for each fixed s.*

Proof Observe that

$$\frac{\partial}{\partial t}P(U_s<v|T>t)=\frac{1}{\left(\int_s^\infty \exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\right)^2}$$
$$\times\left[-\lambda(t)\int_s^v\overline{M}_f(r)\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\cdot\int_s^\infty\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\right.$$
$$\left.+\lambda(t)\int_s^\infty\overline{M}_f(r)\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\cdot\int_s^v\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\right].$$

The numerator of the above equation becomes

$$\left[-\lambda(t)\int_s^v\overline{M}_f(r)\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\cdot\int_v^\infty\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\right.$$
$$\left.+\lambda(t)\int_v^\infty\overline{M}_f(r)\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\cdot\int_s^v\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\right]$$
$$<\left[-\lambda(t)\int_s^v\overline{M}_f(r)\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\cdot\int_v^\infty\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\right.$$
$$\left.+\lambda(t)\int_v^\infty\overline{M}_f(v)\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\cdot\int_s^v\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\right]$$
$$=\left[-\lambda(t)\int_s^v\overline{M}_f(r)\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\cdot\int_v^\infty\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\right.$$
$$\left.+\lambda(t)\int_v^\infty\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\cdot\int_s^v\overline{M}_f(v)\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\right]$$
$$<\left[-\lambda(t)\int_s^v\overline{M}_f(r)\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\cdot\int_v^\infty\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\right.$$
$$\left.+\lambda(t)\int_v^\infty\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\cdot\int_s^v\overline{M}_f(r)\exp\left\{-\overline{M}_f(r)\int_0^t\lambda(x)\mathrm{d}x\right\}g_U(r)\mathrm{d}r\right]=0$$

as $\overline{M}_f(r)$ is strictly decreasing in r. Therefore, $P(U_s<v|T>t)$ is decreasing in $t$and, due to the fact that

$$p(s,t)=\int_0^\infty I(v\in[s,\infty))P(U_s<v|T>t)m_f(v)\mathrm{d}v,$$

$p(s,t)$ is strictly decreasing in t for each fixed s. □

Therefore, the second term of the failure rate in (9.58), $p(s,t)\lambda(t)$ is decreasing in t for each fixed s when $\lambda(t)$ is nonincreasing. Or, even if $\lambda(t)$ is increasing, $p(s,t)\lambda(t)$ can be decreasing in t (for each fixed s) in some cases. Therefore, in this sense, shocks during time burn-in *do contribute* to improvement of reliability characteristics in field use.

Similar considerations can be used for describing the strategy $B(b,s)$. Let $h_2(s,b)$ be the expected burn-in cost. Then by similar arguments as those described in Cha and Finkelstein [7]:

$$h_2(s,b) = \frac{1}{\overline{G}(s)}(c_0 \frac{\int_0^b \overline{F}(t)\mathrm{d}t}{\overline{F}(b)}) + c_s \frac{1}{\overline{G}(s)} + c_{sr} \frac{1}{\overline{F}(b)\overline{G}(s)} - c_{sr}, \tag{9.65}$$

where

$$\overline{F}(t) = \exp\left(-\int_0^t (r_0(u) + p(0,u)\lambda(u))\mathrm{d}u\right).$$

Note that just after time burn-in (before performing the shock burn-in), as follows from Remark 9.14, the initial distribution of U is

$$\overline{G}_U(u;b) = P(U > u | T > b) = \frac{\int_u^\infty \exp\left\{-\overline{M}_f(r)\int_0^b \lambda(x)\mathrm{d}x\right\} \cdot g_U(r)\mathrm{d}r}{\int_0^\infty \exp\left\{-\overline{M}_f(r)\int_0^b \lambda(x)\mathrm{d}x\right\} g_U(r)\mathrm{d}r},$$

and, if we further perform the shock burn-in with the magnitude s, then the resulting pdf for U is

$$\begin{cases} 0, & \text{if } u \le s \\ \frac{g_U(u;b)}{\overline{G}_U(s;b)}, & \text{if } u > s \end{cases},$$

where $g_U(u;b)$ is the pdf which corresponds to $\overline{G}_U(u;b)$:

$$\begin{aligned} g_U(u;b) &= \frac{\exp\left\{-\overline{M}_f(u)\int_0^b \lambda(x)\mathrm{d}x\right\} \cdot g_U(u)}{\int_0^\infty \exp\left\{-\overline{M}_f(r)\int_0^b \lambda(x)\mathrm{d}x\right\} g_U(r)\mathrm{d}r} \\ &= \frac{\exp\left\{-\overline{M}_f(u)\int_0^b \lambda_b(x)\mathrm{d}x\right\} \cdot g_U(u)}{\int_0^\infty \exp\left\{-\overline{M}_f(r)\int_0^b \lambda_b(x)\mathrm{d}x\right\} g_U(r)\mathrm{d}r}. \end{aligned}$$

In accordance with (9.59), the failure probability at the 'field use age' t is

$$p(b,s,t)=\frac{\int_s^\infty \int_s^v \exp\{-\overline{M}_f(r)\int_0^t \lambda_f(x)\mathrm{d}x\}\cdot g_U(r;b)\mathrm{d}r\, m_f(v)\mathrm{d}v}{\int_s^\infty \exp\{-\overline{M}_f(r)\int_0^t \lambda_f(x)\mathrm{d}x\} g_U(r;b)\mathrm{d}r}. \tag{9.66}$$

Finally, from (9.65) and (9.66), the total expected cost $c_2(s,b)$ is

$$c_2(s,b)=\frac{1}{\overline{G}(s)}\left(c_0\frac{\int_0^b \overline{F}(t)\mathrm{d}t}{\overline{F}(b)}\right)+c_s\frac{1}{\overline{G}(s)}+c_{sr}\frac{1}{\overline{F}(b)\overline{G}(s)}-c_{sr}$$
$$-K\left(\int_0^\infty \exp\left\{-\int_0^t (r_0(b+u)+p(b,s,u)\lambda(b+u))\mathrm{d}u\right\}\mathrm{d}t\right).$$

Note that, $p(b,s,u)$ (not $p(s,b+u)$) should be used in $c_2(s,b)$ above. From Theorems 9.11 and 9.12, it is clear that $p(b,s,t)$ is strictly decreasing in both s and t for each fixed b, respectively. By similar procedure as before (Theorem 9.12), it can also be shown that the function $p(b,s,t)$ is strictly decreasing in b for each fixed s and t.

In Cha and Finkelstein [7], two stage optimization procedures for minimizing the cost functions are discussed. Similar approach can be applied to the modified results of the current paper. For example, for obtaining optimal (s_1^*,b_1^*) which minimizes, $c_1(s,b)$ defined by equation (9.64), we can follow the following procedure:

1. Fix $b\geq 0$, then find optimal $s^*(b)$ which satisfies

$$c_1(s^*(b),b)=\min_{0\leq s<\infty} c_1(s,b), \text{for fixed } b\geq 0.$$

Note that, as $c(s,b)$ is eventually increasing in s to infinity for each fixed b, such $s^*(b)$ exists for all b.

2. Find optimal b^* which satisfies

$$c_1(s^*(b^*),b^*)=\min_{0\leq b<\infty} c_1(s^*(b),b).$$

Then, finally, such $(s^*(b^*),b^*)$ is the optimal solution of the problem. However, in this modified model, even if we assume that $r_0(t)$ is the bathtub-shaped failure rate with two change points t_1 and t_2, t_1 is not necessarily the uniform upper bound for the optimal burn-in time. However, if we assume additionally that $r_0(t)$ is *increasing to infinit*y after t_2, there obviously should be the uniform upper bound for the optimal burn-in time and the standard numerical procedures can be used for obtaining optimal solutions in this case.

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