Chapter 7
Burn-in for Repairable Systems

In the previous chapter, the emphasis was made on the burn-in procedures for non-repairable items. If a non-repairable item fails during burn-in, then, obviously, it is just scraped and discarded. However, an expensive, complex product or device will not be discarded on account of failure of its part, but rather a repair will be performed. Therefore, in this chapter, we deal mostly with repairable items. Note that the contents of this chapter are rather technical and it can be skipped by a less mathematically oriented reader.

After the survey provided by Block and Savits [3], there has been much research on burn-in procedures, especially for repairable systems. These studies include: (i) various reliability models which jointly deal with burn-in and maintenance policies; (ii) burn-in procedures for general failure model; (iii) a stochastic model for the accelerated burn-in procedure.

7.1 Burn-in and Maintenance Policies: Initial Models

In this section, reliability models that jointly deal with burn-in and *maintenance policies* will be considered. We describe properties of *joint optimal solutions* for burn-in and replacement times for each of these models. Mi [10] was the first to consider the joint optimization problem for determining optimal burn-in and replacement times.

Let $F(t)$ be the distribution function of the absolutely continuous lifetime X. Mi [10] studied an optimal burn-in and maintenance policy under the assumption that $F(t)$ has a bathtub-shaped failure rate function. The following burn-in procedure was considered.

Burn-in Procedure A
Consider a fixed burn-in time b and begin to burn-in a new device. If the device fails before the burn-in time b, then repair it completely with the shop repair cost $c_s > 0$, then burn-in the repaired device again, and so on. If the device survives the burn-in time b, then it is put into field operation [10].

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We assume here that the repair is complete, i.e., the repaired device is as good as new. Let the cost of burn-in be proportional to the total burn-in time with proportionality constant $c_0 > 0$.

Let $h(b)$ denote the total cost incurred for obtaining the device which survives the burn-in procedure. Then, similar to Sect. 6.3, the mean cost $E[h(b)]$ can be obtained as

$$E[h(b)] = c_0 \frac{\int_0^b \bar{F}(t)\,\mathrm{d}t}{\bar{F}(b)} + c_s \frac{F(b)}{\bar{F}(b)}.$$

7.1.1 Model 1

For field operation, Mi [10] considered two types of replacement policies, depending on whether the device is repairable or not. For a non-repairable device, the age replacement policy is considered. That is, the device is replaced by a new burned-in device at the time of its failure or 'field-use age' T, whichever occurs first. Let c_f denote the cost incurred for each failure in field operation and $c_a\left(0 < c_a < c_f\right)$, the cost incurred for each non-failed item which is replaced by a new burned-in item at its field-use age T. Then, by the theory of renewal reward processes, the long-run average cost rate $c(b, T)$ is given by

$$c(b, T) = \frac{k(b) + c_f F_b(T) + c_a \bar{F}_b(T)}{\int_0^T \bar{F}_b(t)\,\mathrm{d}t},$$

where $\bar{F}_b(t)$ is the conditional survival function, i.e., $\bar{F}_b(t) \equiv \bar{F}(b+t)/\bar{F}(b)$ and $k(b) \equiv E[h(b)]$. Mi [10] have obtained certain results regarding the optimal burn-in time b^* and the optimal age T^* which satisfy

$$c(b^*, T^*) = \min_{b \geq 0,\, T > 0} c(b, T)$$

However, there are several useful 'hidden' properties which can be found in the proof of the corresponding theorem and, therefore, we reformulate the result as follows.

Theorem 7.1 *Suppose that the failure rate function $r(t)$ is bathtub-shaped and differentiable. Let*

$$B_1 \equiv \left\{ b \geq 0 : \mu(b) r(\infty) > \frac{c_f + k(b)}{c_f - c_a} \right\},$$

where $\mu(b) \equiv \int_0^\infty \bar{F}_b(t)\,\mathrm{d}t$, and $B_2 \equiv [0, \infty) \backslash B_1$. Then properties of the optimal burn-in time b^* and of the optimal replacement policy T^* can be stated in detail as follows:

Case 1. $B_1 = [0, \infty), B_2 = \phi$. Let $T^*(b)$ be the unique solution of the equation

$$r(b + T) \int_0^T \frac{\bar{F}(b + t)}{\bar{F}(b)} \mathrm{d}t + \frac{\bar{F}(b + T)}{\bar{F}(b)} = \frac{c_f + k(b)}{c_f - c_a}. \tag{7.1}$$

Then the optimal $(b^*, T^*) = (b^*, T^*(b^*))$, where $0 \le b^* \le t_1$, is the value that satisfies

$$b^* + T^*(b^*) = \min_{0 \le b \le t_1} (b + T^*(b)).$$

Case 2. $B_1 = \phi, B_2 = [0, \infty)$. The optimal $(b^*, T^*) = (b^*, \infty)$, where $0 \le b^* \le t_1$, is the value that satisfies

$$\frac{c_f + k(b^*)}{\mu(b^*)} = \min_{0 \le b \le t_1} \frac{c_f + k(b)}{\mu(b)}.$$

Case 3. $B_1 = \phi$, $B_2 = \phi$. For $b \in B_1$, let $T^*(b)$ be the unique solution of Eq. (7.1). Furthermore, let $b_1^* \in [0, t_1] \cap B_1$ satisfy

$$b_1^* + T^*(b_1^*) = \min_{b \le t_1, b \in B_1} (b + T^*(b)),$$

and $b_2^* \in [0, t_1] \cap B_2$ satisfy

$$\frac{c_f + k(b_2^*)}{\mu(b_2^*)} = \min_{b \le t_1, b \in B_2} \frac{c_f + k(b)}{\mu(b)}.$$

If

$$\left(c_f - c_a\right) r\left(b_1^* + T^*(b_1^*)\right) \le \frac{c_f + k(b_2^*)}{\mu(b_2^*)},$$

then $(b^*, T^*) = \left(b_1^*, T^*(b_1^*)\right)$. Otherwise the optimal (b^*, T^*) is $\left(b_2^*, \infty\right)$.

Proof The proof for a more general model is given in the proof of Theorem 7.4 in this chapter and thus it is omitted. □

7.1.2 Model 2

For a repairable device, applying the same burn-in procedure as before, block replacement with minimal repair on failures is performed in field operation. More precisely, fix a $T > 0$ and replace the component at times $T, 2T, 3T, \ldots$, with a new burned-in component. Also, at each intervening failure, a minimal repair is performed. Let $c_m > 0$ be the cost of a minimal repair, and $c_r > 0$ be the cost of replacement. In this case, the long-run average cost rate is given by

$$c(b,T) = \frac{1}{T}\left(k(b) + c_m \int_b^{b+T} r(t)\,\mathrm{d}t + c_r\right). \tag{7.2}$$

The following theorem [10] provides the properties of optimal (b^*, T^*) minimizing $c(b,T)$.

Theorem 7.2 *Suppose that the failure rate function $r(t)$ is bathtub-shaped and differentiable. Let*

$$B_1 \equiv \left\{ b \geq 0 : \int_b^{\infty} [r(\infty) - r(t)]\,\mathrm{d}t > \frac{1}{c_m \bar{F}(b)}\left[(c_r - c_s)\bar{F}(b) + c_s + c_0 \int_0^b \bar{F}(t)\,\mathrm{d}t\right]\right\},$$

and $B_2 \equiv [0, \infty)\backslash B_1$. Then the properties of the optimal burn-in time b^* and the replacement policy T^* can be stated in detail as follows:

Case 1. $B_1 = [0, \infty),\ B_2 = \phi$. Let $T^*(b)$ be the unique solution of the equation

$$Tr(b + T) - \int_b^{b+T} r(t)\,\mathrm{d}t = \frac{1}{c_m \bar{F}(b)}\left[(c_r - c_s)\bar{F}(b) + c_s + c_0 \int_0^b \bar{F}(t)\,\mathrm{d}t\right]. \tag{7.3}$$

Then, the optimal $(b^*, T^*) = (b^*, T^*(b^*))$, where $0 \leq b^* \leq t_1$, is the value which satisfies

$$b^* + T^*(b^*) = \min_{0 \leq b \leq t_1} (b + T^*(b)).$$

Case 2. $B_1 = \phi,\ B_2 = [0, \infty)$. The optimal $(b^*, T^*) = (b^*, \infty)$, where b^* can be any value in $[0, \infty)$.

Case 3. $B_1 = \phi,\ B_2 = \phi$. For $b \in B_1$, let $T^*(b)$ be the unique solution of the Eq. (7.3). Then, the optimal $(b^*, T^*) = (b^*, T^*(b^*))$, where b^* is the value which satisfies

$$b^* + T^*(b^*) = \min_{b \leq t_1,\ b \in B_1} (b + T^*(b)).$$

Proof The proof for a more general model is given in the proof of Theorem 7.4 in this chapter and thus it is omitted. □

7.1.3 Model 3

In Model 2, Burn-in Procedure A is applied to repairable devices. In many cases, because of practical limitations, products which fail during burn-in are just scraped, regardless of whether the products are repairable or not. In this case, the burn-in procedure A can be applied. However, an expensive, complex product or device will not be discarded on account of failure of its part, but rather a repair will be performed. Cha [4] proposed the following burn-in procedure.

Burn-in Procedure B
Consider the fixed burn-in time b and begin to burn-in a new component. On each component failure, only minimal repair is done with shop minimal repair cost $c_{sm} > 0$. Continue the burn-in procedure for the repaired component. Immediately after the fixed burn-in time b, the component is put into field operation [4].

Note that the total burn-in time for this burn-in procedure is a constant b. For a burned-in component, the block replacement policy with minimal repairs on failures is adopted in field operation as it was in Model 2. Assume $0 < c_{sm} < c_s$, then this means that the cost of a minimal repair during the burn-in process is smaller than that of the complete (perfect) repair, which is a reasonable assumption. Then, the long-run average cost rate is

$$c(b,T) = \frac{1}{T}(c_0 b + c_{sm}\Lambda(b) + c_m(\Lambda(b+T) - \Lambda(b)) + c_r). \qquad (7.4)$$

where $\Lambda(t) \equiv \int_0^t r(u)\,\mathrm{d}u$. It can be shown that

$$c_B(b,\, T) \le c_A(b,\, T), \quad \forall 0 < b < \infty,\ 0 < T < \infty,$$

where $c_A(b,\, T)$ and $c_B(b,T)$ are the cost rate functions in Eqs. (7.2) and (7.4), respectively. This implies that

$$c_B(b_B^*,\, T_B^*) \le c_A(b_A^*,\, T_A^*),$$

where (b_A^*, T_A^*) and $(b_B^*,\, T_B^*)$ are the optimal solutions which minimize $c_A(b,\, T)$ and $c_B(b,\, T)$, respectively. Thus, we can conclude that the burn-in procedure B is always preferable to the burn-in procedure A when the minimal repair policy is applicable.

Let $(b^*,\, T^*)$ be the optimal burn-in time and the optimal replacement time that minimize the cost rate Eq. (7.4). Then the properties of b^* and T^* are given by the following theorem.

Theorem 7.3 *Suppose that the failure rate function $r(t)$ is bathtub-shaped and differentiable. Let*

$$B_1 \equiv \left\{ b \ge 0 : \int_b^\infty [r(\infty) - r(t)]\,\mathrm{d}t > \frac{1}{c_m}[c_r + c_0 b + c_{sm}\Lambda(b)] \right\},$$

and $B_2 \equiv [0, \infty)\backslash B_1$. *Then the properties of the optimal burn-in time* b^* *and of the replacement policy* T^* *can be stated in detail as follows:*

Case 1. $B_1 = [0, \infty),\ B_2 = \phi$. Let $T^*(b)$ be the unique solution of the equation

$$Tr(b + T) - \int_b^{b+T} r(t)\,\mathrm{d}t = \frac{1}{c_m}[c_r + c_0 b + c_{sm}\Lambda(b)]. \tag{7.5}$$

Then, the optimal $(b^*, T^*) = (b^*, T^*(b^*))$, where $0 \le b^* \le t_1$, is the value which satisfies

$$b^* + T^*(b^*) = \min_{0 \le b \le t_1} (b + T^*(b)).$$

Case 2. $B_1 = \phi,\ B_2 = [0, \infty)$. The optimal $(b^*, T^*) = (b^*, \infty)$, where b^* can be any value in $[0, \infty)$.

Case 3. $B_1 = \phi,\ B_2 = \phi$. For $b \in B_1$, let $T^*(b)$ be the unique solution of Eq. (7.5). Then, the optimal $(b^*, T^*) = (b^*, T^*(b^*))$, where b^* is the value which satisfies

$$b^* + T^*(b^*) = \min_{b \le t_1,\ b \in B_1} (b + T^*(b)).$$

Proof Clearly, $b_2^* \ne \infty$, since $c_2(\infty, T) = \infty$ for any $0 < T \le \infty$. For any fixed $0 \le b < \infty$,

$$\frac{\partial c_2}{\partial T} = \frac{c_m}{T^2}\left\{\Psi_b(T) - \frac{1}{c_m}[c_r + c_0 b + c_{sm}\Lambda(b)]\right\},$$

where

$$\Psi_b(T) \equiv Tr(b + T) - \int_b^{b+T} r(t)\,\mathrm{d}t.$$

Hence, $\partial c_2/\partial T = 0$ if and only if

$$\Psi_b(T) = \frac{1}{c_m}[c_r + c_0 b + c_{sm}\Lambda(b)].$$

Note that, $\Psi_b(0) = 0$ and that $\Psi_b(T)$

$$\begin{cases} \text{strictly decreases} & \text{if } 0 \le T \le t_1 - b \\ \text{is a constant} & \text{if } t_1 - b \le T \le t_2 - b \\ \text{strictly increases} & \text{if } t_2 - b \le T \end{cases}$$

Then define

$$B_1 \equiv \left\{ b \geq 0 : \Psi_b(\infty) = \int_b^{\infty} [r(\infty) - r(t)]\,\mathrm{d}t > \frac{1}{c_m}[c_r + c_0 b + c_{sm}\Lambda(b)] \right\}$$

and set $B_2 \equiv [0, \infty)\backslash B_1$.

Now, as in the proof of Theorem 2 in [10], the following three separate cases are considered.

Case 1. $B_1 = [0, \infty),\ B_2 = \phi$.

Case 2. $B_1 = \phi,\ B_2 = [0, \infty)$.

Case 3. $B_1 = \phi,\ B_2 = \phi$.

Case 1 is equivalent to the condition that $\Psi(\infty) \equiv \int_b^{\infty} [r(\infty) - r(t)]\,\mathrm{d}t = \infty$ for at least one $b \geq 0$. In particular, it occurs when $r(\infty) = \infty$ and $r(0) < \infty$. Let $T_2^*(b)$ be the value which satisfies

$$c_2(b, T_2^*(b)) < c_2(b, T),\ \ \forall T \neq T_2^*(b),$$

for all $b \geq 0$. Then for Case 2, it is easy to see that for all $b \geq 0$,

$$c_2(b, T(b)) > c_2(b, \infty),\ \ \forall T > 0,$$

i.e., $T_2^*(b) = \infty$, for $b \geq 0$ and $c_2(b, T_2^*(b)) = c_m r(\infty)$.

For Case 1 and Case 3, it can be shown, as in Case 2, that for every $b' \in B_2$, $T_2^*(b') = \infty$ and $c_2(b', T_2^*(b')) = c_m r(\infty)$. Moreover, for all $b \in B_1$, the following properties can be established:

(i) There exists $T_2^*(b)$, which is the unique solution of Eq. (7.3).
(ii) $t_2 < b + T_2^*(b) < \infty$.
(iii) $c_2(b, T_2^*(b)) = c_m r(b + T_2^*(b))$.
(iv) For all $b' \in B_2$,$c_2(b, T_2^*(b)) = c_m r(b + T_2^*(b)) < c_m r(\infty) = c_2(b', T_2^*(b'))$.
(v) The optimal burn-in time b_2^* satisfies: $0 \leq b_2^* \leq t_1$.

Therefore, $b_2^* \in \{b : 0 \leq b \leq t_1\} \cap B_1$ and b_2^* is the value that satisfies:

$$b_2^* + T_2^*(b_2^*) = \min_{b \leq t_1,\, b \in B_1} (b + T_2^*(b)).$$

7.2 Burn-in Procedures for General Failure Model

In this section, we discuss the burn-in procedures for a general failure model that was partly studied in the previous chapter. Recall that according to this model, when the unit fails, the Type I failure and the Type II failure may occur with some probabilities. We assume that the Type I failure is a minor one and thus can be removed by a minimal repair, whereas Type II failure is a catastrophic one and thus can be removed only by a complete repair. Such models have been considered in the literature (e.g., [1, 2]).

7.2.1 Constant Probability Model

In this model, when the unit fails, Type I failure occurs with probability $1 - p$ and Type II failure occurs with probability p, $0 \leq p \leq 1$. Cha [5] proposed the following burn-in procedure for this model.

Burn-in Procedure C

Consider the fixed burn-in time b and begin to burn-in a new component. On each component failure, only minimal repair is done for the Type I failure with shop minimal repair cost c_{sm}, $0 \leq c_{sm} \leq c_s$, and a complete repair is performed for the Type II failure with shop complete repair cost c_s. Then continue the burn-in procedure for the repaired component [5].

Cha [5] studied optimal burn-in and replacement policy for the burn-in procedures A and C under the general failure model defined above.

Note that the burn-in procedure A stops when there is no failure during the fixed burn-in time $(0, b]$ for the first time, whereas procedure C stops when there is no Type II failure during the fixed burn-in time $(0, b]$ for the first time.

Note that, in field operation, the component is replaced by a new burned-in component at the 'field-use age' T or at the time of the first Type II failure, whichever occurs first. For each Type I failure occurring during field use, only minimal repair is done.

Let Y_b be the time to the first Type II failure of a burned-in component with the fixed burn-in time b. If we define $G_b(t)$ as the distribution function of Y_b and $\overline{G}_b(t)$ as $1 - G_b(t)$, then $\overline{G}_b(t)$ is given by

$$\begin{aligned}\overline{G}_b(t) &= \mathrm{P}(Y_b > t) \\ &= \exp\{-\int_0^t pr(b + u)\,\mathrm{d}u\} \\ &= \exp\{-p[\Lambda(b + t) - \Lambda(b)]\}, \quad \forall t \geq 0, \end{aligned} \tag{7.6}$$

where $\Lambda(t) \equiv \int_0^t r(u)\,\mathrm{d}u$. Let the random variable $N(b;\, T)$ be the total number of minimal repairs of a burned-in component which occur during field operation after the burn-in time b and in accordance with the replacement policy T. Then, using the results of Beichelt [2], it is easy to see that, when $p \neq 0$, the expectation of $N(b;\, T)$ is

$$\begin{aligned} E[N(b;\,T)] &= \frac{1}{G_b(t)}\int_0^T\int_0^t (1-p)r(b+u)\mathrm{d}u\mathrm{d}G_b(t)\cdot G_b(t) \\ &\quad + \int_0^T (1-p)r(b+u)\,\mathrm{d}u\cdot \overline{G}_b(T) \\ &= \left(\frac{1}{p}-1\right)(1-\exp\{-p[\Lambda(b+T)-\Lambda(b)]\}). \end{aligned} \tag{7.7}$$

When $p = 0$ the expectation is given by

$$E[N(b;\,T)] = \Lambda(b+T) - \Lambda(b).$$

Let c_f denote the cost incurred for each Type II failure in field operation and c_a satisfying $0 < c_a < c_f$ be the cost incurred for each non-failed item which is replaced at field use age $T > 0$. Denote also by c_m the cost of a minimal repair which is performed in field operation. When $p = 0$ or $p = 1$, the burn-in and replacement model discussed in this section reduces to that in [10] or [4]. Thus, in the discussion below, we assume that $0 < p < 1$. Then, using the results given by Eqs. (7.6) and (7.7), the long-run average cost rate functions for procedures A and C are given by [5]

$$\begin{aligned} c_A(b,\,T) &= \frac{1}{\int_0^T \overline{G}_b(t)\,\mathrm{d}t}\left(\left[c_0\frac{\int_0^b \overline{F}(t)\,\mathrm{d}t}{\overline{F}(b)} + c_s\frac{F(b)}{\overline{F}(b)}\right]\right. \\ &\quad \left. + c_m\left[\left(\frac{1}{p}-1\right)(1-\exp\{-p[\Lambda(b+T)-\Lambda(b)]\})\right] + c_f G_b(T) + c_a\overline{G}_b(T)\right), \end{aligned} \tag{7.8}$$

and

$$\begin{aligned} c_C(b,T) &= \frac{1}{\int_0^T \overline{G}_b(t)\,\mathrm{d}t}\left(\left[c_0\frac{\int_0^b \overline{G}(t)\,\mathrm{d}t}{\overline{G}(b)} + c_s\frac{G(b)}{\overline{G}(b)} + c_{sm}\left(\frac{1}{p}-1\right)(\exp\{p\Lambda(b)\}-1)\right]\right. \\ &\quad + c_m\left[\left(\frac{1}{p}-1\right)(1-\exp\{-p[\Lambda(b+T)-\Lambda(b)]\})\right] + c_f G_b(T) \\ &\quad \left. + c_a\overline{G}_b(T)\right), \end{aligned} \tag{7.9}$$

where $c_A(b,\,T)$ and $c_C(b,\,T)$ represent the cost rate for the burn-in procedures A and C, respectively.

Cha [5] showed that

(i) $c_C(0,\,T;\,p) = c_A(0,\,T;\,p),\ \forall 0 < T \le \infty,\ 0 < p < 1,$

(ii) $c_C(b,\,T;\,p) < c_A(b,\,T;\,p),\ \forall 0 < b < \infty,\ 0 < p < 1,$

where $c_A(b, T; p)$ and $c_C(b, T; p)$ are the cost rate functions $c_A(b, T)$ and $c_C(b, T)$ when the Type II probability is p, $0 < p < 1$. Then, from the above inequalities, it can be concluded that the burn-in procedure C is always (i.e., for all $0 < p < 1$) preferable to the burn-in procedure A when the minimal repair method is applicable.

Now we discuss the properties of optimal burn-in and of optimal replacement times. Note that the cost rate functions in Eqs. (7.8) and (7.9) can be expressed as

$$c(b, T) = \frac{1}{\int_0^T \overline{G}_b(t)\,\mathrm{d}t}\left(k(b) + c_m\left[\left(\frac{1}{p} - 1\right)(1 - \exp\{-p[\Lambda(b + T) - \Lambda(b)]\})\right] + c_f G_b(T) + c_a\overline{G}_b(T)\right), \tag{7.10}$$

where $k(b)$ is the average cost incurred during the burn-in process for each model. The properties of the optimal (b^*, T^*) which minimizes the cost rate Eq. (7.10) are given by the following theorem.

Theorem 7.4 *Suppose that the failure rate function $r(t)$ is bathtub-shaped and differentiable*. Let

$$B_1 \equiv \left\{ b \geq 0 : pr(\infty)\int_b^\infty \exp\{-p[\Lambda(t) - \Lambda(b)]\}\,\mathrm{d}t - 1 > \frac{1}{\left[c_m\left(\frac{1}{p} - 1\right) + (c_f - c_a)\right]}(c_a + k(b)) \right\},$$

and $B_2 \equiv [0, \infty)\backslash B_1$. *Then the properties of the optimal burn-in time b^* and the replacement policy T^* can be stated in detail as follows:*

Case 1. $B_1 = [0, \infty), B_2 = \phi$. Let $T^*(b)$ be the unique solution of the equation

$$pr(b + T)\int_b^{b+T} \exp\{-p[\Lambda(t) - \Lambda(b)]\}\,\mathrm{d}t + \exp\{-p[\Lambda(b + T) - \Lambda(b)]\} - 1 = \frac{1}{\left[c_m\left(\frac{1}{p} - 1\right) + (c_f - c_a)\right]}(c_a + k(b)). \tag{7.11}$$

Then, the optimal $(b^*, T^*) = (b^*, T^*(b^*))$, where $0 \leq b^* \leq t_1$, is the value which satisfies $b^* + T^*(b^*) = \min_{0 \leq b \leq t_1}(b + T^*(b))$.

Case 2. $B_1 = \phi$, $B_2 = [0, \infty)$. The optimal $(b^*, T^*) = (b^*, \infty)$, where $0 \leq b^* \leq t_1$, is the value which satisfies

$$\frac{1}{\mu(b^*)}\left[c_f + c_m\left(\frac{1}{p} - 1\right) + k(b^*)\right] = \min_{0 \le b \le t_1} \frac{1}{\mu(b)}\left[c_f + c_m\left(\frac{1}{p} - 1\right) + k(b)\right].$$

Case 3. $B_1 = \phi$, $B_2 = \phi$. For $b \in B_1$, let $T^*(b)$ be the unique solution of the Eq. (7.11). Furthermore, let $b_1^* \in [0, t_1] \cap B_1$ satisfy

$$b_1^* + T^*(b_1^*) = \min_{b^* \le t_1,\, b \in B_1} (b + T^*(b)),$$

and $b_2^* \in [0, t_1] \cap B_2$ satisfy

$$\frac{1}{\mu(b_2^*)}\left[c_f + c_m\left(\frac{1}{p} - 1\right) + k(b_2^*)\right] = \min_{b \le t_1,\, b \in B_2} \frac{1}{\mu(b)}\left[c_f + c_m\left(\frac{1}{p} - 1\right) + k(b)\right].$$

If

$$\left[c_m\left(\frac{1}{p} - 1\right) + c_f - c_a\right] pr(b_1^* + T^*(b_1^*)) \le \frac{1}{\mu(b_2^*)}\left[c_f + c_m\left(\frac{1}{p} - 1\right) + k(b_2^*)\right],$$

then the optimal $(b^*, T^*) = (b_1^*, T^*(b_1^*))$. Otherwise the optimal (b^*, T^*) is (b_2^*, ∞).

Proof The cost rate $c(b, T)$ in Eq. (7.10) can be rewritten as

$$c(b, T) = \frac{1}{\int_0^T \overline{G}_b(t)\,\mathrm{d}t}\left(h(b) + c_2 + c_m\left(\frac{1}{p} - 1\right)(1 - \exp\{-p[\Lambda(b + T) - \Lambda(b)]\}) + c_1[1 - \exp\{-p[\Lambda(b + T) - \Lambda(b)]\}]\right), \tag{7.12}$$

where $c_1 \equiv c_f - c_a$ and $c_2 \equiv c_a$. Clearly, $b^* \ne \infty$ since $c(\infty, T) = \infty$ for any $0 < T \le \infty$.Then, for any fixed $0 \le b < \infty$, $\partial c/\partial T = 0$ if and only if

$$\Psi_b(T) = \frac{1}{c_3}(c_2 + h(b)), \tag{7.13}$$

where $c_3 \equiv [c_m(1/p - 1) + c_1]$ and

$$\Psi_b(T) \equiv pr(b + T) \int_b^{b+T} \exp\{-p[\Lambda(t) - \Lambda(b)]\}\,\mathrm{d}t + \exp\{-p[\Lambda(b + T) - \Lambda(b)]\} - 1.$$

Note that $\Psi_b(0) = 0$ and

$$\Psi_b(T)\begin{cases}\text{strictly decreases} & \text{if } 0 \le T \le \mathrm{t}_1 - b,\\ \text{is a constant} & \text{if } t_1 - b \le T \le t_2 - b,\\ \text{strictly increases} & \text{if } t_2 - b \le T.\end{cases}$$

Define

$$B_1 \equiv \Big\{ b \ge 0 :\ \Psi_b(\infty) \equiv \lim_{T\to\infty} \Psi_b(T)$$
$$= pr(\infty)\int_b^\infty \exp\{-p[\Lambda(t) - \Lambda(b)]\}\mathrm{d}t - 1 > \frac{1}{c_3}(c_2 + h(b))\Big\}$$

and set $B_2 \equiv [0, \infty)\backslash B_1$.

We consider now the following three separate cases.

Case 1. $B_1 = [0, \infty)$and $B_2 = \phi$. This is equivalent to the condition that

$$\Psi_b(\infty) = pr(\infty)\int_b^\infty \exp\{-p[\Lambda(t) - \Lambda(b)]\}\,\mathrm{d}t - 1 = \infty$$

for at least one $b \ge 0$. In particular, it occurs when $r(\infty) = \infty$ and $r(0) < \infty$. In this case, Eq. (7.13) has a unique solution for all $b \ge 0$. which we denote by $T^*(b)$. Furthermore, from the fact that $\Psi_b(0) = 0$ and the monotonicity of Ψ_b, we can immediately see that $\Psi_b(T) < 0$, for all $0 < T \le t_2 - b$. This implies that the unique solution $T^*(b)$ of Eq. (7.13) must satisfy $T^*(b) > t_2 - b$ for any given $b \ge 0$. Thus, we have shown that

$$t_2 < T^*(b) + b \le \infty \tag{7.14}$$

As $T^*(b)$ satisfies Eq. (7.13),

$$\begin{aligned} &pr(b + T^*(b)\,)\int_b^{b+\mathrm{T}^*(b)} \exp\{-p[\Lambda(t) - \Lambda(b)]\}\,\mathrm{d}t \\ &\quad + \exp\{-p[\Lambda(b + T^*(b)) - \Lambda(b)]\} - 1 = \frac{1}{c_3}(c_2 + h(b)). \end{aligned} \tag{7.15}$$

Combining Eqs. (7.12) and (7.15), we obtain

$$c(b, T^*(b)) = c_3 pr(b + T^*(b)).$$

Thus, minimizing $c(b, T^*(b))$ is equivalent to minimizing $r(b + T^*(b))$ for $0 \le b < \infty$. By Eq. (7.14), $b + T^*(b) > t_2$, so the problem of finding b^* minimizing $c(b, T^*(b))$ is equivalent to finding b^* which satisfies

$$b + T^*(b) = \min_{b\ge 0}(b + T^*(b)).$$

The inequality $b^* \leq t_1$ is now verified. To prove this inequality, it is sufficient to show that $\partial(b + T^*(b))/\partial b > 0$ for all $b \geq t_1$. From Eq. (7.15),

$$pr(b + T^*(\mathrm{b})) \int_{\mathrm{b}}^{\mathrm{b}+\mathrm{T}^*(b)} \exp\{-p\Lambda(t)\}\,\mathrm{d}t + \exp\{-p\Lambda(b + T^*(b))\} = \exp\{-p\Lambda(b)\}\left[1 + \frac{c_2}{c_3} + \frac{1}{c_3}h(b)\right]. \tag{7.16}$$

Taking the derivative with respect to b on both sides of Eq. (7.16), we obtain

$$\begin{aligned} pr'(b + T^*(\mathrm{b}))(1 + T^{*\prime}(b)) \int_b^{b+\mathrm{T}^*(b)} \exp\{-pA(t)\}\,\mathrm{d}t - pr(b + T^*(b))\exp\{-pA(b)\} \\ = \exp\{-pA(b)\}\frac{1}{c_3}h'(b) - \exp\{-pA(b)\}pr(b)\left(1 + \frac{c_2}{c_3} + \frac{1}{c_3}h(b)\right) \\ > -\exp\{-pA(b)\}pr(b)\left(1 + \frac{c_2}{c_3} + \frac{1}{c_3}h(b)\right), \end{aligned} \tag{7.17}$$

since $h'(b) > 0$. Then, from the Inequality Eq. (7.17),

$$\begin{aligned} & pr'(b + T^*(b))(1 + T^{*\prime}(b)) \int_b^{b+T^*(b)} \exp\{-p\Lambda(t)\mathrm{d}t\} \\ & > pr(b + T^*(b))\exp\{-p\Lambda(b)\} - \exp\left\{-p\Lambda(b)\left(1 + \frac{c_2}{c_2} + \frac{1}{c_3}h(b)\right)\right\}. \end{aligned} \tag{7.18}$$

However, from Eq. (7.15),

$$\begin{aligned} pr(b + T^*(b)) = & \frac{1}{\int_b^{b+T^*(b)} \exp\{-p[\Lambda(t) - \Lambda(b)]\}\,\mathrm{d}t} \\ & \times \left\{1 - \exp\{-p[\Lambda(b + T^*(b)) - \Lambda(b)]\} + \frac{c_2}{c_3} + \frac{1}{c_3}h(b)\right\}, \end{aligned} \tag{7.19}$$

and by the bathtub-shaped assumption, if $b \geq t_1$, it follows that

$$\begin{aligned} pr(b) \int_b^{b+T^*(b)} \exp\{-p[\Lambda(t) - \Lambda(b)]\}\,\mathrm{d}t \leq & \int_b^{b+T^*(b)} pr(t)\exp\{-p[\Lambda(t) - \Lambda(b)]\}\,\mathrm{d}t \\ & = \exp\{p\Lambda(b)\}[-\exp\{-p\Lambda(t)\}]_b^{b+T^*(b)} \\ & = 1 - \exp\{-p[\Lambda(b + T^*(b)) - \Lambda(b)]\} \\ & \leq 1. \end{aligned} \tag{7.20}$$

Then, by combining Eqs. (7.18, 7.19 and 7.20), we obtain

$$pr'(b + T^*(b))(1 + T^{*\prime}(b)) \int_b^{b+T^*(b)} \exp\{-p\Lambda(t)\}\mathrm{d}t > 0,$$

which implies that $\partial(b + T^*(b))/\partial b > 0$ for all $b \geq t_1$. Therefore, $b^* \leq t_1$ holds.

Case 2. $B_1 = \phi$, $B_2 = [0, \infty)$. In this case, it can easily be shown that

$$\Psi_b(T) < \frac{1}{c_3}(c_2 + h(b)), \ \ \forall T \geq 0,$$

which implies that $\partial c/\partial T < 0$, for every $T > 0$ for all fixed $b \geq 0$. Hence, for all $T > 0$ and $b \geq 0$

$$\begin{aligned} c(b, T) &\geq c(b, \infty) \\ &= \frac{1}{\mu(b)}\left[c_1 + c_2 + \mathrm{c_m}\left(\frac{1}{\mathrm{p}} - 1\right) + \mathrm{h(b)}\right], \end{aligned}$$

where $\mu(b)$ is defined by

$$\begin{aligned} \mu(b) &\equiv \int_b^{\infty} \exp\{-p[\Lambda(t) - \Lambda(b)]\}\mathrm{d}t \\ &= \frac{\int_b^{\infty} \overline{G}(t)\,\mathrm{d}t}{\overline{G}(b)}, \end{aligned}$$

which is the MRL. Then, as follows from [2, 7], it is easy to see that $\mu(b)$ strictly decreases for all $b \geq t_1$, whereas the term

$$\left[c_1 + c_2 + \mathrm{c_m}\left(\frac{1}{\mathrm{p}} - 1\right) + \mathrm{h(b)}\right]$$

strictly increases as b increases. Therefore, the inequalities

$$\begin{aligned} c(b, T) \geq c(b, \infty), &\quad \forall T > 0, \forall b \geq 0, \\ > c(t_1, \infty), &\quad \forall b > t_1, \end{aligned}$$

hold and, consequently, in this case, we have $(b^*, T^*) = (b^*, \infty)$, $0 \leq b^* \leq t_1$ and $b^* + T^* > t_2$. Also, the optimal burn-in time b^* is the value which satisfies

$$c(b^*, \infty) = \min_{0 \leq b \leq t_1} c(b, \infty).$$

Case 3. $B_1 = \phi$, $B_2 = \phi$. In advance, note that $\Psi_b(\infty)$ is strictly decreasing in b for $b \geq t_1$ since

$$\Psi_b(\infty) = pr(\infty)\mu(b) - 1,$$

and the function

$$\frac{1}{c_3}[c_2 + h(b)], \tag{7.21}$$

strictly decreases as $b \uparrow \infty$. Then, by similar arguments to those in [10], it can be shown that ∞ cannot be in the closure B_1 and there exists $0 \le s < \infty$ such that $[s, \infty) \subseteq B_2$. If we set

$$\beta \equiv \inf\{t : [t, \infty) \subseteq B_2\},$$

then, clearly, $[\beta, \infty) \subseteq B_2$.

First suppose that $\beta \le t_1$, therefore, obviously $[t_1, \infty) \subseteq B_2$. In this case, by the arguments of Case 2, the set $[t_1, \infty)$ cannot contain the optimal b^*. Hence $b^* \le t_1$.

Suppose now that $\beta > t_1$. Since $\Psi_b(\infty)$ strictly decreases for $b \ge t_1$ and the function in Eq. (7.21) strictly increases, the fact that $\beta > t_1$ yields that $[t_1, \beta) \subseteq B_1$. Then, by the procedure described in Case 2, the relationship

$$\min_{b \in [\beta, \infty), T > 0} c(b, T) = \min_{b \in [\beta, \infty)} c(b, \infty) > c(t_1, \infty)$$

holds, and, therefore, the set $[\beta, \infty)$ cannot contain the optimal b^*. Also, for $b \in [t_1, \beta)$, by the similar arguments to those in Case 1, we can show that $\partial(b + T^*(b))/\partial b > 0$, for all $t_1 \le b < \beta$, and therefore we can conclude that $b^* \le t_1$.

7.2.2 *Time-Dependent Probability Model*

In [6], the Constant Probability Model was further extended to the case when the corresponding probabilities change with operating time. Assume now that, when the unit fails at its age t, Type I failure occurs with probability $1 - p(t)$ and Type II failure occurs with probability $p(t)$, $0 \le p(t) \le 1$.

In this model, we employ the same notations and random variables used before. Also, note that if $p(t) = p$ *a.e.* (w.r.t. Lebesgue measure), $0 \le p \le 1$, the models under consideration can be reduced to those of Mi [10] and Cha [4, 5]. Thus, we only consider the set of functions P as the set of all of the Type II failure probability functions, which is given by

$$P = \{p(\cdot) : 0 \le p(t) \le 1, \ \forall t \ge 0\} \backslash \{p(\cdot) : p(t) = p \ a.e., \ 0 \le p \le 1\}$$

It can be shown that

$$\overline{G}_b(t) = \exp\{-[\Lambda_p(b + t) - \Lambda_p(b)]\}, \ \forall t \ge 0$$

where $\Lambda_p(t) \equiv \int_0^t p(u)\, r(u)\, \mathrm{d}u$, and

$$E[N(b;T)] = \int_0^T r(b+t)\bar{G}_b(t)\,\mathrm{d}t - G_b(T).$$

Then, considering both burn-in procedures A and C for this extended model, the long-run average cost rate functions are given by

$$c_A(b,T) = \frac{1}{\int_0^T \bar{G}_b(t)\,\mathrm{d}t}\left(\left[c_0\int_0^b \exp\{-[\Lambda(t)-\Lambda(b)]\}\,\mathrm{d}t + c_s[\exp\{\Lambda(b)\}-1]\right] + c_m\left[\int_0^T r(b+t)\bar{G}_b(t)\,\mathrm{d}t - G_b(T)\right] + c_f G_b(T) + c_a\bar{G}_b(T)\right), \tag{7.22}$$

where $\Lambda(t) \equiv \int_0^t r(u)\,\mathrm{d}u$, and

$$c_C(b,T) = \frac{1}{\int_0^T \bar{G}_b(t)\mathrm{d}t}\left(\left[c_0\int_0^b \exp\{-[\Lambda_p(t)-\Lambda_p(b)]\}\,\mathrm{d}t + c_s[\exp\{\Lambda_p(b)\}-1] + c_{sm}\int_0^b (1-p(t))r(t)\exp\{-[\Lambda_p(t)-\Lambda_p(b)]\}\,\mathrm{d}t\right] + c_m\left[\int_0^T r(b+t)\bar{G}_b(t)\,\mathrm{d}t - G_b(T)\right] + c_f G_b(T) + c_a\bar{G}_b(T)\right). \tag{7.23}$$

As before, it can be shown that

(i) $c_C(0,T;p(\cdot)) = c_A(0,T;p(\cdot)),\ \forall 0 < T \le \infty,\ p(\cdot) \in P,$
(ii) $c_C(b,T;p(\cdot)) \le c_A(b,T;p(\cdot)),\ \forall 0 < b < \infty,\ 0 < T \le \infty,\ p(\cdot) \in P,$

which ensures the superiority of the burn-in procedure C when the minimal repair method is applicable.

The cost rate functions in Eqs. (7.22) and (7.23) can be rewritten as

$$c(b,T) = \frac{1}{\int_0^T \bar{G}_b(t)\,\mathrm{d}t}\left(k(b) + c_m\left[\int_0^T r(b+t)\bar{G}_b(t)\,\mathrm{d}t - G_b(T)\right]. + c_f G_b(T) + c_a\bar{G}_b(T)\right),$$

where $k(b)$ denotes the average cost incurred during the burn-in process. Then, under the following assumptions, the properties regarding the optimal burn-in time $b^*$and the optimal replacement policy T^* can be obtained.

Assumptions

1. The failure rate function $r(t)$ is differentiable and bathtub shaped with the first change point s_1 and the second change point s_2.
2. The Type II failure probability function $p(t)$ is differentiable and bathtub shaped with the first change point u_1 and the second change point u_2.
3. Let $t_1 \equiv \max(s_1, u_1)$ and $t_2 \equiv \min(s_2, u_2)$ then $t_1 < t_2$ holds.
4. $(c_f - c_a) > c_m$.

Theorem 7.5 *Suppose that assumptions (1)–(4) hold. Let the set* B_1 *be*

$$\begin{aligned} B_1 \equiv \{b \geq 0 : c_m \int_b^\infty [r(\infty) - r(t)] \exp\{-[\Lambda_p(t) - \Lambda_p(b)]\}\mathrm{d}t \\ + ((c_f - c_a) - c_m) \left[p(\infty) r(\infty) \int_b^\infty \exp\{-[\Lambda_p(t) - \Lambda_p(b)]\}\mathrm{d}t - 1 \right] \\ > (c_a + k(b))\}, \end{aligned}$$

and $B_2 \equiv [0, \infty) \backslash B_1$. *Then the properties of the optimal burn-in time* b^* *and replacement policy* T^* *can be stated in detail as follows:*

Case 1. $B_1 = [0, \infty)$, $B_2 = \phi$. Let $T^*(b)$ be the unique solution of the equation,

$$\begin{aligned} c_m \int_b^{b+T} [r(b+T) - r(t)] \exp\{-[\Lambda_p(t) - \Lambda_p(b)]\}\mathrm{d}t + ((c_f - c_a) - c_m) \\ \left[p(b+T) r(b+T) \int_b^{b+T} \exp\{-[\Lambda_p(t) - \Lambda_p(b)]\}\mathrm{d}t - (1 - \exp\{-[\Lambda_p(b+T) - \Lambda_p(b)]\}) \right] \\ = (c_a + k(b)), \end{aligned} \tag{7.24}$$

then the optimal $(b^*, T^*) = (b^*, T^*(b^*))$, where $0 \leq b^* \leq t_1$ is the value which satisfies $(b^* + T^*(b^*)) = \min\limits_{0 \leq b \leq t_1} (b + T^*(b))$.

Case 2. $B_1 = \phi$, $B_2 = [0, \infty)$. The optimal $(b^*, T^*) = (b^*, \infty)$, where $0 \leq b^* \leq t_1$ is the value which satisfies

$$\frac{1}{\mu(b^*)}\left[(c_f - c_m) + c_m\int_{b^*}^{\infty} r(t)\exp\left\{-\left[\Lambda_p(t) - \Lambda_p(b^*)\right]\right\}\mathrm{d}t + k(b^*)\right]$$
$$= \min_{0 \le b \le t_1}\frac{1}{\mu(b)}\left[(c_f - c_m) + c_m\int_{b}^{\infty} r(t)\exp\left\{-\left[\Lambda_p(t) - \Lambda_p(b)\right]\right\}\mathrm{d}t + k(b)\right],$$

where $\mu(b)$ is given by

$$\mu(b) = \int_{b}^{\infty}\exp\left\{-\left[\Lambda_p(t) - \Lambda_p(b)\right]\right\}\mathrm{d}t. \qquad (7.25)$$

Case 3. $B_1 \neq \phi$, $B_2 \neq \phi$. Let $T^*(b)$, $b \in B_1$, be the unique solution of the Eq.(7.24) and $\mu(b)$ be given by Eq. (7.25). Furthermore, let $b_1^* \in [0, t_1] \cap B_1$ be the value which satisfies

$$(b_1^* + T^*(b_1^*)) = \min_{b \le t_1,\, b \in B_1}(b + T^*(b)),$$

and $b_2^* \in [0, t_1] \cap B_2$ be the value which satisfies

$$\frac{1}{\mu(b_2^*)}\left[(c_f - c_m) + c_m\int_{b_2^*}^{\infty} r(t)\exp\left\{-\left[\Lambda_p(t) - \Lambda_p(b_2^*)\right]\right\}\mathrm{d}t + k(b_2^*)\right]$$
$$= \min_{b \le t_1,\, b \in B_2}\frac{1}{\mu(b)}\left[(c_f - c_m) + c_m\int_{b}^{\infty} r(t)\exp\left\{-\left[\Lambda_p(t) - \Lambda_p(b)\right]\right\}\mathrm{d}t + k(b)\right].$$

If

$$c_m r\left(b_1^* + T^*(b_1^*)\right) + \left((c_f - c_a) - c_m\right)p\left(b_1^* + T^*(b_1^*)\right)r\left(b_1^* + T^*(b_1^*)\right)$$
$$\le \frac{1}{\mu(b_2^*)}\left[(c_f - c_m) + c_m\int_{b_2^*}^{\infty} r(t)\exp\left\{-\left[\Lambda_p(t) - \Lambda_p(b_2^*)\right]\right\}\mathrm{d}t + k(b_2^*)\right],$$

then the optimal $(b^*, T^*) = (b_1^*, T^*(b_1^*))$. Otherwise, optimal $(b^*, T^*) = (b_2^*, \infty)$.

Remark 7.1 In this theorem, we assume that both $r(t)$ and $p(t)$ are bathtub-shaped functions. Cha and Mi [7] investigated how this assumption can practically be satisfied when a device is composed of two statistically independent parts (Part A and Part B) in series. Assume that the failure of Part A causes a catastrophic failure, whereas that of Part B causes a minor failure. The failure rate of the device is

$$r(t) = r_1(t) + r_2(t)$$

and the probability of Type II failure $p(t)$ is given by

$$p(t) = \frac{r(t)}{r_1(t) + r_2(t)}$$

where $r_1(t)$ and $r_2(t)$ are the failure rate functions of Parts A and B, respectively (see [7] for a detailed discussion and several examples when $r(t)$ and $p(t)$ have various shapes).

7.3 Accelerated Burn-in and Maintenance Policy

Burn-in is generally considered to be expensive and its duration is typically limited. Stochastic models for accelerated burn-in were introduced in the previous chapter. In this section, we will discuss reliability models that jointly deal with accelerated burn-in and maintenance policies. In [8], the burn-in and replacement models 1, 2, and 3 of Sect. 7.1 were extended to the case when burn-in is performed in an accelerated environment assuming the failure rate model described in Sect. 6.4 of the previous chapter.

7.3.1 Model 1

We consider burn-in and replacement Model 1: the component is burned-in in accordance with the burn-in procedure A under the accelerated environment. The component that had survived burn-in is put into field operation. In field operation, an age replacement policy is applied. We will use the notation of Sects. 6.4 and 7.1.

The corresponding long-run average cost rate is given by (see Sects.6.4 and 7.1)

$$c(b, T) = \frac{1}{\int_0^T \bar{F}_b(t)\,\mathrm{d}t}\left(\left[c_0 \frac{\int_0^b \bar{F}_A(t)\,\mathrm{d}t}{\bar{F}_A(b)} + c_s \frac{F_A(b)}{\bar{F}_A(b)}\right] + c_f F_b(T) + c_a \bar{F}_b(T)\right),$$

where

$$\bar{F}_b(t) \equiv \exp\left(-\int_0^t r(a(b) + u)\,\mathrm{d}u\right) = \frac{\bar{F}(a(b) + t)}{\bar{F}(a(b))},$$

and $F_A(t) = F(\rho(t)),\ \forall t \geq 0$.

Let b^* be the optimal accelerated burn-in time and T^* be the optimal replacement policy which satisfy

$$c(b^*, T^*) = \min_{b \geq 0, T > 0} c(b, T).$$

Then the properties regarding the optimal accelerated burn-in time b^* and the optimal replacement policy T^* are given by the following theorem [8], which is similar in formulation to Theorem 7.1.

Theorem 7.6 *Suppose that the failure rate function $r(t)$ is bathtub-shaped and differentiable. Let the set B_1 be*

$$B_1 \equiv \left\{ b \geq 0 : r(\infty) \int_{a(b)}^{\infty} \exp\{-[\Lambda(t) - \Lambda(a(b))]\} dt - 1 \right.$$
$$> \frac{1}{c_f - c_a} \left[c_a + c_s[\exp\{\Lambda(\rho(b))\} - 1 \right]$$
$$\left. + c_0 \int_0^b \exp\left\{ - \left[\Lambda(\rho(t)) - \Lambda(\rho(b)) \right] \right\} dt \right] \Bigg\},$$

and $B_2 \equiv [0, \infty) \backslash B_1$. Furthermore, let $a^{-1}(t_1) \geq 0$ be the unique solution of the equation $a(t) = t_1$. Then the properties of the optimal accelerated burn-in time b^* and replacement policy T^* can be stated in detail as follows:

Case 1. $B_1 = [0, \infty)$, $B_2 = \phi$. Let $T^*(b)$ be the unique solution of the equation

$$r(a(b) + T) \int_{a(b)}^{a(b)+T} \exp\{-[\Lambda(t) - \Lambda(a(b))]\} dt + \exp\{-[\Lambda(a(b) + T) - \Lambda(a(b))]\} - 1$$
$$= \frac{1}{c_f - c_a} \left[c_a + c_s[\exp\{\Lambda(\rho(b))\} - 1] + c_0 \int_0^b \exp\{-[\Lambda(\rho(t)) - \Lambda(\rho(b))]\} dt \right]. \tag{7.26}$$

Then the optimal $(b^*, T^*) = (b^*, T^*(b^*))$, where $0 \leq b^* \leq a^{-1}(t_1)$, is the value which satisfies $a(b^*) + T^*(b^*) = \min_{0 \leq b \leq a^{-1}(t_1)} (a(b) + T^*(b))$.

Case 2. $B_1 = \phi$, $B_2 = [0, \infty)$. In this case, the optimal $(b^*, T^*) = (b^*, \infty)$, where $0 \leq b^* \leq a^{-1}(t_1)$ is the value which satisfies

$$\frac{1}{\mu(a(b^*))} \left[c_f + c_s[\exp\{\Lambda(\rho(b^*))\} - 1] + c_0 \int_0^{b^*} \exp\{-[\Lambda(\rho(t)) - \Lambda(\rho(b^*))]\} dt \right]$$
$$= \min_{0 \leq b \leq a^{-1}(t_1)} \frac{1}{\mu(a(b))} \left[c_f + c_s[\exp\{\Lambda(\rho(b))\} - 1] + c_0 \int_0^b \exp\{-[\Lambda(\rho(t)) - \Lambda(\rho(b))]\} dt \right],$$

where $\mu(a(b))$ is given by

$$\mu(a(b)) \equiv \int\limits_{a(b)}^{\infty} \exp\{-[\Lambda(t) - \Lambda(a(b))]\}\,\mathrm{d}t. \tag{7.27}$$

Case 3. $B_1 \neq \phi,\ B_2 \neq \phi$ For $b \in B_1$, let $T^*(b)$ be the unique solution of the Eq. (7.26) and let $\mu(a(b))$ be given by Eq. (7.27). Furthermore, let $b_1^* \in [0, a^{-1}(t_1)] \cap B_1$ satisfy

$$a(b_1^*) + T^*(b_1^*) = \min_{b \le a^{-1}(t_1),\, b \in B_1} (a(b) + T^*(b)),$$

and

$$b_2^* \in [0, a^{-1}(t_1)] \cap B_2$$

satisfy

$$\begin{aligned} &\frac{1}{\mu(a(b_2^*))}\left[c_f + c_s\left[\exp\{\Lambda(\rho(b_2^*))\} - 1\right] + c_0 \int\limits_0^{b_2^*} \exp\{-[\Lambda(\rho(t)) - \Lambda(\rho(b_2^*))]\}\,\mathrm{d}t\right] \\ &= \min_{b \le a^{-1}(t_1),\, b \in B_2} \frac{1}{\mu(a(b))}\left[c_f + c_s[\exp\{\Lambda(\rho(b))\} - 1] + c_0 \int\limits_0^{b} \exp\{-[\Lambda(\rho(t)) - \Lambda(\rho(b))]\}\,\mathrm{d}t\right]. \end{aligned}$$

If

$$\begin{aligned} (c_f - c_a)r\big(a(b_1^*) + T^*(b_1^*)\big) \le \frac{1}{\mu\big(a(b_2^*)\big)}\Bigg[&c_f + c_s\left[\exp\{\Lambda\big(\rho(b_2^*)\big)\} - 1\right] \\ &+ c_0 \int\limits_0^{b_2^*} \exp\{-[\Lambda(\rho(t)) - \Lambda\big(\rho(b_2^*)\big)]\}\,\mathrm{d}t\Bigg], \end{aligned}$$

then the optimal (b^*, T^*) is $(b_1^*, T^*(b_1^*))$. Otherwise, the optimal (b^*, T^*) is (b_2^*, ∞).

7.3.2 Model 2

We consider burn-in and replacement model 2: the component is burned-in by the burn-in procedure C and the block replacement with minimal repair at failure is applied to the component in field use.

In this case, the long-run average cost rate is given by

$$c(b, T) = \frac{1}{T}\left(\left[c_0 \frac{\int_0^b \bar{F}_A(t)\,\mathrm{d}t}{\bar{F}_A(b)} + c_s \frac{F_A(b)}{\bar{F}_A(b)}\right] + c_m[\Lambda(a(b) + T) - \Lambda(a(b))] + c_r\right). \tag{7.28}$$

Then properties of the optimal b^* and T^* minimizing $c(b, T)$ in Eq. (7.28) are given by the following theorem [8]

Theorem 7.7 *Suppose that the failure rate function $r(t)$ is bathtub-shaped and differentiable. Let the set B_1 be*

$$B_1 \equiv \left\{ b \geq 0 : \int_{a(b)}^{\infty} [r(\infty) - r(t)]\mathrm{d}t \right.$$
$$\left. > \frac{1}{c_m}\left[c_r + c_s[\exp\{\Lambda(\rho(b))\} - 1] + c_0 \int_0^b \exp\{-[\Lambda(\rho(t)) - \Lambda(\rho(b))]\}\mathrm{d}t\right]\right\},$$

$B_2 \equiv [0, \infty)\backslash B_1$ and $a^{-1}(t_1) \geq 0$ be the unique solution of the equation $a(t) = t_1$. Then the properties of the optimal burn-in time b^* and the replacement policy T^* can be stated in detail as follows:

Case 1. $B_1 = [0, \infty)$, $B_2 = \phi$. Let $T^*(b)$ be the unique solution of the equation

$$Tr(a(b) + T) - \int_{a(b)}^{a(b)+T} r(t)\,\mathrm{d}t$$
$$= \frac{1}{c_m}\left[c_r + c_s[\exp\{\Lambda(\rho(b))\} - 1] + c_0 \int_0^b \exp\{-[\Lambda(\rho(t)) - \Lambda(\rho(b))]\}\,\mathrm{d}t\right]. \tag{7.29}$$

Then the optimal $(b^*, T^*) = (b^*, T^*(b^*))$, where $0 \leq b^* \leq a^{-1}(t_1)$, is the value which satisfies $a(b^*) + T^*(b^*) = \min_{0 \leq b \leq a^{-1}(t_1)} (a(b) + T^*(b))$.

Case 2. $B_1 = \phi$, $B_2 = [0, \infty)$. The optimal $(b^*, T^*) = (b^*, \infty)$, where b^* can be any value in $[0, \infty)$.

Case 3. $B_1 \neq \phi$, $B_2 \neq \phi$. For $b \in B_1$, let $T^*(b)$ be the unique solution of the Eq. (7.29). Then the optimal $(b^*, T^*) = (b^*, T^*(b^*))$, where b^* is the value which satisfies

$$a(b^*) + T^*(b^*) = \min_{b \leq a^{-1}(t_1),\, b \in B_1} (a(b) + T^*(b)).$$

7.3.3 Model 3

We consider burn-in and replacement Model 3: the component is burned-in by the burn-in procedure B and the block replacement with minimal repair at failure is applied to the component in field use. Then, obviously, the long-run average cost rate is given by

$$c(b, T) = \frac{1}{T}\left([c_0 b + c_{sm}\Lambda(\rho(b))] + c_m[\Lambda(a(b) + T) - \Lambda(a(b))] + c_r\right), \tag{7.30}$$

The properties of the optimal b^* and T^* minimizing $c(b, T)$ in Eq. (7.30) are given by the following theorem.

Theorem 7.8 *Suppose that the failure rate function $r(t)$is bathtub-shaped and differentiable. Let*

$$B_1 \equiv \left\{ b \geq 0 : \int_b^\infty [r(\infty) - r(t)]\,\mathrm{d}t > \frac{1}{c_m}[c_r + c_0 b + c_{sm}\Lambda(b)] \right\},$$

$B_2 \equiv [0, \infty)\backslash B_1$ and $a^{-1}(t_1) \geq 0$ be the unique solution of the equation $a(t) = t_1$. Then the properties of the optimal burn-in time b^* and the replacement policy T^* can be stated in detail as follows:

Case 1. $B_1 = [0, \infty)$, $B_2 = \phi$. Let $T^*(b)$ be the unique solution of the equation

$$Tr(a(b) + T) - \int_{a(b)}^{a(b)+T} r(t)\,\mathrm{d}t = \frac{1}{c_m}[c_r + c_0 b + c_{sm}\Lambda(\rho(b))]. \tag{7.31}$$

Then the optimal $(b^*, T^*) = (b^*, T^*(b^*))$, where $0 \leq b^* \leq a^{-1}(t_1)$, is the value which satisfies

$$a(b^*) + T^*(b^*) = \min_{0 \leq b \leq a^{-1}(t_1)} (a(b) + T^*(b)).$$

Case 2. $B_1 = \phi$, $B_2 = [0, \infty)$. The optimal $(b^*, T^*) = (b^*, \infty)$, where b^* can be any value in $[0, \infty)$.

Case 3. $B_1 \neq \phi$, $B_2 \neq \phi$. For $b \in B_1$, let $T^*(b)$ be the unique solution of the Eq. (7.31). Then the optimal $(b^*, T^*) = (b^*, T^*(b^*))$, where b^* is the value which satisfies

$$a(b^*) + T^*(b^*) = \min_{b \leq a^{-1}(t_1),\, b \in B_1} (a(b) + T^*(b)).$$

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